1 Langevin Function

We are for this problem interested in considering a system with N magnetic moments of magnitude μ, where the ith moment is oriented at some angle θ_i from the vertical. Our Hamiltonian has an external field H, but no coupling between moments:

$$\mathcal{H} = -H \sum_{n=1}^{N} \mu \cos\theta_n \tag{1}$$

I'm making a couple assumptions. We're not told that the moments are constrained to rotate within a particular plane, so I'm going to have each moment expressed with polar angle θ, (which assumes the external field is in the z-direction). We want to begin by finding the equilibrium magnetisation M.

(a) Finding the magnetisation

I think we can start by trying to minimise the free energy. We can use $A = -k_{\mathrm{B}} T \ln Q$. To find Q:

$$Q = \underbrace{\int \cdots \int}_{N} d\Omega_1 \cdots d\Omega_N \, e^{-\beta\mathcal{H}} \tag{2}$$

$$Q = \underbrace{\int \cdots \int}_{N} d\Omega_1 \cdots d\Omega_N \, e^{\beta H \sum_{n=1}^{N} \mu \cos\theta_n} \tag{3}$$

$$Q = \underbrace{\int \cdots \int}_{N} d\Omega_1 \cdots d\Omega_N \prod_{n=1}^{N} e^{\beta H \mu \cos\theta_n} \tag{4}$$

All the integrals are independent of one another, because of the lack of coupling. Additionally, each integral is the same. Thus,

$$\begin{aligned} Q &= \left(\int d\Omega \, e^{\beta H \mu \cos\theta} \right)^N & (5) \\ &= \left(\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \, e^{\beta H \mu \cos\theta} \right)^N & (6) \end{aligned}$$

This becomes

$$Q = \left(2\pi \int_{-1}^{1} d(\cos\theta)\, e^{\beta H \mu \cos\theta}\right)^N \tag{7}$$

$$Q = \left(2\pi \frac{1}{\beta H \mu}\; e^{\beta H \mu \cos\theta}\Big|_{-1}^{1}\right)^N \tag{8}$$

$$Q = \left(2\pi \frac{1}{\beta H \mu}\left(e^{\beta H \mu} - e^{-\beta H \mu}\right)\right)^N \tag{9}$$

$$Q = \left(\frac{2\pi}{\beta H \mu}\right)^N \left(e^{\beta H \mu} - e^{-\beta H \mu}\right)^N \tag{10}$$

$$= \left(\frac{2\pi}{\beta H \mu}\right)^N \left(2\sinh(\beta H \mu)\right)^N \tag{11}$$

To find the average magnetisation, we want to compute

$$\langle M \rangle = \frac{1}{Q} \underbrace{\int \cdots \int}_{N} d\Omega_1 \cdots d\Omega_N \sum_{i=1}^{N} \mu \cos\theta_i e^{-\beta \mathcal{H}(\cos\theta)} \tag{12}$$

For each θ_i, we have $N-1$ copies of the integral we already did for Q with no $\cos\theta_i$ in the integrand, and so we can write this as

$$\langle M \rangle = \frac{1}{Q}\left(\frac{2\pi}{\beta H \mu}\right)^{N-1} \left(2\sinh(\beta H \mu)\right)^{N-1} \sum_{i=1}^{N} \int d\Omega_i\, \mu \cos\theta_i e^{-\beta \mathcal{H}_i} \tag{13}$$

There are N identical integrals being summed together, and we can cancel out the terms in Q. Here, $\mathcal{H}_i$ represents the contribution to the Hamiltonian of the ith atom. Every other part of the Hamiltonian has been integrated over.

$$\langle M \rangle = \frac{1}{\frac{2\pi}{\beta H \mu} 2\sinh(\beta H \mu)} N \int d\Omega_i\, \mu \cos\theta_i e^{-\beta \mathcal{H}_i} \tag{14}$$

$$\frac{M}{N} = \frac{\int d\Omega_i\, \mu \cos\theta_i e^{-\beta \mathcal{H}_i}}{\frac{2\pi}{\beta H \mu} 2\sinh(\beta H \mu)} \tag{15}$$

We've dropped the average brackets on M. Let's integrate:

$$\frac{M}{N} = \frac{\int d\phi\, d\theta \sin\theta \mu \cos\theta e^{H\beta\mu\cos\theta}}{\frac{2\pi}{\beta H\mu} 2\sinh(\beta H\mu)} \tag{16}$$

$$= 2\pi\mu \frac{\int_{-1}^{1} d(\cos\theta) \cos\theta e^{H\beta\mu\cos\theta}}{\frac{2\pi}{\beta H\mu} 2\sinh(\beta H\mu)} \tag{17}$$

$$= \frac{2\pi\mu}{H\mu} \frac{\frac{d}{d\beta} \int_{-1}^{1} d(\cos\theta)\, e^{H\beta\mu\cos\theta}}{\frac{2\pi}{\beta H\mu} 2\sinh(\beta H\mu)} \tag{18}$$

$$= \frac{2\pi\mu}{H\mu} \frac{\frac{d}{d\beta} \frac{1}{H\beta\mu} \left(e^{H\beta\mu} - e^{-H\beta\mu}\right)}{\frac{2\pi}{\beta H\mu} 2\sinh(\beta H\mu)} \tag{19}$$

$$= \frac{2\pi\beta H\mu}{H^2\mu} \frac{\frac{d}{d\beta} \frac{1}{\beta} \sinh(\beta H\mu)}{2\pi \sinh(\beta H\mu)} \tag{20}$$

$$= \frac{\beta}{H} \frac{\frac{-1}{\beta^2} \sinh(\beta H\mu) + \frac{1}{\beta} H\mu \cosh(\beta H\mu)}{\sinh(\beta H\mu)} \tag{21}$$

$$= \frac{1}{H} \frac{H\mu \cosh(\beta H\mu) - \frac{1}{\beta} \sinh(\beta H\mu)}{\sinh(\beta H\mu)} \tag{22}$$

$$= \frac{1}{H} \left(H\mu \coth(\beta H\mu) - \frac{1}{\beta} \right) \tag{23}$$

$$M = \mu N \left(\coth(\beta H\mu) - \frac{1}{\beta\mu H} \right) \tag{24}$$

This is what we wanted to show.

(b) Finding the susceptibility

We want to find $\chi = \frac{\partial M}{\partial H}$ for low temperatures. The Taylor series for coth is $\coth(x) \approx \frac{1}{x} + \frac{x}{3}$. Plugging this in, we find

$$M \approx \mu N \left(\frac{1}{\beta H\mu} + \frac{\beta H\mu}{3} - \frac{1}{\beta H\mu} \right) \tag{25}$$

$$M \approx \frac{\mu N \beta H\mu}{3} \tag{26}$$

$$M \approx \frac{\mu^2 N \beta H}{3} \tag{27}$$

This gives us $\chi \approx \frac{\mu^2 N\beta}{3}$.

(c) Finding the Curie's law coefficient)

Rewriting β, we get

$$\chi = \frac{\mu^2 N}{3k_{\mathrm{B}}T}, \tag{28}$$

which satisfies Curie's law for $C = \frac{\mu^2 N}{3k_{\mathrm{B}}}$