Main dielectric notebook

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Inhaltsverzeichnis

1 Drude model parameters

The assumptions of the Drude model are simple: we have interaction-free electrons that occasionally undergo some scattering process during a time dt with probability $\frac{dt}{\tau}$, where τ is some phenomenological parameter. This scattering will randomise electron momentum.

Our ultimate goal will be to find the conductivity σ and the dielectric constant ϵ in the Drude model, with Drude relaxation time τ, electron density n and electron mass m. We'll find

$$\sigma_{\mathrm{DC}} = \frac{ne^2\tau}{m} \tag{1}$$

$$\sigma_{\mathrm{AC}} = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau} \qquad \text{For SI and Gaussian} \tag{2}$$

$$\epsilon_r = 1 + i\frac{4\pi\sigma}{\omega} \qquad \text{Gaussian} \tag{3a}$$

$$\epsilon_r = 1 + i\frac{\sigma}{\omega\epsilon_0} \qquad \text{SI} \tag{3b}$$

Our dielectric constant can be rewritten to plug in for σ, giving us

$$\begin{aligned}
\epsilon &= 1 + i\frac{4\pi\sigma}{\omega} && (4)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau} && (5)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau}\frac{1+i\omega\tau}{1+i\omega\tau} && (6)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1+i\omega\tau}{1+\omega^2\tau^2} && (7)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\omega\tau}{\omega\left(1+\omega^2\tau^2\right)}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) && (8)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\tau}{1+\omega^2\tau^2}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) && (9)
\end{aligned}$$

This lets us write down the explicit real and imaginary of the Drude dielectric function.

1.1 Alternative forms of the Drude model

We can also rewrite the dielectric constant very slightly in terms of the plasma frequency. In Gaussian units:

$$\begin{aligned}\epsilon &= 1 + i\frac{4\pi\sigma}{\omega} && (10)\\ &= 1 + i\frac{4\pi}{\omega}\frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau} && (11)\\ &= 1 + i\frac{4\pi}{\omega}\frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}\frac{i\nu}{i\nu} && (12)\\ &= 1 - \frac{4\pi}{\omega}\frac{ne^2}{m}\frac{1}{i\nu+\omega} && (13)\end{aligned}$$

With $\omega_p^2 = \frac{4\pi ne^2}{m}$ in Gaussian units, this becomes

$$\epsilon = 1 - \frac{\omega_p^2}{\omega(\omega + i\nu)} \qquad (14)$$

We'll see this again later.

1.2 Derivations for Drude model

1.2.1 DC Conductivity

We can start unit-system independently, with the expression

$$\mathbf{j} = \sigma\mathbf{E}. \qquad (15)$$

We can also relate our current to our average electron velocity: $\mathbf{j} = ne\mathbf{v}$. Imagine at time $t = 0$ our electron undergoes a Drude collision, and emerges with $\mathbf{v}_{t=0} = \mathbf{v_0}$. After a time t, the electron will accelerate with acceleration $-\frac{e\mathbf{E}}{m}$ (which fortunately remains unit independent). Because it will only accelerate for a time τ on average before a collision, it will end up with velocity $\mathbf{v} = -\frac{e\mathbf{E}}{m}\tau + \mathbf{v_0}$. The average velocity, and current, will be

$$\begin{aligned}\langle\mathbf{v}\rangle &= -\frac{e\mathbf{E}}{m}\tau + \langle\mathbf{v_0}\rangle && (16)\\ &= -\frac{e\mathbf{E}}{m}\tau && (17)\\ \frac{\mathbf{j}}{ne} &= -\frac{e\mathbf{E}}{m}\tau && (18)\\ \mathbf{j} &= -\frac{ne^2\tau}{m}\mathbf{E}. && (19)\end{aligned}$$

This of course gives us, unit-independently, our DC conductivity $\sigma_{\mathrm{DC}} = \frac{ne^2\tau}{m}$.

1.2.2 AC Conductivity

The AC conductivity is also simple, but we want to be a bit more formal about it. We can write out the contributions to velocity in terms of probabilities. The velocity at a time dt will have probability dt/τ of being 0, and will otherwise be the original velocity minus $a\,dt$:

$$\mathbf{v}(dt) = \left(1 - \frac{dt}{\tau}\right)\left(\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\right) \tag{20}$$

$$= \mathbf{v_0} - \frac{dt}{\tau}\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\,, \tag{21}$$

where we've invoked our inalienable right as physicists to ignore all terms $\mathcal{O}(dt^2)$. This reduces, using the definition of $d\mathbf{v} = \mathbf{v}(dt) - \mathbf{v_0}$, to

$$d\mathbf{v} = \frac{dt}{\tau}\mathbf{v} - \frac{e\mathbf{E}}{m}\,dt \tag{22}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{v}}{\tau} - \frac{e\mathbf{E}}{m} \tag{23}$$

We can quickly Fourier transform this, using $\frac{d}{dt} \to -i\omega$, and we get (after surreptitiously dropping some vector signs)

$$-i\omega v(\omega) = -\frac{v(\omega)}{\tau} - \frac{eE(\omega)}{m} \tag{24}$$

$$v(\omega) = \frac{eE(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{25}$$

$$j(\omega) = \frac{ne^2E(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{26}$$

$$= \frac{ne^2\tau E(\omega)}{m\left(1 - i\omega\tau\right)}, \tag{27}$$

which gives us our AC conductivity in equation (2).

