1 Explicit parts of Lindhard function

We want to find the explicit real and imaginary parts of the Lindhard function. To begin with, we can start with the case where $\nu \to 0$, which means long relaxation times.

We have our Lindhard form

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \tag{1}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1}. \tag{2}$$

1.1 Pines result

From Pines, we have the forms

$$\begin{aligned} \mathrm{Re}[\epsilon_l] = 1 + \frac{k_{TF}^2}{k^2} \left(\frac{1}{2} + \frac{k_F}{4k} \left[\left(1 - \frac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \ln \left[\frac{\omega - kv_F - \frac{\hbar k^2}{2m}}{\omega + kv_F - \frac{\hbar k^2}{2m}} \right] \right.\right. \\ \left.\left. + \left(1 - \frac{\left(\omega + \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \ln \left[\frac{\omega + kv_F + \frac{\hbar k^2}{2m}}{\omega - kv_F + \frac{\hbar k^2}{2m}} \right] \right] \right) \end{aligned} \tag{3}$$

$$\mathrm{Im}[\epsilon_l] = \begin{cases} \dfrac{\pi}{2} \dfrac{\omega}{kv_F} \dfrac{k_{TF}^2}{k^2}, & \omega \leq kv_F - \frac{\hbar k^2}{2m} \\ \dfrac{\pi}{4} \dfrac{k_F}{k} \left(1 - \dfrac{\left(\omega - \frac{\hbar k^2}{2m}\right)^2}{k^2 v_F^2} \right) \dfrac{k_{TF}^2}{k^2}, & kv_F - \frac{\hbar k^2}{2m} \leq \omega \leq kv_F + \frac{\hbar k^2}{2m} \\ 0, & \omega \geq kv_F + \frac{\hbar k^2}{2m} \end{cases} \tag{4}$$