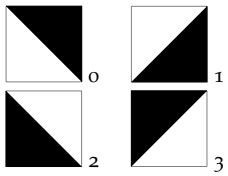
TRUCHET

4×4 patterns with four-fold rotational symmetry

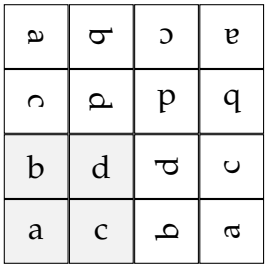
Current printing, February 15, 2025

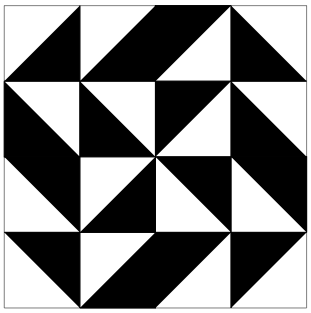
Introduction

Traditionally, Truchet tiles are square tiles that are divided by a diagonal line, and coloured with two colours with a different colour on either side of the diagonal. Each tile can be rotated to one of four positions. Patterns are formed by placing tiles next to each other, rotating individual tiles to create repeated motifs. This booklet presents a complete listing of 4x4 Truchet tile patterns with four-fold (90°) rotational symmetry (256 patterns). Treating these 4x4 tile patterns as tiles themselves allows for larger decorative patterns to be constructed from them. For example, a uniform frieze made from a single 4x4 tile can actually produce interesting secondary patterns which help illustrate some interesting relationships that exist among the tile patterns.



Each 4x4 Truchet tile pattern with rotational symmetry has a core 2x2 pattern in one of its quadrants that is rotated to produce the overall pattern. In this booklet, the core pattern, or prototile, is assumed to be in the lower left. Each pattern can identified as a sequence of 4 digits (a, b, c, d), or more succinctly, *abcd*, that list the rotational positions of each tile in the lower left quadrant. This sequence *abcd* will be referred to as the *signature* of the tile pattern.

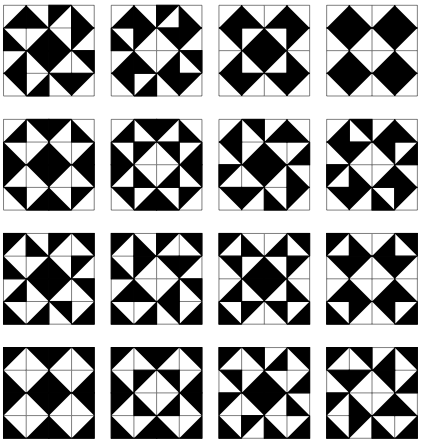




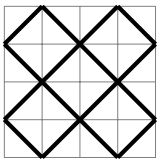
The 0011 pattern

Pattern families

We can group the 4*x*4 Truchet tile patterns with rotational symmetry into families where tile patterns are considered to be in the same family if they would look the same without colour – if each corresponding tile shares the same diagonal direction. The sequence that represents the family of a tile pattern can be found by taking the sequence of the tile pattern *modulo* 2. So, for example, the 16 tile patterns below are all members of the 0110 family.



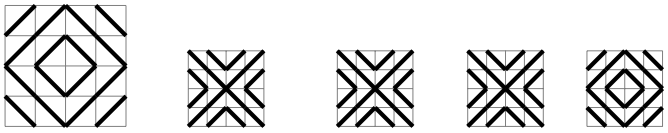
The 0110 pattern family



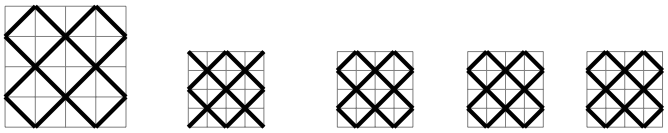
The 0110 family pattern

For a given family, there is corresponding *companion* family, the family of patterns formed by rotating each square in a member of the original family by 90°. There are also two *skew* families, formed by taking the upper left and lower right quadrants of an original family tile pattern as a founding pattern and a *dual* family, formed by taking the upper right quadrant as a founding patterns. A family is always different than its companion, and each family has a distinct companion, but it can happen that skew and duals can coincide. Self-dual families, where the dual family is the same as the original are of particular interest in the frieze patterns of the next chapter.

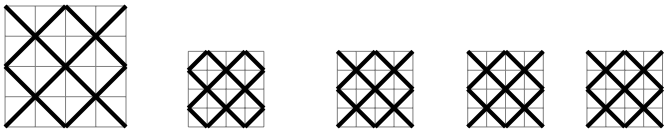
Self-Dual families



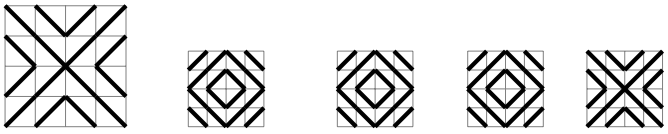
0000	1111	1111	1111	0000
	companion	skew (+)	skew (-)	dual



0110	1001	0110	0110	0110
	companion	skew (+)	skew (-)	dual

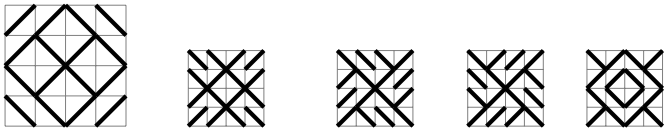


1001	0110	1001	1001	1001
	companion	skew (+)	skew (-)	dual

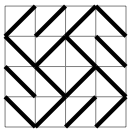


1111	0000	0000	0000	1111
	companion	skew (+)	skew (-)	dual

Non self-dual families



0001	1110	1011	1101	1000
	companion	skew (+)	skew (-)	dual



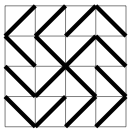
0010

1101
companion

1110
skew (+)

0111
skew (-)

0100
dual



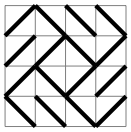
0011

1100
companion

1010
skew (+)

0101
skew (-)

1100
dual



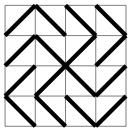
0100

1011
companion

0111
skew (+)

1110
skew (-)

0010
dual



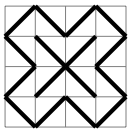
0101

1010
companion

0011
skew (+)

1100
skew (-)

1010
dual



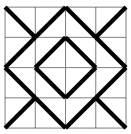
0111

1000
companion

0010
skew (+)

0100
skew (-)

1110
dual



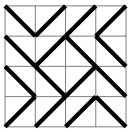
1000

0111
companion

1101
skew (+)

1011
skew (-)

0001
dual



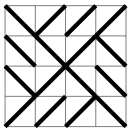
1010

0101
companion

1100
skew (+)

0011
skew (-)

0101
dual



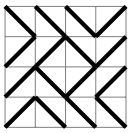
1011

0100
companion

1000
skew (+)

0001
skew (-)

1101
dual



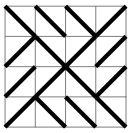
1100

0011
companion

0101
skew (+)

1010
skew (-)

0011
dual



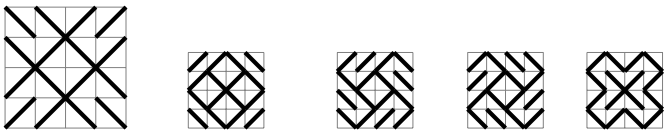
1101

0010
companion

0001
skew (+)

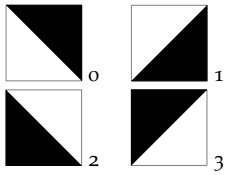
1000
skew (-)

1011
dual



1110 0001 0100 0010 0111
companion skew (+) skew (-) dual

Family and tile pattern mappings

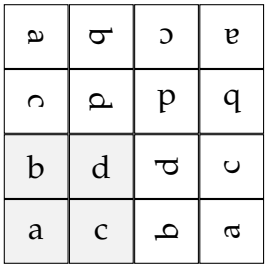


Related families and tiles can be obtained from applying simple mappings on the signature of the tile pattern.

Family mappings

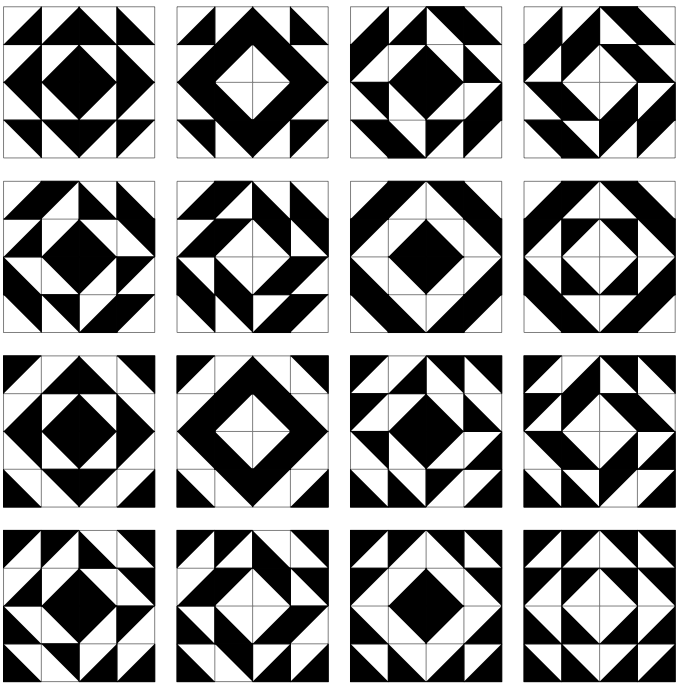
$$\begin{aligned}
\text{companion} &: (a,b,c,d) \mapsto (a+1, b+1, c+1, d+1) \pmod 2;\\
\text{skew}+ &: (a,b,c,d) \mapsto (c+1, a+1, d+1, b+1) \pmod 2;\\
\text{reverse} &: (a,b,c,d) \mapsto (d,c,b,a) \pmod 2;\\
\text{skew}- &: (a,b,c,d) \mapsto (b+1, d+1, a+1, c+1) \pmod 2;
\end{aligned}$$

Tile pattern mappings

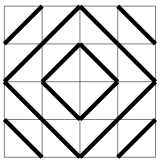


$$\begin{aligned}
\text{skew}+ &: (a,b,c,d) \mapsto (c+1, a+1, d+1, b+1) \pmod 4;\\
\text{dual} &: (a,b,c,d) \mapsto (d+2, c+2, b+2, a+2) \pmod 4;\\
\text{skew}- &: (a,b,c,d) \mapsto (b+3, d+3, a+3, c+3) \pmod 4;\\
\text{opposite} &: (a,b,c,d) \mapsto (a+2, b+2, c+2, d+2) \pmod 4;
\end{aligned}$$

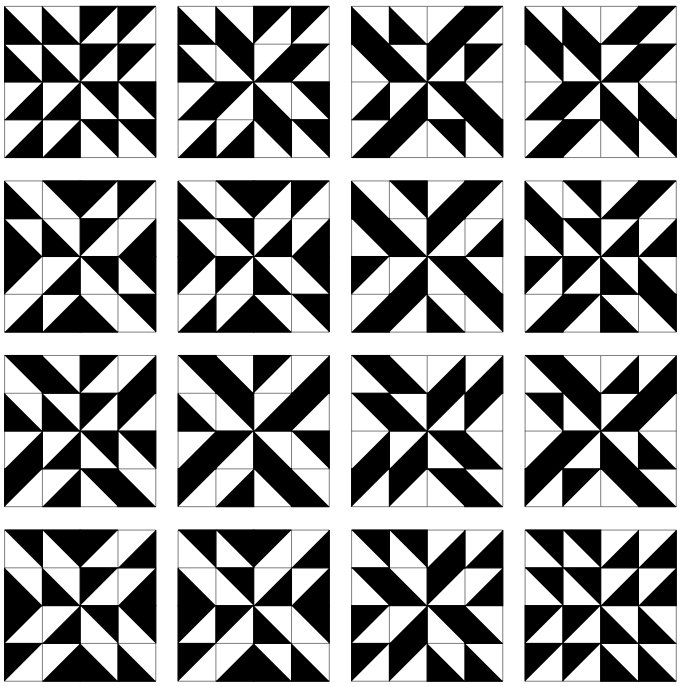
On the following pages each family will be shown along with its corresponding *companion* family, the family of patterns formed by rotating each square in a member of the original family by 90°.



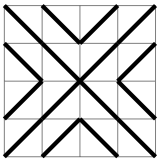
0000



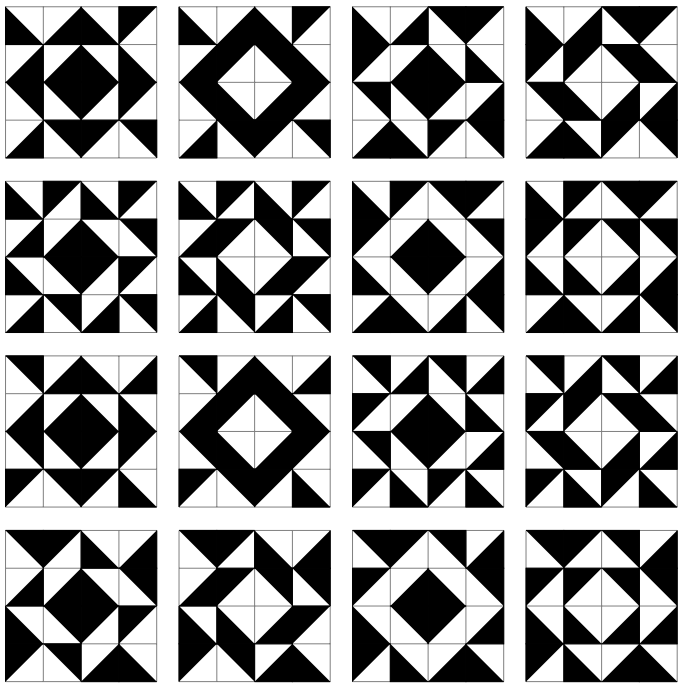
0000 0002 0020 0022
0200 0202 0220 0222
2000 2002 2020 2022
2200 2202 2220 2222



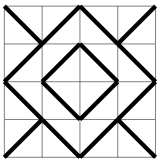
1111



1111 1113 1131 1133
1311 1313 1331 1333
3111 3113 3131 3133
3311 3313 3331 3333



1000



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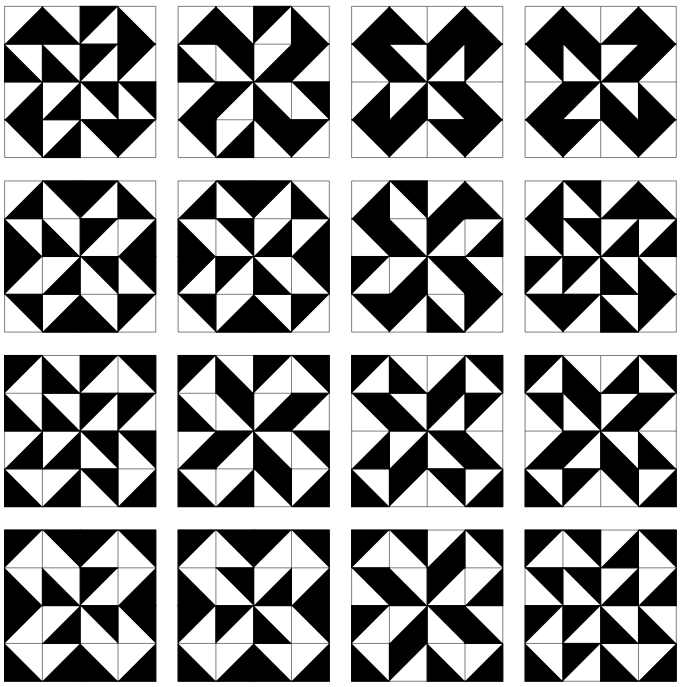
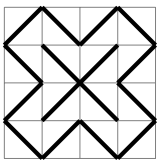
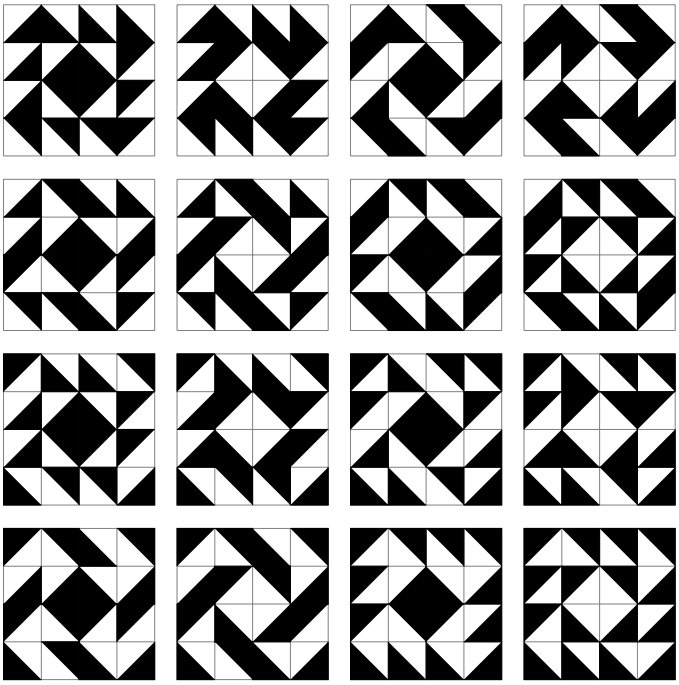
1200	1202	1220	1222
3000	3002	3020	3022
3200	3202	3220	3222