1.2.3 Dielectric constant

Now for our dielectric constant, we have to find some other defining relation on par with (15). The relevant equation we'll want to look at looks like our wave equation

$$-\nabla^2\mathbf{E} = \frac{\omega^2}{v_{\text{phase}}^2}\mathbf{E} \tag{28}$$

The phase velocity would satisfy

$$\frac{1}{v_{\text{phase}}^2} = \mu\epsilon \tag{29}$$

The cleanest thing to do is to consider relative dielectric constants $\epsilon_r = \frac{\epsilon}{\epsilon_0}$. For a non-magnetic material, where $\mu = \mu_0$, and we can see that

$$\begin{align}
\frac{1}{v_{\text{phase}}^2} &= \mu\epsilon \tag{30}\\
&= \mu_0\epsilon_0\epsilon_r \tag{31}\\
&= \frac{1}{c^2}\epsilon_r, \tag{32}
\end{align}$$

which makes our defining relationship

$$-\nabla^2\mathbf{E} = \frac{\omega^2}{c^2}\epsilon_r(\omega)\mathbf{E} \tag{33}$$

For non-magnetic materials, thus, we have our defining relationship. The Drude model falls out of solving for this using Maxwell's equations (in SI units!):

$$\begin{align}
\boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon} \tag{34}\\
\boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \tag{35}\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \tag{36}\\
\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{j} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{37}
\end{align}$$

Now, in order to find $-\nabla^2\mathbf{E}$, we can use everyone's fav identity

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\Upsilon}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\Upsilon}) - \nabla^2\boldsymbol{\Upsilon} \tag{38}$$

Our goal is thus to find $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})$

$$\begin{align}
-\nabla^2\mathbf{E} &= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) \tag{39}\\
&= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \boldsymbol{\nabla}(0) \tag{40}\\
&= \boldsymbol{\nabla}\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \tag{41}
\end{align}$$

Working in Fourier transformed time space, with $\frac{d}{dt} \rightarrow -i\omega$,

$$
\begin{aligned}
-\nabla^2 \mathbf{E} &= i\omega \boldsymbol{\nabla} \times (\mathbf{B}) && (42)\\
&= i\omega \left(\mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) && (43)\\
&= i\omega \left(\frac{1}{c^2 \epsilon_0} \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right) && (44)\\
&= i\frac{\omega}{c^2} \left(\frac{1}{\epsilon_0} \mathbf{j} - i\omega \mathbf{E} \right) && (45)\\
&= \frac{\omega}{c^2} \left(\omega \mathbf{E} + i\frac{1}{\epsilon_0} \mathbf{j} \right) && (46)\\
&= \frac{\omega^2}{c^2} \left(\mathbf{E} + i\frac{1}{\omega \epsilon_0} \mathbf{j} \right) && (47)
\end{aligned}
$$

Up to now, the only real assumptions have been that we're working with a non-magnetic material, and that we're looking at waves specifically. We can now insert the assumption that $\mathbf{j} = \sigma \mathbf{E}$. One important note here is that we're implicitly assuming we have this well defined relationship, which I believe only really works for cases where $\mathbf{E}$ (and thus, $\mathbf{J}$) is constant over scales larger than the electron mean free path. Otherwise, we wouldn't be able to write this down in this way (this is slightly weaker than our earlier assumption of constant field, and explains how to find out how big of field wavelengths the Drude results will be valid over). Plugging this in,

$$
\begin{aligned}
-\nabla^2 \mathbf{E} &= \frac{\omega^2}{c^2} \left(\mathbf{E} + i\frac{1}{\omega \epsilon_0} \sigma \mathbf{E} \right) && (48)\\
&= \frac{\omega^2}{c^2} \left(1 + \frac{i\sigma}{\omega \epsilon_0} \right) \mathbf{E} && (49)
\end{aligned}
$$

Comparing this with (33), we see that we have, in SI units,

$$
\epsilon_r = 1 + \frac{i\sigma}{\omega \epsilon_0}, \qquad (50)
$$

confirming the earlier claim.

Note also that this entire relationship only works for non-zero ω. Essentially, if ω is zero, the Drude model can't say much about a dielectric function; there is simply conduction.

What does this actually mean

2 General discussion of dielectric function

2.1 Fourier transform of dielectric components

We can start by looking at some basic relationships between **D**, **E**. In SI units, and writing ϵ instead of ϵ_r to avoid unnecessary subscripts,

$$\mathbf{D}_\alpha(r) = \int d^d r' \, \epsilon_{\alpha\beta}(r, r') \epsilon_0 \mathbf{E}_\beta(r'). \tag{51}$$

We can start by making the large assumption that the system is isotropic, and thus, the position dependence for ϵ must be of the form $\epsilon_{\alpha\beta}(r - r')$. This justifies the following:

$$\mathbf{D}_\alpha(r) = \int d^d r' \, \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') \tag{52}$$

$$= \int d^d r' \, \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') e^{ikr'} e^{-ikr'} \tag{53}$$

$$= \int d^d r' \, \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') e^{ikr'} e^{-ikr'} \tag{54}$$

$$= \int d^d r' \, \epsilon_{\alpha\beta}(r - r') e^{ikr'} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \tag{55}$$

$$\mathbf{D}_\alpha(r) e^{-ikr} = \int d^d r' \, \epsilon_{\alpha\beta}(r - r') e^{ikr'} e^{-ikr} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \tag{56}$$

$$\mathbf{D}_\alpha(r) e^{-ikr} = \int d^d r' \, \epsilon_{\alpha\beta}(r - r') e^{-ik(r-r')} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \tag{57}$$

If we integrate this over r, and recognise our Fourier transforms, this becomes

$$D_\alpha(k) = \epsilon_{\alpha\beta}(k) \epsilon_0 E_\beta(k) \tag{58}$$

This is a fair result for the assumption of isotropy. We have an implicit sum over β here to look at. This serves as a useful working definition of $\epsilon(k)$.

One more assumption proves useful: for a given $\mathbf{k}$, we can look at the components of **E** and **D** parallel (longitudinal) and perpendicular (transverse) to $\mathbf{k}$. We may assume that only the parallel component of **E** affects the parallel component of **D**, and same for the perpendicular components. Specifically, this leads to

why can we assume this?

include brief derivation of dielectric components

$$\epsilon_{\alpha\beta}(\mathbf{k}) = \epsilon_{\parallel}(\mathbf{k}) \hat{k}_\alpha \hat{k}_\beta + \epsilon_\perp(\mathbf{k})(\delta_{\alpha\beta} - \hat{k}_\alpha \hat{k}_\beta). \tag{59}$$

A moment's reflection shows that this leads to two independent components of $\epsilon_{\alpha\beta}(\mathbf{k})$ (as can be seen by assuming $\mathbf{k}$ points in the x direction). Also note that that for the transverse case, this explicitly only couples y to y and z to z; there are no couplings between y and z.

