1 Nam form of conductivity

Starting with the form in the Nam paper, we have for the conductivity σ:

$$\sigma(q,\omega) = -i\frac{3}{4}\frac{ne^2}{m}\frac{1}{\omega}\left[\int_{\Delta-\omega}^{\Delta} d\omega' \tanh\left(\frac{\omega+\omega'}{2T}\right) I_1 + \int_{\Delta}^{\infty} d\omega' \left(\tanh\left(\frac{\omega+\omega'}{2T}\right) I_1 - \tanh\left(\frac{\omega'}{2T}\right) I_2\right)\right] \tag{1}$$

with

$$\begin{aligned} I_1 &= F(q, \mathrm{Re}[\sqrt{(\omega+\omega')^2-\Delta^2} - \sqrt{\omega'^2-\Delta^2}])(g+1) \\ &\quad + F(q, \mathrm{Re}[-\sqrt{(\omega+\omega')^2-\Delta^2} - \sqrt{\omega'^2-\Delta^2}])(g-1) \qquad (2) \\ I_2 &= F(q, \mathrm{Re}[\sqrt{(\omega+\omega')^2-\Delta^2} - \sqrt{\omega'^2-\Delta^2}])(g+1) \\ &\quad + F(q, \mathrm{Re}[\sqrt{(\omega+\omega')^2-\Delta^2} + \sqrt{\omega'^2-\Delta^2}])(g-1) \qquad (3) \\ F(q,E) &= \frac{1}{qv_0}\left[2S(E) + (1-S(E)^2)\ln\left(\frac{S(E)+1}{S(E)-1}\right)\right] \qquad (4) \\ S(E) &= \frac{1}{qv_0}\left(E - i\left(\mathrm{Im}[\sqrt{(\omega+\omega')^2-\Delta^2} + \sqrt{\omega'^2-\Delta^2}] + \frac{2}{\tau}\right)\right) \qquad (5) \\ g &= \frac{\omega'(\omega+\omega')+\Delta^2}{\sqrt{\omega'^2-\Delta^2}\sqrt{(\omega+\omega')^2-\Delta^2}} \qquad (6) \end{aligned}$$

1.1 Removing units

To remove units, we'll want to represent all the various quantities in terms of Δ:

$$\xi = \frac{\omega}{\Delta} \tag{7}$$

$$\xi' = \frac{\omega'}{\Delta} \tag{8}$$

$$\nu = \frac{1}{\tau\Delta} \tag{9}$$

$$\kappa = \frac{qv_0}{\Delta} \tag{10}$$

$$t = \frac{T}{\Delta} \tag{11}$$

$$\sigma_0 = \frac{ne^2}{m\Delta} \tag{12}$$

This gives us

$$\sigma(\kappa,\xi) = -i\frac{3\sigma_0}{4}\frac{1}{\xi}\left[\int_{1-\xi}^{1} d\xi \tanh\left(\frac{\xi+\xi'}{2t}\right)I_1 + \int_1^\infty d\xi' \left(\tanh\left(\frac{\xi+\xi'}{2t}\right)I_1 - \tanh\left(\frac{\xi'}{2t}\right)I_2\right)\right] \tag{13}$$

with

$$\begin{aligned} I_1 &= F(\kappa, \mathrm{Re}[\sqrt{(\xi+\xi')^2-1} - \sqrt{\xi'^2-1}])(g+1) \\ &\quad + F(\kappa, \mathrm{Re}[-\sqrt{(\xi+\xi')^2-1} - \sqrt{\xi'^2-1}])(g-1) \qquad (14) \\ I_2 &= F(\kappa, \mathrm{Re}[\sqrt{(\xi+\xi')^2-1} - \sqrt{\xi'^2-1}])(g+1) \\ &\quad + F(\kappa, \mathrm{Re}[\sqrt{(\xi+\xi')^2-1} + \sqrt{\xi'^2-1}])(g-1) \qquad (15) \\ F(\kappa,E) &= \frac{1}{\kappa}\left[2S(E) + (1-S(E)^2)\ln\left(\frac{S(E)+1}{S(E)-1}\right)\right] \qquad (16) \\ S(\kappa,E) &= \frac{1}{\kappa}\left(E - i\left(\mathrm{Im}[\sqrt{(\xi+\xi')^2-1} + \sqrt{\xi'^2-1}] + 2\nu\right)\right) \qquad (17) \\ g &= \frac{\xi'(\xi+\xi')+1}{\sqrt{\xi'^2-1}\sqrt{(\xi+\xi')^2-1}} \qquad (18) \end{aligned}$$

For future reference, F carried units, which when included out a Δ in the integrals.

