

1 Sunlight

We want to find the energy density per volume for blackbody radiation for given wavelength λ and temperature T .

(a) $\rho(T, \lambda)$

We can start with the expression for U in terms of an integral over k :

$$U = 2 \frac{V}{(2\pi)^3} 4\pi \hbar c \int_0^\infty dk k^3 \left(e^{\beta \hbar c k} - 1 \right)^{-1} \quad (1)$$

We want this integral in terms of λ . We use $k = \frac{2\pi}{\lambda}$, and $dk = -\frac{2\pi}{\lambda^2} d\lambda$, to rewrite:

$$U = 2 \frac{V}{(2\pi)^3} 4\pi \hbar c \int_{k=0}^{k=\infty} dk \left(\frac{2\pi}{\lambda} \right)^3 \left(e^{\frac{\beta \hbar c 2\pi}{\lambda}} - 1 \right)^{-1} \quad (2)$$

$$U = 2 \frac{V}{(2\pi)^3} 4\pi \hbar c \int_{\lambda=\infty}^{\lambda=0} \left(-\frac{2\pi}{\lambda^2} \right) d\lambda \left(\frac{2\pi}{\lambda} \right)^3 \left(e^{\frac{\beta \hbar c 2\pi}{\lambda}} - 1 \right)^{-1} \quad (3)$$

$$\frac{U}{V} = \frac{(2\pi)^4}{(\pi)^3} \pi \hbar c \int_{\lambda=\infty}^{\lambda=0} \left(-\frac{1}{\lambda^2} \right) d\lambda \left(\frac{1}{\lambda} \right)^3 \left(e^{\frac{\beta \hbar c 2\pi}{\lambda}} - 1 \right)^{-1} \quad (4)$$

$$\frac{U}{V} = \frac{(2\pi)^4}{\pi^2} \hbar c \int_{\lambda=\infty}^{\lambda=0} \left(-\frac{1}{\lambda^2} \right) d\lambda \left(\frac{1}{\lambda} \right)^3 \left(e^{\frac{\beta \hbar c 2\pi}{\lambda}} - 1 \right)^{-1} \quad (5)$$

$$\frac{U}{V} = -16\pi^2 \hbar c \int_{\lambda=\infty}^{\lambda=0} d\lambda \frac{1}{\lambda^5} \left(e^{\frac{\beta \hbar c 2\pi}{\lambda}} - 1 \right)^{-1} \quad (6)$$

$$\frac{U}{V} = 16\pi^2 \hbar c \int_0^\infty d\lambda \frac{1}{\lambda^5} \left(e^{\frac{\beta \hbar c 2\pi}{\lambda}} - 1 \right)^{-1} \quad (7)$$

We want the energy density per wavelength, so we can identify ρ with the integrand:

$$\rho(\lambda, \beta) = 16\pi^2 \hbar c \frac{1}{\lambda^5} \left(e^{\frac{\beta \hbar c 2\pi}{\lambda}} - 1 \right)^{-1} \quad (8)$$

This means

$$\rho(\lambda, T) = \frac{16\pi^2 \hbar c}{\lambda^5} \left(e^{\frac{2\pi \hbar c}{k_B T \lambda}} - 1 \right)^{-1} \quad (9)$$

(b) Finding λ to maximise ρ

Let's rewrite ρ a bit:

$$\rho(\lambda, T) = \frac{16\pi^2\hbar c}{\lambda^5 \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right)} \quad (10)$$

To simplify the math a bit, we can notice that maximising ρ should be equivalent to minimising the denominator. Let's do that:

$$0 = \frac{d}{d\lambda} \left(\lambda^5 \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right) \right) \quad (11)$$

$$0 = \lambda^5 \frac{d}{d\lambda} \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right) + \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right) \frac{d}{d\lambda} \lambda^5 \quad (12)$$

Continuing on, pausing only to admire symmetry,

$$-\lambda^5 \frac{d}{d\lambda} \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right) = \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right) \frac{d}{d\lambda} \lambda^5 \quad (13)$$

$$-\lambda^5 e^{\frac{2\pi\hbar c}{k_B T \lambda}} \left(-\frac{2\pi\hbar c}{k_B T \lambda^2} \right) = \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right) (5\lambda^4) \quad (14)$$

$$\frac{2\pi\hbar c}{k_B T} \lambda^3 e^{\frac{2\pi\hbar c}{k_B T \lambda}} = \left(e^{\frac{2\pi\hbar c}{k_B T \lambda}} - 1 \right) (5\lambda^4) \quad (15)$$

Defining $y = \frac{2\pi\hbar c}{k_B T}$:

$$y e^{\frac{y}{\lambda}} = 5\lambda \left(e^{\frac{y}{\lambda}} - 1 \right) \quad (16)$$

$$\frac{y}{\lambda} = 5 \left(1 - e^{-\frac{y}{\lambda}} \right) \quad (17)$$

I solved this for $\frac{y}{\lambda}$ in Mathematica, giving us two solutions. There's the pathological solution where $\frac{y}{\lambda} = 0$. This corresponds to λ going to infinity. Looking back at (9), we may find ourselves more interested in the other solution, which we get to be $\frac{y}{\lambda} \approx 4.966$. We can solve this for λ_{\max} for a given temperature:

$$\frac{y}{\lambda_{\max}} = 4.966 \quad (18)$$

$$\lambda_{\max} = \frac{y}{4.966} \quad (19)$$

$$\lambda_{\max} = \frac{2\pi\hbar c}{4.966 k_B T} \quad (20)$$

(c) Solar temperature from λ_{\max}

We're given here that λ_{\max} for sunlight is 480 nm, and we want to find the temperature of the radiation-emitting surface of the sun. We can solve (20) for T :

$$T = \frac{2\pi\hbar c}{4.966k_B\lambda_{\max}} \quad (21)$$

I unnecessarily used Mathematica to solve for T , giving $T = 6036$ K, which seems to be around the number I get when searching around.