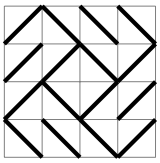
0111



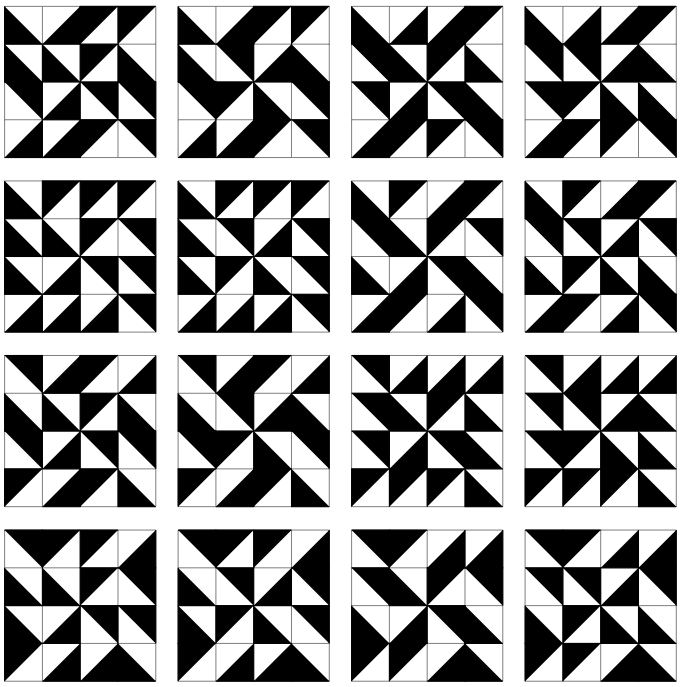
0111	0113	0131	0133
0311	0313	0331	0333
2111	2113	2131	2133
2311	2313	2331	2333



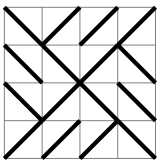
0100



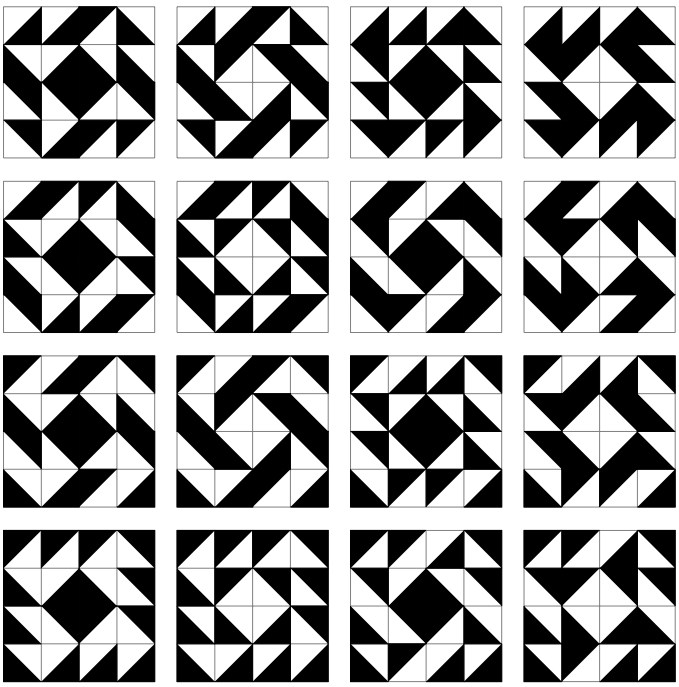
0100	0102	0120	0122
0300	0302	0320	0322
2100	2102	2120	2122
2300	2302	2320	2322



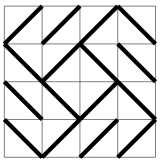
1011



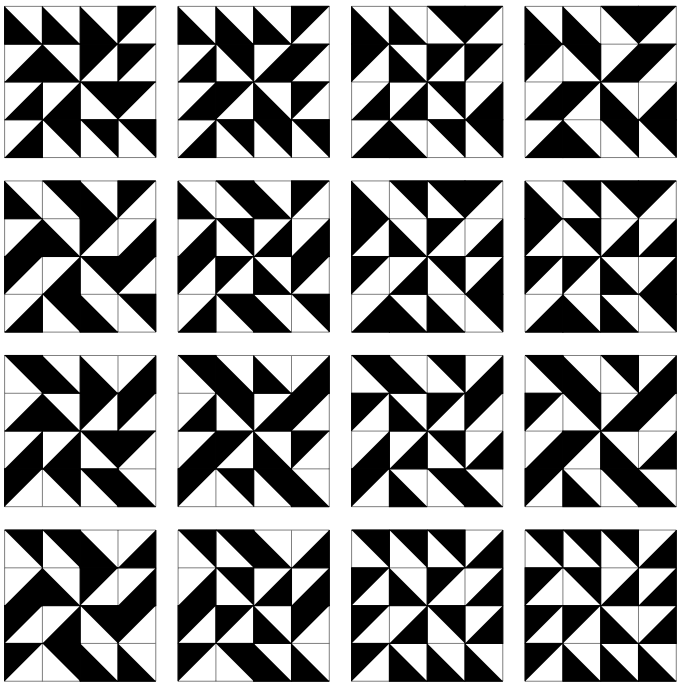
1011	1013	1031	1033
1211	1213	1231	1233
3011	3013	3031	3033
3211	3213	3231	3233



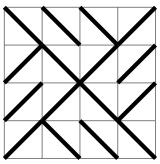
0010



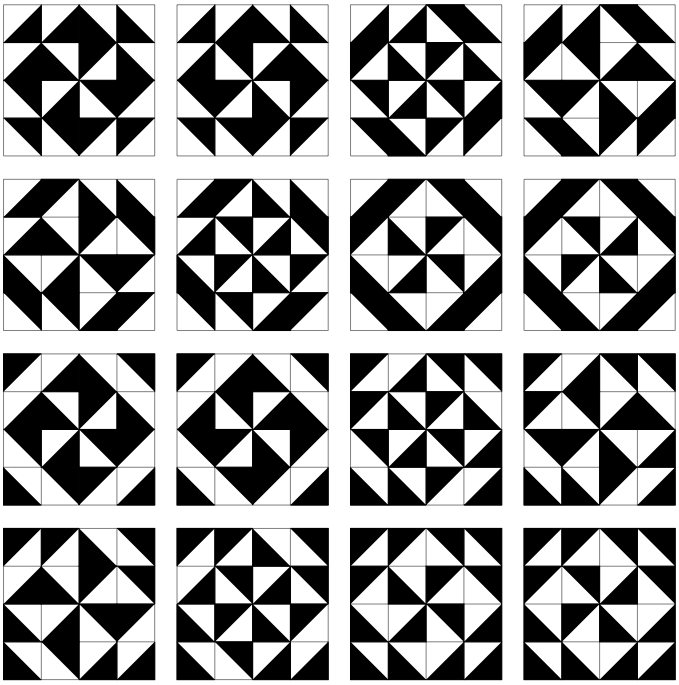
0010	0012	0030	0032
0210	0212	0230	0232
2010	2012	2030	2032
2210	2212	2230	2232



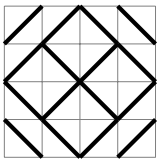
1101



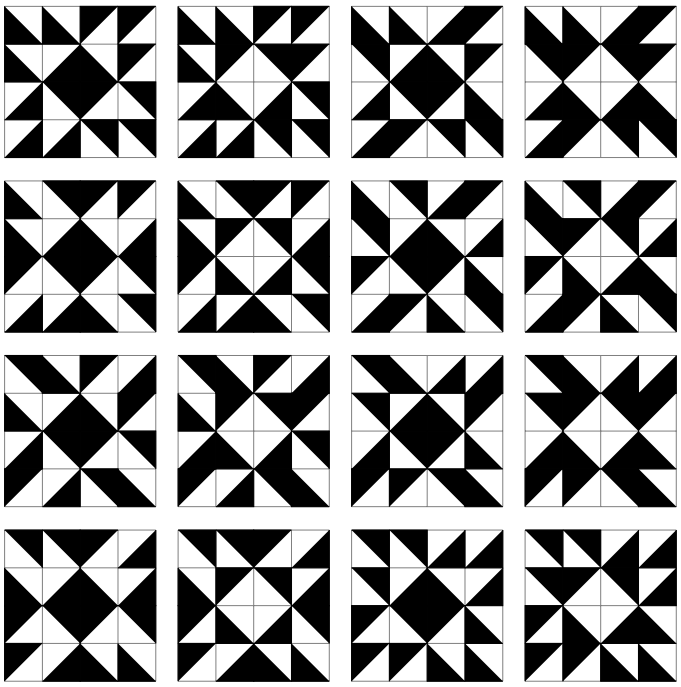
1101	1103	1121	1123
1301	1303	1321	1323
3101	3103	3121	3123
3301	3303	3321	3323



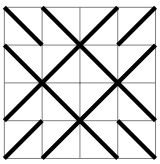
0001



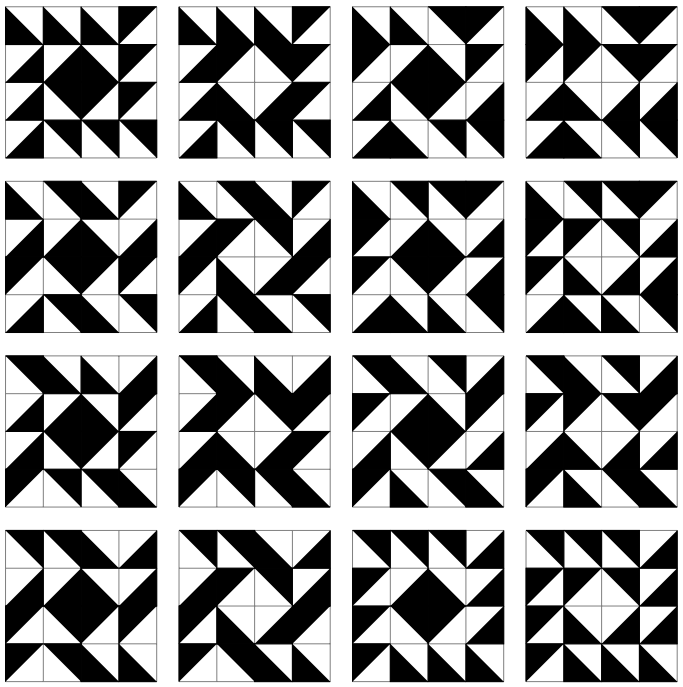
0001 0003 0021 0023
0201 0203 0221 0223
2001 2003 2021 2023
2201 2203 2221 2223



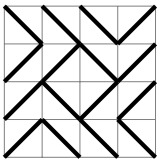
1110



1110 1112 1130 1132
1310 1312 1330 1332
3110 3112 3130 3132
3310 3312 3330 3332



1100



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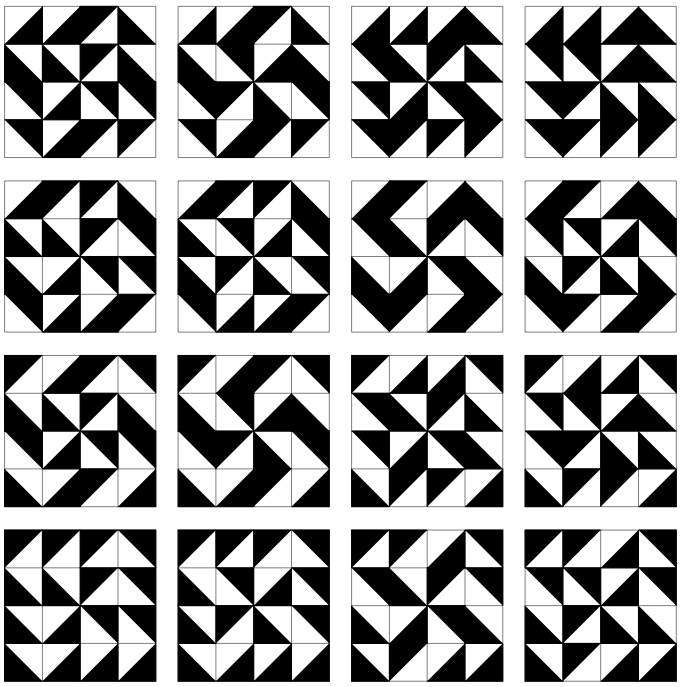

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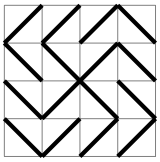

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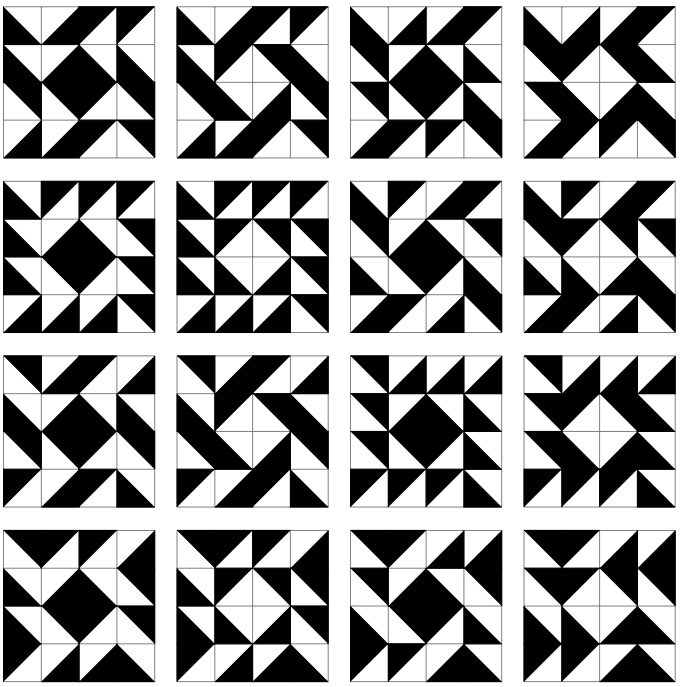

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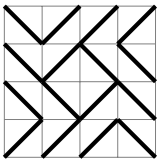

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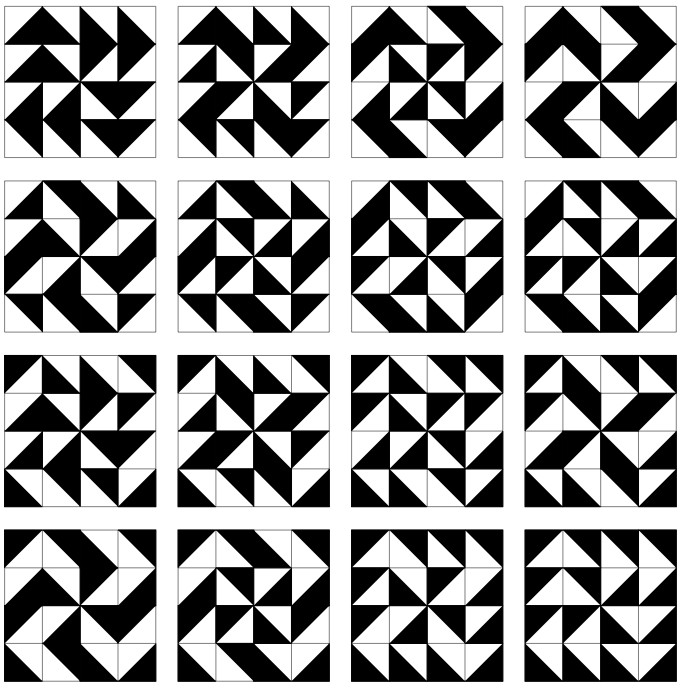

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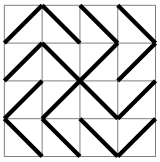

