

Main notebook

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1 Drude model parameters

The assumptions of the Drude model are simple: we have interaction-free electrons that occasionally undergo some scattering process during a time dt with probability $\frac{dt}{\tau}$, where τ is some phenomenological parameter. This scattering will randomise electron momentum.

Our ultimate goal will be to find the conductivity σ and the dielectric constant ϵ in the Drude model, with Drude relaxation time τ , electron density n and electron mass m . We'll find

$$\sigma_{\text{DC}} = \frac{ne^2\tau}{m} \quad (1)$$

$$\sigma_{\text{AC}} = \frac{ne^2\tau}{m} \frac{1}{1 - i\omega\tau} \quad \text{For SI and Gaussian} \quad (2)$$

$$\epsilon_r = 1 + i \frac{4\pi\sigma}{\omega} \quad \text{Gaussian} \quad (3a)$$

$$\epsilon_r = 1 + i \frac{\sigma}{\omega\epsilon_0} \quad \text{SI} \quad (3b)$$

Our dielectric constant can be rewritten to plug in for σ , giving us

$$\epsilon = 1 + i \frac{4\pi\sigma}{\omega} \quad (4)$$

$$= 1 + i \frac{4\pi\sigma_{\text{DC}}}{\omega} \frac{1}{1 - i\omega\tau} \quad (5)$$

$$= 1 + i \frac{4\pi\sigma_{\text{DC}}}{\omega} \frac{1}{1 - i\omega\tau} \frac{1 + i\omega\tau}{1 + i\omega\tau} \quad (6)$$

$$= 1 + i \frac{4\pi\sigma_{\text{DC}}}{\omega} \frac{1 + i\omega\tau}{1 + \omega^2\tau^2} \quad (7)$$

$$= \left(1 - \frac{4\pi\sigma_{\text{DC}}\omega\tau}{\omega(1 + \omega^2\tau^2)}\right) + i \left(\frac{4\pi\sigma_{\text{DC}}}{\omega(1 + \omega^2\tau^2)}\right) \quad (8)$$

$$= \left(1 - \frac{4\pi\sigma_{\text{DC}}\tau}{1 + \omega^2\tau^2}\right) + i \left(\frac{4\pi\sigma_{\text{DC}}}{\omega(1 + \omega^2\tau^2)}\right) \quad (9)$$

This lets us write down the explicit real and imaginary of the Drude dielectric function.

1.1 Limiting forms of the Drude model

We can look at the large and small ω limits for the Drude dielectric function.

1.2 Derivations for Drude model

1.2.1 DC Conductivity

We can start unit-system independently, with the expression

$$\mathbf{j} = \sigma \mathbf{E}. \quad (10)$$

We can also relate our current to our average electron velocity: $\mathbf{j} = ne\mathbf{v}$. Imagine at time $t = 0$ our electron undergoes a Drude collision, and emerges with $\mathbf{v}_{t=0} = \mathbf{v}_0$. After a time t , the electron will accelerate with acceleration $-\frac{e\mathbf{E}}{m}$ (which fortunately remains unit independent). Because it will only accelerate for a time τ on average before a collision, it will end up with velocity

$\mathbf{v} = -\frac{e\mathbf{E}}{m}\tau + \mathbf{v}_0$. The average velocity, and current, will be

$$\langle \mathbf{v} \rangle = -\frac{e\mathbf{E}}{m}\tau + \langle \mathbf{v}_0 \rangle \quad (11)$$

$$= -\frac{e\mathbf{E}}{m}\tau \quad (12)$$

$$\frac{\mathbf{j}}{ne} = -\frac{e\mathbf{E}}{m}\tau \quad (13)$$

$$\mathbf{j} = -\frac{ne^2\tau}{m}\mathbf{E}. \quad (14)$$

This of course gives us, unit-independently, our DC conductivity $\sigma_{\text{DC}} = \frac{ne^2\tau}{m}$.

1.2.2 AC Conductivity

The AC conductivity is also simple, but we want to be a bit more formal about it. We can write out the contributions to velocity in terms of probabilities. The velocity at a time dt will have probability dt/τ of being 0, and will otherwise be the original velocity minus $a dt$:

$$\mathbf{v}(dt) = \left(1 - \frac{dt}{\tau}\right) \left(\mathbf{v}_0 - \frac{e\mathbf{E}}{m} dt\right) \quad (15)$$

$$= \mathbf{v}_0 - \frac{dt}{\tau}\mathbf{v}_0 - \frac{e\mathbf{E}}{m} dt, \quad (16)$$

where we've invoked our inalienable right as physicists to ignore all terms $\mathcal{O}(dt^2)$. This reduces, using the definition of $d\mathbf{v} = \mathbf{v}(dt) - \mathbf{v}_0$, to

$$d\mathbf{v} = \frac{dt}{\tau}\mathbf{v} - \frac{e\mathbf{E}}{m} dt \quad (17)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{v}}{\tau} - \frac{e\mathbf{E}}{m} \quad (18)$$

