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Force, torque, and absorbed energy for a body of arbitrary shape and constitution in an electromagnetic radiation field

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Abstract

The force and torque exerted on a body of arbitrary shape and constitution by a stationary radiation field are in principle given by integrals of Minkowski's stress tensor over a surface surrounding the body. Similarly the absorbed energy is given by an integral of the Poynting vector. These integrals are notoriously difficult to evaluate, and so far only spherical bodies have been considered. It is shown here that the integrals may be cast into a simpler form by use of Debye potentials. General expressions for the integrals are derived as sums of bilinear expressions in the coefficients of the expansion of the incident and scattered waves in terms of vector spherical waves. The expressions are simplified for small particles, such as atoms, for which the electric dipole approximation may be used. It is shown that the calculation is also relevant for bodies with nonlinear electromagnetic response.

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1. Introduction

The force exerted on a neutral polarizable particle by a strongly focused laser beam can be sufficiently strong for trapping and manipulation [1-3]. Many experiments involving trapping of atoms and larger particles have been performed [4]. Additional laser beams can be used for cooling of trapped atoms [5-7]. A calculation of the electromagnetic force on a polarizable particle is therefore of great practical relevance.

In the evaluation of the force a geometrical optics approximation can be used for large bodies [8,9], but for particles with diameter of the order of the wavelength or

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smaller a complete electromagnetic calculation is required. Debye first performed such a calculation for a uniform dielectric sphere placed in a plane wave radiation field [10]. His calculation was generalized by Barton et al. [11] to a uniform sphere in an arbitrary stationary radiation field. Their derivation was sufficiently complicated that no details were given, and only the final results for force and torque were presented. Recently we have checked [12] the calculation using known integrals of products of vector spherical harmonics over the unit sphere [13]. In the following we show that the calculation can be simplified considerably by use of Debye potentials [10,14]. This leads to simpler integrals involving only the scalar Debye potentials.

We derive expressions for force, torque, and absorbed energy in terms of spherical wave expansion coefficients. For force and torque the expressions are identical in form to those derived by Barton et al. [11], apart from correction of minor misprints. However, the calculation shows that the expressions have more general validity, and apply to bodies of arbitrary shape and constitution. The spherical wave scattering coefficients incorporate the response of the body to the incident wave.

In Section 9 we specialize to bodies of size sufficiently small that the electric dipole approximation may be used. This leads to concise expressions for force and torque in terms of the incident field and the induced electric dipole moment. The present derivation differs conceptually from earlier ones [5,15–18].

In Section 10 we show that the calculation is relevant also for bodies with nonlinear electromagnetic response. A spectral decomposition of the time-integrated force and torque leads to the same integrals. The same is true for the absorbed energy. Thus in principle the dynamical effect of a laser pulse on a body of arbitrary shape and constitution can be calculated.

2. Force, torque, and absorbed energy

We consider a uniform medium with dielectric constant $\epsilon_1(\omega)$ and magnetic permeability $\mu_1(\omega)$ at frequency ω surrounding a body of arbitrary shape and constitution, centered at the origin. A stationary radiation field, oscillating at frequency ω, is incident on the body, and is partly scattered, partly absorbed. It is convenient to use complex notation, and drop a common time factor $\exp(-i\omega t)$. The incident electric and magnetic fields $\boldsymbol{E}^i_\omega(\boldsymbol{r})$ and $\boldsymbol{H}^i_\omega(\boldsymbol{r})$ satisfy the homogeneous Maxwell equations

$$\begin{aligned} &\nabla\cdot\boldsymbol{E}^i_\omega = 0, \qquad \nabla\times\boldsymbol{E}^i_\omega = i\mu_1 k\boldsymbol{H}^i_\omega, \\ &\nabla\cdot\boldsymbol{H}^i_\omega = 0, \qquad \nabla\times\boldsymbol{H}^i_\omega = -i\epsilon_1 k\boldsymbol{E}^i_\omega, \end{aligned} \tag{2.1}$$

everywhere in space. Here $k=\omega/c$, with c the velocity of light. We use Gaussian units. For simplicity of notation we shall omit the subscript ω henceforth. We assume that the uniform medium is transparent at the frequency considered, so that waves propagate without damping, and $\epsilon_1(\omega)$ and $\mu_1(\omega)$ are positive.

The total fields in the presence of the body can be expressed as

$$E(r) = E^i(r) + E^s(r), \quad H(r) = H^i(r) + H^s(r), \tag{2.2}$$

with scattering fields $E^s(r)$ and $H^s(r)$ defined everywhere in space. The fields E and H induce an electric polarization $P(r)$ and magnetization $M(r)$. For a conducting body the free charge and current density are included in the polarization. The dielectric displacement $D(r)$ and magnetic induction $B(r)$ are defined by

$$D = E + 4\pi P, \quad B = H + 4\pi M. \tag{2.3}$$

Maxwell's equations read

$$\begin{aligned} \nabla \cdot D &= 0, \quad \nabla \times E = ikB, \\ \nabla \cdot B &= 0, \quad \nabla \times H = -ikD. \end{aligned} \tag{2.4}$$

These must be supplemented with constitutive equations relating P and M to E and H. We denote the volume of the body as V_0, and the complementary space as $\overline{V}$. The constitutive equations may be nonlocal, but by definition they reduce to

$$D = \epsilon_1 E, \quad B = \mu_1 H, \quad r \in \overline{V}, \tag{2.5}$$

outside the body. A solution of the complete set of equations is considered for which the scattering fields $E^s(r)$ and $H^s(r)$ satisfy an outgoing wave condition at large distance from the body.

The time-averaged electromagnetic force and torque exerted on the body may be evaluated from an integral of the Minkowski tensor over a surface surrounding the body [19,20]. We choose a spherical surface S of radius R, centered at the origin O, and surrounding the body. We average over a period $2\pi/\omega$. The time-averaged force may be expressed as

$$\begin{aligned} \overline{F} = \frac{\epsilon_1}{8\pi}\mathrm{Re} \int_S \left[\left(E^* \cdot e_r\right) E - \tfrac{1}{2} e_r |E|^2 \right] dS \\ + \frac{\mu_1}{8\pi}\mathrm{Re} \int_S \left[\left(H^* \cdot e_r\right) H - \tfrac{1}{2} e_r |H|^2 \right] dS, \end{aligned} \tag{2.6}$$

where e_r is the unit vector in the radial direction. Similarly, the time-averaged torque may be expressed as

$$\begin{aligned} \overline{N} = \frac{\epsilon_1}{8\pi}\mathrm{Re} \int_S \left(E^* \cdot e_r\right) (r \times E)\, dS \\ + \frac{\mu_1}{8\pi}\mathrm{Re} \int_S \left(H^* \cdot e_r\right) (r \times H)\, dS. \end{aligned} \tag{2.7}$$

We consider also the absorbed energy. The time-averaged Poynting-vector is

$$\overline{S} = \frac{c}{8\pi}\mathrm{Re}\, E^* \times H. \tag{2.8}$$

The absorbed energy is therefore

$$W^a = -\frac{c}{8\pi}\mathrm{Re}\int_S \left(\boldsymbol{E}^* \times \boldsymbol{H}\right)\cdot \boldsymbol{e}_r\, dS. \tag{2.9}$$

It follows from conservation of momentum, angular momentum, and energy in the uniform medium that the time-averaged force $\overline{\boldsymbol{F}}$, torque $\overline{\boldsymbol{N}}$, and absorbed energy W^a are independent of the radius R of the spherical surface, provided the surface completely surrounds the body. These quantities are therefore calculated conveniently in the limit $R \to \infty$, where the fields become simple.

3. Extinction and scattering

The decomposition (2.2) of the fields into an incident and scattered wave leads to a similar decomposition of force, torque, and absorbed energy. It is clear that in the expressions (2.6), (2.7), and (2.9) the contributions bilinear in the incident fields vanish, by conservation of momentum, angular momentum, and energy in the uniform medium. Correspondingly we have for the force

$$\overline{\boldsymbol{F}} = \overline{\boldsymbol{F}}^{\mathrm{ext}} - \overline{\boldsymbol{F}}^{\mathrm{sca}}, \tag{3.1}$$

where the extinction term $\overline{\boldsymbol{F}}^{\mathrm{ext}}$ consists of the cross terms of the incident and scattered fields, and $\overline{\boldsymbol{F}}^{\mathrm{sca}}$ is bilinear in the scattered fields. Explicitly the extinction term is given by

$$\begin{aligned}\overline{\boldsymbol{F}}^{\mathrm{ext}} = {} & \frac{\epsilon_1}{8\pi}\mathrm{Re}\int_S \left[\left(\boldsymbol{E}^{i^*}\cdot \boldsymbol{e}_r\right)\boldsymbol{E}^s + \left(\boldsymbol{E}^{s^*}\cdot \boldsymbol{e}_r\right)\boldsymbol{E}^i - \tfrac{1}{2}\boldsymbol{e}_r\left(\boldsymbol{E}^{i^*}\cdot \boldsymbol{E}^s + \boldsymbol{E}^{s^*}\cdot \boldsymbol{E}^i\right)\right] dS \\ & + \frac{\mu_1}{8\pi}\mathrm{Re}\int_S \left[\left(\boldsymbol{H}^{i^*}\cdot \boldsymbol{e}_r\right)\boldsymbol{H}^s + \left(\boldsymbol{H}^{s^*}\cdot \boldsymbol{e}_r\right)\boldsymbol{H}^i - \tfrac{1}{2}\boldsymbol{e}_r\left(\boldsymbol{H}^{i^*}\cdot \boldsymbol{H}^s + \boldsymbol{H}^{s^*}\cdot \boldsymbol{H}^i\right)\right] dS,\end{aligned} \tag{3.2}$$

and the scattering term is given by

$$\begin{aligned}\overline{\boldsymbol{F}}^{\mathrm{sca}} = {} & -\frac{\epsilon_1}{8\pi}\mathrm{Re}\int_S \left[\left(\boldsymbol{E}^{s^*}\cdot \boldsymbol{e}_r\right)\boldsymbol{E}^s - \tfrac{1}{2}\boldsymbol{e}_r\left|\boldsymbol{E}^s\right|^2\right] dS \\ & -\frac{\mu_1}{8\pi}\mathrm{Re}\int_S \left[\left(\boldsymbol{H}^{s^*}\cdot \boldsymbol{e}_r\right)\boldsymbol{H}^s - \tfrac{1}{2}\boldsymbol{e}_r\left|\boldsymbol{H}^s\right|^2\right] dS.\end{aligned} \tag{3.3}$$