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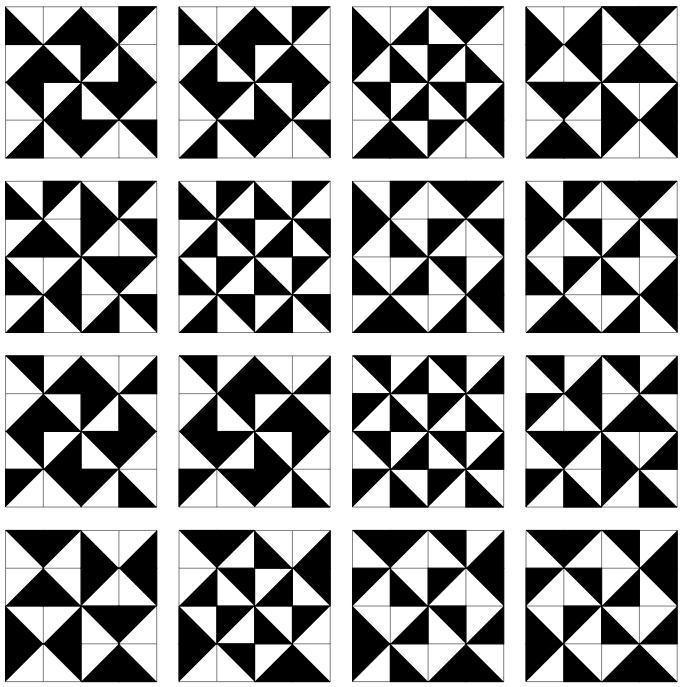

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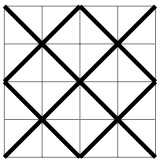

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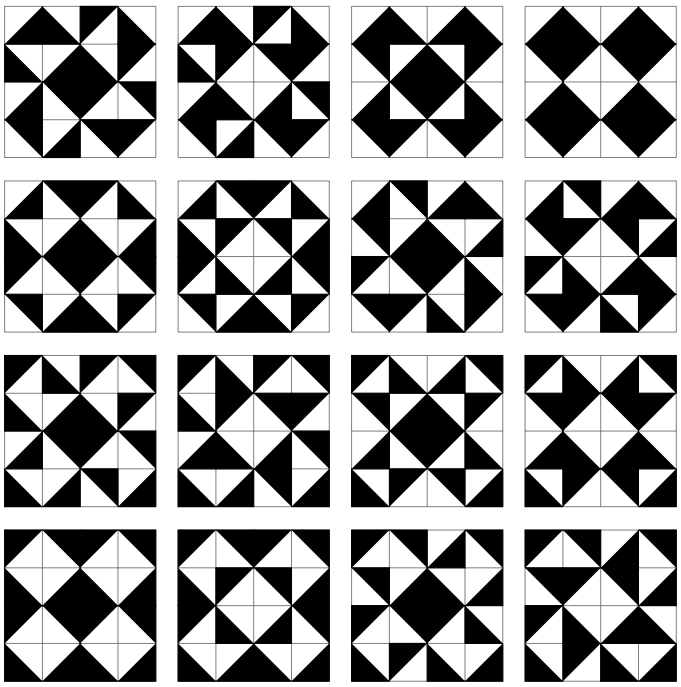

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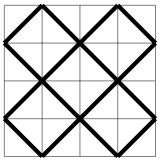

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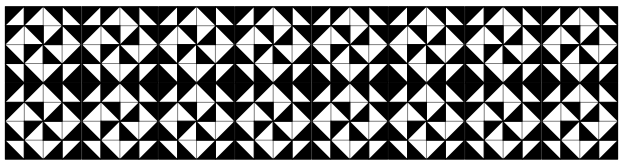
Uniform friezes

Each 4*x*4 Truchet pattern can be treated like a tile and used in a larger pattern. A uniform *frieze* is a horizontal strip of the same tile pattern repeated. Friezes of 4*x*4 Truchet pattern tiles with rotational symmetry can be quite striking, and have some interesting characteristics.

A frieze of more than one row of a primary tile reveals a secondary tile pattern that appears as another horizontal strip of 4*x*4 Truchet tile patterns nestled between the rows of primary tiles. Below, a frieze of 2223 tiles has a secondary pattern of 1000 tiles.



2223

1000

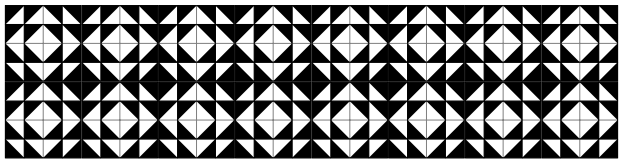
The secondary tile in a frieze pattern is the pattern that has been referred to previously as the *dual* of the original pattern. The dual of a tile pattern is the pattern formed by taking the top right quadrant of the original tile as the prototile of the new tile.

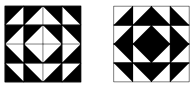
Some tiles are self-dual, and frieze patterns formed by self-dual tiles show a much more uniform pattern, as the extra rows of tiles seemingly nestled between the rows of the original tile are made up of the same original tile. Friezes of self-dual tiles have a third *tertiary* tile pattern with four-fold rotational symmetry that appears to overlap between adjacent tiles of the original tile. These tertiary tile patterns are the *skew* of the original tile pattern. Some self-dual friezes are also self-skew, leading to even more uniform patterns.

We can consider the uniform friezes formed by the dual tiles as the same pattern. There are 6 pairs of families where the original and dual are not the same, and these pairs of families yield 16 patterns

each. The 4 remaining families contain some self-dual patterns, and some patterns that are *opp-dual* (the secondary tile is the opposite tile of the original), also reducing the number of patterns. These 4 remaining families provide 10 distinct frieze patterns each. This means that the 256 tile patterns generate 136 distinct friezes.

Frieze patterns for family oooo (secondary, oooo)





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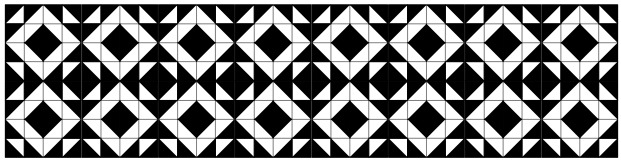

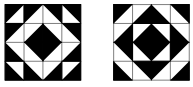

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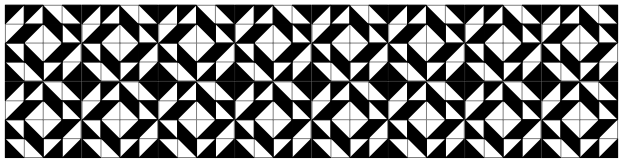

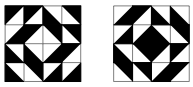

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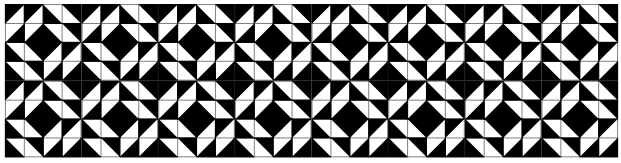

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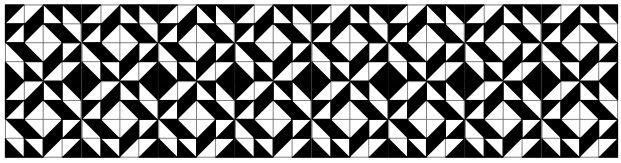

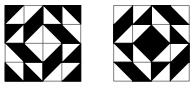

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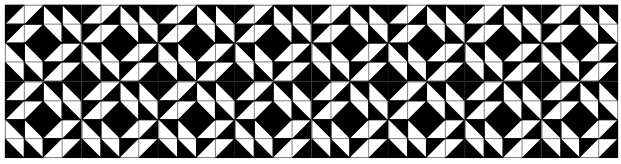

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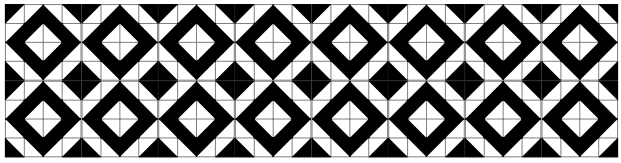

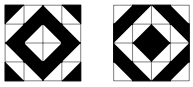

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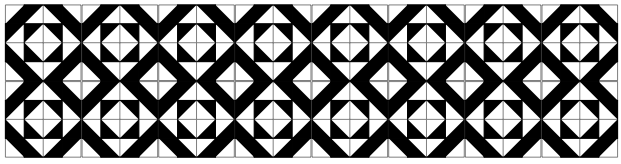

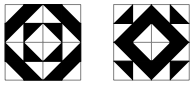

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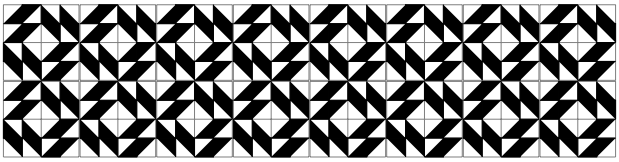

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0202

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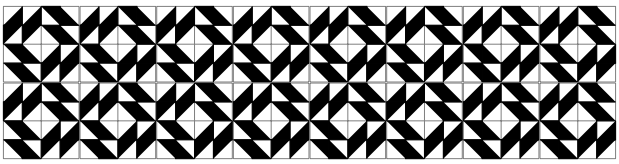
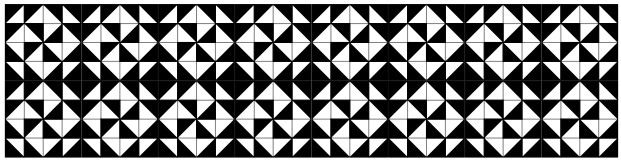


0022

0022

3131

Frieze patterns for family 0001 (secondary, 1000)



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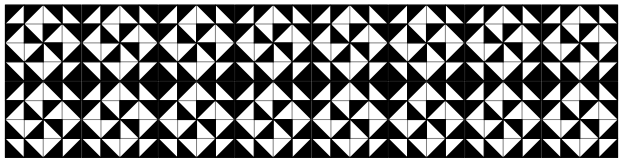

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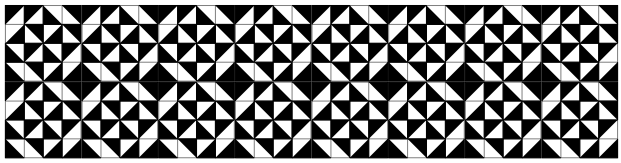

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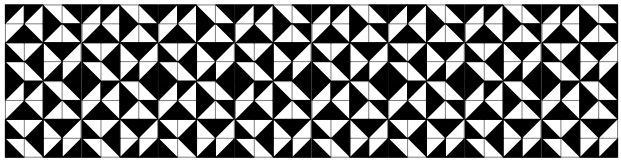

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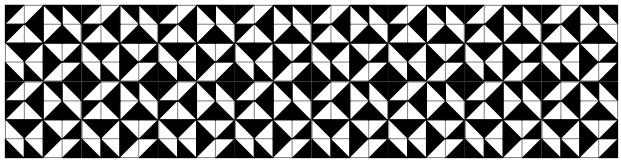

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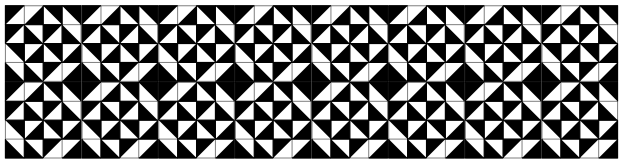

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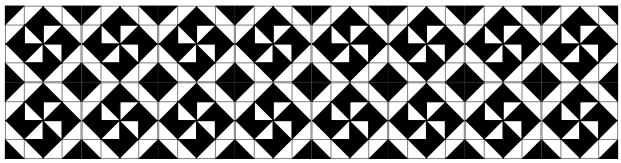

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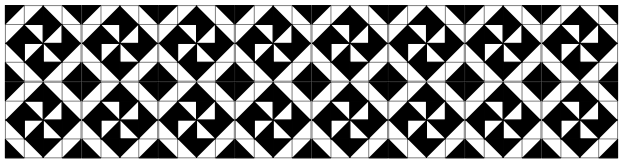

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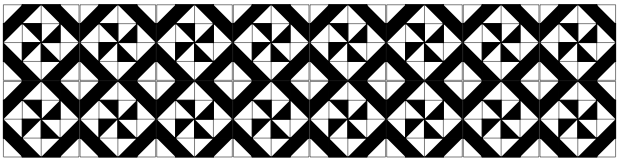

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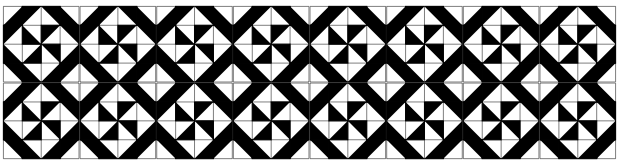

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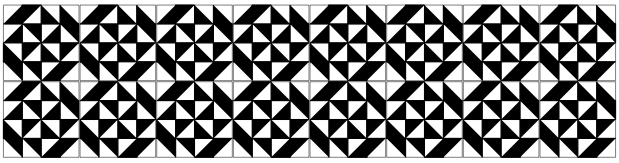

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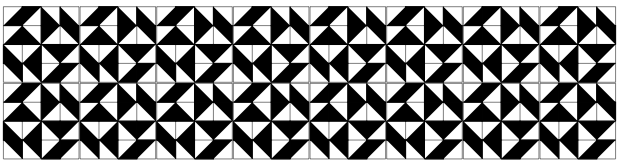

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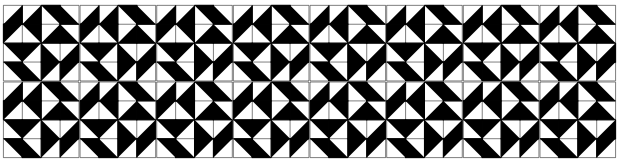

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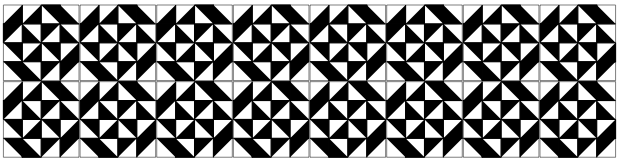

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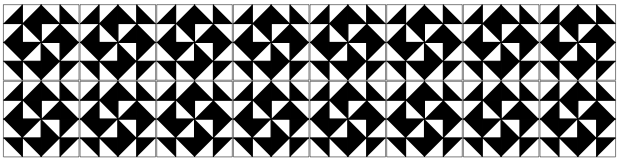

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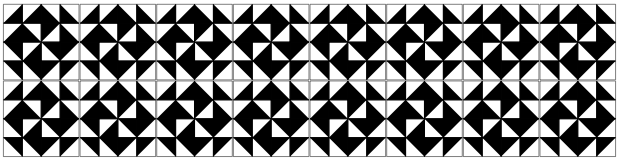

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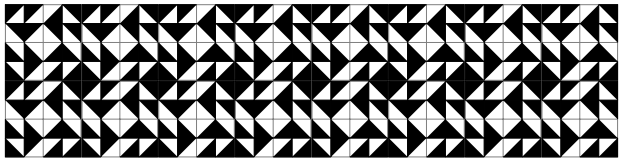
Frieze patterns for family 0010 (secondary, 0100)



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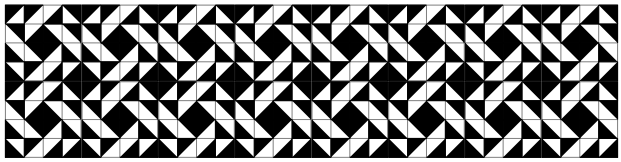

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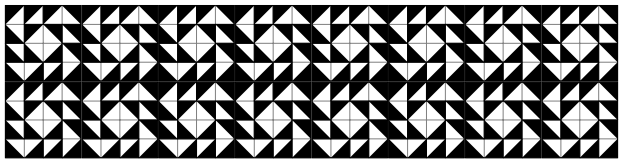

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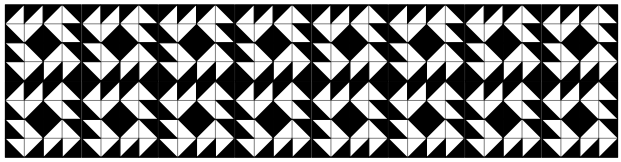

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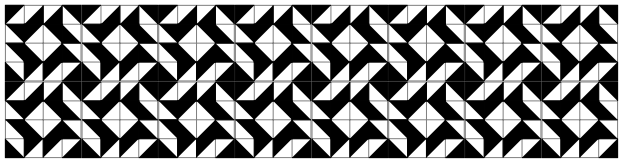