2.2 Electron gas densities and longitudinal eqs

To begin looking at an actual electron gas, with some external charge density ρ_{ext} imposed on it, we can start with Gauss's law:

$$\boldsymbol{\nabla} \cdot \mathbf{D}(\mathbf{r}) = \rho_{ext}(r) \tag{60}$$

This external charge density induces a charge density in the gas:

$$\rho(r) = \rho_{ind}(r) + \rho_{ext}(r) \tag{61}$$

We can define some scalar potentials Φ and Φ_{ext}, and therewith write

$$\mathbf{E}(r) = -\boldsymbol{\nabla}\Phi(r) \tag{62}$$

$$\mathbf{D}(r) = -\epsilon_0 \boldsymbol{\nabla}\Phi_{ext}(r) \tag{63}$$

Going Fourier with it leads to

$$\mathbf{E}(k) = -i\mathbf{k}\Phi(k) \tag{64}$$

$$\mathbf{D}(k) = -i\epsilon_0 \mathbf{k}\Phi_{ext}(k) \tag{65}$$

These are the components of $\mathbf{E}$ and $\mathbf{D}$ parallel to $\mathbf{k}$, which means that these are really the *longitudinal* components.

Relating Gauss's laws to the scalar components, we get

$$\rho_{ext}(\mathbf{k}) = \boldsymbol{\nabla} \cdot \mathbf{D}(\mathbf{k}) \tag{66}$$

$$= \boldsymbol{\nabla} \cdot (-i\epsilon_0 \mathbf{k}\Phi_{ext}(\mathbf{k})) \tag{67}$$

$$\rho_{ext}(\mathbf{k}) = \epsilon_0 k^2 \Phi_{ext}(\mathbf{k}) \tag{68}$$

Similarly, for $\mathbf{E}$ and ρ, we get

$$\rho(\mathbf{k}) = \epsilon_0 k^2 \Phi(\mathbf{k}). \tag{69}$$

Because of (58), we should be able to write

$$\Phi_{ext}(\mathbf{k}) = \epsilon_{\|}(\mathbf{k})\Phi(\mathbf{k}) \tag{70}$$

Skipping over some details to fill in later (which involve defining potential energies $V = -e\Phi$ and using number densities defined by $\rho = -en$, we end up with

Fill in these details

$$\frac{1}{\epsilon_{\|}(k,\omega)} = 1 + \frac{4\pi e^2}{k^2}\Pi(k,\omega), \tag{71}$$

where $\Pi(k, \omega)$ is a response function satisfying

$$n_{ind}(k, \omega) = \Pi(k, \omega) V_f(k, \omega) \tag{72}$$

Here n_{ind} is the number density of induced electrons, and V_f is the voltage created by any free electrons in the metal (which isn't quite the same as an external voltage, but I think you might be able to ignore that difference).

There's no reason for $\Pi(\mathbf{k}, \omega)$ to be simple. In general, it describes the complete induced density reponse to any external potential.

3 Reducing Lindhard to Drude

3.1 The longitudinal case

We want to see how we can reduce the longitudinal Lindhard dielectric function

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l(\,(\omega + i\nu)/kv_f\,)}{\omega + i\nu f_l(\,(\omega + i\nu)/kv_f\,)}, \tag{73}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1}. \tag{74}$$

We can reduce things in the $k \to 0$ limit. The first half of (73) has the simple $\frac{1}{k^2}$ dependence, so we can look at how the rest of it behaves to start with.

3.2 f

$$f_l(\,(\omega + i\nu)/kv_F\,) = 1 - \frac{(\omega + i\nu)/kv_F}{2} \ln \frac{(\omega + i\nu)/kv_F\, + 1}{(\omega + i\nu)/kv_F\, - 1} \tag{75}$$

Defining $\eta = \omega + i\nu$:

$$f_l(\,(\omega + i\nu)/kv_f\,) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f\, + 1}{(\omega + i\nu)/kv_f\, - 1} \tag{76}$$

$$f_l(\,\eta/kv_F\,) = 1 - \frac{\eta}{2kv_F} \ln \frac{\eta + kv_F}{\eta - kv_F} \tag{77}$$

$$f_l = 1 - \frac{\eta}{2v_F} \frac{\ln \frac{\eta + kv_F}{\eta - kv_F}}{k} \tag{78}$$

$$\lim_{k \to 0} f_l = 1 - \frac{\eta}{2v_F} \frac{d \ln \frac{\eta + kv_F}{\eta - kv_F}}{dk} \tag{79}$$

$$= 1 - \frac{\eta}{2v_F}\frac{\eta - kv_F}{\eta + kv_F}\frac{v_F(\eta - kv_F) + v_F(\eta + kv_F)}{(\eta - kv_F)^2} \tag{80}$$

$$= 1 - \frac{\eta}{2}\frac{\eta - kv_F}{\eta + kv_F}\frac{(\eta - kv_F) + (\eta + kv_F)}{(\eta - kv_F)^2} \tag{81}$$

$$= 1 - \frac{\eta}{2}\frac{1}{\eta + kv_F}\frac{2\eta}{\eta - kv_F} \tag{82}$$

$$= 1 - \frac{\eta^2}{\eta^2 - k^2 v_F^2} \tag{83}$$

$$= \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2} \tag{84}$$

$$\lim_{k \to 0} f_l = 0 \tag{85}$$

Note that this goes to 0 for $k \to 0$.