1.2 Comparing to normal conductivity

We can also compare this to the normal conductivity: $\sigma_N = \frac{ne^2\tau}{m}$:

$$\begin{aligned} \sigma_N &= \frac{ne^2\tau}{m} \qquad (19) \\ &= \frac{ne^2}{m\Delta}\tau\Delta \qquad (20) \\ &= \sigma_0\frac{1}{\nu} \qquad (21) \end{aligned}$$

This, we can find the ratio $\Sigma = \frac{\sigma}{\sigma_N} = \frac{\sigma\nu}{\sigma_0}$,

$$\Sigma(\kappa,\xi) = -i\frac{3}{4}\frac{\nu}{\xi}\left[\int_{1-\xi}^{1} d\xi \tanh\left(\frac{\xi+\xi'}{2t}\right)I_1 + \int_1^\infty d\xi' \left(\tanh\left(\frac{\xi+\xi'}{2t}\right)I_1 - \tanh\left(\frac{\xi'}{2t}\right)I_2\right)\right] \tag{22}$$

1.3 Verifying small κ dependence

We should expect that the conductivity reaches a finite value as $\kappa \to 0$. To verify this, we'll want to actually take that limit. All of the dependence on momentum comes in through the function F and S, so we can begin by writing S as $S = \frac{\eta}{\kappa}$, which means that

$$F = \frac{1}{\kappa}\left[2S(E) + (1 - S(E)^2)\ln\left(\frac{S(E)+1}{S(E)-1}\right)\right] \tag{23}$$

$$F = \frac{1}{\kappa}\left[2\frac{\eta}{\kappa} + (1 - \frac{\eta^2}{\kappa^2})\ln\left(\frac{\frac{\eta}{\kappa}+1}{\frac{\eta}{\kappa}-1}\right)\right] \tag{24}$$

We can then expand out the log term:

$$\ln\left(\frac{\frac{\eta}{\kappa}+1}{\frac{\eta}{\kappa}-1}\right) = \ln\left(\frac{\eta+\kappa}{\eta-\kappa}\right) \tag{25}$$

$$= 2\frac{\kappa}{\eta} + \frac{2}{3}\left(\frac{\kappa}{\eta}\right)^3 + \frac{2}{5}\left(\frac{\kappa}{\eta}\right)^5 + \mathcal{O}\left(\left(\frac{\kappa}{\eta}\right)^7\right) \tag{26}$$

Plugging the first two terms into (24) gives us

$$F = \frac{1}{\kappa}\left[2\frac{\eta}{\kappa} + (1 - \frac{\eta^2}{\kappa^2})\ln\left(\frac{\frac{\eta}{\kappa}+1}{\frac{\eta}{\kappa}-1}\right)\right] \tag{27}$$

$$= \frac{1}{\kappa}\left[2\frac{\eta}{\kappa} + (1 - \frac{\eta^2}{\kappa^2})\left(2\frac{\kappa}{\eta} + \frac{2}{3}\left(\frac{\kappa}{\eta}\right)^3\right)\right] \tag{28}$$

$$= \frac{1}{\kappa}\left[2\frac{\eta}{\kappa} + 2\frac{\kappa}{\eta} - 2\frac{\eta}{\kappa} + \frac{2}{3}\frac{\kappa^3}{\eta^3} - \frac{2}{3}\frac{\kappa}{\eta}\right] \tag{29}$$

$$= \frac{1}{\kappa}\left[\frac{4}{3}\frac{\kappa}{\eta}\right] \tag{30}$$

$$= \frac{4}{3}\frac{1}{\eta} \tag{31}$$

Here we dropped the second leading term in κ^3 before simplifying, to find that F does indeed approximate a constant value depending on F.