Similarly, the torque can be expressed as

$$\overline{\boldsymbol{N}} = \overline{\boldsymbol{N}}^{\mathrm{ext}} - \overline{\boldsymbol{N}}^{\mathrm{sca}}, \tag{3.4}$$

with the extinction term

$$\overline{N}^{\text{ext}} = \frac{\epsilon_1}{8\pi}\text{Re}\int_S \left[\left(E^{i^*}\cdot e_r\right)\left(r\times E^s\right) + \left(E^{s^*}\cdot e_r\right)\left(r\times E^i\right)\right]dS$$
$$+\frac{\mu_1}{8\pi}\text{Re}\int_S \left[\left(H^{i^*}\cdot e_r\right)\left(r\times H^s\right) + \left(H^{s^*}\cdot e_r\right)\left(r\times H^i\right)\right]dS, \qquad (3.5)$$

and the scattering term

$$\overline{N}^{\text{sca}} = -\frac{\epsilon_1}{8\pi}\text{Re}\int_S \left(E^{s^*}\cdot e_r\right)\left(r\times E^s\right)dS$$
$$-\frac{\mu_1}{8\pi}\text{Re}\int_S \left(H^{s^*}\cdot e_r\right)\left(r\times H^s\right)dS. \qquad (3.6)$$

We can also write the absorbed energy as [21]

$$W^a = W^{\text{ext}} - W^{\text{sca}}, \qquad (3.7)$$

with the extinction term

$$W^{\text{ext}} = -\frac{c}{8\pi}\text{Re}\int_S \left(E^{i^*}\times H^s + E^{s^*}\times H^i\right)\cdot e_r\, dS, \qquad (3.8)$$

and the scattering term

$$W^{\text{sca}} = \frac{c}{8\pi}\text{Re}\int_S \left(E^{s^*}\times H^s\right)\cdot e_r\, dS. \qquad (3.9)$$

The extinction terms in Eqs. (3.1), (3.4), and (3.7) correspond to the total loss of momentum, angular momentum, and energy from the incident wave upon interaction with the body. The scattering terms correspond to the electromagnetic momentum, angular momentum, and energy that are carried away by the scattered wave. The difference of the two terms corresponds to absorption by the body, and gives force, torque, and absorbed energy.

4. Debye potentials

The various surface integrals occurring in the preceding section can be evaluated by expansion of the fields in vector spherical harmonics [22]. However, because of the vector character the calculation is cumbersome. We show in the following that it can be simplified considerably by expressing the fields in terms of scalar Debye potentials [10,14]. The electric and magnetic fields are separated into two contributions

$$E = E_e + E_h, \quad H = H_e + H_h, \qquad (4.1)$$

with E_e and E_h defined by

$$E_e = \nabla\times\left[\nabla\times(r\psi_e)\right], \quad E_h = i\mu_1 k\nabla\times(r\psi_h), \qquad (4.2)$$

and $\boldsymbol{H}_e$ and $\boldsymbol{H}_h$ defined by

$$\boldsymbol{H}_e = -i\epsilon_1 k\nabla \times (\boldsymbol{r}\psi_e), \quad \boldsymbol{H}_h = \nabla \times \left[\nabla \times (\boldsymbol{r}\psi_h)\right], \tag{4.3}$$

with potentials ψ_e and ψ_h that satisfy the scalar wave equation

$$\nabla^2\psi_{e,h} + k_1^2\psi_{e,h} = 0, \quad \boldsymbol{r} \in \overline{V}, \tag{4.4}$$

with wavenumber $k_1 = \sqrt{\epsilon_1\mu_1}k$. The fields $\boldsymbol{E}_h$ and $\boldsymbol{H}_e$ have no radial component. The fields $\boldsymbol{E}_e$ and $\boldsymbol{E}_h$ can be expressed as

$$\begin{aligned} \boldsymbol{E}_e &= k_1^2\boldsymbol{r}\psi_e + \nabla\left[\psi_e + (\boldsymbol{r}\cdot\nabla)\psi_e\right], \\ \boldsymbol{E}_h &= -ik\mu_1\boldsymbol{r} \times \nabla\psi_h. \end{aligned} \tag{4.5}$$

The fields $\boldsymbol{H}_e$ and $\boldsymbol{H}_h$ can be expressed as

$$\begin{aligned} \boldsymbol{H}_e &= i\epsilon_1 k\boldsymbol{r} \times \nabla\psi_e, \\ \boldsymbol{H}_h &= k_1^2\boldsymbol{r}\psi_h + \nabla\left[\psi_h + (\boldsymbol{r}\cdot\nabla)\psi_h\right]. \end{aligned} \tag{4.6}$$

The potentials ψ_e, ψ_h are identical to the conventionally defined [23] potentials ${}^e\Pi$, ${}^m\Pi$. The incident and scattered fields can be decomposed correspondingly with potentials ψ_e^i, ψ_h^i and ψ_e^s, ψ_h^s. It has been shown by Nisbet [24] how the scattering potentials ψ_e^s and ψ_h^s can be expressed as integrals of the current distribution in the body.

It will be useful to introduce the gradient operator ∇_t that differentiates tangentially to a spherical surface of radius r. In spherical coordinates (r, θ, φ),

$$\nabla_t = \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{4.7}$$

We also introduce the angular momentum operator

$$\boldsymbol{L} = -i\boldsymbol{r} \times \nabla_t = i\boldsymbol{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - i\boldsymbol{e}_\varphi \frac{\partial}{\partial\theta}, \tag{4.8}$$

and the notation

$$\hat{\psi} = \psi + (\boldsymbol{r}\cdot\nabla)\psi. \tag{4.9}$$

With these definitions the fields $\boldsymbol{E}_e$ and $\boldsymbol{E}_h$ in Eq. (4.2) can be rewritten as

$$\boldsymbol{E}_e = \nabla_t\hat{\psi}_e + \frac{\boldsymbol{r}}{r^2}L^2\psi_e, \quad \boldsymbol{E}_h = \mu_1 k\boldsymbol{L}\psi_h, \tag{4.10}$$

without use of the wave equation (4.4). In the same way the fields $\boldsymbol{H}_e$ and $\boldsymbol{H}_h$ in Eq. (4.3) can be expressed as

$$\boldsymbol{H}_e = -\epsilon_1 k\boldsymbol{L}\psi_e, \quad \boldsymbol{H}_h = \nabla_t\hat{\psi}_h + \frac{\boldsymbol{r}}{r^2}L^2\psi_h. \tag{4.11}$$

The identities (4.10) and (4.11) are useful in the manipulation of the surface integrals.

As Debye noted [10], his potentials are not unique, since one can find nonvanishing functions for which the electric and magnetic fields vanish identically. In the following we make the same choice as Debye and require the potentials to be regular functions of the spatial coordinates, possessing a convergent expansion in spherical harmonics. It has been suggested by Stratton [14] that the use of Debye potentials is restricted to spherical coordinates. However, this is not the case, as shown for example by the work of Meixner [25].

5. Surface integrals

In this section we derive a number of integral identities that are useful in the calculation of force, torque, and absorbed energy. The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are decomposed as in Eq. (4.1), and expressed in terms of Debye potentials ψ_e and ψ_h by Eqs. (4.10) and (4.11). To see how the integrals can be transformed, consider first the second term in the first integral in Eq. (2.6) for a field $\boldsymbol{E}_h$. Thus we study

$$\boldsymbol{P}^E_{hh}(R) = \int_S \boldsymbol{r}\,|\boldsymbol{E}_h|^2\,dS = \mu_1^2 k^2 \int_S \boldsymbol{r}(\boldsymbol{L}\psi_h)^* \cdot \boldsymbol{L}\psi_h dS. \tag{5.1}$$

Using the fact that $\boldsymbol{L}$ is Hermitian we transform the integral on the right to

$$\int_S \boldsymbol{r}(\boldsymbol{L}\psi_h)^* \cdot \boldsymbol{L}\psi_h dS = \sum_\alpha \int_S \psi_h^* L_\alpha \boldsymbol{r} L_\alpha \psi_h dS. \tag{5.2}$$

The linear operator appearing here can be expressed as

$$\sum_\alpha L_\alpha \boldsymbol{r} L_\alpha = \boldsymbol{r}L^2 - r^2\nabla_t, \tag{5.3}$$

so that

$$\int_S \boldsymbol{r}(\boldsymbol{L}\psi_h)^* \cdot \boldsymbol{L}\psi_h dS = \int_S \boldsymbol{r}\psi_h^* L^2 \psi_h dS - R^2 \int_S \psi_h^* \nabla_t \psi_h dS. \tag{5.4}$$

The real part of the last integral can be transformed to

$$\mathrm{Re} \int_S \psi_h^* \nabla_t \psi_h dS = \frac{1}{R^2} \int_S |\psi_h|^2\,\boldsymbol{r}\,dS. \tag{5.5}$$

We therefore obtain

$$\boldsymbol{P}^E_{hh}(R) = \mu_1^2 k^2 \mathrm{Re} \int_S \boldsymbol{r}\psi_h^* (L^2 - 1)\psi_h dS. \tag{5.6}$$

This integral is relatively easy to do by expansion of $\psi_h(\boldsymbol{r})$ in spherical harmonics.

