1 Sunlight

We want to find the energy density per volume for blackbody radiation for given wavelength λ and temperature T.

(a) $\rho(T, \lambda)$

We can start with the expression for U in terms of an integral over k:

$$U = 2\frac{V}{(2\pi)^3} 4\pi\hbar c \int_0^\infty dk\, k^3 \left(e^{\beta\hbar ck} - 1\right)^{-1} \tag{1}$$

We want this integral in terms of λ. We use $k = \frac{2\pi}{\lambda}$, and $dk = -\frac{2\pi}{\lambda^2}\, d\lambda$, to rewrite:

$$U = 2\frac{V}{(2\pi)^3} 4\pi\hbar c \int_{k=0}^{k=\infty} dk \left(\frac{2\pi}{\lambda}\right)^3 \left(e^{\frac{\beta\hbar c 2\pi}{\lambda}} - 1\right)^{-1} \tag{2}$$

$$U = 2\frac{V}{(2\pi)^3} 4\pi\hbar c \int_{\lambda=\infty}^{\lambda=0} \left(-\frac{2\pi}{\lambda^2}\right) d\lambda \left(\frac{2\pi}{\lambda}\right)^3 \left(e^{\frac{\beta\hbar c 2\pi}{\lambda}} - 1\right)^{-1} \tag{3}$$

$$\frac{U}{V} = \frac{(2\pi)^4}{(\pi)^3} \pi\hbar c \int_{\lambda=\infty}^{\lambda=0} \left(-\frac{1}{\lambda^2}\right) d\lambda \left(\frac{1}{\lambda}\right)^3 \left(e^{\frac{\beta\hbar c 2\pi}{\lambda}} - 1\right)^{-1} \tag{4}$$

$$\frac{U}{V} = \frac{(2\pi)^4}{\pi^2} \hbar c \int_{\lambda=\infty}^{\lambda=0} \left(-\frac{1}{\lambda^2}\right) d\lambda \left(\frac{1}{\lambda}\right)^3 \left(e^{\frac{\beta\hbar c 2\pi}{\lambda}} - 1\right)^{-1} \tag{5}$$

$$\frac{U}{V} = -16\pi^2\hbar c \int_{\lambda=\infty}^{\lambda=0} d\lambda\, \frac{1}{\lambda^5} \left(e^{\frac{\beta\hbar c 2\pi}{\lambda}} - 1\right)^{-1} \tag{6}$$

$$\frac{U}{V} = 16\pi^2\hbar c \int_0^\infty d\lambda\, \frac{1}{\lambda^5} \left(e^{\frac{\beta\hbar c 2\pi}{\lambda}} - 1\right)^{-1} \tag{7}$$

We want the energy density per wavelength, so we can identify ρ with the integrand:

$$\rho(\lambda, \beta) = 16\pi^2\hbar c \frac{1}{\lambda^5} \left(e^{\frac{\beta\hbar c 2\pi}{\lambda}} - 1\right)^{-1} \tag{8}$$

This means

$$\rho(\lambda, T) = \frac{16\pi^2\hbar c}{\lambda^5} \left(e^{\frac{2\pi\hbar c}{k_B T\lambda}} - 1\right)^{-1} \tag{9}$$

(b) Finding $\boldsymbol{\lambda}$ to maximise $\boldsymbol{\rho}$

Let's rewrite ρ a bit:

$$\rho(\lambda, T) = \frac{16\pi^2 \hbar c}{\lambda^5 \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right)} \tag{10}$$

To simplify the math a bit, we can notice that maximising ρ should be equivalent to minimising the denominator. Let's do that:

$$0 = \frac{d}{d\lambda} \left(\lambda^5 \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right) \right) \tag{11}$$

$$0 = \lambda^5 \frac{d}{d\lambda} \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right) + \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right) \frac{d}{d\lambda} \lambda^5 \tag{12}$$

Continuing on, pausing only to admire symmetry,

$$-\lambda^5 \frac{d}{d\lambda} \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right) = \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right) \frac{d}{d\lambda} \lambda^5 \tag{13}$$

$$-\lambda^5 e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} \left(-\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda^2} \right) = \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right) \left(5\lambda^4 \right) \tag{14}$$

$$\frac{2\pi\hbar c}{k_{\mathrm{B}} T} \lambda^3 e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} = \left(e^{\frac{2\pi\hbar c}{k_{\mathrm{B}} T \lambda}} - 1 \right) \left(5\lambda^4 \right) \tag{15}$$

Defining $y = \frac{2\pi\hbar c}{k_{\mathrm{B}} T}$:

$$y e^{\frac{y}{\lambda}} = 5\lambda \left(e^{\frac{y}{\lambda}} - 1 \right) \tag{16}$$

$$\frac{y}{\lambda} = 5 \left(1 - e^{-\frac{y}{\lambda}} \right) \tag{17}$$

Listing 1: Mathematica script

```
(* ::Package:: *)

BeginPackage["PS3Prob1Script`"]

CurrentDir = DirectoryName[FileNameJoin[{Directory[], $ScriptCommandLine[[1]]}]]
outFile = OpenWrite[FileNameJoin[{CurrentDir, "prob1ScriptOutput.txt"}]]


Print["Solving equation for Problem 1"]

sols = NSolve[ x == 5(1 - E^(-x)), x]

Print[StringTemplate["Found solutions: `1`"][x /. sols]]
Write[outFile, StringTemplate["Solutions: `1`"][x /. sols]]

EndPackage[]
```

Listing 2: Mathematica output

```
"Solutions: {0., 4.96511}"
```