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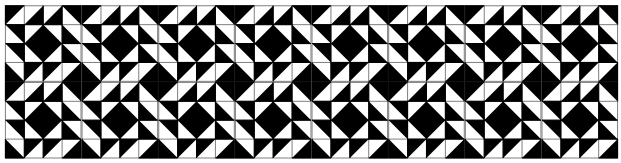

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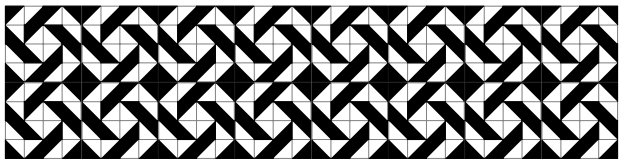
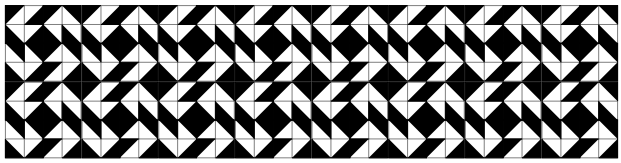
2120



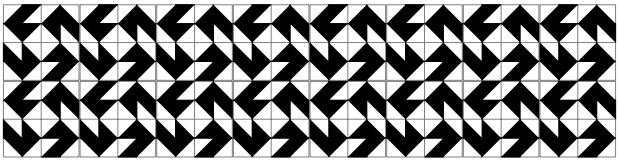
2012

0320



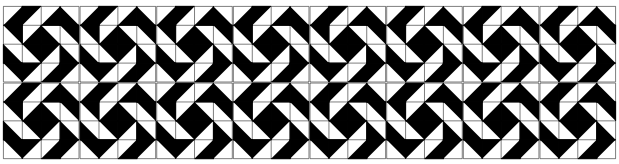
2010

2320



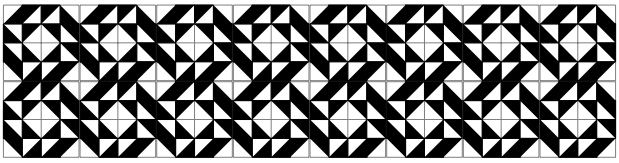
0232

0102



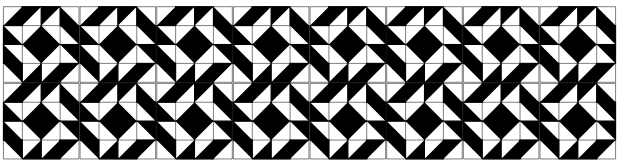
0230

2102



0212

0302



0210

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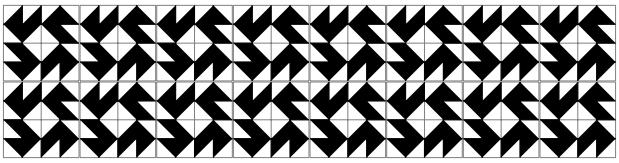
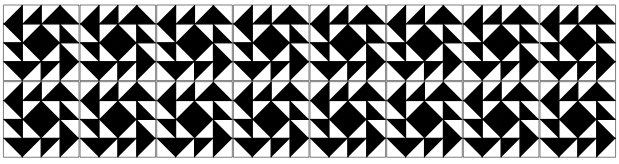


0032

0122



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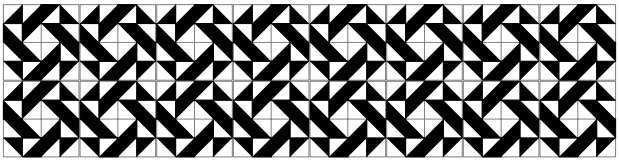
2122



0012

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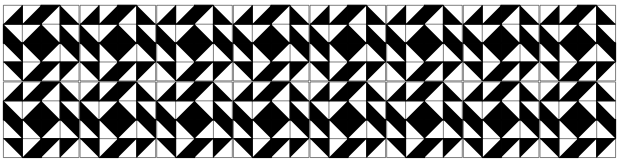
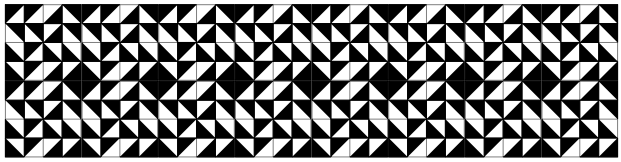


0010

2322

Frieze patterns for family 0011 (secondary, 1100)



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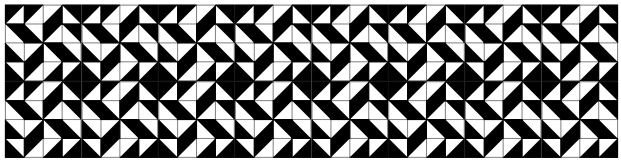
1100



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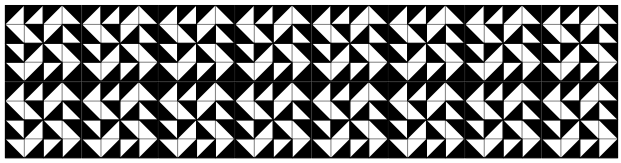

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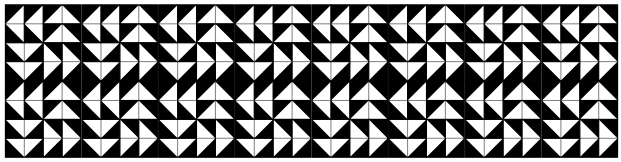

2211

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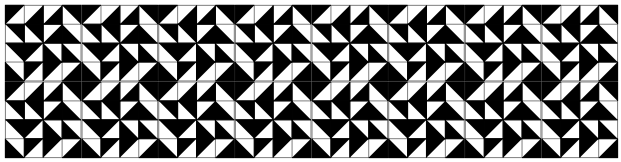
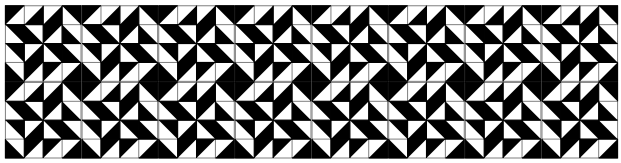


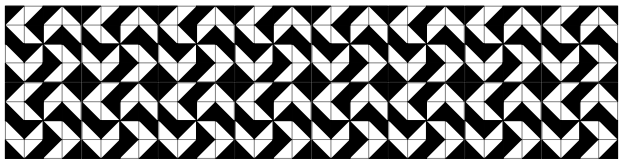
2033

1120



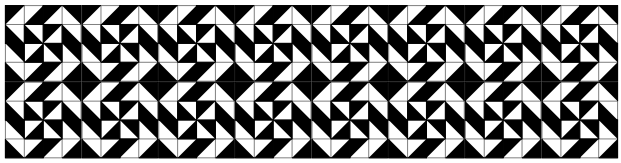
2031

3120



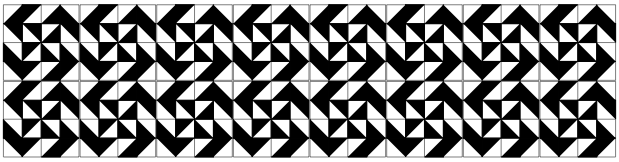
2013

1320



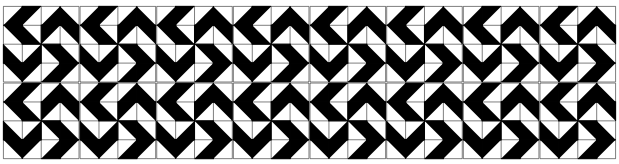
2011

3320



0233

1102



0231

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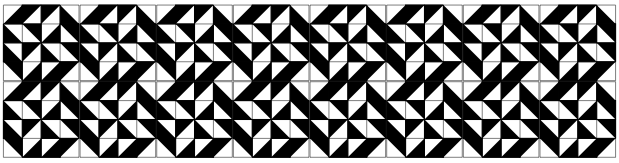

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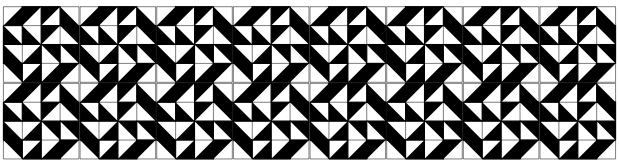

0211

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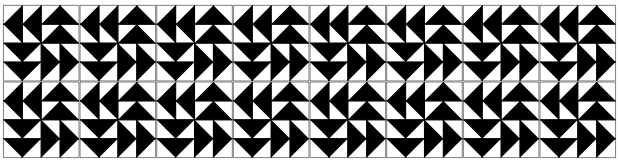

0033

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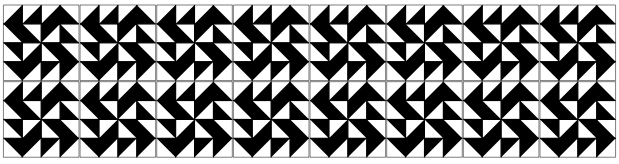

0031

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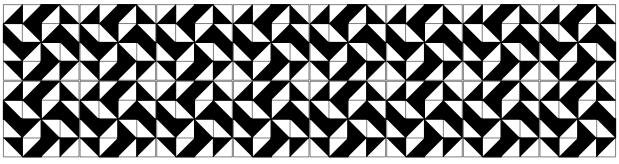
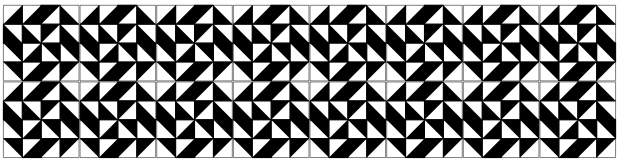


0013

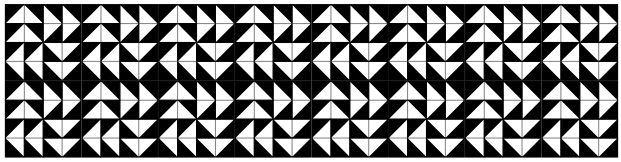
1322



0011

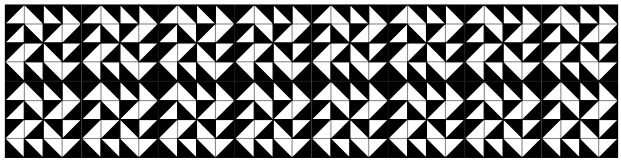
3322

Frieze patterns for family 0101 (secondary, 1010)



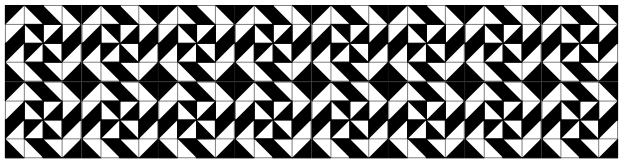
2323

1010



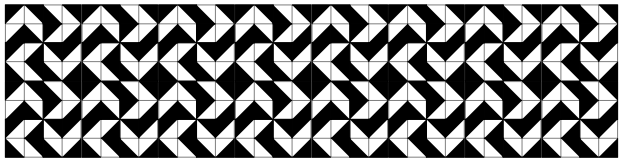
2321

3010



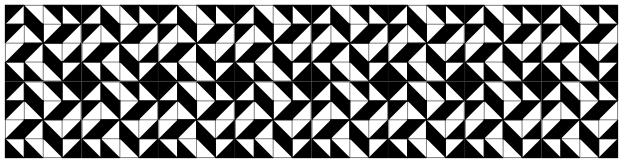
2303

1210



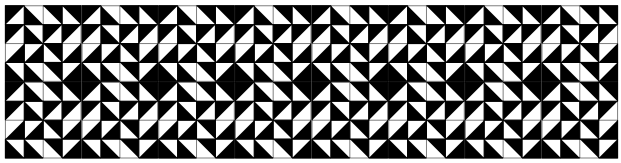
2301

3210



2123

1030



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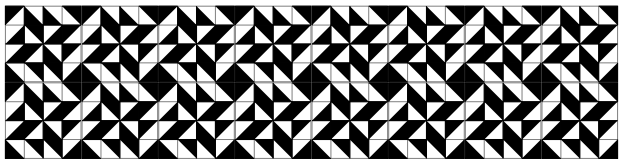

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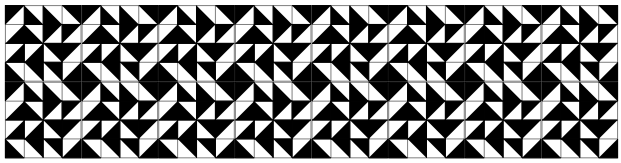

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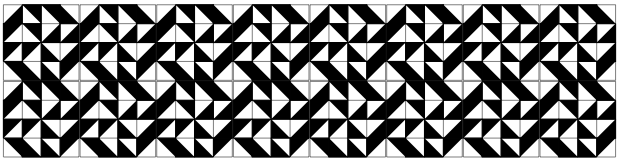
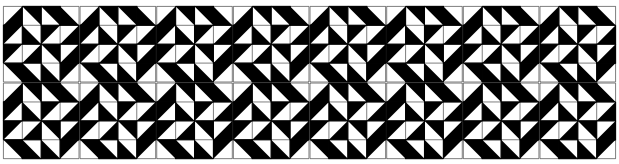


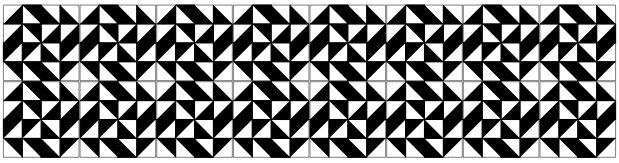
0323

1012



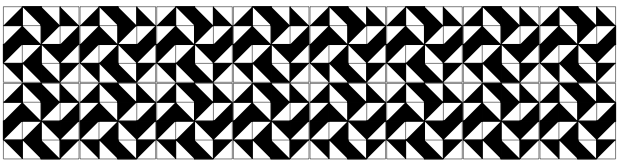
0321

3012



0303

1212



0301

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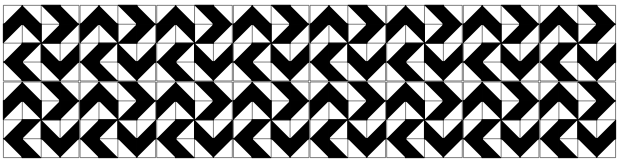
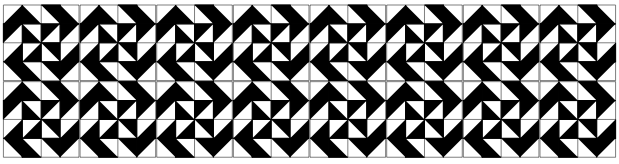


0123

1032



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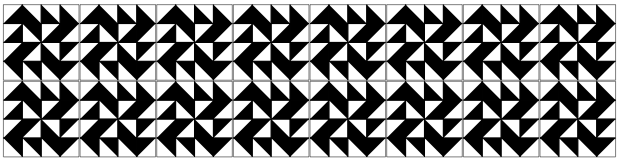
3032