Next we consider the same type of integral for the field $\boldsymbol{E}_e$,

$$\boldsymbol{P}^E_{ee}(R) = \int_S \boldsymbol{r}\,|\boldsymbol{E}_e|^2\,dS. \tag{5.7}$$

From Eq. (4.10) we find

$$|\boldsymbol{E}_e|^2 = \nabla_t\hat{\psi}_e^* \cdot \nabla_t\hat{\psi}_e + \frac{1}{r^2}(L^2\psi_e^*)(L^2\psi_e). \tag{5.8}$$

In the limit $R \to \infty$ the second term can be omitted from the integral. The Debye potentials fall off as $1/r$, but $\hat{\psi}$ is of order unity for large r, because of the operator $\boldsymbol{r}\cdot\nabla$ in Eq. (4.9). Hence the second term in Eq. (5.8) decays a factor $1/r^2$ faster than the first. We therefore find

$$\lim_{R\to\infty} \frac{1}{R}\boldsymbol{P}_{ee}^E(R) = \lim_{R\to\infty} \frac{1}{R^3}\mathrm{Re}\int_S \boldsymbol{r}\hat{\psi}_e^*(L^2-1)\hat{\psi}_e dS, \tag{5.9}$$

by use of the analysis from Eq. (5.2) to Eq. (5.6).

Next we consider the cross term

$$\boldsymbol{P}_{eh}^E(R) = \int_S \boldsymbol{r}\boldsymbol{E}_e^* \cdot \boldsymbol{E}_h dS. \tag{5.10}$$

Substituting Eq. (4.10) we find after an integration by parts,

$$\boldsymbol{P}_{eh}^E(R) = \mu_1 k \int_S \boldsymbol{r}\left(\nabla_t\hat{\psi}_e^*\right)\cdot \boldsymbol{L}\psi_h dS = -\mu_1 k \int_S \hat{\psi}_e^* \boldsymbol{L}\psi_h dS, \tag{5.11}$$

where we have used $\nabla_t \cdot \boldsymbol{L} = 0$. The other cross term is found from the relation

$$\boldsymbol{P}_{he}^E(R) = \left[\boldsymbol{P}_{eh}^E(R)\right]^*. \tag{5.12}$$

The second term in the second integral in Eq. (2.6) can be handled in the same way as above, since by Eqs. (4.10) and (4.11) we need merely interchange the potentials ψ_e and ψ_h, apart from trivial constant factors. The first term in the two integrals in Eq. (2.6) does not contribute in the limit $R \to \infty$. In the scalar product $\boldsymbol{E}\cdot\boldsymbol{e}_r$ we can replace $\boldsymbol{E}$ by $\boldsymbol{E}_e$, since $\boldsymbol{E}_h$ is tangential. Therefore consider the two integrals

$$\boldsymbol{Q}_{ee}^E(R) = \int_S \left(\boldsymbol{E}_e^*\cdot\boldsymbol{r}\right)\boldsymbol{E}_e dS, \quad \boldsymbol{Q}_{eh}^E(R) = \int_S \left(\boldsymbol{E}_e^*\cdot\boldsymbol{r}\right)\boldsymbol{E}_h dS. \tag{5.13}$$

The first integral becomes in terms of the Debye potential ψ_e,

$$\boldsymbol{Q}_{ee}^E(R) = \int_S (L^2\psi_e)^*\left(\nabla_t\hat{\psi}_e + \boldsymbol{r}\frac{L^2}{r^2}\psi_e\right)dS. \tag{5.14}$$

This tends to a constant vector as $R \to \infty$. Hence

$$\lim_{R\to\infty} \frac{1}{R}\boldsymbol{Q}_{ee}^E(R) = 0. \tag{5.15}$$

The second integral can be written

$$\boldsymbol{Q}_{eh}^E(R) = \mu_1 k \int_S (L^2\psi_e)^* \boldsymbol{L}\psi_h dS. \tag{5.16}$$

This also tends to a constant vector, so that

$$\lim_{R\to\infty} \frac{1}{R} \boldsymbol{Q}^E_{eh}(R) = 0. \tag{5.17}$$

The integral with the magnetic field $\boldsymbol{H}$ in Eq. (2.6) can be transformed the same way.

For the calculation of the torque in Eq. (2.7) we must consider the integrals

$$\boldsymbol{T}^E_{ee}(R) = \int_S \left(\boldsymbol{E}^*_e \cdot \boldsymbol{r}\right) (\boldsymbol{r} \times \boldsymbol{E}_e) dS, \quad \boldsymbol{T}^E_{eh}(R) = \int_S \left(\boldsymbol{E}^*_e \cdot \boldsymbol{r}\right) (\boldsymbol{r} \times \boldsymbol{E}_h) dS. \tag{5.18}$$

Using Eq. (4.10) again we find for the first integral

$$\boldsymbol{T}^E_{ee}(R) = i \int_S (L^2\psi_e)^* \boldsymbol{L}\hat{\psi}_e dS. \tag{5.19}$$

The second integral yields

$$\boldsymbol{T}^E_{eh}(R) = i\mu_1 k R^2 \int_S (L^2\psi_e)^* \nabla_t \psi_h dS. \tag{5.20}$$

The integral with the magnetic field in Eq. (2.7) can be transformed the same way. It is easy to show that

$$\mu_1 \mathrm{Re}\, \boldsymbol{T}^H_{he}(R) = -\epsilon_1 \mathrm{Re}\, \boldsymbol{T}^E_{eh}(R). \tag{5.21}$$

Finally we consider the absorbed energy given by Eq. (2.9). The first integral that occurs is

$$U_{ee}(R) = \int_S \left(\boldsymbol{E}^*_e \times \boldsymbol{H}_e\right) \cdot \boldsymbol{r} dS. \tag{5.22}$$

This can be written as

$$U_{ee}(R) = -\epsilon_1 k \int_S \left(\nabla_t \hat{\psi}^*_e \times \boldsymbol{L}\psi_e\right) \cdot \boldsymbol{r} dS = -i\epsilon_1 k \int_S (\boldsymbol{L}\hat{\psi}^*_e) \cdot (\boldsymbol{L}\psi_e) dS, \tag{5.23}$$

or equivalently

$$U_{ee}(R) = i\epsilon_1 k \int_S \hat{\psi}^*_e L^2 \psi_e dS. \tag{5.24}$$

A second integral to consider is

$$U_{eh}(R) = \int_S \left(\boldsymbol{E}^*_e \times \boldsymbol{H}_h\right) \cdot \boldsymbol{r} dS. \tag{5.25}$$

This becomes

$$U_{eh}(R) = \int_S \left(\nabla_t\hat{\psi}_e^* \times \nabla_t\hat{\psi}_h\right) \cdot \boldsymbol{r} dS = -i \int_S \hat{\psi}_e^* \boldsymbol{L} \cdot \left(\nabla_t\hat{\psi}_h\right) dS. \tag{5.26}$$

This vanishes since $\boldsymbol{L} \cdot \nabla_t = 0$. The third integral is

$$U_{he}(R) = \int_S \left(\boldsymbol{E}_h^* \times \boldsymbol{H}_e\right) \cdot \boldsymbol{r} dS. \tag{5.27}$$

This becomes

$$U_{he}(R) = -k_1^2 \int_S \left[(\boldsymbol{L}\psi_h)^* \times \boldsymbol{L}\psi_e\right] \cdot \boldsymbol{r} dS = k_1^2 R^2 \int_S \left(\nabla_t\psi_h^* \times \nabla_t\psi_e\right) \cdot \boldsymbol{r} dS. \tag{5.28}$$

This vanishes for the same reason as the integral in Eq. (5.26). The last integral is

$$U_{hh}(R) = \int_S \left(\boldsymbol{E}_h^* \times \boldsymbol{H}_h\right) \cdot \boldsymbol{r} dS. \tag{5.29}$$

This can be written as

$$U_{hh}(R) = \mu_1 k \int_S \left[(\boldsymbol{L}\psi_h)^* \times \nabla_t\hat{\psi}_h\right] \cdot \boldsymbol{r} dS = -i\mu_1 k \int_S \psi_h^* L^2 \hat{\psi}_h dS \tag{5.30}$$

by the same transformation that leads to Eq. (5.24).

6. Expressions for force, torque, and absorbed energy

We collect results and find expressions for force, torque, and absorbed energy in terms of the Debye potentials. By use of the asymptotic behavior at large distance the integrals can be transformed into integrals over solid angle of bilinear expressions in the angular amplitude of the potentials.