We can quickly Fourier transform this, using $\frac{d}{dt} \rightarrow -i\omega$, and we get (after surreptitiously dropping some vector signs)

$$-i\omega v(\omega) = -\frac{v(\omega)}{\tau} - \frac{eE(\omega)}{m} \quad (19)$$

$$v(\omega) = \frac{eE(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \quad (20)$$

$$j(\omega) = \frac{ne^2E(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \quad (21)$$

$$= \frac{ne^2\tau E(\omega)}{m(1 - i\omega\tau)}, \quad (22)$$

which gives us our AC conductivity in equation (2).

Now for our dielectric constant, we have to find some other defining relation on par with (10).

2 Reducing Lindhard to Drude

We want to see how we can reduce the Lindhard dielectric function

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \quad (23)$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1} \quad (24)$$

We can reduce things in the $k \rightarrow 0$ limit. The first half of (23) has the simple $\frac{1}{k^2}$ dependence, so we can look at how the rest of it behaves to start with.

2.1 f

$$f_l((\omega + i\nu)/kv_F) = 1 - \frac{(\omega + i\nu)/kv_F}{2} \ln \frac{(\omega + i\nu)/kv_F + 1}{(\omega + i\nu)/kv_F - 1} \quad (25)$$

Defining $\eta = \omega + i\nu$:

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \quad (26)$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2kv_F} \ln \frac{\eta + kv_F}{\eta - kv_F} \quad (27)$$

$$f_l = 1 - \frac{\eta}{2v_F} \frac{\ln \frac{\eta + kv_F}{\eta - kv_F}}{k} \quad (28)$$

$$\lim_{k \rightarrow 0} f_l = 1 - \frac{\eta}{2v_F} \frac{d \ln \frac{\eta + kv_F}{\eta - kv_F}}{dk} \quad (29)$$

$$= 1 - \frac{\eta}{2v_F} \frac{\eta - kv_F}{\eta + kv_F} \frac{v_F (\eta - kv_F) + v_F (\eta + kv_F)}{(\eta - kv_F)^2} \quad (30)$$

$$= 1 - \frac{\eta}{2} \frac{\eta - kv_F}{\eta + kv_F} \frac{(\eta - kv_F) + (\eta + kv_F)}{(\eta - kv_F)^2} \quad (31)$$

$$= 1 - \frac{\eta}{2} \frac{1}{\eta + kv_F} \frac{2\eta}{\eta - kv_F} \quad (32)$$

$$= 1 - \frac{\eta^2}{\eta^2 - k^2 v_F^2} \quad (33)$$

$$= \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2} \quad (34)$$

$$\lim_{k \rightarrow 0} f_l = 0 \quad (35)$$

Note that this goes to 0 for $k \rightarrow 0$.

2.2 Series expansion of f

The previous section gives the limit, but having the actual series expansion is probably more valuable. Again, with $\eta = \omega + i\nu$,

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \quad (36)$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2v_F k} \ln \frac{\eta + kv_F}{\eta - kv_F} \quad (37)$$

We want to expand up to k^2 overall, to cancel out the k^2 in the denominator of (23). We're looking at the function.

$$\frac{\ln(\eta + kv_F)}{k} - \frac{\ln(\eta - kv_F)}{k} \quad (38)$$

Generally, the derivatives of $\frac{g(a \pm x)}{x}$ are

$$\left(\frac{g(x)}{x} \right)' = \frac{\pm g'}{x} - \frac{g}{x^2} \quad (39)$$

$$\left(\frac{g(x)}{x} \right)'' = \frac{g''}{x} - \frac{\pm 2g'}{x^2} + \frac{2g}{x^3} \quad (40)$$

2.3 Back to dielectric

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)} \quad (41)$$

In the denominator, we can note that ω should dominate $i\nu f$, because f goes to zero, so we can simplify that.

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega} \quad (42)$$

Using (34), we get

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega} \quad (43)$$

$$= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2}}{\omega} \quad (44)$$

$$= 1 - 3\omega_p^2 \frac{(\omega + i\nu) \frac{1}{\eta^2}}{\omega} \quad (45)$$

$$= 1 - \frac{3\omega_p^2}{\omega(\omega + i\nu)} \quad (46)$$

This is the Drude limit.