(d) Differences between real spectrum and blackbody

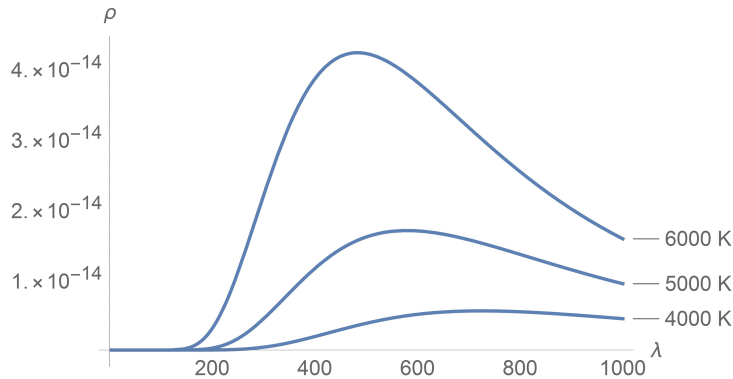


Figure 1: Spectra for 4000 K, 5000 K and 6000 K

The mention of the surface of the sun is important for the earlier part, as different layers of the sun are at different temperatures. Some brief searching suggests that the photosphere ranges from 4000 K to 6000 K. Figure 1 shows some spectra for this temperature range. The sum of these spectra (depending on the total power at each temperature for the overall scale) may probably look different than the spectrum from just one temperature.

Listing 1: Mathematica script

```
1 (* ::Package:: *)
2
3 BeginPackage["PS3Prob1Script`"]
4
5 CurrentDir = DirectoryName[FileNameJoin[{Directory[],
6   $ScriptCommandLine[[1]]}]]
7 ImageDir = FileNameJoin[{CurrentDir, "images"}]
```

```

7  outFile = OpenWrite[FileNameJoin[{CurrentDir, "problScriptOutput.
   txt"}]]
8
9  Print["All_the_stuff_for_problem_1"]
10
11  (* Part b *)
12  sols = NSolve[ x == 5(1 - E(-x)), x]
13
14  Print[StringTemplate["Got_solutions:_\`1\`"][x /. sols]]
15  WriteString[outFile, StringTemplate["Got_solutions_for_part_b:_
   \`1\`\n"][x /. sols]]
16
17  (* Part c *)
18  T[\[Lambda]_] := UnitSimplify[(2 * Pi * Quantity[1, "
   ReducedPlanckConstant"] * Quantity[1, "SpeedOfLight"]) /
   (4.966 * Quantity[1, "BoltzmannConstant"] * \[Lambda])]
19  Print[StringTemplate["Temperature:_\`1\`"][T[ Quantity[480, "
   Nanometers"]]]]
20  WriteString[outFile, StringTemplate["Solar_temperature_for_480_nm:
   _\`1\`\n"][T[ Quantity[480, "Nanometers"]]]]
21
22  (* Part d *)
23
24  rho[\[Lambda]_, T_] := (16 * Pi2 * Quantity[1, "
   ReducedPlanckConstant"] * Quantity[1, "SpeedOfLight"]) / (\[
   Lambda]5 (E((2 * Pi * Quantity[1, "ReducedPlanckConstant"] *
   Quantity[1, "SpeedOfLight"])/( Quantity[1, "BoltzmannConstant"]
   * T * \[Lambda])) - 1))
25
26  Export[FileNameJoin[{ImageDir, "4000Spectrum.jpg"}],
27    Plot[rho[Quantity[1, "Nanometers"], Quantity[4000, "
   Kelvins"]], {1, 1, 1000}, AxesLabel->{\[Lambda], \[Rho]}],
28    ImageResolution -> 1000
29  ]
30
31  Export[FileNameJoin[{ImageDir, "6000Spectrum.jpg"}],
32    Plot[rho[Quantity[1, "Nanometers"], Quantity[6000, "
   Kelvins"]], {1, 1, 1000}, AxesLabel->{\[Lambda], \[Rho]}],
33    ImageResolution -> 1000
34  ]
35
36  Export[FileNameJoin[{ImageDir, "4000And6000Spectrum.jpg"}],
37    Show[
38      Plot[rho[Quantity[1, "Nanometers"], Quantity[4000,
   "Kelvins"]], {1, 1, 1000}, AxesLabel->{\[Lambda], \[Rho]},
   PlotLabels->"4000_K",
39      Plot[rho[Quantity[1, "Nanometers"], Quantity[5000,
   "Kelvins"]], {1, 1, 1000}, AxesLabel->{\[Lambda], \[Rho]},
   PlotLabels->"5000_K"],

```

```

40      Plot[rho[Quantity[1, "Nanometers"], Quantity[6000,
      "Kelvins"]], {1, 1, 1000}, AxesLabel->{\[Lambda], \[Rho]},
      PlotLabels->"6000_K",
41      PlotRange->All, PlotLabels->Automatic
42    ],
43    ImageResolution -> 1000
44 ]
45
46
47 EndPackage[]

```

Listing 2: Mathematica output

```

1 Got solutions for part b: {0., 4.96511}
2 Solar temperature for 480 nm: 6035.95 kelvins

```