For the time-averaged force we find from Eq. (2.6) and the integrals in the preceding section,

$$\overline{\boldsymbol{F}} = \overline{\boldsymbol{F}}_{ee} + \overline{\boldsymbol{F}}_{eh} + \overline{\boldsymbol{F}}_{hh}, \tag{6.1}$$

with the contributions

$$\begin{aligned}
\overline{\boldsymbol{F}}_{ee} &= -\frac{1}{16\pi} \lim_{R\to\infty} \frac{1}{R} \left[\epsilon_1 \boldsymbol{P}_{ee}^E(R) + \mu_1 \boldsymbol{P}_{ee}^H(R)\right], \\
\overline{\boldsymbol{F}}_{eh} &= -\frac{1}{8\pi} \lim_{R\to\infty} \operatorname{Re} \frac{1}{R} \left[\epsilon_1 \boldsymbol{P}_{eh}^E(R) + \mu_1 \boldsymbol{P}_{eh}^H(R)\right], \\
\overline{\boldsymbol{F}}_{hh} &= -\frac{1}{16\pi} \lim_{R\to\infty} \left[\epsilon_1 \boldsymbol{P}_{hh}^E(R) + \mu_1 \boldsymbol{P}_{hh}^H(R)\right].
\end{aligned} \tag{6.2}$$

The Debye potentials ψ_e and ψ_h behave asymptotically as

$$\psi(\boldsymbol{r}) \approx f_S(\hat{\boldsymbol{r}}) \frac{\sin k_1 r}{k_1 r} + f_C(\hat{\boldsymbol{r}}) \frac{\cos k_1 r}{k_1 r} \quad \text{as} \quad r \to \infty, \tag{6.3}$$

with complex amplitudes $f_S(\hat{\boldsymbol{r}})$ and $f_C(\hat{\boldsymbol{r}})$ that depend only on angle. Correspondingly the potentials $\hat{\psi}_e$ and $\hat{\psi}_h$ defined in Eq. (4.9) behave asymptotically as

$$\hat{\psi}(\boldsymbol{r}) \approx f_S(\hat{\boldsymbol{r}}) \cos k_1 r - f_C(\hat{\boldsymbol{r}}) \sin k_1 r. \tag{6.4}$$

By use of Eq. (5.7) for $\boldsymbol{P}^E_{ee}(R)$ and the analogue of Eq. (5.4) for $\boldsymbol{P}^H_{ee}(R)$ we find that asymptotically for large R the integrand in the integral for $\overline{\boldsymbol{F}}_{ee}$ behaves as a constant. We obtain

$$\overline{\boldsymbol{F}}_{ee} = -\frac{\epsilon_1}{16\pi}\mathrm{Re}\int \left[f^*_{eS}(L^2-1)f_{eS} + f^*_{eC}(L^2-1)f_{eC}\right]\boldsymbol{e}_r d\Omega, \tag{6.5}$$

where the integral is over solid angle $d\Omega = \sin\theta d\theta d\varphi$. By the same analysis

$$\overline{\boldsymbol{F}}_{eh} = \frac{\sqrt{\epsilon_1\mu_1}}{8\pi}\mathrm{Re}\int \left(f^*_{eS}\boldsymbol{L}f_{hC} - f^*_{eC}\boldsymbol{L}f_{hS}\right)d\Omega. \tag{6.6}$$

The third contribution to the force is

$$\overline{\boldsymbol{F}}_{hh} = -\frac{\mu_1}{16\pi}\mathrm{Re}\int \left[f^*_{hS}(L^2-1)f_{hS} + f^*_{hC}(L^2-1)f_{hC}\right]\boldsymbol{e}_r d\Omega. \tag{6.7}$$

In the same way the time-averaged torque can be expressed as

$$\overline{\boldsymbol{N}} = \overline{\boldsymbol{N}}_{ee} + \overline{\boldsymbol{N}}_{hh}. \tag{6.8}$$

It follows from Eq. (5.21) that the cross term $\overline{\boldsymbol{N}}_{eh}$ vanishes. The first contribution in Eq. (6.8) is given by

$$\overline{\boldsymbol{N}}_{ee} = \frac{\epsilon_1}{8\pi k_1}\mathrm{Im}\int f^*_{eS}\boldsymbol{L}L^2 f_{eC} d\Omega, \tag{6.9}$$

and the second by

$$\overline{\boldsymbol{N}}_{hh} = \frac{\mu_1}{8\pi k_1}\mathrm{Im}\int f^*_{hS}\boldsymbol{L}L^2 f_{hC} d\Omega. \tag{6.10}$$

The absorbed energy can be decomposed as

$$W^a = W^a_{ee} + W^a_{hh}, \tag{6.11}$$

with first contribution

$$W^a_{ee} = \frac{c}{8\pi}\sqrt{\frac{\epsilon_1}{\mu_1}}\mathrm{Im}\int f^*_{eS}L^2 f_{eC} d\Omega, \tag{6.12}$$

and second contribution

$$W^a_{hh} = \frac{c}{8\pi}\sqrt{\frac{\mu_1}{\epsilon_1}}\mathrm{Im}\int f^*_{hS}L^2 f_{hC} d\Omega. \tag{6.13}$$

Thus all required integrals can be expressed as integrals of bilinear forms in the asymptotic angular amplitudes of the Debye potentials.

7. Expansion in spherical waves

The angular integrals derived in the preceding section can be evaluated by expansion of the asymptotic amplitudes in spherical harmonics. Each of the amplitudes has an expansion of the form

$$f(\hat{\boldsymbol{r}}) = \sum_{\ell m} c_{\ell m} Y_{\ell m}(\theta, \varphi), \tag{7.1}$$

with spherical harmonics $Y_{\ell m}$ in the notation of Edmonds [22].

The Debye potentials of the incident wave can be expanded as

$$\psi^i_\sigma(\boldsymbol{r}) = \sum_{\ell m} c^i_{\sigma\ell m} j_\ell(k_1 r) Y_{\ell m}(\theta, \varphi), \quad \sigma = e, h. \tag{7.2}$$

The sum starts at $\ell = 1$. Using the asymptotic behavior of the spherical Bessel function $j_\ell(k_1 r)$ we find

$$\begin{aligned} c^i_{\sigma S\ell m} &= \tfrac{1}{2} i^\ell \left(1 + (-1)^\ell\right) c^i_{\sigma\ell m}, \\ c^i_{\sigma C\ell m} &= \tfrac{1}{2} i^{\ell+1} \left(1 - (-1)^\ell\right) c^i_{\sigma\ell m}, \quad \sigma = e, h. \end{aligned} \tag{7.3}$$

This shows that the sine amplitude $f^i_{\sigma S}(\hat{\boldsymbol{r}})$ has even parity, and the cosine amplitude $f^i_{\sigma C}(\hat{\boldsymbol{r}})$ has odd parity,

$$f^i_{\sigma S}(-\hat{\boldsymbol{r}}) = f^i_{\sigma S}(\hat{\boldsymbol{r}}), \quad f^i_{\sigma C}(-\hat{\boldsymbol{r}}) = -f^i_{\sigma C}(\hat{\boldsymbol{r}}), \tag{7.4}$$

for both e and h polarization. Hence all integrals in the preceding section vanish for the incident wave. The integrals can be written as the difference of an extinction and a scattering contribution, as shown in Section 3. The scattered wave can be decomposed as

$$\psi^s_\sigma(\boldsymbol{r}) = \sum_{\ell m} c^s_{\sigma\ell m} h^{(1)}_\ell(k_1 r) Y_{\ell m}(\theta, \varphi), \quad r > R_0, \tag{7.5}$$

where R_0 is the radius of the smallest sphere centered at O that surrounds the body. From the asymptotic behavior of the spherical Bessel function $h^{(1)}_\ell(k_1 r)$ we find

$$c^s_{\sigma S\ell m} = i\, c^s_{\sigma C\ell m} = i^{-\ell} c^s_{\sigma\ell m}, \quad \sigma = e, h. \tag{7.6}$$

The scalar integrals for the absorbed energy are the simplest. Thus we find from Eq. (6.12),

$$W^a_{ee} = \frac{c}{8\pi}\sqrt{\frac{\epsilon_1}{\mu_1}} \operatorname{Im} \sum_{\ell m} \ell(\ell+1) c^*_{eS\ell m} c_{eC\ell m}. \tag{7.7}$$

From Eqs. (7.3) and (7.6) we find that the extinction contribution is given by

$$W^{\text{ext}}_{ee} = -\frac{c}{8\pi}\sqrt{\frac{\epsilon_1}{\mu_1}} \sum_{\ell m} \ell(\ell+1) \operatorname{Re} c^{i*}_{e\ell m} c^s_{e\ell m}, \tag{7.8}$$

and the scattering contribution by

$$W_{ee}^{\rm sca} = \frac{c}{8\pi}\sqrt{\frac{\epsilon_1}{\mu_1}}\sum_{\ell m}\ell(\ell+1)\,|c_{e\ell m}^{s}|^2. \tag{7.9}$$

Similarly, we find from Eq. (6.13) the extinction contribution

$$W_{hh}^{\rm ext} = -\frac{c}{8\pi}\sqrt{\frac{\mu_1}{\epsilon_1}}\sum_{\ell m}\ell(\ell+1){\rm Re}\,c_{h\ell m}^{i*}c_{h\ell m}^{s}, \tag{7.10}$$

and the scattering contribution

$$W_{hh}^{\rm sca} = \frac{c}{8\pi}\sqrt{\frac{\mu_1}{\epsilon_1}}\sum_{\ell m}\ell(\ell+1)\,|c_{h\ell m}^{s}|^2. \tag{7.11}$$

The expressions for force and torque can be converted in the same way into bilinear forms of the spherical wave coefficients. Two basic integrals over spherical harmonics occur repeatedly. The first is

$$\int Y_{\ell m}^{*}\boldsymbol{e}_q\cdot\boldsymbol{e}_r Y_{\ell' m'}\,d\Omega = R_q(\ell m,\ell' m'),\quad q=0,\pm 1, \tag{7.12}$$

where $\boldsymbol{e}_q$ is a spherical unit vector in standard notation [22]. The integral can be expressed in terms of Wigner $3j$-symbols, and is given by [13]

$$R_q(\ell m,\ell' m') = (-1)^m\sqrt{(2\ell+1)(2\ell'+1)}\begin{pmatrix}\ell & \ell' & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell & \ell' & 1\\ -m & m' & q\end{pmatrix}. \tag{7.13}$$

The second basic integral is

$$\int Y_{\ell m}^{*}\boldsymbol{e}_q\cdot\boldsymbol{L}Y_{\ell' m'}\,d\Omega = L_q(\ell m,\ell' m'),\quad q=0,\pm 1. \tag{7.14}$$

The integral is given by

$$L_q(\ell m,\ell' m') = L_q(\ell m m')\delta_{\ell\ell'}, \tag{7.15}$$

with factor

$$L_q(\ell m m') = (-1)^{\ell+m+1}\sqrt{\ell(\ell+1)(2\ell+1)}\begin{pmatrix}\ell & \ell & 1\\ -m & m' & q\end{pmatrix}. \tag{7.16}$$

In the appendix we list the values of the $3j$-symbols that occur.