3.3 Series expansion of f

The previous section gives the limit, but having the actual series expansion is probably more valuable. Again, with $\eta = \omega + i\nu$,

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \tag{86}$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2v_F k} \ln \frac{\eta + kv_F}{\eta - kv_F} \tag{87}$$

We want to expand up to k^2 overall, to cancel out the k^2 in the denominator of (73). We're looking at the function.

$$\frac{\ln(\eta + kv_F)}{k} - \frac{\ln(\eta - kv_F)}{k} \tag{88}$$

Generally, the derivatives of $\frac{g(a \pm x)}{x}$ are

$$\left(\frac{g(x)}{x}\right)' = \frac{\pm g'}{x} - \frac{g}{x^2} \tag{89}$$

$$\left(\frac{g(x)}{x}\right)'' = \frac{g''}{x} - \frac{\pm 2g'}{x^2} + \frac{2g}{x^3} \tag{90}$$

$$\left(\frac{g(x)}{x}\right)''' = \frac{\pm g'''}{x} - \frac{3g''}{x^2} + \frac{\pm 6g'}{x^3} - \frac{6g}{x^4} \tag{91}$$

When we take the difference, we see that we'll only end up keeping (and doubling) the terms of odd derivatives. Thus, up to this order, the series for $\frac{g(a+x)-g(a-x)}{x}$ will look like:

$$\frac{1}{2}\frac{g(a+x)-g(a-x)}{x} = x\frac{g'}{x} - \frac{1}{2}x^2\frac{2g'}{x^2} + \frac{1}{6}x^3\left(\frac{g'''}{x} + \frac{6g'}{x^3}\right) \quad (92)$$

$$= g' - g' + \frac{1}{6}x^2 g''' + g' \quad (93)$$

$$\frac{g(a+x)-g(a-x)}{x} = 2g' + \frac{1}{3}x^2 g''' + \mathcal{O}(x^4) \quad (94)$$

This type of result is to be expected: we are starting with an even function. For $g = \ln(\eta + kv_F)$, we have

$$g'(k=0) = \frac{v_F}{\eta} \quad (95)$$

$$g'''(k=0) = \frac{2v_F^3}{\eta^3} \quad (96)$$

Plugging these into (87) gives us:

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2v_F k}\ln\frac{\eta + kv_F}{\eta - kv_F} \quad (97)$$

$$= 1 - \frac{\eta}{2v_F}\left(2\frac{v_F}{\eta} + \frac{1}{3}k^2\frac{2v_F^3}{\eta^3}\right) \quad (98)$$

$$= 1 - 1 - \frac{1}{3}k^2\frac{v_F^2}{\eta^2} \quad (99)$$

$$= -\frac{k^2 v_F^2}{3\eta^2} \quad (100)$$

This gives us a simple approximation for f_l in the long wavelength limit.

3.4 Back to dielectric

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2}\frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)} \quad (101)$$

In the denominator, we can note that ω should dominate $i\nu f$, because f goes to zero, so we can simplify that.

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2}\frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} \quad (102)$$

Using (100), we get

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} \tag{103}$$

$$= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, \frac{-k^2 v_F^2}{3\eta^2}}{\omega} \tag{104}$$

$$= 1 - 3\omega_p^2 \frac{\eta \frac{1}{3\eta^2}}{\omega} \tag{105}$$

$$= 1 - \frac{\omega_p^2}{\omega\,(\omega + i\nu)} \tag{106}$$

This is the Drude limit, keeping in mind that $\omega_p^2 = \frac{4\pi n e^2}{m}$ in Gaussian units.

3.5 Looking at transverse Lindhard form

It's possible also to show that the transverse Lindhard dielectric function ends up also going to the Drude form in the $k \to 0$ limit.

Writing out ϵ_t:

$$\epsilon_t = 1 - \frac{\omega_p^2}{\omega\eta} \left[\frac{3}{2} \frac{\eta^2}{v_F^2 k^2} - \frac{3}{4} \frac{\eta}{v_F k} \left(\frac{\eta^2}{v_F^2 k^2} - 1 \right) \ln \frac{\eta + v_F k}{\eta - v_F k} \right] \tag{107}$$

We can also simplify this slightly, to highlight the similarities between this and the longitudinal forms:

$$\epsilon_t = 1 - \frac{3\omega_p^2}{2\omega\eta} \left[\frac{\eta^2}{v_F^2 k^2} - \frac{1}{2} \frac{\eta}{v_F k} \left(\frac{\eta^2}{v_F^2 k^2} - 1 \right) \ln \frac{\eta + v_F k}{\eta - v_F k} \right] \tag{108}$$

$$= 1 - \frac{3\omega_p^2}{2\omega\eta} \left[\frac{\eta^2}{v_F^2 k^2} - \frac{1}{2} \frac{\eta^2}{v_F^2 k^2} \left(\frac{\eta}{v_F k} - \frac{v_F k}{\eta} \right) \ln \frac{\eta + v_F k}{\eta - v_F k} \right] \tag{109}$$

$$= 1 - \frac{3\omega_p^2}{2\omega\eta} \frac{\eta^2}{v_F^2 k^2} \left[1 - \frac{1}{2} \left(\frac{\eta}{v_F k} - \frac{v_F k}{\eta} \right) \ln \frac{\eta + v_F k}{\eta - v_F k} \right] \tag{110}$$

$$= 1 - \frac{3\omega_p^2 \eta}{2\omega v_F^2 k^2} \left[1 - \frac{1}{2} \left(\frac{\eta}{v_F k} - \frac{v_F k}{\eta} \right) \ln \frac{\eta + v_F k}{\eta - v_F k} \right] \tag{111}$$

There is an extra term within the brackets, as well as an extra factor of $-\frac{1}{2}$ on the outside.

