The attached code models a massive spring pendulum system. The mass of the spring can be specified. Motion is constrained to a plane, the spring is assumed to not bend, and damping is neglected.

Define the origin as the point where the pendulum is attached to the support. The generalized coordinates are r and θ or, letting $r = l + x$ where l is the rest length of the spring, the generalized coordinates are x and θ. Also note $\dot{r} = \dot{x}$. The total Lagrangian is Lagrangian of the mass plus that of the spring.

For the mass, the Lagrangian is given by

$$\begin{aligned} L_{mass} &= T - V && (1)\\ &= \frac{1}{2}Mv^2 + Mg(l+x)\cos(\theta) - \frac{1}{2}kx^2 && (2)\\ &= \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2)^2 + Mg(l+x)\cos(\theta) - \frac{1}{2}kx^2 && (3)\\ &= \frac{1}{2}M(\dot{x}^2 + (l+x)^2\dot{\theta}^2)^2 + Mg(l+x)\cos(\theta) - \frac{1}{2}kx^2 && (4) \end{aligned}$$

To find the Lagrangian for the spring, first find the kinetic energy:

$$T = \int_m \frac{1}{2}u^2 dm \tag{5}$$

Where u is the velocity of a differential element of mass along the spring. For a uniform spring, $dm = m\left(\frac{dy}{L}\right)$. Performing a change of variables yields:

$$\begin{aligned} T &= \int_0^L \frac{1}{2}u^2 m\left(\frac{dy}{L}\right) && (6)\\ &= \frac{1}{2}\frac{m}{L}\int_0^L u^2 dy && (7) \end{aligned}$$

At the point where the pendulum is attached, u=0. At the other end, u=v, the velocity of the mass that's attached. Assuming the spring doesn't bend like a slinky, the velocity of each differential element is proportional to it's distance from where the pendulum is attached, $u = \frac{vy}{L}$. Therefore,

$$\begin{aligned} T &= \frac{1}{2}\frac{m}{L}\int_0^L \left(\frac{vy}{L}\right)^2 dy && (8)\\ &= \frac{1}{2}\frac{m}{3}v^2 && (9) \end{aligned}$$

Also, knowing the internal potential energy of the spring is given by $-\frac{1}{2}kx^2$, the

Lagrangian is given by:

$$L_{spring} = \frac{1}{2}\frac{m}{3}v^2 + mg\frac{(l+x)}{2}\cos(\theta) \tag{10}$$

$$= \frac{1}{2}\frac{m}{3}(\dot{x}^2 + (l+x)^2\dot{\theta}^2)^2 + mg\frac{(l+x)}{2}\cos(\theta) \tag{11}$$

Letting $L_{total} = L_{mass} + L_{spring}$ and plugging into the Euler-Lagrange Equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$, the following equations of motion are obtained:

$$\ddot{x} = (l+x)\dot{\theta}^2 + \left(\frac{m/2+M}{m/3+M}\right) g\cos(\theta) - \left(\frac{1}{m/3+M}\right) kx \tag{12}$$

$$\ddot{\theta} = -\frac{1}{(l+x)}\left[2\dot{\theta}\dot{x} + g\left(\frac{m/2+M}{m/3+M}\right) g\sin(\theta)\right] \tag{13}$$

These are the equations used in the Python SciPy ODE solver used to generate the animation.