By the same method as before we find for the contribution $\overline{\boldsymbol{F}}_{\sigma\sigma}^{\rm ext}$ to the force

$$\overline{\boldsymbol{F}}_{\sigma\sigma}^{\rm ext} = -\frac{\epsilon_\sigma}{16\pi}{\rm Re}\sum_{\substack{\ell m\\ \ell' m'}} i^{\ell-\ell'}(\ell'^2+\ell'-1)\left(c_{\sigma\ell m}^{i*}c_{\sigma\ell' m'}^{s}+c_{\sigma\ell m}^{s*}c_{\sigma\ell' m'}^{i}\right)$$
$$\times R_q(\ell m,\ell' m')\boldsymbol{e}_q^{*}, \tag{7.17}$$

where $\epsilon_e\equiv\epsilon_1$ and $\epsilon_h\equiv\mu_1$. The corresponding scattering contribution is

$$\overline{\boldsymbol{F}}_{\sigma\sigma}^{\rm sca} = \frac{\epsilon_\sigma}{8\pi}{\rm Re}\sum_{\substack{\ell m\\ \ell' m'}} i^{\ell-\ell'}(\ell'^2+\ell'-1)c_{\sigma\ell m}^{s*}c_{\sigma\ell m}^{s}R_q(\ell m,\ell' m')\boldsymbol{e}_q^{*}. \tag{7.18}$$

From the cross terms between electric and magnetic Debye potentials we find the extinction contribution

$$\overline{\boldsymbol{F}}_{eh}^{\rm ext} = \frac{\sqrt{\epsilon_1\mu_1}}{8\pi}{\rm Im}\sum_{\ell mm'}\left(c^{i*}_{e\ell m}c^{s}_{h\ell m'} + c^{s*}_{e\ell m}c^{i}_{h\ell m'}\right)L_q(\ell mm')\boldsymbol{e}_q^*, \tag{7.19}$$

and the scattering contribution

$$\overline{\boldsymbol{F}}_{eh}^{\rm sca} = -\frac{\sqrt{\epsilon_1\mu_1}}{4\pi}{\rm Im}\sum_{\ell mm'}c^{s*}_{e\ell m}c^{s}_{h\ell m'}L_q(\ell mm')\boldsymbol{e}_q^*. \tag{7.20}$$

Finally we get corresponding expressions for the torque. An extinction contribution

$$\overline{\boldsymbol{N}}_{\sigma\sigma}^{\rm ext} = \frac{-\epsilon_\sigma}{16\pi k_1}{\rm Re}\sum_{\ell mm'}\ell(\ell+1)\left[(1+(-1)^\ell)c^{i*}_{\sigma\ell m}c^{s}_{\sigma\ell m'} + (1-(-1)^\ell)c^{s*}_{\sigma\ell m}c^{i}_{\sigma\ell m'}\right]$$
$$\times L_q(\ell mm')\boldsymbol{e}_q^*, \tag{7.21}$$

and a scattering contribution

$$\overline{\boldsymbol{N}}_{\sigma\sigma}^{\rm sca} = \frac{\epsilon_\sigma}{8\pi k_1}{\rm Re}\sum_{\ell mm'}\ell(\ell+1)c^{s*}_{\sigma\ell m}c^{s}_{\sigma\ell m'}L_q(\ell mm')\boldsymbol{e}_q^*. \tag{7.22}$$

The extinction contribution can be written more concisely by use of the relations

$$L_1(\ell,m+1,m) = L_{-1}(\ell,m,m+1) = \frac{-1}{\sqrt{2}}\sqrt{(\ell-m)(\ell+m+1)},$$
$$L_0(\ell,m,m) = m, \tag{7.23}$$

and the fact that all other coefficients $L_q(\ell,m,m')$ vanish. Hence the terms with $(-1)^\ell$ in Eq. (7.21) cancel, and we find

$$\overline{\boldsymbol{N}}_{\sigma\sigma}^{\rm ext} = \frac{-\epsilon_\sigma}{16\pi k_1}{\rm Re}\sum_{\ell mm'}\ell(\ell+1)\left[c^{i*}_{\sigma\ell m}c^{s}_{\sigma\ell m'} + c^{s*}_{\sigma\ell m}c^{i}_{\sigma\ell m'}\right]L_q(\ell mm')\boldsymbol{e}_q^*. \tag{7.24}$$

For completeness, and for ease of comparison with results derived earlier for a spherical particle [11], we present explicit expressions for the vector components of force and torque. Using the expression Eq. (7.12) and the values of the $3j$-symbols listed in the appendix we find for the x, y components of the force contribution in Eq. (7.17),

$$\overline{F}_{\sigma\sigma x}^{\rm ext} + i\overline{F}_{\sigma\sigma y}^{\rm ext} = i\frac{\epsilon_\sigma}{16\pi}\sum_{\ell m}\frac{\ell(\ell+2)}{\sqrt{(2\ell+1)(2\ell+3)}}$$
$$\times\left[\sqrt{(\ell+m+1)(\ell+m+2)}\left(c^{i*}_{\sigma\ell+1,m+1}\,c^{s}_{\sigma\ell m} + c^{s*}_{\sigma\ell+1,m+1}\,c^{i}_{\sigma\ell m}\right)\right.$$
$$\left.+\sqrt{(\ell-m+1)(\ell-m+2)}\left(c^{i*}_{\sigma\ell m}\,c^{s}_{\sigma\ell+1,m-1} + c^{s*}_{\sigma\ell m}\,c^{i}_{\sigma\ell+1,m-1}\right)\right], \tag{7.25}$$

and for the z component,

$$\overline{F}^{\text{ext}}_{\sigma\sigma z} = -\frac{\epsilon_\sigma}{8\pi}\text{Im}\sum_{\ell m}\ell(\ell+2)\sqrt{\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}}$$
$$\times\left(c^{i*}_{\sigma\ell m}\,c^{s}_{\sigma\ell+1,m}+c^{s*}_{\sigma\ell m}\,c^{i}_{\sigma\ell+1,m}\right). \tag{7.26}$$

For the cross term in Eq. (7.19) we find the x, y components

$$\overline{F}^{\text{ext}}_{ehx}+i\overline{F}^{\text{ext}}_{ehy} = i\frac{\sqrt{\epsilon_1\mu_1}}{16\pi}\sum_{\ell m}\sqrt{(\ell-m)(\ell+m+1)}$$
$$\times\left[c^{s*}_{h\ell,m+1}\,c^{i}_{e\ell m}+c^{i*}_{h\ell,m+1}\,c^{s}_{e\ell m}-c^{s*}_{e\ell,m+1}\,c^{i}_{h\ell m}-c^{i*}_{e\ell,m+1}\,c^{s}_{h\ell m}\right], \tag{7.27}$$

and the z component

$$\overline{F}^{\text{ext}}_{ehz} = \frac{\sqrt{\epsilon_1\mu_1}}{8\pi}\text{Im}\sum_{\ell m} m\left(c^{i*}_{e\ell m}\,c^{s}_{h\ell m}+c^{s*}_{e\ell m}\,c^{i}_{h\ell m}\right). \tag{7.28}$$

For the x, y components of the torque in Eq. (7.24) we find

$$\overline{N}^{\text{ext}}_{\sigma\sigma x}+i\overline{N}^{\text{ext}}_{\sigma\sigma y} = -\frac{\epsilon_\sigma}{16\pi k_1}\sum_{\ell m}\ell(\ell+1)\sqrt{(\ell-m)(\ell+m+1)}$$
$$\times\left(c^{i*}_{\sigma\ell,m+1}\,c^{s}_{\sigma\ell m}+c^{s*}_{\sigma\ell,m+1}\,c^{i}_{\sigma\ell m}\right), \tag{7.29}$$

and for the z component

$$\overline{N}^{\text{ext}}_{\sigma\sigma z} = -\frac{\epsilon_\sigma}{8\pi k_1}\text{Re}\sum_{\ell m}\ell(\ell+1)m\,c^{i*}_{\sigma\ell m}\,c^{s}_{\sigma\ell m}. \tag{7.30}$$

One obtains the Cartesian components of the scattering contributions $\overline{F}^{\text{sca}}_{\sigma\sigma}$, $\overline{F}^{\text{sca}}_{eh}$, and $\overline{N}^{\text{sca}}_{\sigma\sigma}$ from the expressions in Eqs. (7.25)-(7.30) by replacing the superscript i by s and multiplying by -1.

