1 Qubit Relaxation Time

1.1 The quasi-static limit

We can start by looking at

$$\chi_{zz}^{E}(z,z,\omega) = \frac{\hbar}{\epsilon_0}\,\mathrm{Re}\int_0^\infty dp\,\frac{p^3}{q}e^{2iqz}r_p(p) \tag{1}$$

Here, we have

$$q = \begin{cases} \sqrt{\frac{\omega^2}{c^2}-p^2}, & p^2 \le \frac{\omega^2}{c^2} \\ i\sqrt{p^2-\frac{\omega^2}{c^2}}, & p^2 > \frac{\omega^2}{c^2} \end{cases} \tag{2}$$

If we look at the case where $\omega = 6\pi \times 10^8\,\mathrm{s}^{-1}$, the cutoff for real or imaginary q will be when $p = 2\pi\,\mathrm{m}^{-1}$.

If we assume that $\mathrm{Im}\, r_p$ doesn't decay too quickly, this integral will be dominated by values of p larger than this, which lets us make the substitution that $\frac{\omega}{c} \to 0$, which means this integral will reduce to

$$\chi_{zz}^{E}(z,z,\omega) = \frac{\hbar}{\epsilon_0}\int_0^\infty dp\,p^2 e^{-2pz}\,\mathrm{Im}\,r_p(p,\omega) \tag{3}$$

The note that we're effectively taking $c \to \infty$ is important, as we still shouldn't necessarily assume that we can take $\omega \to 0$ in r_p.

1.2 Non-local reflection coefficient

We can look specifically at what r_p will be in the non-local case:

$$r_p(p,\omega) = \frac{1 - \frac{2p}{\pi}\int_0^\infty d\kappa\,\frac{1}{k^2\epsilon_l(k\omega)}}{1 + \frac{2p}{\pi}\int_0^\infty d\kappa\,\frac{1}{k^2\epsilon_l(k\omega)}} \tag{4}$$

where $k^2 = p^2 + \kappa^2$ and

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2}\,\frac{(\omega + i\nu)\,f_l((\omega+i\nu)/kv_f)}{\omega + i\nu f_l((\omega+i\nu)/kv_f)}. \tag{5}$$

All of the interesting behaviour here comes from the integral, which we might name $I = \int_0^\infty d\kappa\,\frac{1}{k^2\epsilon_l(k,m\omega)}$. Knowing that we will eventually need to find $\mathrm{Im}\,r_p$, we might find utility in also writing $I = I_1 + iI_2$ and noting that

$$\mathrm{Im}\,r_p = \mathrm{Im}\,\frac{1-I}{1+I} \tag{6}$$

$$
\begin{aligned}
&= \operatorname{Im} \frac{1 - I_1 - iI_2}{1 + I_1 + iI_2} && (7)\\
&= \operatorname{Im} \frac{1 - I_1 - iI_2}{1 + I_1 + iI_2} \frac{I + I_1 - iI_2}{I + I_1 - iI_2} && (8)\\
&= \operatorname{Im} \frac{(1 - I_1)(1 + I_1) - I_2^2 - iI_2(1 - I_1 + 1 + I_1)}{(1 + I_1)^2 + I_2^2} && (9)\\
&= \operatorname{Im} \frac{(1 - I_1)(1 + I_1) - I_2^2 - 2iI_2}{(1 + I_1)^2 + I_2^2} && (10)\\
&= \frac{-2I_2}{(1 + I_1)^2 + I_2^2}. && (11)
\end{aligned}
$$

This gives us some idea of how $\operatorname{Im} r_p$ should behave, at least once we can write out the integral I.

$$
I = \int_0^\infty d\kappa \, \frac{1}{k^2 \epsilon_l(k, \omega)} \tag{12}
$$