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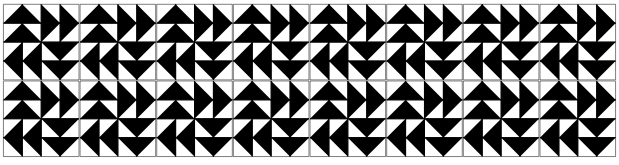
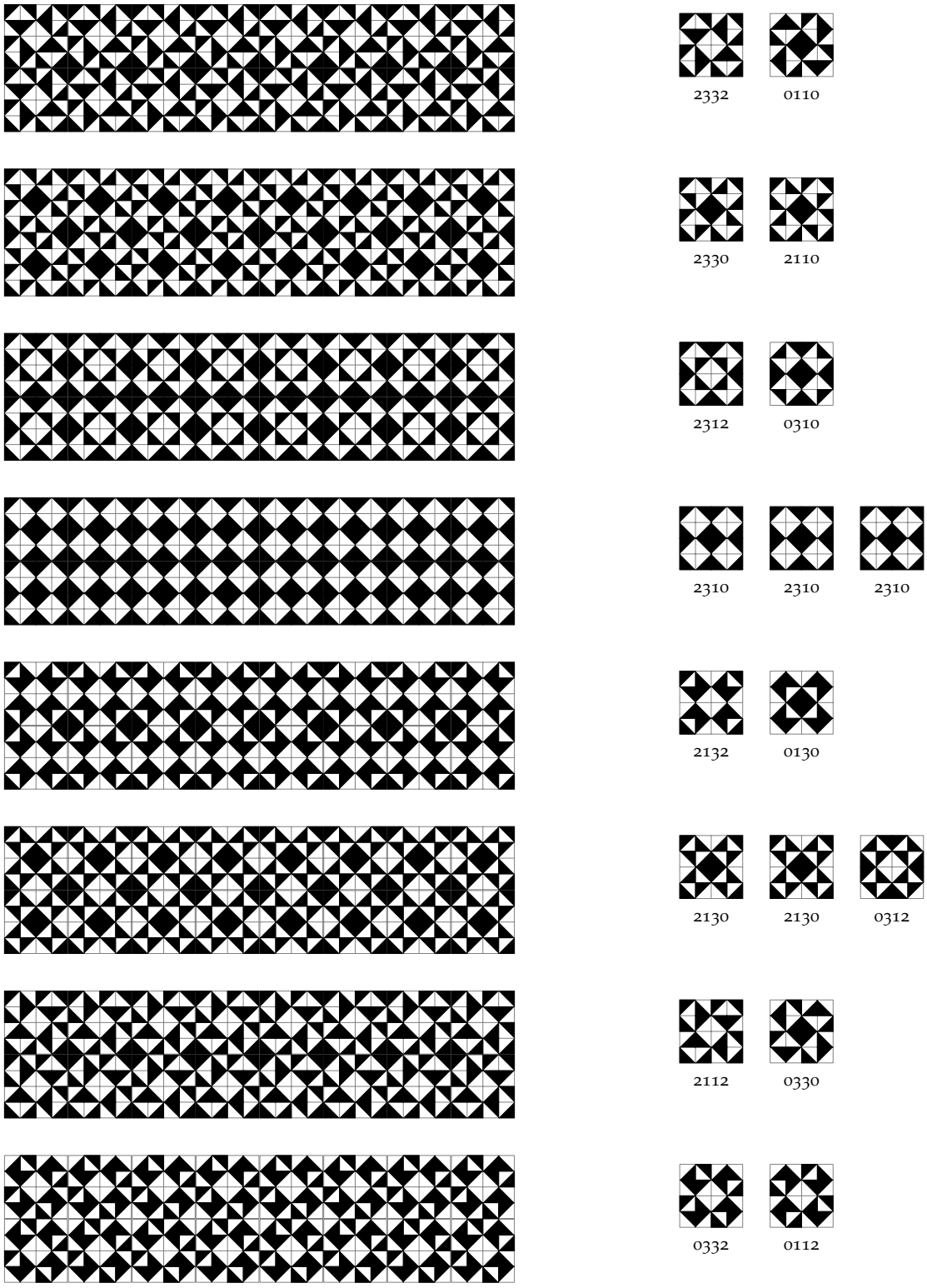
1232

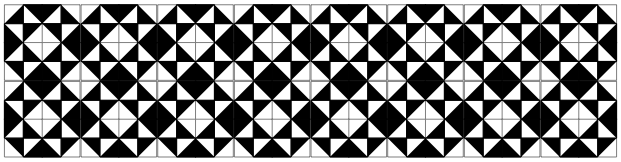


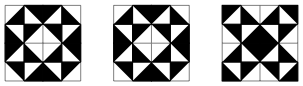
0101

3232

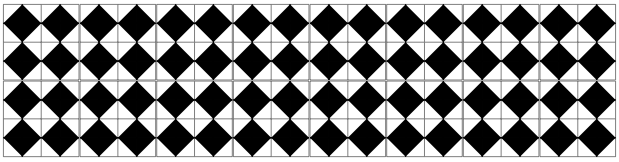
Frieze patterns for family 0110 (secondary, 0110)

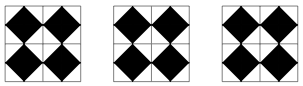






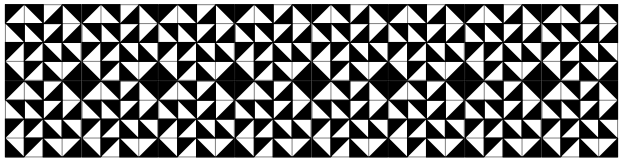
0312 0312 2130





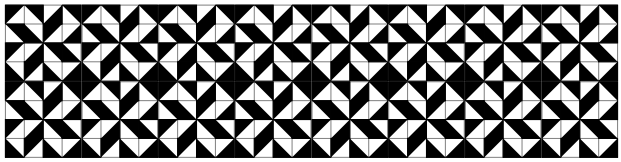
0132 0132 0132

Frieze patterns for family 0111 (secondary, 1110)



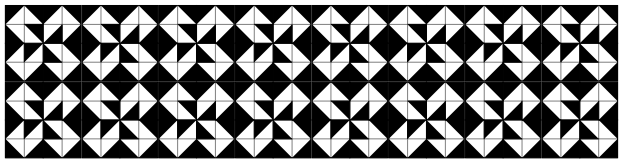
2333

1110



2331

3110



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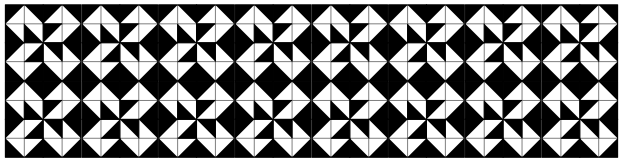
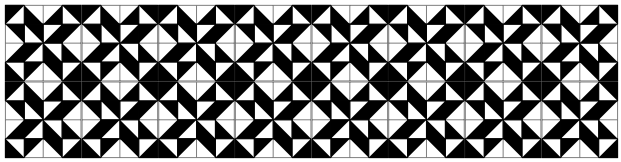
1310



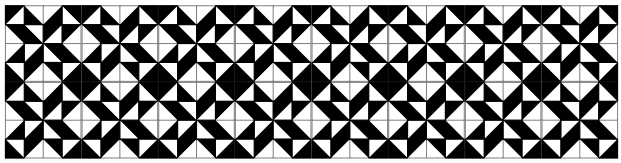
2311

3310



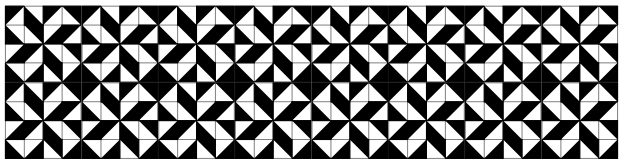
2133

1130



2131

3130



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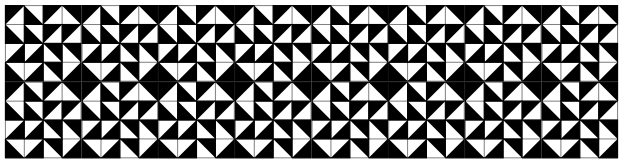

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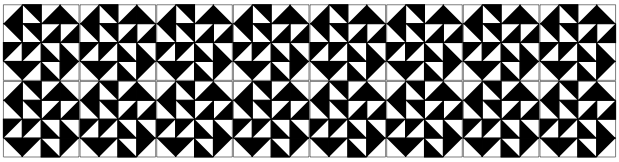

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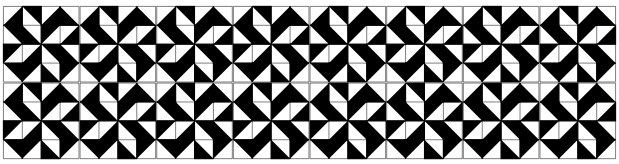

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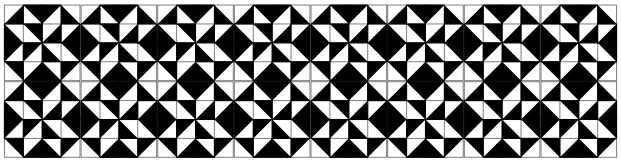

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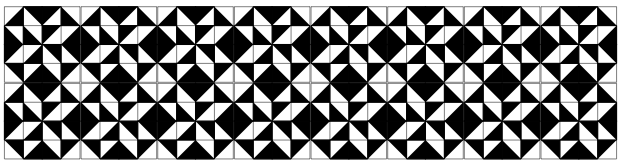

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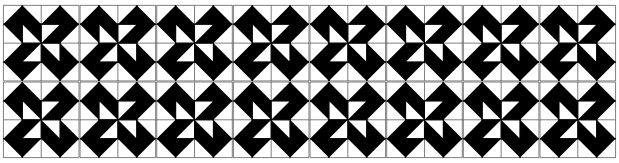

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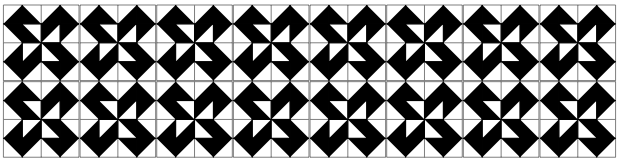

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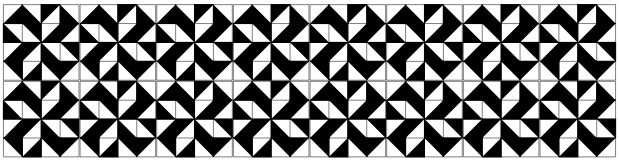

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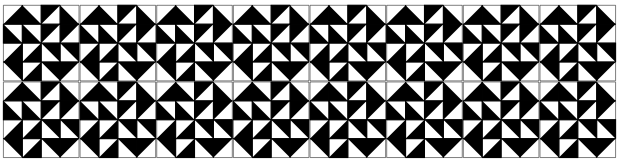

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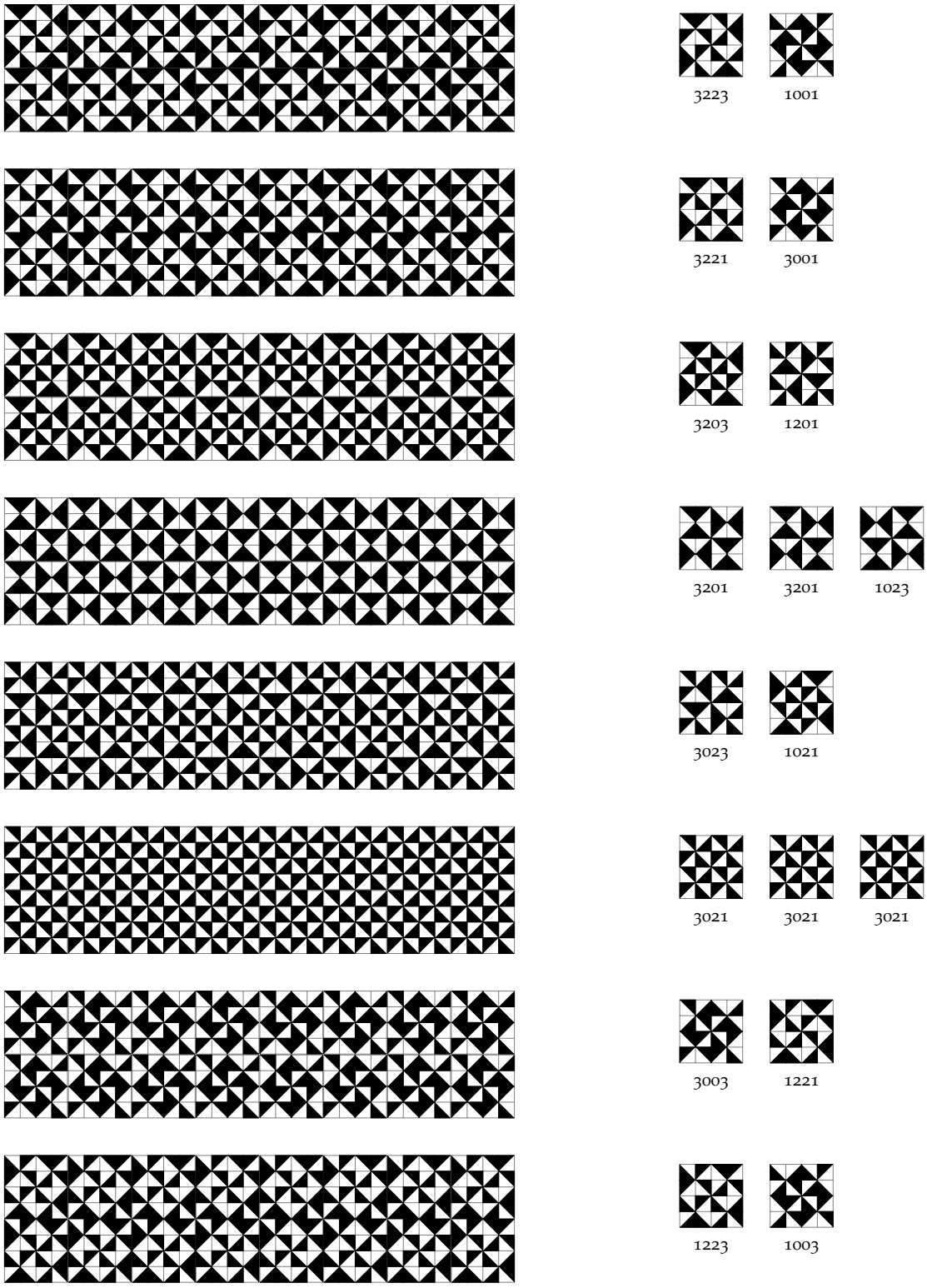
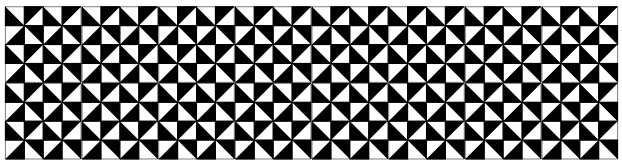
Frieze patterns for family 1001 (secondary, 1001)





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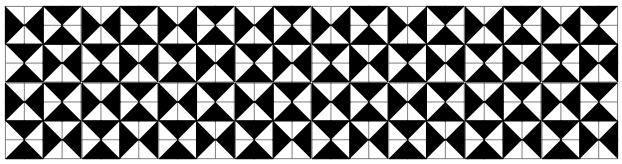

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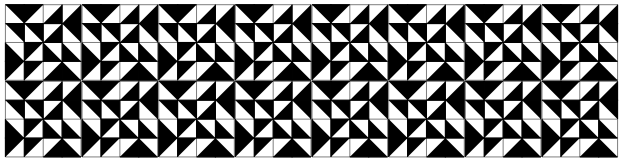

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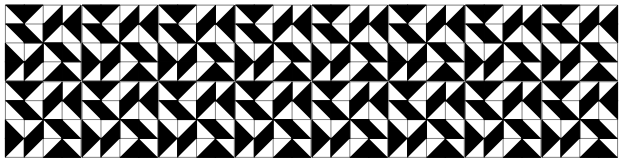

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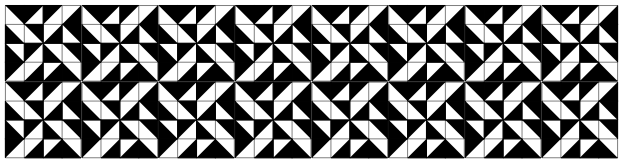

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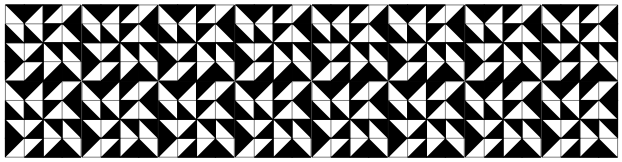

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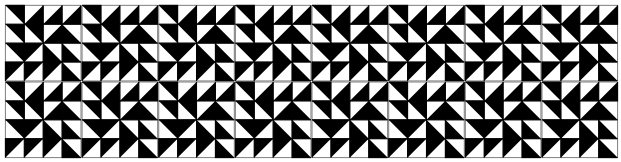

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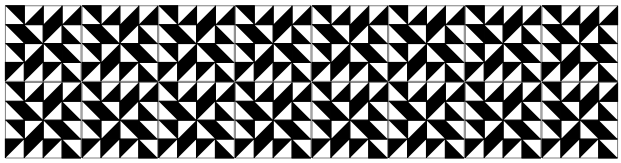

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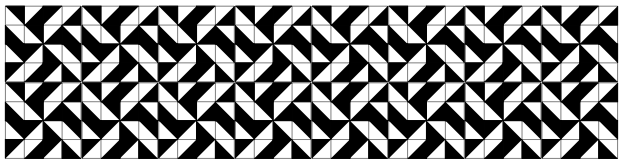

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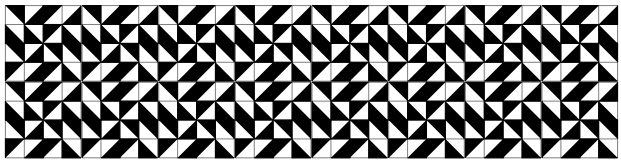