8. Comparison

The expressions obtained in the preceding section for the Cartesian components of force and torque can be compared directly with corresponding expressions derived by Barton et al. [11] for the case of a spherical particle. Generalizing their expansions of the incident and scattered wave to the situation considered here, where the ambient medium has magnetic permeability μ_1 rather than unity, we get the relations

$$\begin{aligned} c^{i}_{e\ell m} &= \alpha a\,A_{\ell m}\,E_0, & c^{i}_{h\ell m} &= \alpha a\,B_{\ell m}\,E_0,\\ c^{s}_{e\ell m} &= \alpha a\,a_{\ell m}\,E_0, & c^{s}_{h\ell m} &= \alpha a\,b_{\ell m}\,E_0, \end{aligned} \tag{8.1}$$

where a is the radius of the spherical particle, $\alpha = k_1 a$, and E_0 is an electric field amplitude characteristic of the incident beam. Substituting these relations into the expressions (7.25)-(7.28) for the extinction contribution to the force and into the corresponding expressions for the scattering contribution we get a precise agreement with Eqs. (5), (6) of Barton et al., except for one obvious misprint in their first line. Similarly, our expressions for the torque in Eqs. (7.29) and (7.30) agree precisely with Eqs. (10)–(12) of Barton et al. [11], except that in their Eq. (12) a factor ℓ is missing. We note that the standard scattering coefficients $\{a_\ell, b_\ell\}$ for a uniform sphere with permeabilities ϵ_2, μ_2 are related to the coefficients in Eq. (8.1) by

$$c^s_{e\ell m} = -a_\ell\, c^i_{e\ell m}, \quad c^s_{h\ell m} = -b_\ell\, c^i_{h\ell m}. \tag{8.2}$$

The size parameter in expression Eq. (4.56) of Bohren and Huffman [21] is $x = k_1 a$, and the complex refractive index is $m = \sqrt{\epsilon_2\mu_2/\epsilon_1\mu_1}$.

We also relate the expansion of the Debye potentials in spherical harmonics to the expansion of the electric and magnetic fields in vector spherical waves. The incident radiation field can be expanded as [14,26]

$$\begin{aligned} \boldsymbol{E}^i(\boldsymbol{r}) &= \sum_{\ell m}\left[c^+_{1\ell m}\boldsymbol{M}^+_{\ell m}(k_1,\boldsymbol{r}) + c^+_{2\ell m}\boldsymbol{N}^+_{\ell m}(k_1,\boldsymbol{r})\right], \\ \boldsymbol{H}^i(\boldsymbol{r}) &= -i\sqrt{\frac{\epsilon_1}{\mu_1}}\sum_{\ell m}\left[c^+_{1\ell m}\boldsymbol{N}^+_{\ell m}(k_1,\boldsymbol{r}) + c^+_{2\ell m}\boldsymbol{M}^+_{\ell m}(k_1,\boldsymbol{r})\right], \end{aligned} \tag{8.3}$$

and the scattered wave can be expanded as

$$\begin{aligned} \boldsymbol{E}^s(\boldsymbol{r}) &= \sum_{\ell m}\left[c^-_{1\ell m}\boldsymbol{M}^-_{\ell m}(k_1,\boldsymbol{r}) + c^-_{2\ell m}\boldsymbol{N}^-_{\ell m}(k_1,\boldsymbol{r})\right], \\ \boldsymbol{H}^s(\boldsymbol{r}) &= -i\sqrt{\frac{\epsilon_1}{\mu_1}}\sum_{\ell m}\left[c^-_{1\ell m}\boldsymbol{N}^-_{\ell m}(k_1,\boldsymbol{r}) + c^-_{2\ell m}\boldsymbol{M}^-_{\ell m}(k_1,\boldsymbol{r})\right]. \end{aligned} \tag{8.4}$$

With vector spherical waves defined as in Ref. [26] we find the coefficient relations

$$\begin{aligned} c^+_{1\ell m} &= i\mu_1 k\, c^i_{h\ell m}, \quad & c^+_{2\ell m} &= k_1\, c^i_{e\ell m}, \\ c^-_{1\ell m} &= i\mu_1 k\, c^s_{h\ell m}, \quad & c^-_{2\ell m} &= k_1\, c^s_{e\ell m}. \end{aligned} \tag{8.5}$$

In particular this yields expressions for force, torque, and absorbed energy for a spherical particle in an incident plane wave by use of the known plane wave expansion [26,27].

The scattering coefficients $\{c^-_{1\ell m}, c^-_{2\ell m}\}$ can be expressed as integrals of the polarization induced in the body. We define the additional polarization $\boldsymbol{P}'(\boldsymbol{r})$ as

$$\boldsymbol{P}'(\boldsymbol{r}) = \boldsymbol{P}(\boldsymbol{r}) - \frac{\epsilon_1 - 1}{4\pi}\boldsymbol{E}(\boldsymbol{r}). \tag{8.6}$$

In contrast to the polarization $\boldsymbol{P}(\boldsymbol{r})$ this vanishes outside the body. The scattered electric field for $r > R_0$ can be expressed with the retarded Green function in terms of the additional polarization,

$$\boldsymbol{E}^s(\boldsymbol{r}) = \int \mathsf{G}_>(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{P}'(\boldsymbol{r}')d\boldsymbol{r}', \quad r > R_0. \tag{8.7}$$

For $r > r'$ the Green function has the expansion [26]

$$\mathsf{G}_>(\boldsymbol{r}-\boldsymbol{r}') = \frac{4\pi i}{\epsilon_1} k_1^3 \sum_{\ell m} \left[\boldsymbol{M}_{\ell m}^-(k_1,\boldsymbol{r})\boldsymbol{M}_{\ell m}^+(k_1,\boldsymbol{r}')^* + \boldsymbol{N}_{\ell m}^-(k_1,\boldsymbol{r})\boldsymbol{N}_{\ell m}^+(k_1,\boldsymbol{r}')^* \right]. \tag{8.8}$$

We have used that in our case k_1 is real. Substituting into Eq. (8.7) we find the expansion in Eq. (8.4) with coefficients

$$\begin{aligned} c_{1\ell m}^- &= \frac{4\pi i}{\epsilon_1} k_1^3 \frac{1}{\ell(\ell+1)} \int \boldsymbol{M}_{\ell m}^+(k_1,\boldsymbol{r})^* \cdot \boldsymbol{P}'(\boldsymbol{r}) d\boldsymbol{r}, \\ c_{2\ell m}^- &= \frac{4\pi i}{\epsilon_1} k_1^3 \frac{1}{\ell(\ell+1)} \int \boldsymbol{N}_{\ell m}^+(k_1,\boldsymbol{r})^* \cdot \boldsymbol{P}'(\boldsymbol{r}) d\boldsymbol{r}. \end{aligned} \tag{8.9}$$

Thus the scattering coefficients may be regarded as multipole expansion coefficients of the additional polarization $\boldsymbol{P}'(\boldsymbol{r})$. For $\epsilon_1 = \mu_1 = 1$ the expressions reduce to those derived by Bouwkamp and Casimir [28], and by Nisbet [24].

9. Electric dipole approximation

In this section we consider a body of size much smaller than the wavelength of the incident light, such that the scattered field can be approximated by the electric dipole contribution. The induced electric dipole moment is given by

$$\boldsymbol{d} = \int \boldsymbol{P}'(\boldsymbol{r}) d\boldsymbol{r}. \tag{9.1}$$

The force and torque on the dipole, as well as the absorbed energy, can be expressed in terms of the dipole moment and the incident electric field and its gradient at the position of the dipole. The incident electric and magnetic fields can be expanded as

$$\begin{aligned} \boldsymbol{E}^i(\boldsymbol{r}) &= \boldsymbol{E}^i(\boldsymbol{0}) + (\boldsymbol{r}\cdot\nabla)\boldsymbol{E}^i|_0 + O(r^2), \\ \boldsymbol{H}^i(\boldsymbol{r}) &= \boldsymbol{H}^i(\boldsymbol{0}) + (\boldsymbol{r}\cdot\nabla)\boldsymbol{H}^i|_0 + O(r^2). \end{aligned} \tag{9.2}$$

This can be decomposed into e and h polarizations as

$$\boldsymbol{E}_e^i(\boldsymbol{r}) = \boldsymbol{E}^i(\boldsymbol{0}) + \overset{\sqcap}{\mathsf{T}}{}_e^i \cdot \boldsymbol{r} + O(r^2), \quad \boldsymbol{E}_h^i(\boldsymbol{r}) = -\tfrac{1}{2} i\mu_1 k\boldsymbol{r} \times \boldsymbol{H}^i(\boldsymbol{0}) + O(r^2), \tag{9.3}$$

$$\boldsymbol{H}_e^i(\boldsymbol{r}) = \tfrac{1}{2} i\epsilon_1 k\boldsymbol{r} \times \boldsymbol{E}^i(\boldsymbol{0}) + O(r^2), \quad \boldsymbol{H}_h^i(\boldsymbol{r}) = \boldsymbol{H}^i(\boldsymbol{0}) + \overset{\sqcap}{\mathsf{T}}{}_h^i \cdot \boldsymbol{r} + O(r^2), \tag{9.4}$$

where $\overset{\sqcap}{\mathsf{T}}{}_e^i$ and $\overset{\sqcap}{\mathsf{T}}{}_h^i$ are the symmetric parts of the field gradient tensors

$$\mathsf{T}_e^i = (\nabla \boldsymbol{E}^i)|_0, \quad \mathsf{T}_h^i = (\nabla \boldsymbol{H}^i)|_0. \tag{9.5}$$

