1   Explicit parts of Lindhard function

We want to find the explicit real and imaginary parts of the Lindhard function. To begin with, we can start with the case where $\nu \to 0$, which means long relaxation times.

We have our Lindhard form

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/k v_f)}{\omega + i\nu\, f_l((\omega + i\nu)/k v_f)}, \tag{1}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1}. \tag{2}$$

1.1   Pines result

From Pines, we have the forms

$$\mathrm{Re}[\epsilon_l] = 1 + \frac{k_{TF}^2}{k^2} \left(\frac{1}{2} + \frac{k_F}{4k} \left[\left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \ln \left[\frac{\omega - k v_F - \frac{\hbar k^2}{2m}}{\omega + k v_F - \frac{\hbar k^2}{2m}} \right] \right. \right.$$

$$\left. \left. + \left(1 - \frac{\left(\omega + \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \ln \left[\frac{\omega + k v_F + \frac{\hbar k^2}{2m}}{\omega - k v_F + \frac{\hbar k^2}{2m}} \right] \right] \right) \tag{3}$$

$$\mathrm{Im}[\epsilon_l] = \begin{cases} \dfrac{\pi}{2} \dfrac{\omega}{k v_F} \dfrac{k_{TF}^2}{k^2}, & \omega \leq k v_F - \frac{\hbar k^2}{2m} \\[2ex] \dfrac{\pi}{4} \dfrac{k_F}{k} \left(1 - \dfrac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \dfrac{k_{TF}^2}{k^2}, & k v_F - \frac{\hbar k^2}{2m} \leq \omega \leq k v_F + \frac{\hbar k^2}{2m} \\[2ex] 0, & \omega \geq k v_F + \frac{\hbar k^2}{2m} \end{cases} \tag{4}$$

1.2   Long relaxation time forms of the logs

In order to analyse the $\nu \to 0$ limit, we can start by looking at what happens to the logarithms in the Lindhard function:

$$\ln \frac{\omega + i\nu + k v_F}{\omega + i\nu - k v_F} \tag{5}$$

The first thing to note is that the numerator will always have a very small, positive argument, while the denominator will have a small argument which may be positive or negative. As Lindhard mentions, these logarithms should all have imaginary parts between $\pm i\pi$. This effectively means we can treat each logarithm as giving the principal value, which give us a result that looks like

$$\ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right), \tag{6}$$

where θ_+ and θ_- are the arguments of the numerator and denominator. For small ν, θ_+ is proportional to ν, as it'll be determined by an arcsine. However, the denominator may be negative, which would contribute a factor of $\theta_- = +i\pi$ (with a plus sign because ν would be just above the real line).

$$\ln \frac{\omega + i\nu + kv_F}{\omega + i\nu - kv_F} = \ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right) \tag{7}$$

$$= \ln \frac{\sqrt{(\omega + kv_F)^2}}{\sqrt{(\omega - kv_F)^2}} - \sigma i\pi \tag{8}$$

$$= \ln \left| \frac{\omega + kv_F}{\omega - kv_F} \right| - \sigma i\pi, \tag{9}$$

where $\sigma = 1$ if $\omega < kv_F$, and 0 otherwise.

1.3 Taking long scattering time limit

To make our constants line up with the Pines form, we'll use the definition $k_{TF}^2 = \frac{3\omega_p^2}{v_F^2}$, giving us

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \tag{10}$$

We can't simply set $\nu = 0$, as we need to respect the relation

$$\frac{1}{x + i\delta} = \frac{1}{x} - i\pi\delta(x) \tag{11}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}. \tag{12}$$

We can eliminate all the terms proportional to ν in the numerator, as they will disappear as we take our limit:

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)} \tag{13}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega\left(1 - \frac{\omega}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)} \tag{14}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \frac{2kv_F - \omega\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| + i\omega\sigma}{\omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)} \tag{15}$$

We can now look at just the denominator:

$$= \omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right) \tag{16}$$

$$= \omega + i\nu - i\nu\frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right) \tag{17}$$

$$= \omega + i\nu + \frac{\nu^2 - i\nu\omega}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right) \tag{18}$$