1 Reducing Lindhard to Drude

We want to see how we can reduce the Lindhard dielectric function

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \tag{1}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1} \tag{2}$$

We can reduce things in the $k \to 0$ limit. The first half of (1) has the simple $\frac{1}{k^2}$ dependence, so we can look at how the rest of it behaves to start with.

1.1 f

$$f_l((\omega + i\nu)/kv_F) = 1 - \frac{(\omega + i\nu)/kv_F}{2} \ln \frac{(\omega + i\nu)/kv_F + 1}{(\omega + i\nu)/kv_F - 1} \tag{3}$$

Defining $\eta = \omega + i\nu$:

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \tag{4}$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2kv_F} \ln \frac{\eta + kv_F}{\eta - kv_F} \tag{5}$$

$$f_l = 1 - \frac{\eta}{2v_F} \frac{\ln \frac{\eta + kv_F}{\eta - kv_F}}{k} \tag{6}$$

$$\lim_{k \to 0} f_l = 1 - \frac{\eta}{2v_F} \frac{d \ln \frac{\eta + kv_F}{\eta - kv_F}}{dk} \tag{7}$$

$$= 1 - \frac{\eta}{2v_F} \frac{\eta - kv_F}{\eta + kv_F} \frac{v_F(\eta - kv_F) + v_F(\eta + kv_F)}{(\eta - kv_F)^2} \tag{8}$$

$$= 1 - \frac{\eta}{2} \frac{\eta - kv_F}{\eta + kv_F} \frac{(\eta - kv_F) + (\eta + kv_F)}{(\eta - kv_F)^2} \tag{9}$$

$$= 1 - \frac{\eta}{2} \frac{1}{\eta + kv_F} \frac{2\eta}{\eta - kv_F} \tag{10}$$

$$= 1 - \frac{\eta^2}{\eta^2 - k^2 v_F^2} \tag{11}$$

$$= \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2} \tag{12}$$

$$\lim_{k\to 0} f_l = 0 \tag{13}$$

Note that this goes to 0 for $k \to 0$.

1.2 Series expansion of f

The previous section gives the limit, but having the actual series expansion is probably more valuable. Again, with $\eta = \omega + i\nu$,

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \tag{14}$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2v_F k} \ln \frac{\eta + kv_F}{\eta - kv_F} \tag{15}$$

We want to expand up to k^2 overall, to cancel out the k^2 in the denominator of (1). We're looking at the function.

$$\frac{\ln(\eta + kv_F)}{k} - \frac{\ln(\eta - kv_F)}{k} \tag{16}$$

Generally, the derivatives of $\frac{g(a\pm x)}{x}$ are

$$\left(\frac{g(x)}{x}\right)' = \frac{\pm g'}{x} - \frac{g}{x^2} \tag{17}$$

$$\left(\frac{g(x)}{x}\right)'' = \frac{g''}{x} - \frac{\pm 2g'}{x^2} + \frac{2g}{x^3} \tag{18}$$

1.3 Back to dielectric

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)} \tag{19}$$

In the denominator, we can note that ω should dominate $i\nu f$, because f goes to zero, so we can simplify that.

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} \tag{20}$$

Using (12), we get

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} \tag{21}$$

$$= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2}}{\omega} \tag{22}$$

$$= 1 - 3\omega_p^2 \frac{(\omega + i\nu) \frac{1}{\eta^2}}{\omega} \tag{23}$$

$$= 1 - \frac{3\omega_p^2}{\omega (\omega + i\nu)} \tag{24}$$

This is the Drude limit.