These tensors are automatically traceless because of Eqs. (2.1). The corresponding expansion of the Debye potentials reads

$$\psi_e^i(\boldsymbol{r}) = \tfrac{1}{2}\boldsymbol{E}^i(\boldsymbol{0})\cdot\boldsymbol{r} + \tfrac{1}{6}\overset{\sqcap}{\mathsf{T}}{}_e^i : \boldsymbol{r}\boldsymbol{r} + O(r^3),$$

$$\psi_h^i(\boldsymbol{r}) = \tfrac{1}{2}\boldsymbol{H}^i(\boldsymbol{0}) \cdot \boldsymbol{r} + \tfrac{1}{6}\overleftrightarrow{T}{}_h^i : \boldsymbol{r}\boldsymbol{r} + O(r^3). \tag{9.6}$$

The incident potentials vanish at the origin, since the sum in Eq. (7.2) starts at $\ell = 1$. From the expansion Eq. (9.6) one can easily express the coefficients c_{e1m}^i, c_{e2m}^i, c_{h1m}^i, c_{h2m}^i in terms of the fields $\boldsymbol{E}^i(\boldsymbol{0})$, $\boldsymbol{H}^i(\boldsymbol{0})$, and the tensors $\overleftrightarrow{T}{}_e^i$, $\overleftrightarrow{T}{}_h^i$. For e polarization the coefficients are

$$\begin{aligned}
c_{e10}^i &= \frac{1}{k_1}\sqrt{3\pi}E_z^i(\boldsymbol{0}), \\
c_{e1\pm1}^i &= \mp\frac{1}{k_1}\sqrt{\tfrac{3}{2}\pi}\left[E_x^i(\boldsymbol{0}) \mp iE_y^i(\boldsymbol{0})\right], \\
c_{e20}^i &= \frac{1}{k_1^2}\sqrt{5\pi}\,\overleftrightarrow{T}{}_{ezz}^i, \\
c_{e2\pm1} &= \mp\frac{1}{k_1^2}\sqrt{\tfrac{10}{3}\pi}\left[\overleftrightarrow{T}{}_{exz}^i \mp i\,\overleftrightarrow{T}{}_{eyz}^i\right], \\
c_{e2\pm2}^i &= \frac{1}{k_1^2}\sqrt{\tfrac{10}{3}\pi}\left[\overleftrightarrow{T}{}_{exx}^i \mp i\,\overleftrightarrow{T}{}_{exy}^i + \tfrac{1}{2}\overleftrightarrow{T}{}_{ezz}^i\right].
\end{aligned} \tag{9.7}$$

For the h polarization one finds the same expressions for c_{h1m}^i, c_{h2m}^i with $\boldsymbol{E}^i(0)$ replaced by $\boldsymbol{H}^i(\boldsymbol{0})$, and $\overleftrightarrow{T}{}_e^i$ replaced by $\overleftrightarrow{T}{}_h^i$.

For a body of small size we can take the limit $k_1 \to 0$ in the integrals in Eq. (8.9). In this limit the electric dipole moment dominates. The corresponding scattering coefficients are given by

$$c_{e1m}^s = \sqrt{\tfrac{4}{3}\pi}i\mu_1 k^2 d_m, \tag{9.8}$$

where d_m are the spherical components of the dipole

$$d_{\pm1} = \mp\frac{1}{\sqrt{2}}(d_x \mp id_y), \quad d_0 = d_z. \tag{9.9}$$

By substitution into Eqs. (7.25) and (7.26) one finds the extinction force

$$\overline{\boldsymbol{F}}_{ee}^{\text{ext}} = \tfrac{1}{2}\text{Re}\,\boldsymbol{d}\cdot\overleftrightarrow{T}{}_e^{i*}. \tag{9.10}$$

From Eq. (7.27) one finds the force

$$\overline{\boldsymbol{F}}_{eh}^{\text{ext}} = \tfrac{1}{4}\mu_1 k\text{Im}\,\boldsymbol{d} \times \boldsymbol{H}^i(\boldsymbol{0})^*. \tag{9.11}$$

The corresponding contributions to the scattering force vanish. Therefore one finds for the total force in electric dipole approximation,

$$\overline{\boldsymbol{F}} = \overline{\boldsymbol{F}}_{ee}^{\text{ext}} + \overline{\boldsymbol{F}}_{eh}^{\text{ext}}. \tag{9.12}$$

Similarly we find from Eqs. (7.29) and (7.30) in electric dipole approximation

$$\overline{\boldsymbol{N}}_{ee}^{\text{ext}} = \tfrac{1}{2}\text{Re}\,\boldsymbol{d} \times \boldsymbol{E}^i(\boldsymbol{0})^*. \tag{9.13}$$

The corresponding scattering torque is

$$\overline{N}^{\rm sca}_{ee} = \tfrac{2}{3}\sqrt{\epsilon_1\mu_1}\mu_1 k^3 \mathrm{Im}\, \boldsymbol{d}^* \times \boldsymbol{d}. \tag{9.14}$$

The total torque in the electric dipole approximation is the difference $\overline{N} = \overline{N}^{\rm ext}_{ee} - \overline{N}^{\rm sca}_{ee}$.

From Eq. (7.8) we find for the extinction contribution to the absorbed energy,

$$W^{\rm ext}_{ee} = \tfrac{1}{4}\omega \mathrm{Im}\, \boldsymbol{d} \cdot \boldsymbol{E}^i(\boldsymbol{0})^*. \tag{9.15}$$

From Eq. (7.9) we find the scattering contribution

$$W^{\rm sca}_{ee} = \tfrac{1}{3}\omega\sqrt{\epsilon_1\mu_1}\mu_1 k^3 |\boldsymbol{d}|^2. \tag{9.16}$$

The total absorbed energy in electric dipole approximation is the difference $W^a = W^{\rm ext}_{ee} - W^{\rm sca}_{ee}$.

By use of the field equations (2.1) the force contribution in Eq. (9.10) can be expressed as

$$\overline{\boldsymbol{F}}^{\rm ext}_{ee} = \overline{\boldsymbol{F}}_d + \overline{\boldsymbol{F}}^{\rm ext}_{eh}, \tag{9.17}$$

where $\overline{\boldsymbol{F}}_d$ is the electric dipole force

$$\overline{\boldsymbol{F}}_d = \tfrac{1}{2}\mathrm{Re}\,(\boldsymbol{d}\cdot\nabla)\boldsymbol{E}^{i^*}|_0. \tag{9.18}$$

The total force in Eq. (9.12) is given by

$$\overline{\boldsymbol{F}} = \overline{\boldsymbol{F}}_d + \overline{\boldsymbol{F}}_L, \tag{9.19}$$

where $\overline{\boldsymbol{F}}_L$ is the Lorentz force

$$\overline{\boldsymbol{F}}_L = 2\overline{\boldsymbol{F}}^{\rm ext}_{eh}. \tag{9.20}$$

The total force can also be expressed as

$$\overline{\boldsymbol{F}} = \tfrac{1}{2}\mathrm{Re}\,(\nabla\boldsymbol{E}^i)^*|_0 \cdot \boldsymbol{d}. \tag{9.21}$$

It is of interest to consider the special case of an isotropic particle with scalar polarizability $\alpha(\omega)$, defined by

$$\boldsymbol{d} = \alpha(\omega)\boldsymbol{E}^i(\boldsymbol{0}). \tag{9.22}$$

For this case the total force is

$$\overline{\boldsymbol{F}} = \tfrac{1}{4}[\mathrm{Re}\,\alpha(\omega)]\nabla|\boldsymbol{E}^i|^2|_0 + \tfrac{1}{2}[\mathrm{Im}\,\alpha(\omega)]\mathrm{Im}\,(\nabla\boldsymbol{E}^i)\cdot\boldsymbol{E}^{i^*}|_0. \tag{9.23}$$

The first term is the gradient of the ponderomotive potential [18]

$$\Phi_p(\boldsymbol{r}) = -\tfrac{1}{4}[\mathrm{Re}\,\alpha(\omega)]|\boldsymbol{E}^i(\boldsymbol{r})|^2, \tag{9.24}$$

taken at the particle center. The second term becomes important near resonance. For a plane wave the first term vanishes, so that the force calculated by Debye [10] corresponds to the second term. For a plane wave this term is identical with the Lorentz force

$$\overline{\boldsymbol{F}}_L = \tfrac{1}{2}|\boldsymbol{E}^i(\boldsymbol{0})|^2 \mathrm{Im}\,\alpha(\omega)\cdot \boldsymbol{k}_1. \tag{9.25}$$

For a linearly polarized plane wave the torque, given by Eqs. (9.13) and (9.14), vanishes. For a circularly polarized plane wave $\boldsymbol{E}^i(\boldsymbol{r}) = E_0\boldsymbol{e}_1 \exp(ik_1 z)$ with real amplitude E_0 the torque is

$$\overline{\boldsymbol{N}} = E_0^2\left[\tfrac{1}{2}\mathrm{Im}\,\alpha(\omega) - \tfrac{2}{3}\sqrt{\epsilon_1\mu_1}\mu_1 k^3|\alpha(\omega)|^2\right]\boldsymbol{e}_z. \tag{9.26}$$

This can be understood as being due to absorption and scattering of circularly polarized photons. A circularly polarized wave also exerts an extinction torque on a transparent gyrotropic medium. This was first demonstrated experimentally by Beth [29] and by Holbourn [30].

10. Nonlinear response

So far we have considered a stationary radiation field and a body with linear electromagnetic properties. We show in this section that our calculation is also relevant for an incident radiation field with arbitrary time-dependence and a body with nonlinear electromagnetic response. This allows us, for example, to calculate the momentum and angular momentum imparted to a nonlinear body by a laser pulse. We limit our considerations to the case where the external space is vacuum, so that $\epsilon_1 = 1$, $\mu_1 = 1$ at any frequency. It is assumed that the electromagnetic response of the body to a stationary incident radiation field at any frequency is known, so that the outgoing wave can in principle be calculated.