To find the $k \to 0$ limit here, we can do the same series expansion as we did for the longitudinal case. The relevant series to find is for the bracketed portion, which we can break into two parts:

$$1 - \frac{1}{2}\frac{\eta}{v_F k}\ln\frac{\eta + v_F k}{\eta - v_F k} - \frac{1}{2}\frac{v_F k}{\eta}\ln\frac{\eta + v_F k}{\eta - v_F k} \tag{112}$$

We have already done the expansion of the first two terms earlier, and we found that that should equal $-\frac{k^2 v_F^2}{3\eta^2}$. We now need the series expansion of the third term. Like earlier, we can write out the derivatives of $kg(a \pm k)$, keeping in mind that the log will become a difference:

$$(kg)' = g \pm kg' \tag{113}$$
$$(kg)'' = \pm 2g' + kg'' \tag{114}$$

After $k \to 0$ and the subtraction, the only term remaining will be the $2 \pm g'$ term. This will end up giving us

$$\frac{1}{2}\frac{v_F}{\eta}k\ln\frac{\eta + v_F k}{\eta - v_F k} = \frac{1}{2}\frac{v_F}{\eta}\frac{1}{2}k^2 4\frac{v_F}{\eta} \tag{115}$$
$$= \frac{k^2 v_F^2}{\eta^2} \tag{116}$$

Adding this to (112), we end up with

$$1 - \frac{1}{2}\left(\frac{\eta}{v_F k} - \frac{v_F k}{\eta}\right)\ln\frac{\eta + v_F k}{\eta - v_F k} = -\frac{k^2 v_F^2}{3\eta^2} + \frac{k^2 v_F^2}{\eta^2} \tag{117}$$
$$= \frac{2}{3}\frac{k^2 v_F^2}{\eta^2} \tag{118}$$

Plugging this into (111) gives us

$$\epsilon_t = 1 - \frac{3\omega_p^2 \eta}{2\omega v_F^2 k^2}\left[1 - \frac{1}{2}\left(\frac{\eta}{v_F k} - \frac{v_F k}{\eta}\right)\ln\frac{\eta + v_F k}{\eta - v_F k}\right] \tag{119}$$
$$= 1 - \frac{3\omega_p^2 \eta}{2\omega v_F^2 k^2}\frac{2}{3}\frac{k^2 v_F^2}{\eta^2} \tag{120}$$
$$= 1 - \frac{\omega_p^2}{\omega\eta} \tag{121}$$
$$= 1 - \frac{\omega_p^2}{\omega(\omega + i\nu)} \tag{122}$$

So to lowest order, both the longitudinal and transverse dielectric functions reduce to the Drude case in the $k \to 0$ limit. This makes intuitive sense: The Drude derivation assumes an isotropic $\mathbf{E}$. There should remain no distinction between transverse and longitudinal in this limit.

4 Reducing Lindhard to Thomas-Fermi

We can also try to recover the Thomas-Fermi dielectric function

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \tag{123}$$

We'll see this in the static limit $\omega \to 0$.

4.1 Longitudinal Lindhard form going to TF form

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l(\,(\omega + i\nu)/kv_f\,)}{\omega + i\nu f_l(\,(\omega + i\nu)/kv_f\,)}, \tag{124}$$

This limit is simpler to take:

$$\begin{aligned}
\epsilon_l &= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l(\,(\omega + i\nu)/kv_f\,)}{\omega + i\nu f_l(\,(\omega + i\nu)/kv_f\,)} && (125)\\
&= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{i\nu f_l(\,(\omega + i\nu)/kv_f\,)}{i\nu f_l(\,(\omega + i\nu)/kv_f\,)} && (126)\\
&= 1 + \frac{3\omega_p^2}{k^2 v_F^2} && (127)
\end{aligned}$$

Using the definition $k_{TF}^2 = \frac{3\omega_p^2}{v_F^2}$, we rather easily get the Thomas-Fermi form.

It's worth noting that we get some disagreement between the Thomas-Fermi form and the Drude form around the region where both k and ω go to zero.

5 Explicit parts of Lindhard function

We want to find the explicit real and imaginary parts of the Lindhard function. To begin with, we can start with the case where $\nu \to 0$, which means long relaxation times.

We have our Lindhard form

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \tag{128}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1}. \tag{129}$$

5.1 Pines result

From Pines, we have the forms

$$\begin{aligned}\mathrm{Re}[\epsilon_l] = 1 + \frac{k_{TF}^2}{k^2}\Bigg(\frac{1}{2} + \frac{k_F}{4k}\Bigg[\left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\right) \ln\left|\frac{\omega - kv_F - \frac{\hbar k^2}{2m}}{\omega + kv_F - \frac{\hbar k^2}{2m}}\right| \\ + \left(1 - \frac{\left(\omega + \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\right) \ln\left|\frac{\omega + kv_F + \frac{\hbar k^2}{2m}}{\omega - kv_F + \frac{\hbar k^2}{2m}}\right|\Bigg]\Bigg)\end{aligned} \tag{130}$$

$$\mathrm{Im}[\epsilon_l] = \begin{cases} \frac{\pi}{2} \frac{\omega}{kv_F} \frac{k_{TF}^2}{k^2}, & \omega \leq kv_F - \frac{\hbar k^2}{2m} \\ \frac{\pi}{4} \frac{k_F}{k} \left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2}\right) \frac{k_{TF}^2}{k^2}, & kv_F - \frac{\hbar k^2}{2m} \leq \omega \leq kv_F + \frac{\hbar k^2}{2m} \\ 0, & \omega \geq kv_F + \frac{\hbar k^2}{2m} \end{cases} \tag{131}$$

5.2 Long relaxation time forms of the logs

In order to analyse the $\nu \to 0$ limit, we can start by looking at what happens to the logarithms in the Lindhard function:

$$\ln \frac{\omega + i\nu + kv_F}{\omega + i\nu - kv_F} \tag{132}$$

The first thing to note is that the numerator will always have a very small, positive argument, while the denominator will have a small argument which may be positive or negative. As Lindhard mentions, these logarithms should all have imaginary parts between $\pm i\pi$. This effectively means we can treat each logarithm as giving the principal value, which give us a result that looks like

$$\ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right), \tag{133}$$

where θ_+ and θ_- are the arguments of the numerator and denominator. For small ν, θ_+ is proportional to ν, as it'll be determined by an arcsine. However, the denominator may be negative, which would contribute a factor of $\theta_- = +i\pi$ (with a plus sign because ν would be just above the real line).

$$\ln \frac{\omega + i\nu + kv_F}{\omega + i\nu - kv_F} = \ln \frac{\sqrt{(\omega + kv_F)^2 + \nu^2}}{\sqrt{(\omega - kv_F)^2 + \nu^2}} + i\left(\theta_+ - \theta_-\right) \tag{134}$$

$$= \ln \frac{\sqrt{(\omega + kv_F)^2}}{\sqrt{(\omega - kv_F)^2}} - \sigma i\pi \tag{135}$$

$$= \ln \left| \frac{\omega + kv_F}{\omega - kv_F} \right| - \sigma i\pi, \tag{136}$$

where $\sigma = 1$ if $\omega < kv_F$, and 0 otherwise.

