

## 1 Reducing Lindhard to Drude

We want to see how we can reduce the Lindhard dielectric function

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \quad (1)$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1} \quad (2)$$

We can reduce things in the  $k \rightarrow 0$  limit. The first half of (1) has the simple  $\frac{1}{k^2}$  dependence, so we can look at how the rest of it behaves to start with.

### 1.1 f

$$f_l((\omega + i\nu)/kv_F) = 1 - \frac{(\omega + i\nu)/kv_F}{2} \ln \frac{(\omega + i\nu)/kv_F + 1}{(\omega + i\nu)/kv_F - 1} \quad (3)$$

Defining  $\eta = \omega + i\nu$ :

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \quad (4)$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2kv_F} \ln \frac{\eta + kv_F}{\eta - kv_F} \quad (5)$$

$$f_l = 1 - \frac{\eta}{2v_F} \frac{\ln \frac{\eta + kv_F}{\eta - kv_F}}{k} \quad (6)$$

$$\lim_{k \rightarrow 0} f_l = 1 - \frac{\eta}{2v_F} \frac{d \ln \frac{\eta + kv_F}{\eta - kv_F}}{dk} \quad (7)$$

$$= 1 - \frac{\eta}{2v_F} \frac{\eta - kv_F}{\eta + kv_F} \frac{v_F (\eta - kv_F) + v_F (\eta + kv_F)}{(\eta - kv_F)^2} \quad (8)$$

$$= 1 - \frac{\eta}{2} \frac{\eta - kv_F}{\eta + kv_F} \frac{(\eta - kv_F) + (\eta + kv_F)}{(\eta - kv_F)^2} \quad (9)$$

$$= 1 - \frac{\eta}{2} \frac{1}{\eta + kv_F} \frac{2\eta}{\eta - kv_F} \quad (10)$$

$$= 1 - \frac{\eta^2}{\eta^2 - k^2 v_F^2} \quad (11)$$

$$= \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2} \quad (12)$$

$$\lim_{k \rightarrow 0} f_l = 0 \quad (13)$$

Note that this goes to 0 for  $k \rightarrow 0$ .

## 1.2 Series expansion of $f$

The previous section gives the limit, but having the actual series expansion is probably more valuable. Again, with  $\eta = \omega + i\nu$ ,

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \quad (14)$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2v_F k} \ln \frac{\eta + kv_F}{\eta - kv_F} \quad (15)$$

We want to expand up to  $k^2$  overall, to cancel out the  $k^2$  in the denominator of (1). We're looking at the function.

$$\frac{\ln(\eta + kv_F)}{k} - \frac{\ln(\eta - kv_F)}{k} \quad (16)$$

Generally, the derivatives of  $\frac{g(a \pm x)}{x}$  are

$$\left( \frac{g(x)}{x} \right)' = \frac{\pm g'}{x} - \frac{g}{x^2} \quad (17)$$

$$\left( \frac{g(x)}{x} \right)'' = \frac{g''}{x} - \frac{\pm 2g'}{x^2} + \frac{2g}{x^3} \quad (18)$$

$$\left( \frac{g(x)}{x} \right)''' = \frac{\pm g'''}{x} - \frac{3g''}{x^2} + \frac{\pm 6g'}{x^3} - \frac{6g}{x^4} \quad (19)$$

When we take the difference, we see that we'll only end up keeping (and doubling) the terms of odd derivatives. Thus, up to this order, the series for  $\frac{g(a+x) - g(a-x)}{x}$  will look like:

$$\frac{1}{2} \frac{g(a+x) - g(a-x)}{x} = x \frac{g'}{x} - \frac{1}{2} x^2 \frac{2g'}{x^2} + \frac{1}{6} x^3 \left( \frac{g'''}{x} + \frac{6g'}{x^3} \right) \quad (20)$$

$$= g' - g' + \frac{1}{6} x^2 g''' + g' \quad (21)$$

$$\frac{g(a+x) - g(a-x)}{x} = 2g' + \frac{1}{3} x^2 g''' + \mathcal{O}(x^4) \quad (22)$$

This type of result is to be expected: we are starting with an even function. For  $g = \ln(\eta + kv_F)$ , we have

$$g'(k=0) = \frac{v_F}{\eta} \quad (23)$$

$$g'''(k=0) = \frac{2v_F^3}{\eta^3} \quad (24)$$

Plugging these into (15) gives us:

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2v_F k} \ln \frac{\eta + kv_F}{\eta - kv_F} \quad (25)$$

$$= 1 - \frac{\eta}{2v_F} \left( 2\frac{v_F}{\eta} + \frac{1}{3}k^2 \frac{2v_F^3}{\eta^3} \right) \quad (26)$$

$$= 1 - 1 - \frac{1}{3}k^2 \frac{v_F^2}{\eta^2} \quad (27)$$

$$= -\frac{k^2 v_F^2}{3\eta^2} \quad (28)$$

This gives us a simple approximation for  $f_l$  in the long wavelength limit.

### 1.3 Back to dielectric

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)} \quad (29)$$

In the denominator, we can note that  $\omega$  should dominate  $i\nu f$ , because  $f$  goes to zero, so we can simplify that.

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega} \quad (30)$$

Using (28), we get

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega} \quad (31)$$

$$= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) \frac{-k^2 v_F^2}{3\eta^2}}{\omega} \quad (32)$$

$$= 1 - 3\omega_p^2 \frac{\eta \frac{1}{3\eta^2}}{\omega} \quad (33)$$

$$= 1 - \frac{\omega_p^2}{\omega(\omega + i\nu)} \quad (34)$$

This is the Drude limit.