

1 Spin-1 Partition functions

We're told for this problem that we have a 1-D array of N spins s_1, \dots, s_N , where $s_i \in \{-1, 0, 1\}$. We're also, importantly, given periodic boundary conditions $s_{N+1} = s_1$. Our Hamiltonian has no external field:

$$\mathcal{H} = -J \sum_{i=1}^N s_i s_{i+1} \quad (1)$$

We're asked to effectively find the transfer matrix, and show that the Helmholtz free energy A satisfies

$$A = -Nk_B T \log \lambda_+, \quad (2)$$

where λ_+ is the largest eigenvalue of the transfer matrix \mathcal{T} .

(a) Helmholtz free energy

We want to start by finding the partition function Q , as we know that $A = -k_B T \log Q$. For Q , summing over all values of each spin:

$$Q = \sum_{s_1} \dots \sum_{s_N} e^{-\beta \mathcal{H}} \quad (3)$$

$$= \sum_{s_1} \dots \sum_{s_N} e^{-\beta (-J \sum_{i=1}^N s_i s_{i+1})} \quad (4)$$

$$= \sum_{s_1} \dots \sum_{s_N} e^{\sum_{i=1}^N \beta J s_i s_{i+1}} \quad (5)$$

$$= \sum_{s_1} \dots \sum_{s_N} \prod_{i=1}^N e^{\beta J s_i s_{i+1}} \quad (6)$$

This is the point where we can define a matrix \mathcal{T} such that

$$\mathcal{T}_{ab} = e^{\beta J ab}, \quad (7)$$

where the indices run over possible spin values. Substituting this in for Q leads to:

$$Q = \sum_{s_1} \dots \sum_{s_N} \prod_{i=1}^N \mathcal{T}_{s_i s_{i+1}} \quad (8)$$

$$= \sum_{s_1} \dots \sum_{s_N} \mathcal{T}_{s_1 s_2} \mathcal{T}_{s_2 s_3} \dots \mathcal{T}_{s_{N-1} s_N} \mathcal{T}_{s_N s_1} \quad (9)$$

We recognise this sum over repeated indices as basic matrix multiplication:

$$Q = \sum_{s_1} \mathcal{T}_{s_1 s_1}^N \quad (10)$$

It's important here that we had a periodic boundary condition, or we'd have some extra work to do for the first and last spin and the next step wouldn't be possible. We have only the s_1 index left to sum over, and we recognise here the trace of \mathcal{T}^N :

$$Q = \text{Tr } \mathcal{T}^N \quad (11)$$

Why's this good? We know that $\text{Tr } \mathcal{T} = \sum \lambda_i$, where the λ are the eigenvalues of \mathcal{T} , and we also know that $\text{Tr } \mathcal{T}^N = \sum \lambda_i^N$. As with the examples of the transfer matrix method, we can make the assumption that the three eigenvalues for this Hamiltonian's transfer matrix have a T - and J - independent order, and name them λ_+ , λ_0 and λ_- (where $\lambda_+ > \lambda_0 > \lambda_-$). Thus we write

$$Q = \text{Tr } \mathcal{T}^N \quad (12)$$

$$= \lambda_+^N + \lambda_0^N + \lambda_-^N \quad (13)$$

$$= \lambda_+^N \left(1 + \left(\frac{\lambda_0}{\lambda_+} \right)^N + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right) \quad (14)$$

For sufficiently large N , the quantity in parentheses goes to 1 (because each parenthesised quantity therein contained goes to 0), so

$$Q = \lambda_+^N \quad (15)$$

The Helmholtz free energy is trivial:

$$A = -k_B T \log Q \quad (16)$$

$$A = -k_B T \log (\lambda_+^N) \quad (17)$$

$$A = -N k_B T \log \lambda_+ \quad (18)$$

This is what we wanted to show for this part.

(b) The components of the transfer matrix

We can go back to (7):, $\mathcal{T}_{ab} = e^{\beta J_{ab}}$, and recall that a and b ran over the values for the spins, $\{-1, 0, 1\}$. If either a or b is 0, $\mathcal{T}_{ab} = 1$, and the other

components are only slightly more complicated:

$$\mathcal{T} = \begin{pmatrix} s=1 & s=0 & s=-1 \\ e^{\beta J} & 1 & e^{-\beta J} \\ 1 & 1 & 1 \\ e^{-\beta J} & 1 & e^{\beta J} \end{pmatrix} \begin{matrix} s=1 \\ s=0 \\ s=-1 \end{matrix} \quad (19)$$

(c) Equation for λ_+

We want to find the eigenvalues of \mathcal{T} . Let's solve our standard eigenvalue equation:

$$0 = \det(\mathcal{T} - \lambda I) \quad (20)$$

$$= \det \begin{pmatrix} e^{\beta J} - \lambda & 1 & e^{-\beta J} \\ 1 & 1 - \lambda & 1 \\ e^{-\beta J} & 1 & e^{\beta J} - \lambda \end{pmatrix} \quad (21)$$

To spare my fingers the extra typing, we'll define $a = e^{\beta J}$ and $b = e^{-\beta J}$.

$$0 = \det \begin{pmatrix} a - \lambda & 1 & b \\ 1 & 1 - \lambda & 1 \\ b & 1 & a - \lambda \end{pmatrix} \quad (22)$$

$$= (a - \lambda)((1 - \lambda)(a - \lambda) - 1) - ((a - \lambda) - b) + b(1 - (1 - \lambda)b) \quad (23)$$

$$= (a - \lambda)(a + \lambda^2 - \lambda(1 + a) - 1) - a + \lambda + b + b(1 - b + b\lambda) \quad (24)$$

$$= (a - \lambda)(\lambda^2 - \lambda(1 + a) + a - 1) - a + \lambda + b + b - b^2 + b^2\lambda \quad (25)$$

$$= (a - \lambda)(\lambda^2 - \lambda(1 + a) + a - 1) + \lambda(1 + b^2) - a + 2b - b^2 \quad (26)$$

$$= a\lambda^2 - \lambda(a + a^2) + a^2 - a - \lambda^3 + \lambda^2(1 + a) - \lambda a + \lambda + \lambda(1 + b^2) - a + 2b - b^2 \quad (27)$$

$$= -\lambda^3 + \lambda^2(2a + 1) + \lambda(-a - a^2 - a + 1 + 1 + b^2) + a^2 - a - a + 2b - b^2 \quad (28)$$

$$= -\lambda^3 + \lambda^2(2a + 1) + \lambda(b^2 - a^2 - 2a + 2) + a^2 - b^2 - 2a + 2b \quad (29)$$

This cubic equation seems rather ugly, but not unexpectedly so. I'm sure this can be simplified further by plugging in for a and b and writing some hyperbolic trig functions, but I don't think that helps, particularly given that we're not going to be solving this. It's exactly solvable though, and that's quite cool.