1 Langevin Function

We are for this problem interested in considering a system with N magnetic moments of magnitude μ, where the ith moment is oriented at some angle θ_i from the vertical. Our Hamiltonian has an external field H, but no coupling between moments:

$$\mathcal{H} = -H \sum_{n=1}^{N} \mu \cos\theta_n \tag{1}$$

I'm making a couple assumptions. We're not told that the moments are constrained to rotate within a particular plane, so I'm going to have each moment expressed with polar angle θ, (which assumes the external field is in the z-direction). We want to begin by finding the equilibrium magnetisation M.

(a) Finding the magnetisation

I think we can start by trying to minimise the free energy. We can use $A = -k_\mathrm{B} T \ln Q$. To find Q:

$$Q = \underbrace{\int \cdots \int}_{N} d\Omega_1 \cdots d\Omega_N \, e^{-\beta\mathcal{H}} \tag{2}$$

$$Q = \underbrace{\int \cdots \int}_{N} d\Omega_1 \cdots d\Omega_N \, e^{\beta H \sum_{n=1}^{N} \mu \cos\theta_n} \tag{3}$$

$$Q = \underbrace{\int \cdots \int}_{N} d\Omega_1 \cdots d\Omega_N \prod_{n=1}^{N} e^{\beta H \mu \cos\theta_n} \tag{4}$$

All the integrals are independent of one another, because of the lack of coupling. Additionally, each integral is the same. Thus,

$$Q = \left(\int d\Omega \, e^{\beta H \mu \cos\theta} \right)^N \tag{5}$$

$$= \left(\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \, e^{\beta H \mu \cos\theta} \right)^N \tag{6}$$

This becomes

$$Q = \left(2\pi \int_{-1}^{1} d(\cos\theta)\, e^{\beta H\mu\cos\theta}\right)^N \tag{7}$$

$$Q = \left(2\pi \frac{1}{\beta H\mu}\; e^{\beta H\mu\cos\theta}\Big|_{-1}^{1}\right)^N \tag{8}$$

$$Q = \left(2\pi \frac{1}{\beta H\mu}\left(e^{\beta H\mu} - e^{-\beta H\mu}\right)\right)^N \tag{9}$$

$$Q = \left(\frac{2\pi}{\beta H\mu}\right)^N \left(e^{\beta H\mu} - e^{-\beta H\mu}\right)^N \tag{10}$$

This gives us a plausible Helmholtz free energy:

$$\begin{aligned} A &= -\frac{1}{\beta}\ln Q && (11)\\ &= -\frac{1}{\beta}\ln\left(\left(\frac{2\pi}{\beta H\mu}\right)^N \left(e^{\beta H\mu} - e^{-\beta H\mu}\right)^N\right) && (12)\\ &= -\frac{N}{\beta}\left(\ln\left(\frac{2\pi}{\beta H\mu}\right) + \ln\left(e^{\beta H\mu} - e^{-\beta H\mu}\right)\right) && (13) \end{aligned}$$

I think that implicit in the question is that we want to find the equilibrium M for fixed H, which means that our natural potential will be the Gibbs free energy $G(T, H)$ rather than $A(T, M)$. We use $G = A - HM$:

$$G = -\frac{N}{\beta}\left(\ln\left(\frac{2\pi}{\beta H\mu}\right) + \ln\left(e^{\beta H\mu} - e^{-\beta H\mu}\right)\right) - HM \tag{14}$$

Now, we can minimise this with respect to M:

$$\begin{aligned} 0 &= \frac{dG}{dM} && (15)\\ &= \frac{d}{dM}\left(-\frac{N}{\beta}\left(\ln\left(\frac{2\pi}{\beta H\mu}\right) + \ln\left(e^{\beta H\mu} - e^{-\beta H\mu}\right)\right) - HM\right) && (16)\\ &= \frac{N}{\beta}\frac{d}{dM}\left(\ln\left(\frac{2\pi}{\beta H\mu}\right) + \ln\left(e^{\beta H\mu} - e^{-\beta H\mu}\right)\right) + H + M\frac{dH}{dM} && (17)\\ &= \frac{N}{\beta}\frac{d}{dM}\left(\ln\left(\frac{1}{H}\right) + \ln(2\sinh(\beta H\mu))\right) + H + M\frac{dH}{dM} && (18)\\ &= \frac{N}{\beta}\frac{d}{dM}\left(\ln\left(\frac{1}{H}\right) + \ln(\sinh(\beta H\mu))\right) + H + M\frac{dH}{dM} && (19) \end{aligned}$$

This simplifies things a bit, noting that we can always drop the derivatives of constant factors in logarithms.

$$0 = \frac{N}{\beta}\frac{d}{dM}\left(\ln\left(\frac{1}{H}\right) + \ln(\sinh(\beta H\mu))\right) + H + M\frac{dH}{dM} \tag{20}$$

$$0 = \frac{N}{\beta}\left(H\frac{-1}{H^2}\frac{dH}{dM} + \frac{\beta\mu}{\sinh(\beta H\mu)}\cosh(\beta H\mu)\frac{dH}{dM}\right) + H + M\frac{dH}{dM} \tag{21}$$

$$0 = \frac{N}{\beta}\frac{dH}{dM}\left(-\frac{1}{H} + \beta\mu\coth(\beta H\mu)\right) + H + M\frac{dH}{dM} \tag{22}$$

Solving for M:

$$M\frac{dH}{dM} = -\frac{N}{\beta}\frac{dH}{dM}\left(-\frac{1}{H} + \beta\mu\coth(\beta H\mu)\right) + H \tag{23}$$