5.3 Taking long scattering time limit

To make our constants line up with the Pines form, we'll use the definition $k_{TF}^2 = \frac{3\omega_p^2}{v_F^2}$, giving us

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\, f_l(\,(\omega + i\nu)/kv_f\,)}{\omega + i\nu f_l(\,(\omega + i\nu)/kv_f\,)}, \tag{137}$$

We can't simply set $\nu = 0$, as we need to respect the relation

$$\frac{1}{x + i\delta} = \frac{1}{x} - i\pi\delta(x) \tag{138}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}. \tag{139}$$

We can eliminate all the terms proportional to ν in the numerator, as they will disappear as we take our limit:

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{(\omega + i\nu)\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega + i\nu}{2kv_F}\left(\ln\left|\frac{\omega + kv_F}{\omega - kv_F}\right| - \sigma i\pi\right)\right)} \tag{140}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega \left(1 - \frac{\omega}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)}{\omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)} \tag{141}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \frac{2kv_F - \omega \ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| + i\omega\pi\sigma}{\omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)} \tag{142}$$

We can now look at just the denominator:

$$= \omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right) \tag{143}$$

$$= \omega + i\nu - i\nu\frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right) \tag{144}$$

$$= \omega + i\nu + \frac{\nu^2 - i\nu\omega}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right) \tag{145}$$

$$= \omega + \frac{\nu^2}{2kv_F}L - \frac{\nu\omega\pi\sigma}{2kv_F} + i\nu - i\frac{\nu\omega}{2kv_F}L - i\frac{\nu^2\sigma\pi}{2kv_F}, \tag{146}$$

where $L = \ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right|$. We can notice here that all the imaginary terms are proportional to ν. We can ignore the ν^2 term, as it will go to zero faster than the other terms. We can see that we essentially have two regimes: if $\omega < 2kv_F$, this will have a positive imaginary part, and if $\omega > 2kv_F$, the imaginary part will be negative.

We can also notice that the only real part that will survive the limiting process is simply ω (which is of course clear from the original form of the denominator anyway). This lets us essentially write the denominator as

$$\omega + i\nu\zeta C, \tag{147}$$

where I'm defining ζ as 1 if $\omega < 2kv_F$, and -1 otherwise. We also can note that C is an irrelevant positive constant; when we take the long scattering time limit, all that matters is that the imaginary part is proportional to ν. Plugging this back into (142) gives us

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \frac{2kv_F - \omega \ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| + i\omega\pi\sigma}{\omega + i\nu\left(1 - \frac{\omega+i\nu}{2kv_F}\left(\ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| - \sigma i\pi\right)\right)} \tag{148}$$

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \frac{2kv_F - \omega \ln\left|\frac{\omega+kv_F}{\omega-kv_F}\right| + i\omega\pi\sigma}{\omega + i\nu\zeta C} \tag{149}$$

$$= 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \left(2kv_F - \omega \ln \left| \frac{\omega + kv_F}{\omega - kv_F} \right| + i\omega\pi\sigma \right) \left(\frac{1}{\omega} - i\pi\zeta\delta(\omega) \right) \tag{150}$$

$$= 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \left(\frac{2kv_F}{\omega} - L + i\pi\sigma - i\pi\zeta 2kv_F\delta(\omega) + (iL\zeta + \sigma\zeta)\,\pi\omega\delta(\omega) \right) \tag{151}$$

We can eliminate the terms proportional to $\omega\delta(\omega)$:

$$\epsilon_l = 1 + \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \left(\frac{2kv_F}{\omega} - L + i\pi\sigma - i\pi\zeta 2kv_F\delta(\omega) \right) \tag{152}$$

$$= 1 + \frac{k_{TF}^2}{k^2} \left(1 - \frac{\omega}{2kv_F} L + \frac{\omega}{2kv_F} i\pi\sigma - i\pi\zeta\omega\delta(\omega) \right) \tag{153}$$

$$= 1 + \frac{k_{TF}^2}{k^2} \left(1 - \frac{\omega}{2kv_F} L + \frac{\omega}{2kv_F} i\pi\sigma \right) \tag{154}$$

$$= 1 + \frac{k_{TF}^2}{k^2} \left(1 - \frac{\omega}{2kv_F} L \right) + i \frac{k_{TF}^2}{k^2} \frac{\omega}{2kv_F} \pi\sigma \tag{155}$$

$$= 1 + \frac{k_{TF}^2}{k^2} \left(1 - \frac{\omega}{2kv_F} L \right) + i \frac{\pi}{2} \frac{\omega}{kv_F} \frac{k_{TF}^2}{k^2} \sigma \tag{156}$$

5.4 Looking at classical limit of Pines result

As Ford and Weber mention, the result we've been looking at should correspond to a classical $\hbar \to 0$ limit. In order to verify that we match the Pines result, we can look at this limit. The imaginary part is trivially true, as in this classical limit the middle region of (131) goes away, and we see that the imaginary part of (156) matches, because of the behaviour we encapsulated within σ.