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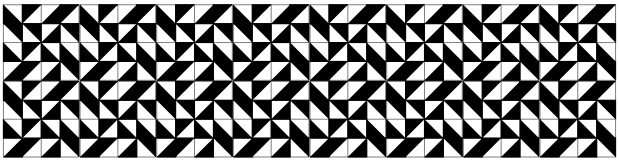

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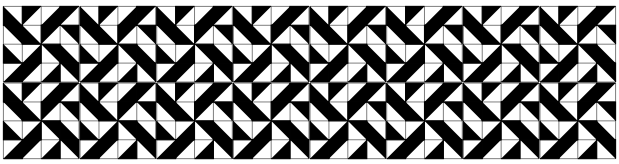

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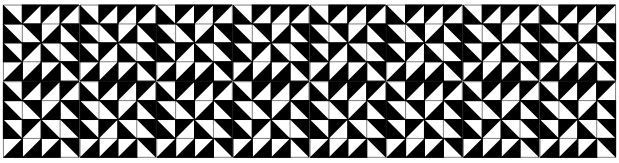

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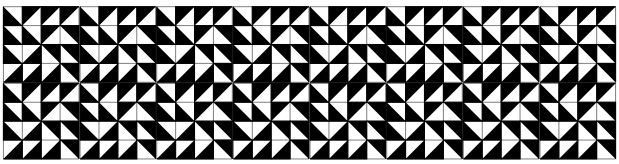

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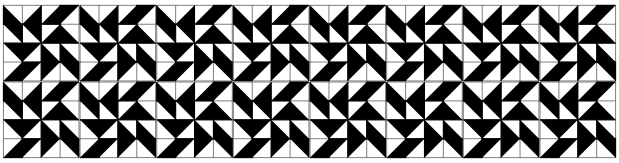

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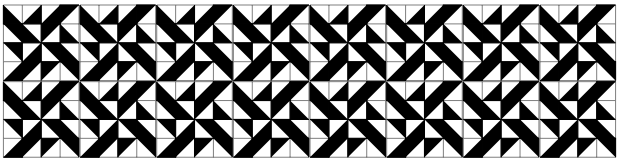

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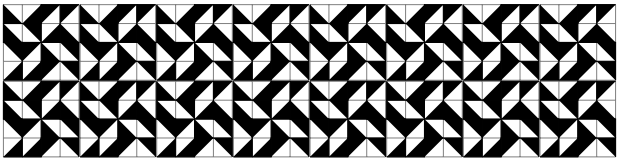

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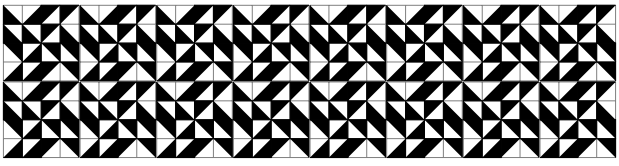

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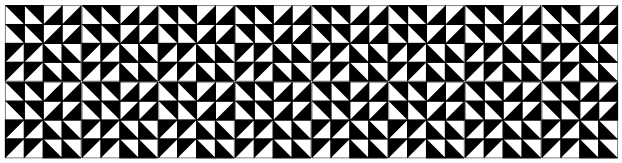
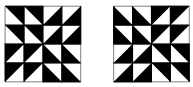
Frieze patterns for family 1111 (secondary, 1111)





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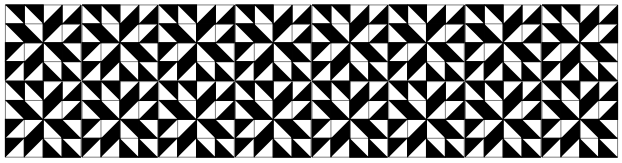

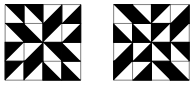

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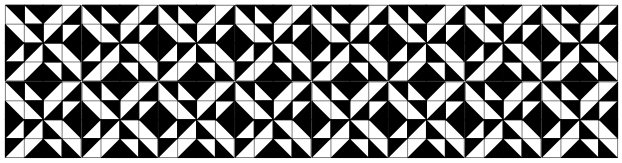

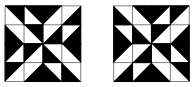

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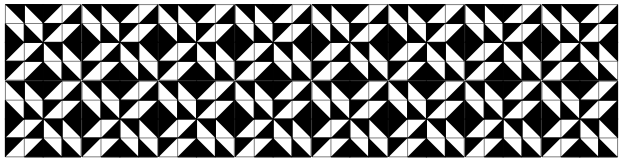

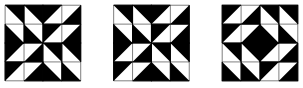

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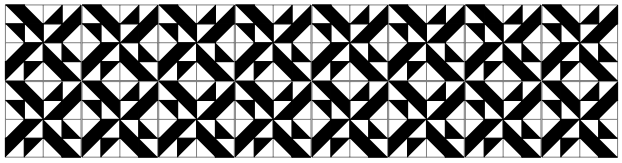

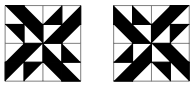

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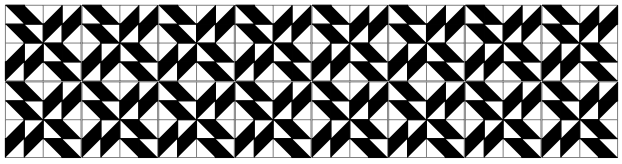

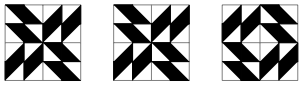

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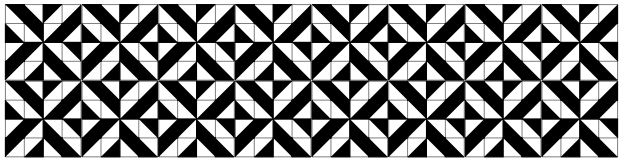

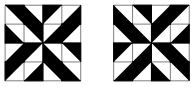

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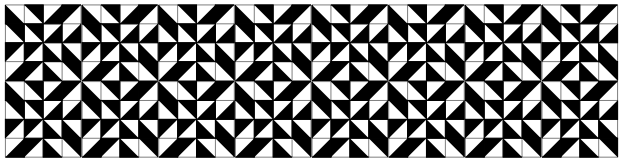

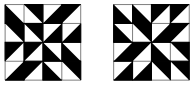

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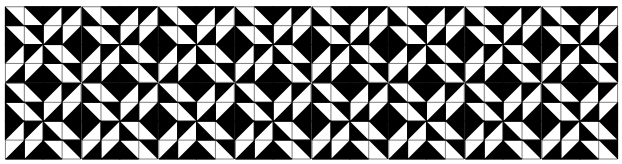

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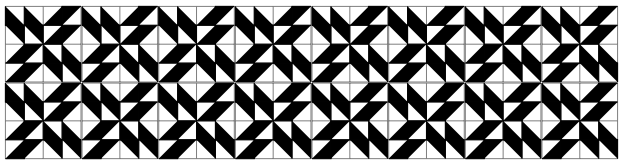

1133

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0202

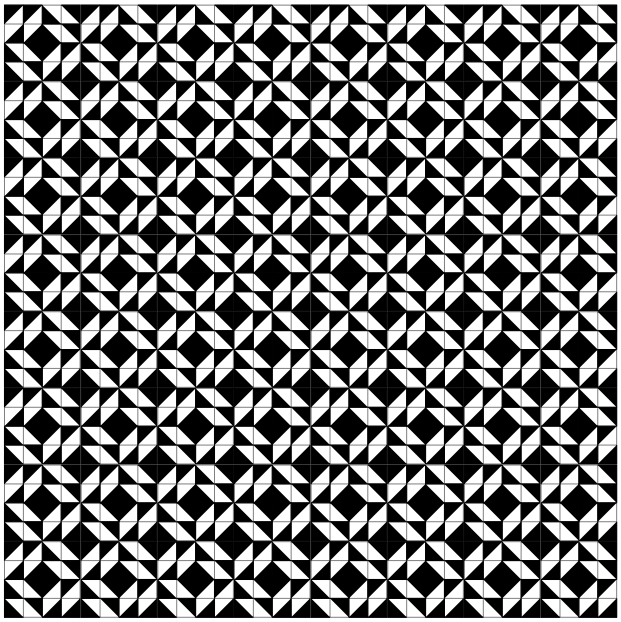
Some designs

This chapter presents some designs made using one or two types of the 4×4 Truchet tile patterns with four-fold rotational symmetry.

Self-dual designs

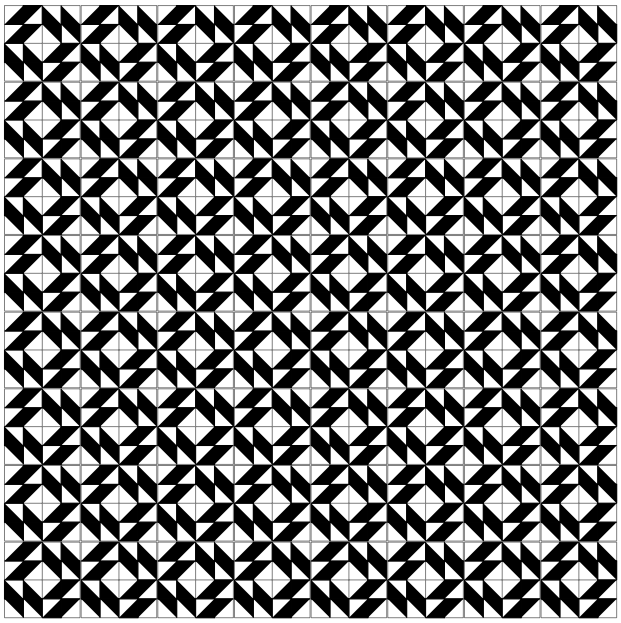
These designs extend the friezes of self-dual patterns to create patterns where copies of the main tile appear as secondary tiles. The other patterns you can see are provided by the tertiary skew tile patterns that seem to overlap betwen the adjacent pairs of tiles.

Design using 2200



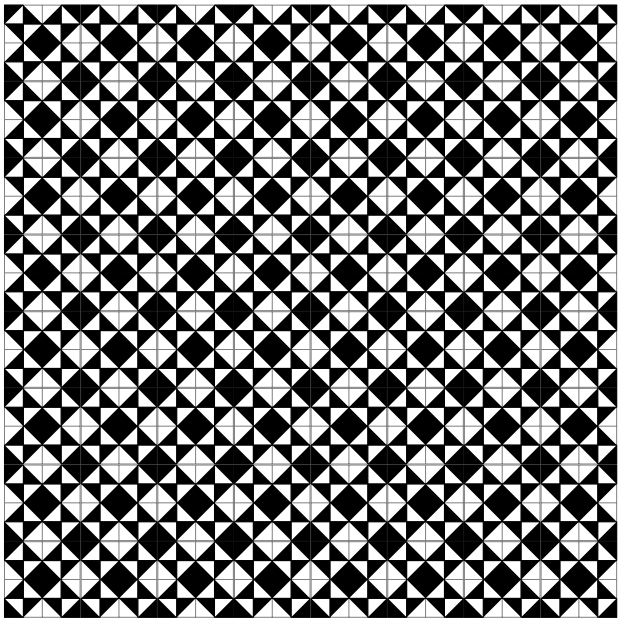
2200

Design using 0202



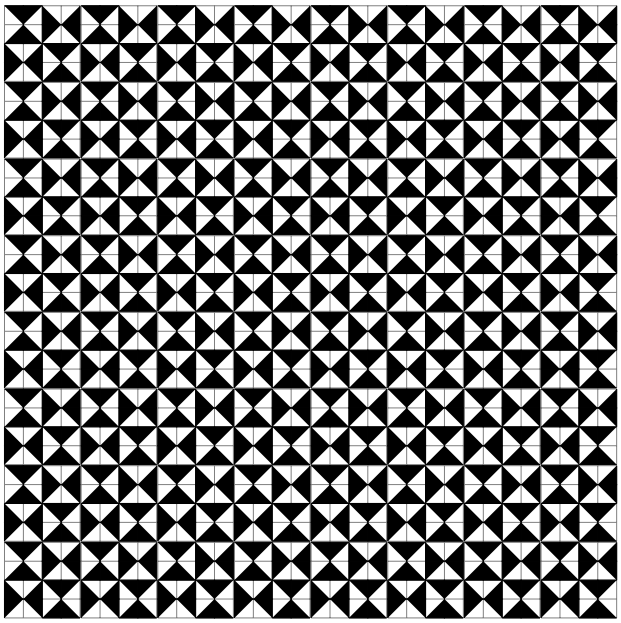
0202

Design using 2130



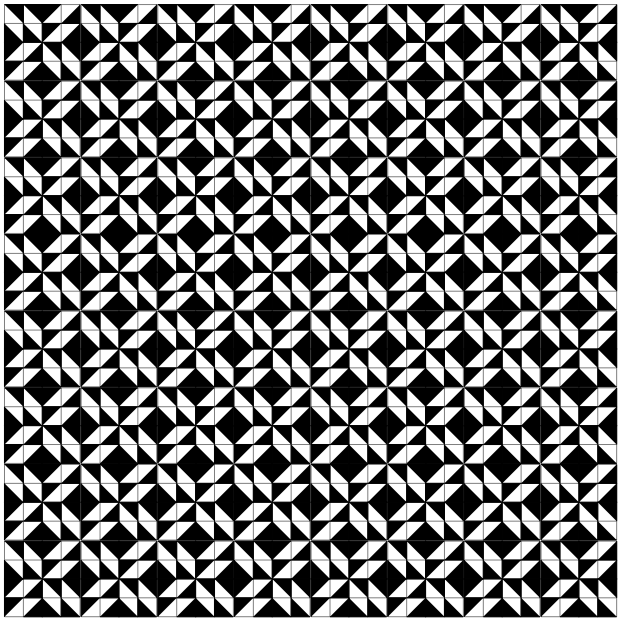
2130

Design using 3201



3201

Design using 3311

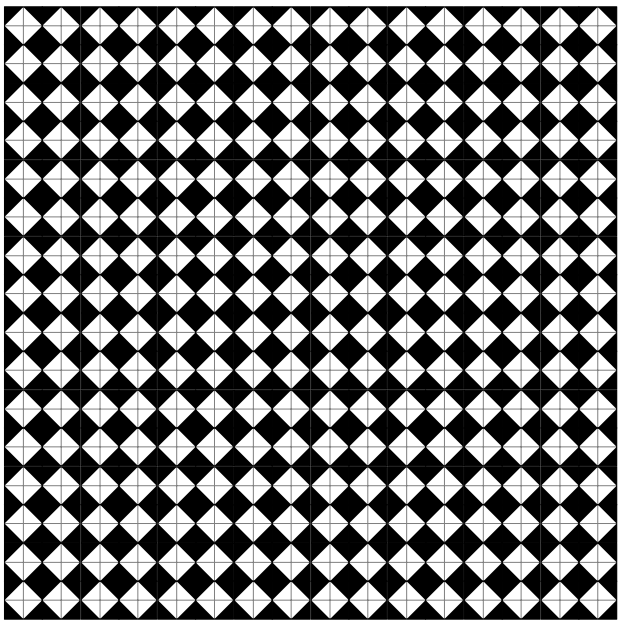


3311

Strongly uniform designs

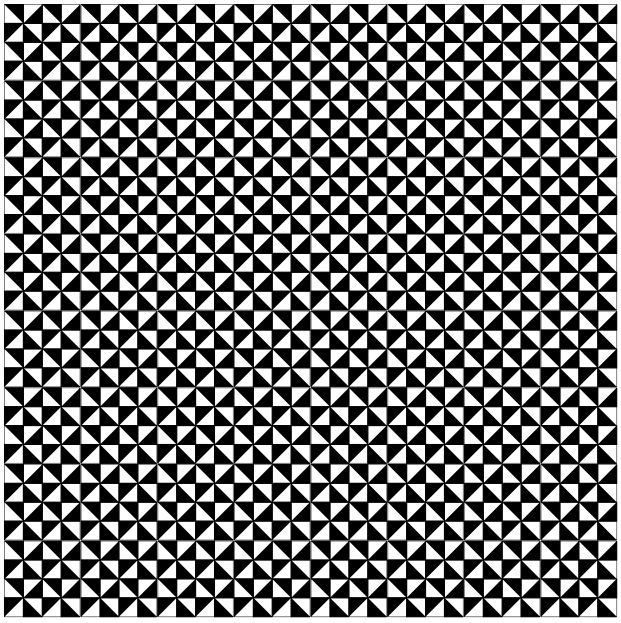
These self-dual tiles are also self-skew, so the pattern that emerges from placing them together is strongly uniform.

