1 Explicit parts of Lindhard function

We want to find the explicit real and imaginary parts of the Lindhard function. To begin with, we can start with the case where $\nu \to 0$, which means long relaxation times.

We have our Lindhard form

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l(\,(\omega + i\nu)/kv_f\,)}{\omega + i\nu f_l(\,(\omega + i\nu)/kv_f\,)}, \tag{1}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1}. \tag{2}$$

1.1 Pines result

From Pines, we have the forms

$$\begin{aligned} \mathrm{Re}[\epsilon_l] = 1 + \frac{k_{TF}^2}{k^2}\Bigg(\frac{1}{2} + \frac{k_F}{4k}\Bigg[\Bigg(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\Bigg) \ln\left[\frac{\omega - kv_F - \frac{\hbar k^2}{2m}}{\omega + kv_F - \frac{\hbar k^2}{2m}}\right] \\ + \Bigg(1 - \frac{\left(\omega + \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\Bigg) \ln\left[\frac{\omega + kv_F + \frac{\hbar k^2}{2m}}{\omega - kv_F + \frac{\hbar k^2}{2m}}\right]\Bigg]\Bigg) \end{aligned} \tag{3}$$

$$\mathrm{Im}[\epsilon_l] = \begin{cases} \frac{\pi}{2} \frac{\omega}{kv_F} \frac{k_{TF}^2}{k^2}, & \omega \le kv_F - \frac{\hbar k^2}{2m} \\ \frac{\pi}{4} \frac{k_F}{k} \left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\right) \frac{k_{TF}^2}{k^2}, & kv_F - \frac{\hbar k^2}{2m} \le \omega \le kv_F + \frac{\hbar k^2}{2m} \\ 0, & \omega \ge kv_F + \frac{\hbar k^2}{2m} \end{cases} \tag{4}$$

1.2 Long relaxation time forms of the logs

In order to analyse the $\nu \to 0$ limit, we can start by looking at what happens to the logarithms in the Lindhard function:

$$\ln \frac{\omega + i\nu + kv_F}{\omega + i\nu - kv_F} \tag{5}$$

The first thing to note is that the numerator will always have a very small, positive argument, while the denominator will have a small argument which may be positive or negative. As Lindhard mentions, these logarithms should all have imaginary parts between $\pm i\pi$. This effectively means we can treat each logarithm as giving the principal value, which give us a result that looks like

$$\ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right), \tag{6}$$

where θ_+ and θ_- are the arguments of the numerator and denominator. For small ν, θ_+ is proportional to ν, as it'll be determined by an arcsine. However, the denominator may be negative, which would contribute a factor of $\theta_- = +i\pi$ (with a plus sign because ν would be just above the real line).

$$\begin{aligned}
\ln \frac{\omega + i\nu + kv_F}{\omega + i\nu - kv_F} &= \ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right) &(7)\\
&= \ln \frac{\sqrt{(\omega + kv_F)^2}}{\sqrt{(\omega - kv_F)^2}} - \sigma i\pi &(8)\\
&= \ln \left| \frac{\omega + kv_F}{\omega - kv_F} \right| - \sigma i\pi, &(9)
\end{aligned}$$

where $\sigma = 1$ if $\omega < kv_F$, and 0 otherwise.