

# 1 General discussion of dielectric function

## 1.1 Fourier transform of dielectric components

We can start by looking at some basic relationships between  $\mathbf{D}$ ,  $\mathbf{E}$ . In SI units, and writing  $\epsilon$  instead of  $\epsilon_r$  to avoid unnecessary subscripts,

$$\mathbf{D}_\alpha(r) = \int d^d r' \epsilon_{\alpha\beta}(r, r') \epsilon_0 \mathbf{E}_\beta(r'). \quad (1)$$

We can start by making the large assumption that the system is isotropic, and thus, the position dependence for  $\epsilon$  must be of the form  $\epsilon_{\alpha\beta}(r - r')$ . This justifies the following:

$$\mathbf{D}_\alpha(r) = \int d^d r' \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') \quad (2)$$

$$= \int d^d r' \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') e^{ikr'} e^{-ikr'} \quad (3)$$

$$= \int d^d r' \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') e^{ikr'} e^{-ikr'} \quad (4)$$

$$= \int d^d r' \epsilon_{\alpha\beta}(r - r') e^{ikr'} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \quad (5)$$

$$\mathbf{D}_\alpha(r) e^{-ikr} = \int d^d r' \epsilon_{\alpha\beta}(r - r') e^{ikr'} e^{-ikr} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \quad (6)$$

$$\mathbf{D}_\alpha(r) e^{-ikr} = \int d^d r' \epsilon_{\alpha\beta}(r - r') e^{-ik(r-r')} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \quad (7)$$

If we integrate this over  $r$ , and recognise our Fourier transforms, this becomes

$$D_\alpha(k) = \epsilon_{\alpha\beta}(k) \epsilon_0 E_\beta(k) \quad (8)$$

This is a fair result for the assumption of isotropy. We have an implicit sum over  $\beta$  here to look at. This serves as a useful working definition of  $\epsilon(k)$ .

One more assumption proves useful: for a given  $\mathbf{k}$ , we can look at the components of  $\mathbf{E}$  and  $\mathbf{D}$  parallel (longitudinal) and perpendicular (transverse) to  $\mathbf{k}$ . We may assume that only the parallel component of  $\mathbf{E}$  affects the parallel component of  $\mathbf{D}$ , and same for the perpendicular components. Specifically, this leads to

$$\epsilon_{\alpha\beta}(\mathbf{k}) = \epsilon_{\parallel}(\mathbf{k}) \hat{k}_\alpha \hat{k}_\beta + \epsilon_{\perp}(\mathbf{k}) (\delta_{\alpha\beta} - \hat{k}_\alpha \hat{k}_\beta). \quad (9)$$

A moment's reflection shows that this leads to two independent components of  $\epsilon_{\alpha\beta}(\mathbf{k})$  (as can be seen by assuming  $\mathbf{k}$  points in the  $x$  direction). Also note that that for the transverse case, this explicitly only couples  $y$  to  $y$  and  $z$  to  $z$ ; there are no couplings between  $y$  and  $z$ .

why can we assume this?

include brief derivation of dielectric components

## 1.2 Electron gas densities and longitudinal eqs

To begin looking at an actual electron gas, with some external charge density  $\rho_{ext}$  imposed on it, we can start with Gauss's law:

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{ext}(r) \quad (10)$$

This external charge density induces a charge density in the gas:

$$\rho(r) = \rho_{ind}(r) + \rho_{ext}(r) \quad (11)$$

We can define some scalar potentials  $\Phi$  and  $\Phi_{ext}$ , and therewith write

$$\mathbf{E}(r) = -\nabla\Phi(r) \quad (12)$$

$$\mathbf{D}(r) = -\epsilon_0 \nabla\Phi_{ext}(r) \quad (13)$$

Going Fourier with it leads to

$$\mathbf{E}(k) = -i\mathbf{k}\Phi(k) \quad (14)$$

$$\mathbf{D}(k) = -i\epsilon_0\mathbf{k}\Phi_{ext}(k) \quad (15)$$

These are the components of  $\mathbf{E}$  and  $\mathbf{D}$  parallel to  $\mathbf{k}$ , which means that these are really the *longitudinal* components.

Relating Gauss's laws to the scalar components, we get

$$\rho_{ext}(\mathbf{k}) = \nabla \cdot \mathbf{D}(\mathbf{k}) \quad (16)$$

$$= \nabla \cdot (-i\epsilon_0\mathbf{k}\Phi_{ext}(\mathbf{k})) \quad (17)$$

$$\rho_{ext}(\mathbf{k}) = \epsilon_0 k^2 \Phi_{ext}(\mathbf{k}) \quad (18)$$

Similarly, for  $\mathbf{E}$  and  $\rho$ , we get

$$\rho(\mathbf{k}) = \epsilon_0 k^2 \Phi(\mathbf{k}). \quad (19)$$

Because of (8), we should be able to write

$$\Phi_{ext}(\mathbf{k}) = \epsilon_{\parallel}(\mathbf{k})\Phi(\mathbf{k}) \quad (20)$$

Skipping over some details to fill in later (which involve defining potential energies  $V = -e\Phi$  and using number densities defined by  $\rho = -en$ , we end up with

Fill in these details

$$\frac{1}{\epsilon_{\parallel}(k, \omega)} = 1 + \frac{4\pi e^2}{k^2} \Pi(k, \omega), \quad (21)$$

where  $\Pi(k, \omega)$  is a response function satisfying

$$n_{ind}(k, \omega) = \Pi(k, \omega) V_f(k, \omega) \quad (22)$$

Here  $n_{ind}$  is the number density of induced electrons, and  $V_f$  is the voltage created by any free electrons in the metal (which isn't quite the same as an external voltage, but I think you might be able to ignore that difference).

There's no reason for  $\Pi(\mathbf{k}, \omega)$  to be simple. In general, it describes the complete induced density response to any external potential.