Main notebook

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1 Drude model parameters

The assumptions of the Drude model are simple: we have interaction-free electrons that occasionally undergo some scattering process during a time dt with probability $\frac{dt}{\tau}$, where τ is some phenomenological parameter. This scattering will randomise electron momentum.

Our ultimate goal will be to find the conductivity σ and the dielectric constant ϵ in the Drude model, with Drude relaxation time τ, electron density n and electron mass m. We'll find

$$\sigma_{\mathrm{DC}} = \frac{ne^2\tau}{m} \tag{1}$$

$$\sigma_{\mathrm{AC}} = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau} \qquad \text{For SI and Gaussian} \tag{2}$$

$$\epsilon_r = 1 + i\frac{4\pi\sigma}{\omega} \qquad \text{Gaussian} \tag{3a}$$

$$\epsilon_r = 1 + i\frac{\sigma}{\omega\epsilon_0} \qquad \text{SI} \tag{3b}$$

Our dielectric constant can be rewritten to plug in for σ, giving us

$$\begin{aligned}
\epsilon &= 1 + i\frac{4\pi\sigma}{\omega} &(4)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau} &(5)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau}\frac{1+i\omega\tau}{1+i\omega\tau} &(6)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1+i\omega\tau}{1+\omega^2\tau^2} &(7)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\omega\tau}{\omega\left(1+\omega^2\tau^2\right)}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) &(8)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\tau}{1+\omega^2\tau^2}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) &(9)
\end{aligned}$$

This lets us write down the explicit real and imaginary of the Drude dielectric function.

1.1 Limiting forms of the Drude model

We can look at the large and small ω limits for the Drude dielectric function.

1.2 Derivations for Drude model

1.2.1 DC Conductivity

We can start unit-system independently, with the expression

$$\mathbf{j} = \sigma\mathbf{E}. \tag{10}$$

We can also relate our current to our average electron velocity: $\mathbf{j} = ne\mathbf{v}$. Imagine at time $t = 0$ our electron undergoes a Drude collision, and emerges with $\mathbf{v}_{t=0} = \mathbf{v_0}$. After a time t, the electron will accelerate with acceleration $-\frac{e\mathbf{E}}{m}$ (which fortunately remains unit independent). Because it will only accelerate for a time τ on average before a collision, it will end up with velocity

$\mathbf{v} = -\frac{e\mathbf{E}}{m}\tau + \mathbf{v_0}$. The average velocity, and current, will be

$$\langle \mathbf{v} \rangle = -\frac{e\mathbf{E}}{m}\tau + \langle \mathbf{v_0} \rangle \tag{11}$$
$$= -\frac{e\mathbf{E}}{m}\tau \tag{12}$$
$$\frac{\mathbf{j}}{ne} = -\frac{e\mathbf{E}}{m}\tau \tag{13}$$
$$\mathbf{j} = -\frac{ne^2\tau}{m}\mathbf{E}. \tag{14}$$

This of course gives us, unit-independently, our DC conductivity $\sigma_{\mathrm{DC}} = \frac{ne^2\tau}{m}$.

1.2.2 AC Conductivity

The AC conductivity is also simple, but we want to be a bit more formal about it. We can write out the contributions to velocity in terms of probabilities. The velocity at a time dt will have probability dt/τ of being 0, and will otherwise be the original velocity minus $a\,dt$:

$$\mathbf{v}(dt) = \left(1 - \frac{dt}{\tau}\right)\left(\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\right) \tag{15}$$
$$= \mathbf{v_0} - \frac{dt}{\tau}\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\,, \tag{16}$$

where we've invoked our inalienable right as physicists to ignore all terms $\mathcal{O}(dt^2)$. This reduces, using the definition of $d\mathbf{v} = \mathbf{v}(dt) - \mathbf{v_0}$, to

$$d\mathbf{v} = \frac{dt}{\tau}\mathbf{v} - \frac{e\mathbf{E}}{m}\,dt \tag{17}$$
$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{v}}{\tau} - \frac{e\mathbf{E}}{m} \tag{18}$$

We can quickly Fourier transform this, using $\frac{d}{dt} \to -i\omega$, and we get (after surreptitiously dropping some vector signs)

$$-i\omega v(\omega) = -\frac{v(\omega)}{\tau} - \frac{eE(\omega)}{m} \tag{19}$$
$$v(\omega) = \frac{eE(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{20}$$
$$j(\omega) = \frac{ne^2E(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{21}$$
$$= \frac{ne^2\tau E(\omega)}{m\,(1 - i\omega\tau)}, \tag{22}$$

which gives us our AC conductivity in equation (2).

Now for our dielectric constant, we have to find some other defining relation on par with (10).

2 Reducing to Drude

We want to see how we can reduce the Lindhard dielectric function

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l(\,(\omega + i\nu)/kv_f\,)}{\omega + i\nu f_l(\,(\omega + i\nu)/kv_f\,)}, \tag{23}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1} \tag{24}$$

We can reduce things in the $k \to 0$ limit. The first half of (23) has the simple $\frac{1}{k^2}$ dependence, so we can look at how the rest of it behaves to start with.

2.1 f

$$f_l(\,(\omega + i\nu)/kv_F\,) = 1 - \frac{(\omega + i\nu)/kv_F}{2} \ln \frac{(\omega + i\nu)/kv_F\, + 1}{(\omega + i\nu)/kv_F\, - 1} \tag{25}$$

Defining $\eta = \omega + i\nu$:

$$\begin{aligned}
f_l(\,(\omega + i\nu)/kv_f\,) &= 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f\, + 1}{(\omega + i\nu)/kv_f\, - 1} && (26)\\
f_l(\eta/kv_F\,) &= 1 - \frac{\eta}{2kv_F} \ln \frac{\eta + kv_F}{\eta - kv_F} && (27)\\
f_l &= 1 - \frac{\eta}{2v_F} \frac{\ln \frac{\eta + kv_F}{\eta - kv_F}}{k} && (28)\\
\lim_{k \to 0} f_l &= 1 - \frac{\eta}{2v_F} \frac{d \ln \frac{\eta + kv_F}{\eta - kv_F}}{dk} && (29)\\
&= 1 - \frac{\eta}{2v_F} \frac{\eta - kv_F}{\eta + kv_F} \frac{v_F\,(\eta - kv_F) + v_F\,(\eta + kv_F)}{(\eta - kv_F)^2} && (30)\\
&= 1 - \frac{\eta}{2} \frac{\eta - kv_F}{\eta + kv_F} \frac{(\eta - kv_F) + (\eta + kv_F)}{(\eta - kv_F)^2} && (31)\\
&= 1 - \frac{\eta}{2} \frac{1}{\eta + kv_F} \frac{2\eta}{\eta - kv_F} && (32)
\end{aligned}$$

$$= 1 - \frac{\eta^2}{\eta^2 - k^2 v_F^2} \tag{33}$$

$$= \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2} \tag{34}$$

$$\lim_{k \to 0} f_l = 0 \tag{35}$$

Note that this goes to 0 for $k \to 0$.

2.2 Back to dielectric

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)} \tag{36}$$

In the denominator, we can note that ω should dominate $i\nu f$, because f goes to zero, so we can simplify that.

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} \tag{37}$$

Using (34), we get

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} \tag{38}$$

$$= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2}}{\omega} \tag{39}$$

$$= 1 - 3\omega_p^2 \frac{(\omega + i\nu)\, \frac{1}{\eta^2}}{\omega} \tag{40}$$

$$= 1 - \frac{3\omega_p^2}{\omega\,(\omega + i\nu)} \tag{41}$$

This is the Drude limit.