We can look more closely at (130) to verify that the real part also matches up:

$$\mathrm{Re}[\epsilon_l] = 1 + \frac{k_{TF}^2}{k^2} \left(\frac{1}{2} + \frac{k_F}{4k} \left[\left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \ln \left| \frac{\omega - kv_F - \frac{\hbar k^2}{2m}}{\omega + kv_F - \frac{\hbar k^2}{2m}} \right| + \left(1 - \frac{\left(\omega + \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \ln \left| \frac{\omega + kv_F + \frac{\hbar k^2}{2m}}{\omega - kv_F + \frac{\hbar k^2}{2m}} \right| \right] \right) \tag{157}$$

I believe that implicit in this limit will also need to be that k_F will change in this limit as well. We can write this in terms of $\hbar$, as $k_F = \frac{m}{\hbar} v_F$. Thus, we want to find

$$\lim_{\hbar\to 0} 1+\frac{k_{TF}^2}{k^2}\left(\frac{1}{2}+\frac{mv_F}{4\hbar k}\left[\left(1-\frac{\left(\omega-\frac{\hbar k^2}{2m}\right)^2}{k^2v_F^2}\right)\ln\left|\frac{\omega-kv_F-\frac{\hbar k^2}{2m}}{\omega+kv_F-\frac{\hbar k^2}{2m}}\right|\right.\right.$$
$$\left.\left.+\left(1-\frac{\left(\omega+\frac{\hbar k^2}{2m}\right)^2}{k^2v_F^2}\right)\ln\left|\frac{\omega+kv_F+\frac{\hbar k^2}{2m}}{\omega-kv_F+\frac{\hbar k^2}{2m}}\right|\right]\right) \tag{158}$$

We can do a series expansion on the entire term in brackets, which (need to verify) should give

$$\frac{4\frac{k^2}{2m}\left(kv_F-\omega L\right)\hbar}{k^2v_F^2} \tag{159}$$

Plugging this in gives us (writing only the terms multiplying in the $\frac{k_{TF}^2}{k^2}$ parentheses):

$$=\frac{1}{2}+\frac{mv_F}{4\hbar k}\frac{4\frac{k^2}{2m}\left(kv_F-\omega L\right)\hbar}{k^2v_F^2} \tag{160}$$

$$=\frac{1}{2}+\frac{v_F}{\hbar k}\frac{\frac{k^2}{2}\left(kv_F-\omega L\right)\hbar}{k^2v_F^2} \tag{161}$$

$$=\frac{1}{2}+\frac{v_F}{k}\frac{\frac{k^2}{2}\left(kv_F-\omega L\right)}{k^2v_F^2} \tag{162}$$

Our series expansion gives us no terms proportional to $\frac{1}{\hbar}$, and we also fortunately have a term that will survive the $\hbar\to 0$ limit.

$$=\frac{1}{2}+\frac{\frac{k^2}{2}\left(kv_F-\omega L\right)}{k^3v_F} \tag{163}$$

$$=\frac{1}{2}+\frac{1}{2}\frac{\left(kv_F-\omega L\right)}{kv_F} \tag{164}$$

$$=\frac{1}{2}+\frac{1}{2}\left(1-\frac{\omega L}{kv_F}\right) \tag{165}$$

$$=1-\frac{1}{2}\frac{\omega L}{kv_F} \tag{166}$$

Thus, in this classical limit, (130) reduces to

$$\mathrm{Re}[\epsilon_l]=1+\frac{k_{TF}^2}{k^2}\left(1-\frac{1}{2}\frac{\omega L}{kv_F}\right) \tag{167}$$

We note that this matches the real part of (156). This means that our original Lindhard form corresponds to the classical limit of the Pines result.

6 Qubit Relaxation Time

6.1 The quasi-static limit

We can start by looking at

$$\chi_{zz}^{E}(z,z,\omega)=\frac{\hbar}{\epsilon_0}\,\mathrm{Re}\int_0^\infty dp\,\frac{p^3}{q}e^{2iqz}r_p(p) \tag{168}$$

Here, we have

$$q=\begin{cases}\sqrt{\frac{\omega^2}{c^2}-p^2}, & p^2\le\frac{\omega^2}{c^2}\\ i\sqrt{p^2-\frac{\omega^2}{c^2}}, & p^2>\frac{\omega^2}{c^2}\end{cases} \tag{169}$$

If we look at the case where $\omega=6\pi\times10^8\,\mathrm{s}^{-1}$, the cutoff for real or imaginary q will be when $p=2\pi\,\mathrm{m}^{-1}$.

If we assume that $\mathrm{Im}\,r_p$ doesn't decay too quickly, this integral will be dominated by values of p larger than this, which lets us make the substitution that $\frac{\omega}{c}\to0$, which means this integral will reduce to

$$\chi_{zz}^{E}(z,z,\omega)=\frac{\hbar}{\epsilon_0}\int_0^\infty dp\,p^2e^{-2pz}\,\mathrm{Im}\,r_p(p,\omega) \tag{170}$$

The note that we're effectively taking $c\to\infty$ is important, as we still shouldn't necessarily assume that we can take $\omega\to0$ in r_p.

6.2 Non-local reflection coefficient

We can look specifically at what r_p will be in the non-local case:

$$r_p(p,\omega)=\frac{1-\frac{2p}{\pi}\int_0^\infty d\kappa\,\frac{1}{k^2\epsilon_l(k\omega)}}{1+\frac{2p}{\pi}\int_0^\infty d\kappa\,\frac{1}{k^2\epsilon_l(k\omega)}} \tag{171}$$

where $k^2=p^2+\kappa^2$ and

$$\epsilon_l=1+\frac{3\omega_p^2}{k^2v_F^2}\frac{(\omega+i\nu)\,f_l((\omega+i\nu)/kv_f)}{\omega+i\nu f_l((\omega+i\nu)/kv_f)}. \tag{172}$$

All of the interesting behaviour here comes from the integral, which we might name $I = \int_0^\infty d\kappa \, \frac{1}{k^2 \epsilon_l(k, m\omega)}$. Knowing that we will eventually need to find $\operatorname{Im} r_p$, we might find utility in also writing $I = I_1 + iI_2$ and noting that

$$\begin{aligned}
\operatorname{Im} r_p &= \operatorname{Im} \frac{1-I}{1+I} && (173)\\
&= \operatorname{Im} \frac{1 - I_1 - iI_2}{1 + I_1 + iI_2} && (174)\\
&= \operatorname{Im} \frac{1 - I_1 - iI_2}{1 + I_1 + iI_2} \frac{I + I_1 - iI_2}{I + I_1 - iI_2} && (175)\\
&= \operatorname{Im} \frac{(1 - I_1)(1 + I_1) - I_2^2 - iI_2(1 - I_1 + 1 + I_1)}{(1 + I_1)^2 + I_2^2} && (176)\\
&= \operatorname{Im} \frac{(1 - I_1)(1 + I_1) - I_2^2 - 2iI_2}{(1 + I_1)^2 + I_2^2} && (177)\\
&= \frac{-2I_2}{(1 + I_1)^2 + I_2^2}. && (178)
\end{aligned}$$

