1 Explicit parts of Lindhard function

We want to find the explicit real and imaginary parts of the Lindhard function. To begin with, we can start with the case where $\nu \to 0$, which means long relaxation times.

We have our Lindhard form

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \tag{1}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1}. \tag{2}$$

1.1 Pines result

From Pines, we have the forms

$$\begin{aligned} \mathrm{Re}[\epsilon_l] = 1 + \frac{k_{TF}^2}{k^2}\left(\frac{1}{2} + \frac{k_F}{4k}\left[\left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\right) \ln\left[\frac{\omega - kv_F - \frac{\hbar k^2}{2m}}{\omega + kv_F - \frac{\hbar k^2}{2m}}\right]\right.\right. \\ \left.\left. + \left(1 - \frac{\left(\omega + \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\right) \ln\left[\frac{\omega + kv_F + \frac{\hbar k^2}{2m}}{\omega - kv_F + \frac{\hbar k^2}{2m}}\right]\right]\right) \end{aligned} \tag{3}$$

$$\mathrm{Im}[\epsilon_l] = \begin{cases} \frac{\pi}{2} \frac{\omega}{kv_F} \frac{k_{TF}^2}{k^2}, & \omega \le kv_F - \frac{\hbar k^2}{2m} \\ \frac{\pi}{4} \frac{k_F}{k} \left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\right) \frac{k_{TF}^2}{k^2}, & kv_F - \frac{\hbar k^2}{2m} \le \omega \le kv_F + \frac{\hbar k^2}{2m} \\ 0, & \omega \ge kv_F + \frac{\hbar k^2}{2m} \end{cases} \tag{4}$$

1.2 Long relaxation time forms of the logs

In order to analyse the $\nu \to 0$ limit, we can start by looking at what happens to the logarithms in the Lindhard function:

$$\ln \frac{\omega + i\nu + kv_F}{\omega + i\nu - kv_F} \tag{5}$$

The first thing to note is that the numerator will always have a very small, positive argument, while the denominator will have a small argument which may be positive or negative. As Lindhard mentions, these logarithms should all have imaginary parts between $\pm i\pi$. This effectively means we can treat each logarithm as giving the principal value, which give us a result that looks like

$$\ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right), \tag{6}$$

where θ_+ and θ_- are the arguments of the numerator and denominator. For small ν, θ_+ is proportional to ν, as it'll be determined by an arcsine. However, the denominator may be negative, which would contribute a factor of $\theta_- = +i\pi$ (with a plus sign because ν would be just above the real line).

$$\begin{aligned}
\ln \frac{\omega + i\nu + kv_F}{\omega + i\nu - kv_F} &= \ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right) && (7)\\
&= \ln \frac{\sqrt{(\omega + kv_F)^2}}{\sqrt{(\omega - kv_F)^2}} - \sigma i\pi && (8)\\
&= \ln \left| \frac{\omega + kv_F}{\omega - kv_F} \right| - \sigma i\pi, && (9)
\end{aligned}$$

where $\sigma = 1$ if $\omega < kv_F$, and 0 otherwise.

1.3 Taking long scattering time limit

To make our constants line up with the Pines form, we'll use the definition $k_{TF}^2 = \frac{3\omega_p^2}{v_F^2}$, giving us

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\, f_l(\,(\omega + i\nu)/kv_f\,)}{\omega + i\nu f_l(\,(\omega + i\nu)/kv_f\,)}, \tag{10}$$

We can't simply set $\nu = 0$, as we need to respect the relation

$$\frac{1}{x + i\delta} = \frac{1}{x} - i\pi\delta(x) \tag{11}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}. \tag{12}$$

We can eliminate all the terms proportional to ν in the numerator, as they will disappear as we take our limit:

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)} \tag{13}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega\left(1 - \frac{\omega}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)} \tag{14}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \frac{2kv_F - \omega\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| + i\omega\pi\sigma}{\omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)} \tag{15}$$

We can now look at just the denominator:

$$= \omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right) \tag{16}$$

$$= \omega + i\nu - i\nu\frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right) \tag{17}$$

$$= \omega + i\nu + \frac{\nu^2 - i\nu\omega}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right) \tag{18}$$

$$= \omega + \frac{\nu^2}{2kv_F}L - \frac{\nu\omega\pi\sigma}{2kv_F} + i\nu - i\frac{\nu\omega}{2kv_F}L - i\frac{\nu^2\sigma\pi}{2kv_F}, \tag{19}$$

where $L = \ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right|$. We can notice here that all the imaginary terms are proportional to ν. We can ignore the ν^2 term, as it will go to zero faster than the other terms. We can see that we essentially have two regimes: if $\omega < 2kv_F$, this will have a positive imaginary part, and if $\omega > 2kv_F$, the imaginary part will be negative.

We can also notice that the only real part that will survive the limiting process is simply ω (which is of course clear from the original form of the denominator anyway). This lets us essentially write the denominator as

$$\omega + i\nu\zeta C, \tag{20}$$

where I'm defining ζ as 1 if $\omega < 2kv_F$, and -1 otherwise. We also can note that C is an irrelevant positive constant; when we take the long scattering time limit, all that matters is that the imaginary part is proportional to ν.

Plugging this back into (15) gives us

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2}\frac{\omega}{2kv_F}\frac{2kv_F - \omega\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| + i\omega\pi\sigma}{\omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)} \tag{21}$$

$$\begin{aligned}\epsilon_l &= 1 + \frac{k_{TF}^2}{k^2}\frac{\omega}{2kv_F}\frac{2kv_F - \omega\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| + i\omega\pi\sigma}{\omega + i\nu\zeta C} && (22)\\ &= 1 + \frac{k_{TF}^2}{k^2}\frac{\omega}{2kv_F}\left(2kv_F - \omega\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| + i\omega\pi\sigma\right)\left(\frac{1}{\omega} - i\pi\zeta\delta(\omega)\right) && (23)\\ &= 1 + \frac{k_{TF}^2}{k^2}\frac{\omega}{2kv_F}\left(\frac{2kv_F}{\omega} - L + i\pi\sigma - i\pi\zeta 2kv_F\delta(\omega) + (iL\zeta + \sigma\zeta)\,\pi\omega\delta(\omega)\right) && (24)\end{aligned}$$

We can eliminate the terms proportional to $\omega\delta(\omega)$:

$$\begin{aligned}\epsilon_l &= 1 + \frac{k_{TF}^2}{k^2}\frac{\omega}{2kv_F}\left(\frac{2kv_F}{\omega} - L + i\pi\sigma - i\pi\zeta 2kv_F\delta(\omega)\right) && (25)\\ &= 1 + \frac{k_{TF}^2}{k^2}\left(1 - \frac{\omega}{2kv_F}L + \frac{\omega}{2kv_F}i\pi\sigma - i\pi\zeta\omega\delta(\omega)\right) && (26)\\ &= 1 + \frac{k_{TF}^2}{k^2}\left(1 - \frac{\omega}{2kv_F}L + \frac{\omega}{2kv_F}i\pi\sigma\right) && (27)\end{aligned}$$