At time t the radiation field exerts a force $\boldsymbol{F}(t)$ and a torque $\boldsymbol{N}(t)$ on the body, as defined from integrals of the Maxwell stress tensor over a surface S_b immediately surrounding the body. The total momentum imparted to the body is the time integral of the force

$$\Delta\boldsymbol{P} = \int_{-\infty}^{\infty} \boldsymbol{F}(t)\,dt. \tag{10.1}$$

The total imparted angular momentum is the time integral of the torque

$$\Delta\boldsymbol{J} = \int_{-\infty}^{\infty} \boldsymbol{N}(t)\,dt. \tag{10.2}$$

Both $\boldsymbol{F}(t)$ and $\boldsymbol{N}(t)$ are given by integrals over S_b of a bilinear expression in the electromagnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$. The Fourier transforms of the fields are

$$\boldsymbol{E}_\omega(\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},t)e^{i\omega t}\,d\omega,$$

$$H_\omega(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(r,t)e^{i\omega t}d\omega. \tag{10.3}$$

By Parseval's theorem [31] the imparted momentum and angular momentum can be expressed as

$$\Delta P = 4\pi\int_0^\infty \overline{F}(\omega)d\omega, \quad \Delta J = 4\pi\int_0^\infty \overline{N}(\omega)d\omega, \tag{10.4}$$

where $\overline{F}(\omega)$ and $\overline{N}(\omega)$ are given by Eqs. (2.6) and (2.7) with $\epsilon_1 = \mu_1 = 1$. The surface integrals may be extended over the surface S instead of S_b, because of conservation of electromagnetic momentum and angular momentum in vacuum, and the fact that for large negative and positive times there is zero electromagnetic radiation in the volume enclosed by S_b and S. We note that we consider only the coherent radiation field generated by external sources of finite duration. The thermal radiation field and spontaneous emission from the body do not contribute to this field.

By the same argument the total energy absorbed by the body is given by

$$\Delta W = 4\pi\int_0^\infty W^a(\omega)d\omega, \tag{10.5}$$

where $4\pi W^a(\omega)$ is the absorbed energy per frequency interval given by Eq. (2.9) with the Fourier transforms $E_\omega(r)$ and $H_\omega(r)$. We extend the integral only over positive frequencies, as in Eq. (10.4).

The spectral components $\overline{F}(\omega)$, $\overline{N}(\omega)$, and $W^a(\omega)$ in Eqs. (10.4) and (10.5) may be expressed as before in terms of a spherical wave decomposition. The linear or nonlinear response of the body is assumed known, so that in principle the spherical wave scattering coefficients can be expressed in terms of the coefficients of the incident wave. Spontaneous emission contributes to the response. Its influence can be seen in the natural linewidth of spectral lines.

11. Discussion

We have shown that the force, torque, and absorbed energy for a body with linear electromagnetic response in a stationary radiation field can be calculated by means of an expansion in spherical waves. Our expressions for force and torque are identical with those found by Barton et al. [11] for a uniform isotropic sphere, apart from correction of minor misprints. This shows that in their calculation the particular shape and nature of the body were not used. Thus the expressions are valid for bodies of any shape and constitution. In particular, the body may have nonlocal material properties. The electromagnetic response may even be nonlinear, as shown in Section 10.

In Section 9 we have considered bodies sufficiently small in comparison with the wavelength of light that the electric dipole approximation applies. In this case the lengthy expansions in spherical waves simplify. The calculation is particularly relevant for the trapping force on atoms [5-7]. The force, torque, and absorbed energy are expressed in terms of the induced dipole moment. The latter must be found from a separate quantummechanical calculation.

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Appendix A

In this appendix we list the values of the Wigner $3j$-symbols that have been used in the calculation. These are

$$\begin{pmatrix} \ell & \ell-1 & 1 \\ -m & m' & 1 \end{pmatrix} = (-1)^{\ell+m} \left[\frac{(\ell+m-1)(\ell+m)}{2\ell(2\ell-1)(2\ell+1)}\right]^{1/2} \delta_{m',m-1}, \tag{A.1}$$

$$\begin{pmatrix} \ell & \ell & 1 \\ -m & m' & 1 \end{pmatrix} = (-1)^{\ell+m} \left[\frac{(\ell+m)(\ell-m+1)}{2\ell(\ell+1)(2\ell+1)}\right]^{1/2} \delta_{m',m-1}, \tag{A.2}$$

$$\begin{pmatrix} \ell & \ell+1 & 1 \\ -m & m' & 1 \end{pmatrix} = (-1)^{\ell+m} \left[\frac{(\ell-m+1)(\ell-m+2)}{2(\ell+1)(2\ell+1)(2\ell+3)}\right]^{1/2} \delta_{m',m-1}, \tag{A.3}$$

$$\begin{pmatrix} \ell & \ell-1 & 1 \\ -m & m' & 0 \end{pmatrix} = (-1)^{\ell+m} \left[\frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)(2\ell+1)}\right]^{1/2} \delta_{m',m}, \tag{A.4}$$

$$\begin{pmatrix} \ell & \ell & 1 \\ -m & m' & 0 \end{pmatrix} = (-1)^{\ell+m+1} \left[\frac{m^2}{\ell(\ell+1)(2\ell+1)}\right]^{1/2} \delta_{m',m}, \tag{A.5}$$

$$\begin{pmatrix} \ell & \ell+1 & 1 \\ -m & m' & 0 \end{pmatrix} = (-1)^{\ell+m+1} \left[\frac{(\ell+m+1)(\ell-m+1)}{(\ell+1)(2\ell+1)(2\ell+3)}\right]^{1/2} \delta_{m',m}, \tag{A.6}$$

$$\begin{pmatrix} \ell & \ell-1 & 1 \\ -m & m' & -1 \end{pmatrix} = (-1)^{\ell+m} \left[\frac{(\ell-m-1)(\ell-m)}{2\ell(2\ell-1)(2\ell+1)}\right]^{1/2} \delta_{m',m+1}, \tag{A.7}$$

$$\begin{pmatrix} \ell & \ell & 1 \\ -m & m' & -1 \end{pmatrix} = (-1)^{\ell+m+1} \left[\frac{(\ell-m)(\ell+m+1)}{2\ell(\ell+1)(2\ell+1)}\right]^{1/2} \delta_{m',m+1}, \tag{A.8}$$

$$\begin{pmatrix} \ell & \ell+1 & 1 \\ -m & m' & -1 \end{pmatrix} = (-1)^{\ell+m} \left[\frac{(\ell+m+1)(\ell+m+2)}{2(\ell+1)(2\ell+1)(2\ell+3)}\right]^{1/2} \delta_{m',m+1}, \tag{A.9}$$

as follows from Table 2 in Ref. 22.

References

[1] A. Ashkin, Phys. Rev. Lett. 24 (1970) 156.
[2] A. Ashkin, Science 210 (1980) 1081.
[3] E. Arimondo, W.D. Phillips and F. Strumia, eds., Laser Manipulation of Atoms and Ions, Proceedings Summer School Varenna, 1991 (North-Holland, Amsterdam, 1992).
[4] S. Chu, Scientific American (February 1992) 49.
[5] S. Stenholm, Rev. Mod. Phys. 58 (1986) 699.
[6] C.N. Cohen-Tannoudji and W.D. Phillips, Physics Today (October 1990) 33.
[7] C.N. Cohen-Tannoudji, Phys. Blätt. 51 (1995) 91.
[8] R. Gussgard, T. Lindmo and I. Brevik, J. Opt. Soc. Am. B9 (1992) 1922.
[9] A. Ashkin, Biophys. J. 61 (1992) 569.
[10] P. Debye, Ann. Phys. 30 (1909) 57.
[11] J.P. Barton, D.R. Alexander and S.A. Schaub, J. Appl. Phys. 66 (1989) 4594.
[12] Ø. Farsund, Diploma thesis N.T.H. Trondheim (1994).
[13] B.U. Felderhof and R.B. Jones, Physica A 202 (1994) 119.
[14] J.A. Stratton, Electromagnetic Theory (McGraw-Hill, New York, 1941).
[15] J.P. Gordon and A. Ashkin, Phys. Rev. A 21 (1980) 1606.
[16] V.S. Letokhov and V.G. Minogin, Physics Reports 73 (1981) 1.
[17] P. Mulser, J. Opt. Soc. Am. B 2 (1985) 1814.
[18] P. Mulser, S. Uryupin, R. Sauerbrey and B. Wellegehausen, Phys. Rev. A 48 (1993) 4547.
[19] B.U. Felderhof, J. Phys. C 10 (1977) 4605.
[20] I.H. Brevik, Physics Reports 52 (1979) 133.
[21] C.F. Bohren and D.R. Huffman, Absorption and Scattering of Light by Small Particles (Wiley, New York, 1983).
[22] A.R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton Univ. Press, Princeton, 1957).
[23] M. Born and E. Wolf, Principles of Optics (Pergamon, Oxford, 1975) p. 638.
[24] A. Nisbet, Physica 21 (1955) 799.
[25] J. Meixner, Z. Naturf. 1 (1946) 496; 3 (1948) 506.
[26] B. Ersfeld and B.U. Felderhof, Physica A 173 (1991) 561.
[27] R.G. Newton, Scattering Theory of Waves and Particles (McGraw-Hill, New York, 1966) p. 39.
[28] C.J. Bouwkamp and H.B.G. Casimir, Physica 20 (1954) 539.
[29] R.A. Beth, Phys. Rev. 50 (1936) 115.
[30] A.H.S. Holbourn, Nature 137 (1936) 31.
[31] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis (Cambridge Univ. Press, Cambridge, 1958) p. 182.