This gives us some idea of how $\operatorname{Im} r_p$ should behave, at least once we can write out the integral I.

$$I = \int_0^\infty d\kappa \, \frac{1}{k^2 \epsilon_l(k, \omega)} \tag{179}$$

Starting with the form in the Nam paper, we have for the conductivity σ:

$$\sigma(q, \omega) = -i \frac{3}{4} \frac{ne^2}{m} \frac{1}{\omega} \left[\int_{\Delta-\omega}^{\Delta} d\omega' \tanh\left(\frac{\omega + \omega'}{2T}\right) I_1 + \int_{\Delta}^{\infty} d\omega' \left(\tanh\left(\frac{\omega + \omega'}{2T}\right) I_1 + \tanh\left(\frac{\omega'}{2T}\right) I_2 \right) \right] \tag{180}$$

with

$$\begin{aligned}
I_1 = {} & F(q, \sqrt{(\omega + \omega')^2 - \Delta^2} - \sqrt{\omega'^2 - \Delta^2})(g + 1) \\
& + F(q, -\sqrt{(\omega + \omega')^2 - \Delta^2} - \sqrt{\omega'^2 - \Delta^2})(g - 1) && (181)\\
I_2 = {} & F(q, \sqrt{(\omega + \omega')^2 - \Delta^2} - \sqrt{\omega'^2 - \Delta^2})(g + 1) \\
& + F(q, \sqrt{(\omega + \omega')^2 - \Delta^2} + \sqrt{\omega'^2 - \Delta^2})(g - 1) && (182)\\
F(q, E) = {} & \frac{1}{qv_0} \left[S(E) + (1 - S(E)^2) \ln\left(\frac{S(E) + 1}{S(E) - 1}\right) \right] && (183)\\
S(E) = {} & \frac{1}{qv_0} \left(E - \frac{i}{\tau} \right) && (184)
\end{aligned}$$

$$g = \frac{\omega' \left(\omega + \omega'\right)}{\sqrt{\omega'^2 - \Delta^2}\sqrt{(\omega + \omega')^2 - \Delta^2}} \tag{185}$$

6.3 Removing units

To remove units, we'll want to represent all the various quantities in terms of Δ:

$$\xi = \frac{\omega}{\Delta} \tag{186}$$

$$\xi' = \frac{\omega'}{\Delta} \tag{187}$$

$$\nu = \frac{1}{\tau\Delta} \tag{188}$$

$$\kappa = \frac{qv_0}{\Delta} \tag{189}$$

$$t = \frac{T}{\Delta} \tag{190}$$

$$\sigma_0 = \frac{ne^2}{m\Delta} \tag{191}$$

This gives us

$$\sigma(\kappa, \xi) = -i\frac{3\sigma_0}{4}\frac{1}{\xi}\left[\int_{1-\xi}^{1} d\xi \tanh\left(\frac{\xi + \xi'}{2t}\right) I_1 + \int_1^\infty d\xi' \left(\tanh\left(\frac{\xi + \xi'}{2t}\right) I_1 + \tanh\left(\frac{\xi'}{2t}\right) I_2\right)\right] \tag{192}$$

with

$$\begin{aligned} I_1 &= F(\kappa, \sqrt{(\xi + \xi')^2 - 1} - \sqrt{\xi'^2 - 1})(g + 1) \\ &\quad + F(\kappa, -\sqrt{(\xi + \xi')^2 - 1} - \sqrt{\xi'^2 - 1})(g - 1) && (193) \\ I_2 &= F(\kappa, \sqrt{(\xi + \xi')^2 - 1} - \sqrt{\xi'^2 - 1})(g + 1) \\ &\quad + F(\kappa, \sqrt{(\xi + \xi')^2 - 1} + \sqrt{\xi'^2 - 1})(g - 1) && (194) \\ F(\kappa, E) &= \frac{1}{\kappa}\left[S(E) + (1 - S(E)^2)\ln\left(\frac{S(E) + 1}{S(E) - 1}\right)\right] && (195) \\ S(\kappa, E) &= \frac{1}{\kappa}\left(E - i\nu\right) && (196) \\ g &= \frac{\xi' \left(\xi + \xi'\right)}{\sqrt{\xi'^2 - 1}\sqrt{(\xi + \xi')^2 - 1}} && (197) \end{aligned}$$

For future reference, F carried units, which when included out a Δ in the integrals.

6.4 Comparing to normal conductivity

We can also compare this to the normal conductivity: $\sigma_N = \frac{ne^2\tau}{m}$:

$$\sigma_N = \frac{ne^2\tau}{m} \tag{198}$$

$$= \frac{ne^2}{m\Delta}\tau\Delta \tag{199}$$

$$= \sigma_0 \frac{1}{\nu} \tag{200}$$

This, we can find the ratio $\Sigma = \frac{\sigma}{\sigma_N} = \frac{\sigma\nu}{\sigma_0}$,

$$\Sigma(\kappa, \xi) = -i\frac{3}{4}\frac{\nu}{\xi}\left[\int_{1-\xi}^{1} d\xi \tanh\left(\frac{\xi+\xi'}{2t}\right) I_1 + \int_1^\infty d\xi' \left(\tanh\left(\frac{\xi+\xi'}{2t}\right) I_1 + \tanh\left(\frac{\xi'}{2t}\right) I_2\right)\right] \tag{201}$$

7 todos

Liste der noch zu erledigenden Punkte