Design using 2310



2310

Design using 3021



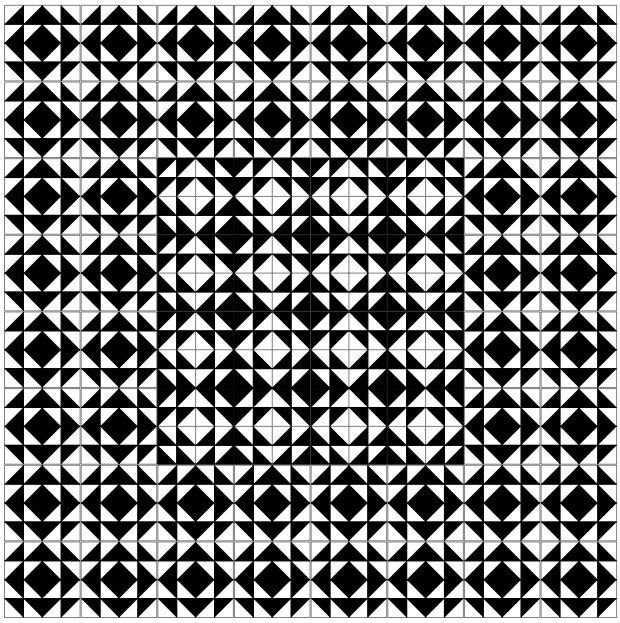
3021

Op-dual designs

These designs include two primary tiles, One tile type is placed in the center of the design in a 4×4 square, and a second tile type is used to create a 2-tile wide border around the central square. The tiles

chosen are opposites of each other, and they are also opp-dual tiles, so we see secondary effects where each tile type appears in between the rows of the other.

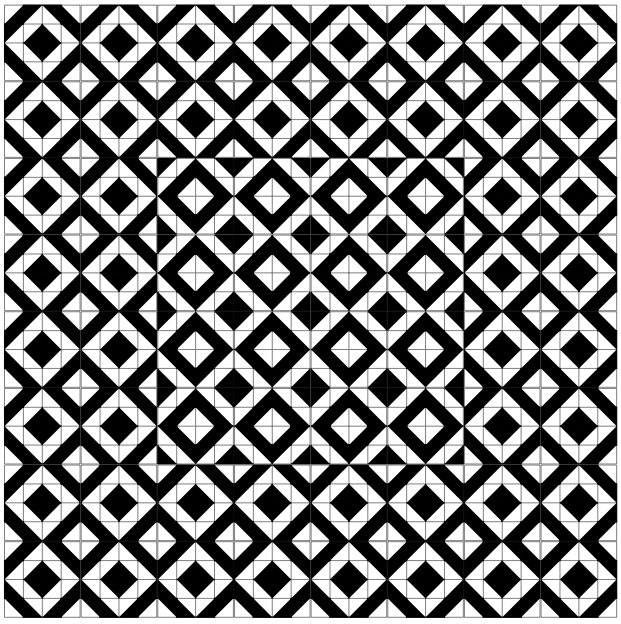
Design using 2222 and 0000



2222

0000

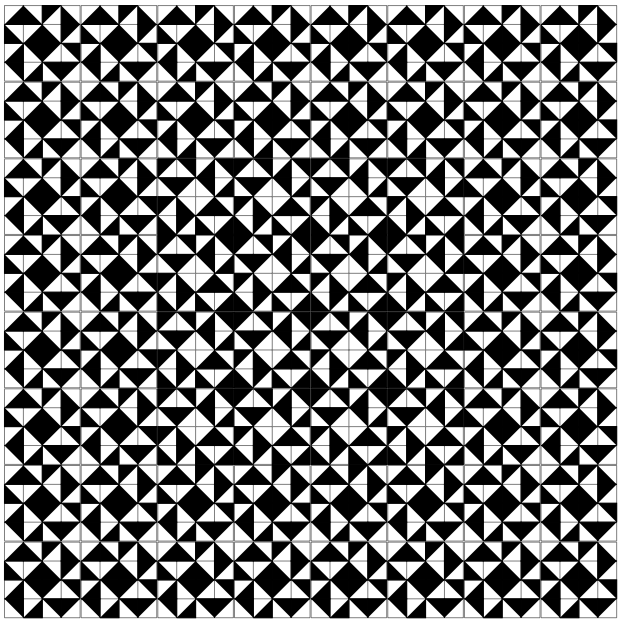
Design using 2002 and 0220



2002

0220

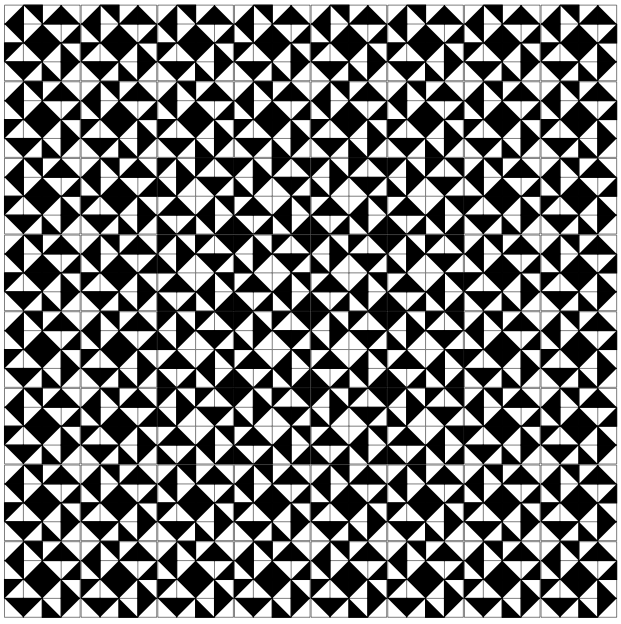
Design using 2332 and 0110



2332

0110

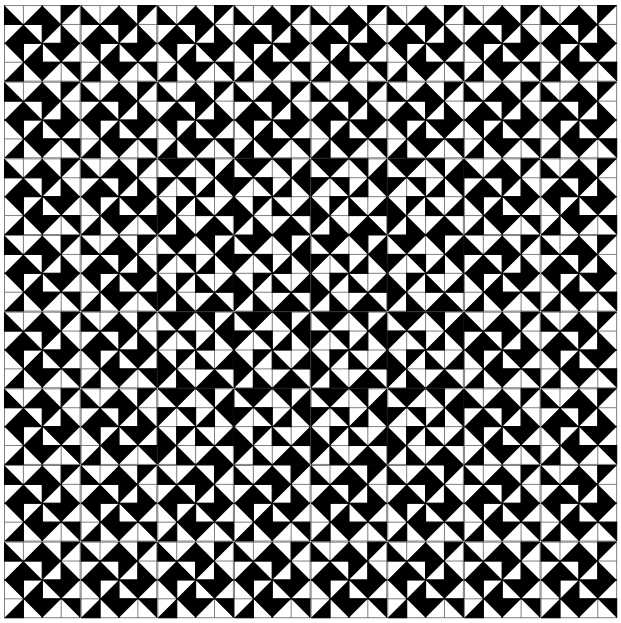
Design using 2112 and 0330



2112

0330

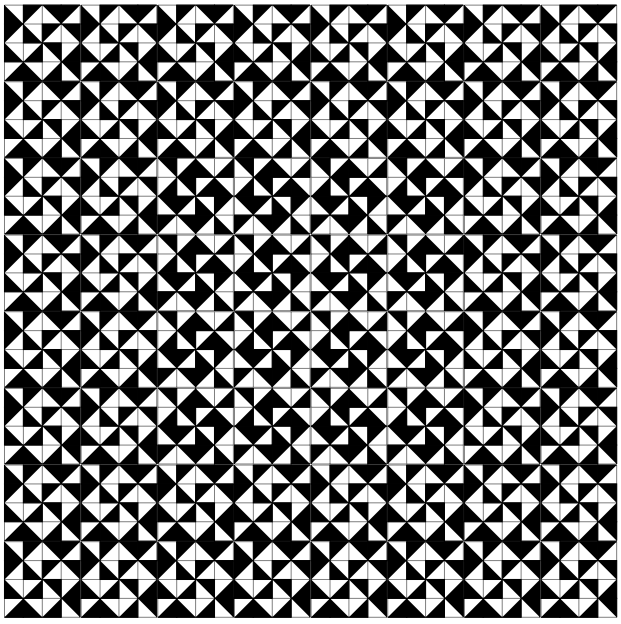
Design using 3223 and 1001



3223

1001

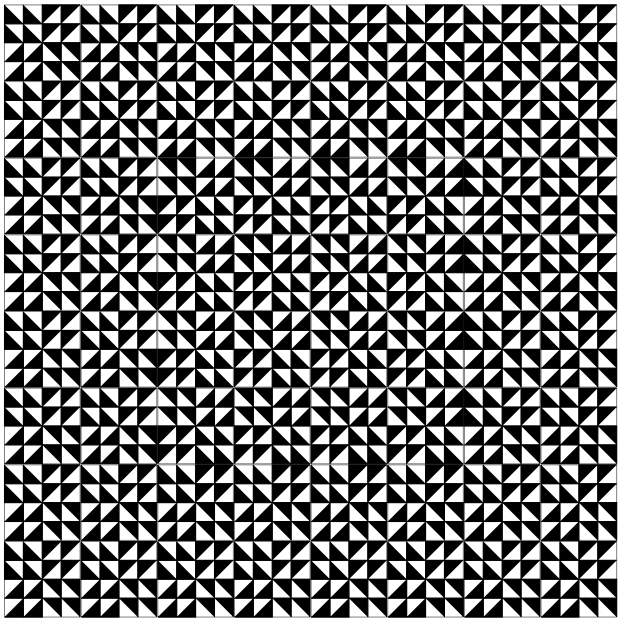
Design using 3003 and 1221



3003

1221

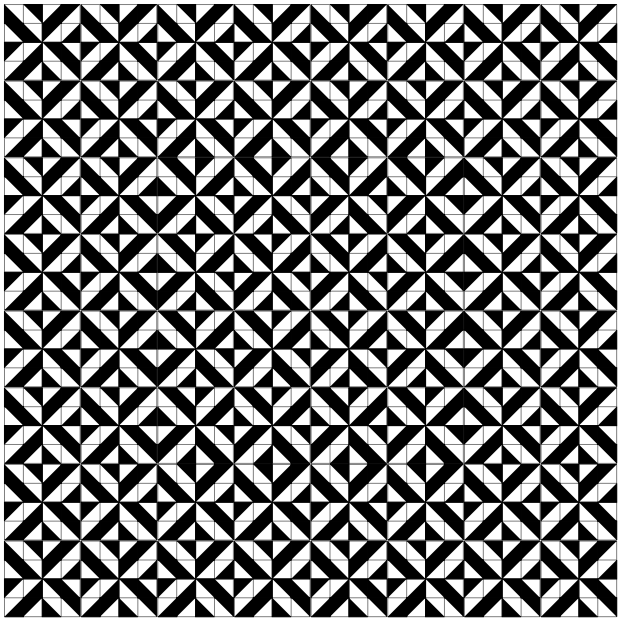
Design using 3333 and 1111



3333

1111

Design using 3113 and 1331



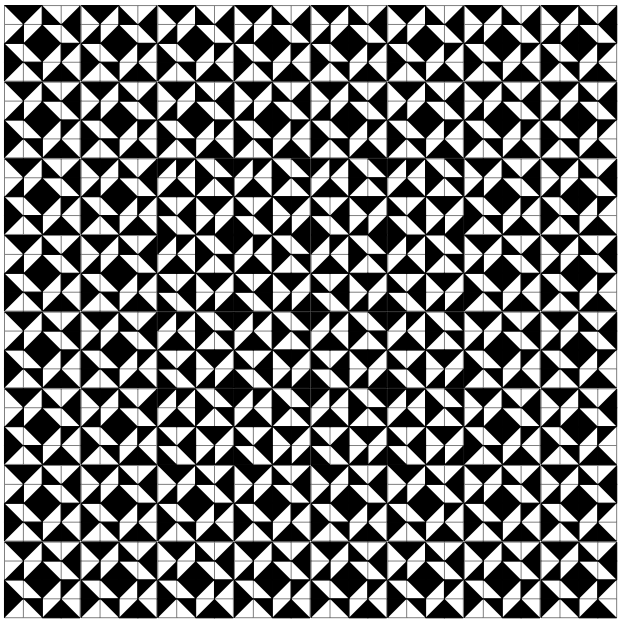
3113

1331

Designs with duals

Like the previous section, these patterns feature a square of one tile type surrounded by another. These tiles are duals of each other, but are not opposite.

