0.1 General discussion of dielectric function

We can start by looking at some basic relationships between **D**, **E**. In SI units, and writing ϵ instead of ϵ_r to avoid unnecessary subscripts,

$$\mathbf{D}_\alpha(r) = \int d^d r' \, \epsilon_{\alpha\beta}(r, r')\epsilon_0 \mathbf{E}_\beta(r'). \tag{1}$$

We can start by making the large assumption that the system is isotropic, and thus, the position dependence for ϵ must be of the form $\epsilon_{\alpha\beta}(r - r')$. This justifies the following:

$$\begin{aligned}
\mathbf{D}_\alpha(r) &= \int d^d r' \, \epsilon_{\alpha\beta}(r - r')\epsilon_0 \mathbf{E}_\beta(r') && (2)\\
&= \int d^d r' \, \epsilon_{\alpha\beta}(r - r')\epsilon_0 \mathbf{E}_\beta(r') e^{iqr'} e^{-iqr'} && (3)\\
&= \int d^d r' \, \epsilon_{\alpha\beta}(r - r')\epsilon_0 \mathbf{E}_\beta(r') e^{iqr'} e^{-iqr'} && (4)\\
&= \int d^d r' \, \epsilon_{\alpha\beta}(r - r') e^{iqr'} \epsilon_0 \mathbf{E}_\beta(r') e^{-iqr'} && (5)\\
\mathbf{D}_\alpha(r) e^{-iqr} &= \int d^d r' \, \epsilon_{\alpha\beta}(r - r') e^{iqr'} e^{-iqr} \epsilon_0 \mathbf{E}_\beta(r') e^{-iqr'} && (6)\\
\mathbf{D}_\alpha(r) e^{-iqr} &= \int d^d r' \, \epsilon_{\alpha\beta}(r - r') e^{-iq(r-r')} \epsilon_0 \mathbf{E}_\beta(r') e^{-iqr'} && (7)
\end{aligned}$$

If we integrate this over r, and recognise our Fourier transforms, this becomes

$$D_\alpha(q) = \epsilon_{\alpha\beta}(q) E_\beta(q) \tag{8}$$

This is a fair result for the assumption of isotropy. We have an implicit sum over β here to look at.

Skipping over some details to fill in later, we end up with

$$\frac{1}{\epsilon_r(q, \omega)} = 1 + \frac{4\pi e^2}{q^2}\Pi(q, \omega), \tag{9}$$

where $\Pi(q, \omega)$ is a response function satisfying

$$n_{ind}(q, \omega) = \Pi(q, \omega) V_f(q, \omega) \tag{10}$$

Here $n_i nd$ is the number density of induced electrons, and V_f is the voltage created by any free electrons in the metal (which isn't quite the same as an external voltage, but I think you might be able to ignore that difference).