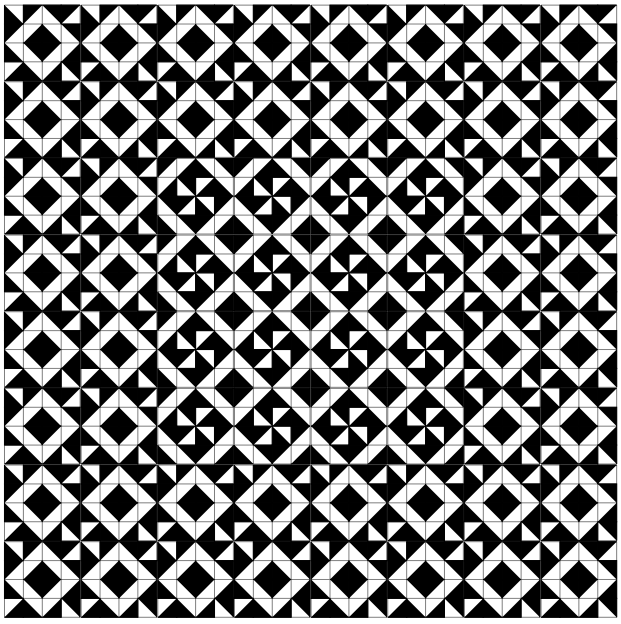
Design using 2201 and 3200



2201

3200

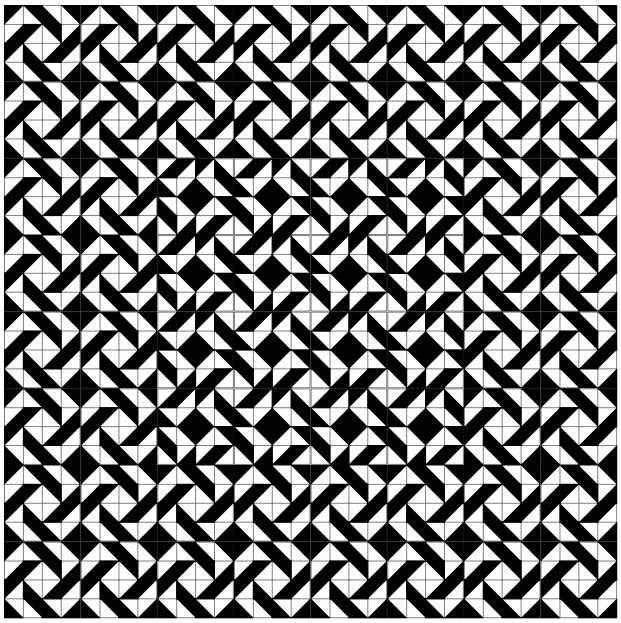
Design using 2003 and 1220



2003

1220

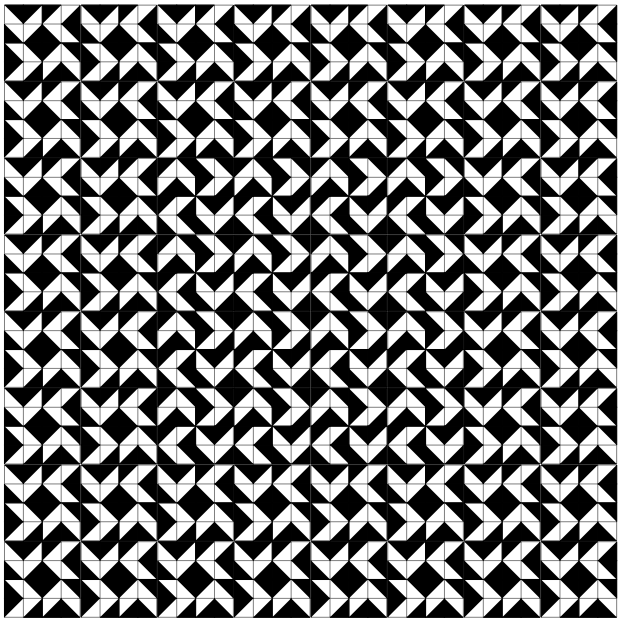
Design using 0210 and 2302



0210

2302

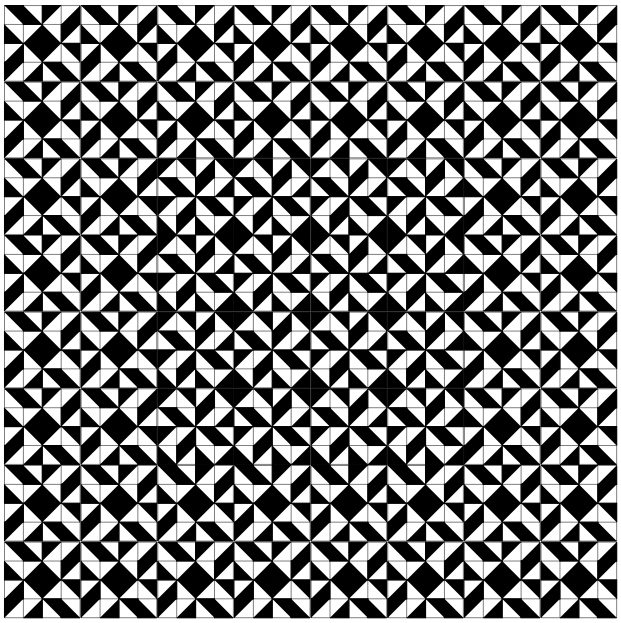
Design using 2301 and 3210



2301

3210

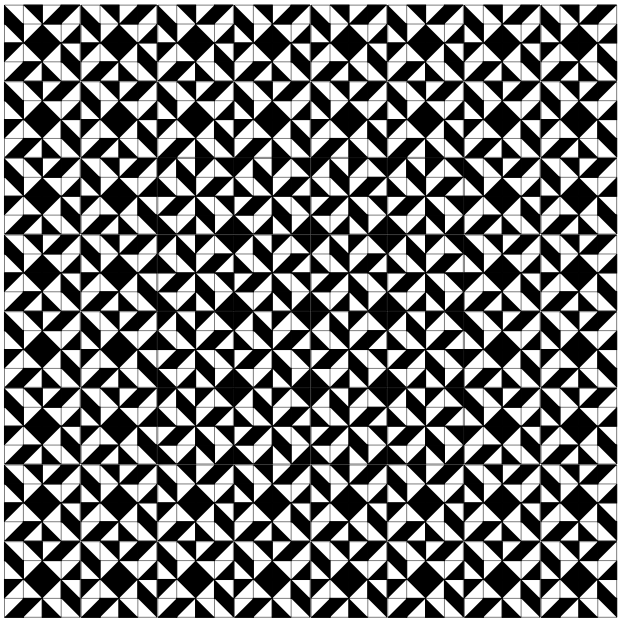
Design using 2331 and 3110



2331

3110

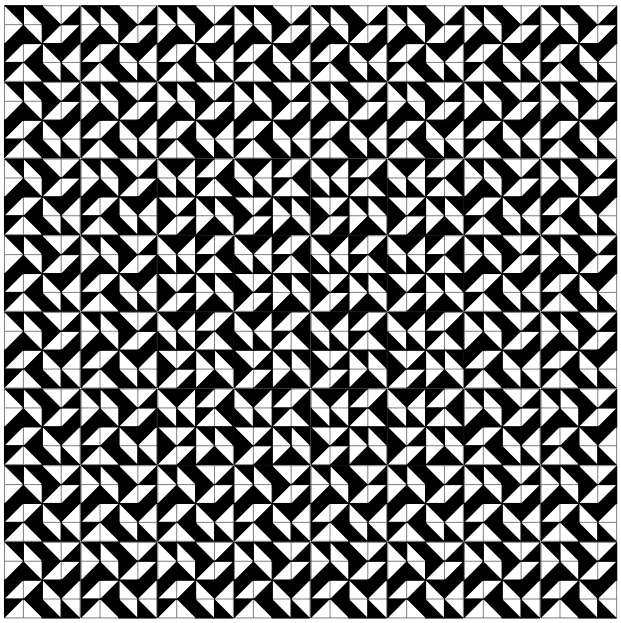
Design using 2113 and 1330



2113

1330

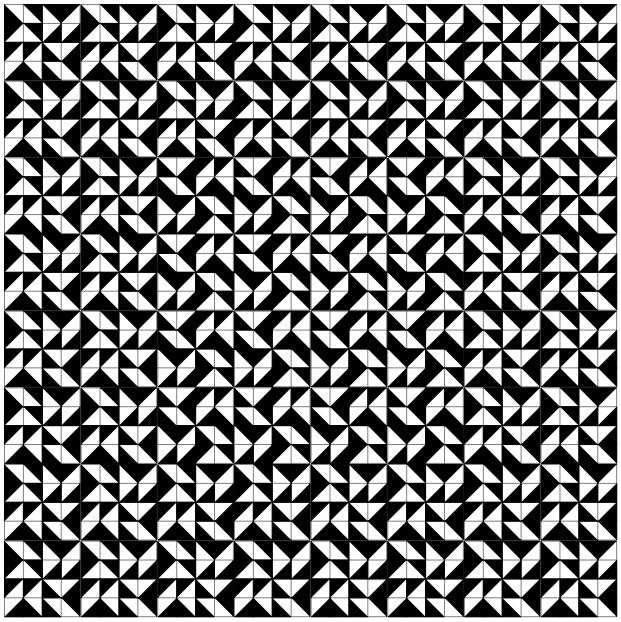
Design using 3211 and 3301



3211

3301

Design using 1013 and 1323



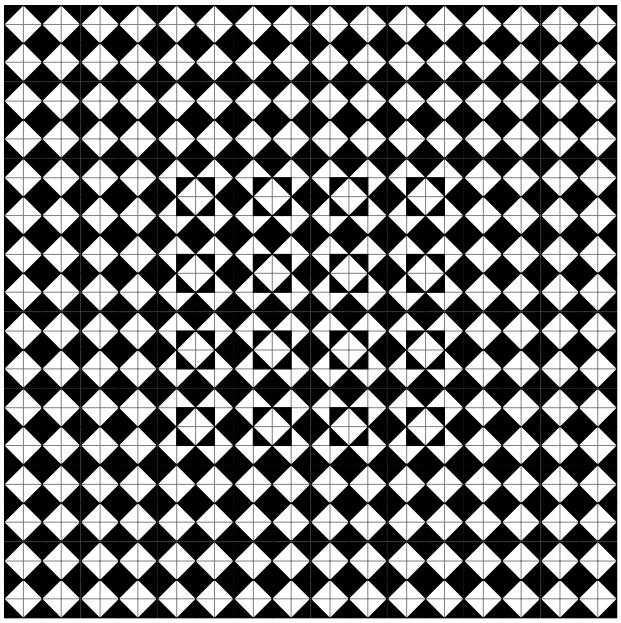
1013

1323

Contrasting designs

The tiles in these square and frame patterns were chosen for their contrast with one another.

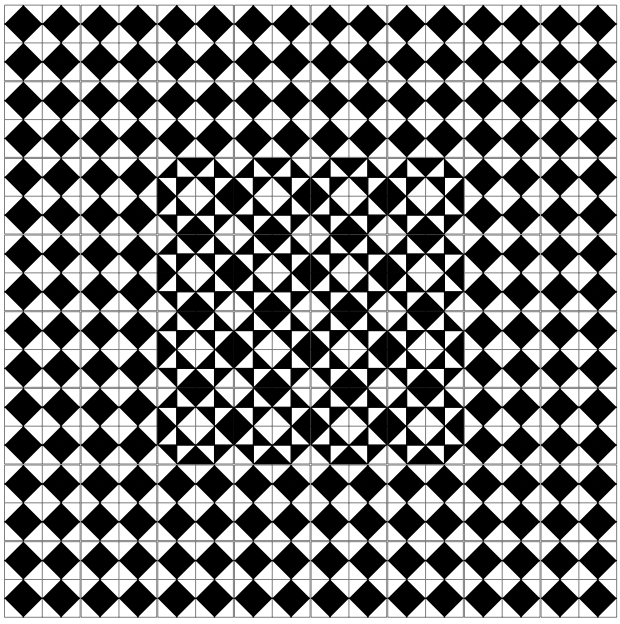
Design using 2312 and 2310



2312

2310

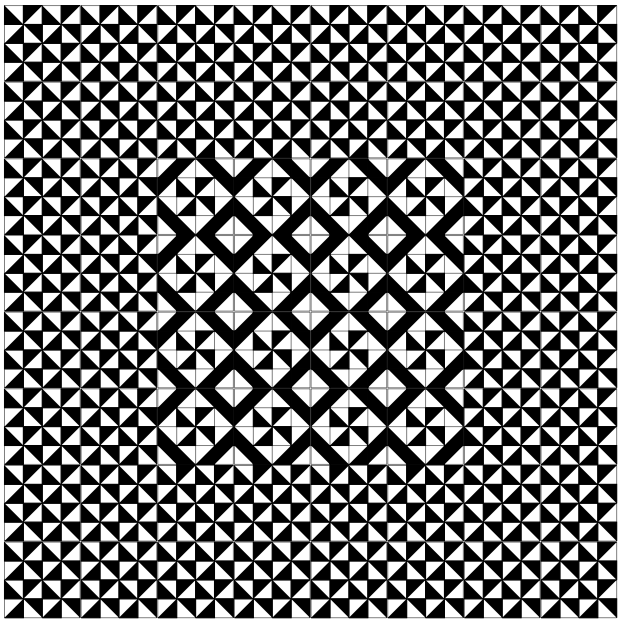
Design using 0312 and 0132



0312

0132

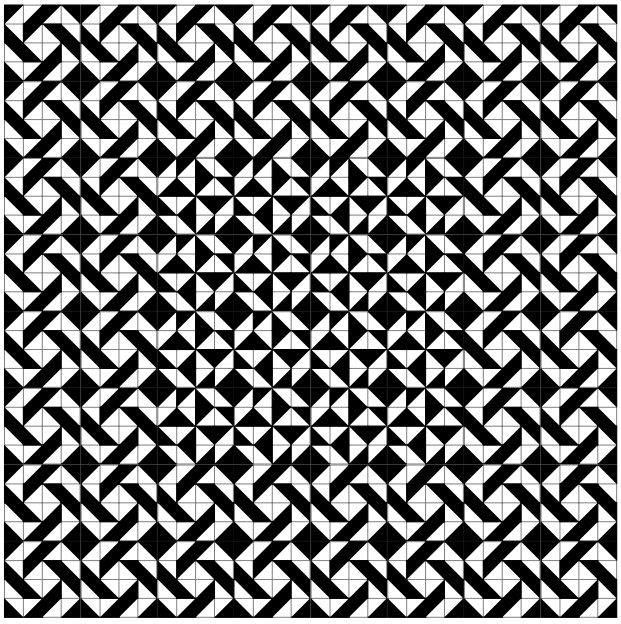
Design using 0221 and 1203



0221

1203

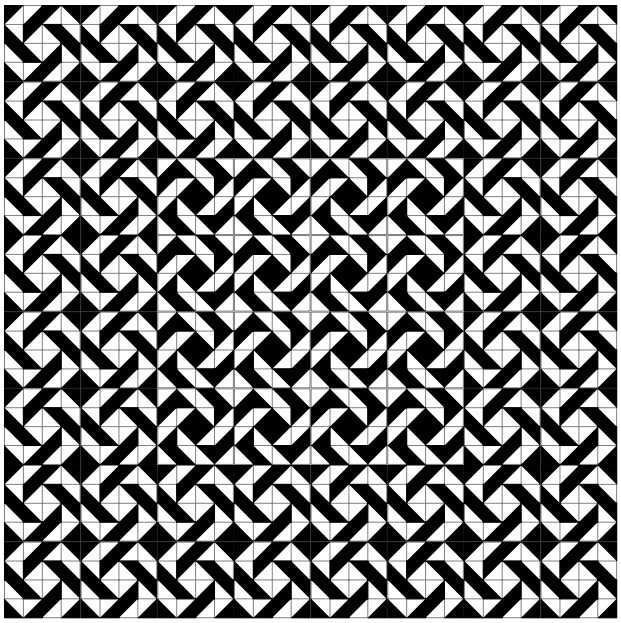
Design using 2201 and 2012



2201

2012

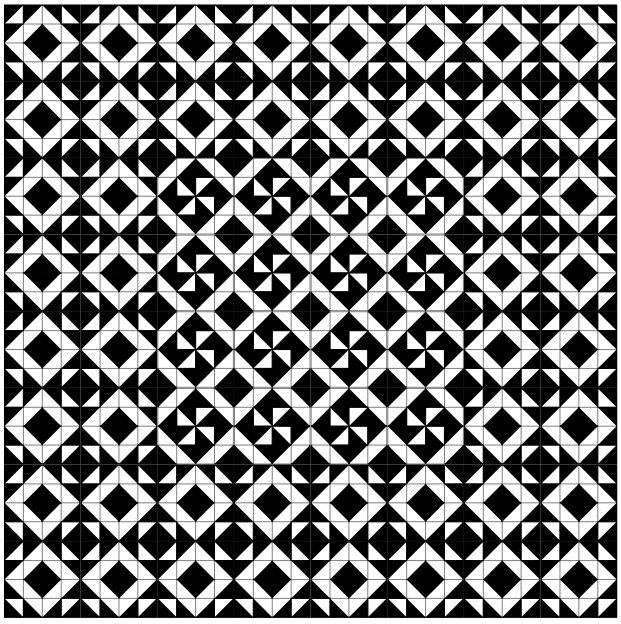
Design using 0120 and 2012



0120

2012

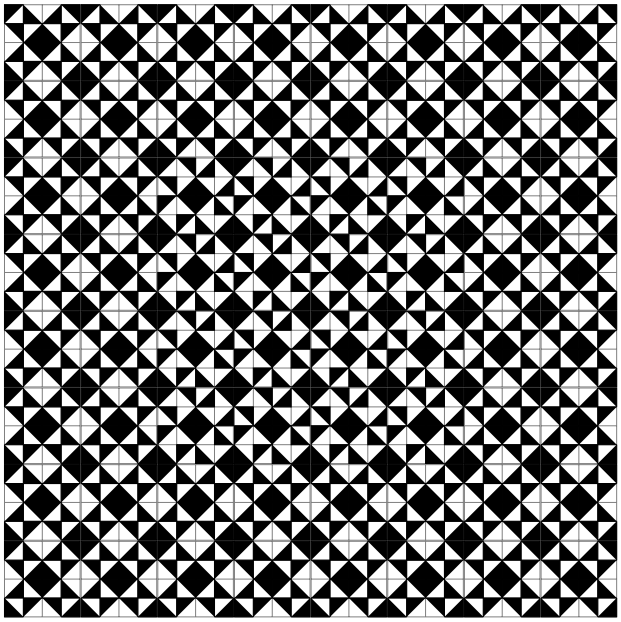
Design using 2003 and 2220



2003

2220

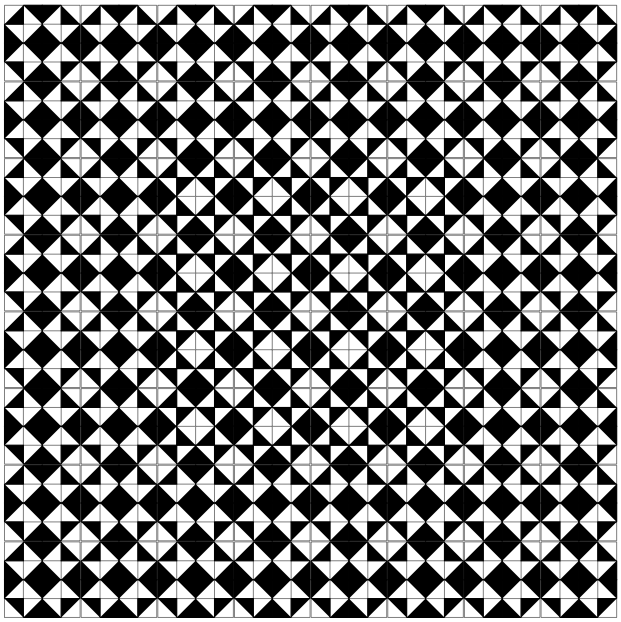
Design using 2330 and 2130



2330

2130

Design using 0312 and 0310



0312

0310

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Sébastien Truchet. *Memoir on combinations*. Memoirs of the Royal Academy of Sciences, 1704.