1 Reducing Lindhard to Drude

We want to see how we can reduce the Lindhard dielectric function

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \tag{1}$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1} \tag{2}$$

We can reduce things in the $k \to 0$ limit. The first half of (1) has the simple $\frac{1}{k^2}$ dependence, so we can look at how the rest of it behaves to start with.

1.1 f

$$f_l((\omega + i\nu)/kv_F) = 1 - \frac{(\omega + i\nu)/kv_F}{2} \ln \frac{(\omega + i\nu)/kv_F + 1}{(\omega + i\nu)/kv_F - 1} \tag{3}$$

Defining $\eta = \omega + i\nu$:

$$\begin{aligned}
f_l((\omega + i\nu)/kv_f) &= 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} && (4)\\
f_l(\eta/kv_F) &= 1 - \frac{\eta}{2kv_F} \ln \frac{\eta + kv_F}{\eta - kv_F} && (5)\\
f_l &= 1 - \frac{\eta}{2v_F} \frac{\ln \frac{\eta + kv_F}{\eta - kv_F}}{k} && (6)\\
\lim_{k \to 0} f_l &= 1 - \frac{\eta}{2v_F} \frac{d \ln \frac{\eta + kv_F}{\eta - kv_F}}{dk} && (7)\\
&= 1 - \frac{\eta}{2v_F} \frac{\eta - kv_F}{\eta + kv_F} \frac{v_F(\eta - kv_F) + v_F(\eta + kv_F)}{(\eta - kv_F)^2} && (8)\\
&= 1 - \frac{\eta}{2} \frac{\eta - kv_F}{\eta + kv_F} \frac{(\eta - kv_F) + (\eta + kv_F)}{(\eta - kv_F)^2} && (9)\\
&= 1 - \frac{\eta}{2} \frac{1}{\eta + kv_F} \frac{2\eta}{\eta - kv_F} && (10)\\
&= 1 - \frac{\eta^2}{\eta^2 - k^2 v_F^2} && (11)\\
&= \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2} && (12)
\end{aligned}$$

$$\lim_{k\to 0} f_l = 0 \tag{13}$$

Note that this goes to 0 for $k \to 0$.

1.2 Series expansion of f

The previous section gives the limit, but having the actual series expansion is probably more valuable. Again, with $\eta = \omega + i\nu$,

$$f_l(\,(\omega+i\nu)/kv_f\,) = 1 - \frac{(\omega+i\nu)/kv_f}{2} \ln \frac{(\omega+i\nu)/kv_f\,+1}{(\omega+i\nu)/kv_f\,-1} \tag{14}$$

$$f_l(\,\eta/kv_F\,) = 1 - \frac{\eta}{2v_F k} \ln \frac{\eta + kv_F}{\eta - kv_F} \tag{15}$$

We want to expand up to k^2 overall, to cancel out the k^2 in the denominator of (1). We're looking at the function.

$$\frac{\ln(\eta + kv_F)}{k} - \frac{\ln(\eta - kv_F)}{k} \tag{16}$$

Generally, the derivatives of $\frac{g(a\pm x)}{x}$ are

$$\left(\frac{g(x)}{x}\right)' = \frac{\pm g'}{x} - \frac{g}{x^2} \tag{17}$$

$$\left(\frac{g(x)}{x}\right)'' = \frac{g''}{x} - \frac{\pm 2g'}{x^2} + \frac{2g}{x^3} \tag{18}$$

$$\left(\frac{g(x)}{x}\right)''' = \frac{\pm g'''}{x} - \frac{3g''}{x^2} + \frac{\pm 6g'}{x^3} - \frac{6g}{x^4} \tag{19}$$

When we take the difference, we see that we'll only end up keeping (and doubling) the terms of odd derivatives. Thus, up to this order, the series for $\frac{g(a+x)-g(a-x)}{x}$ will look like:

$$\frac{1}{2}\frac{g(a+x)-g(a-x)}{x} = x\frac{g'}{x} - \frac{1}{2}x^2\frac{2g'}{x^2} + \frac{1}{6}x^3\left(\frac{g'''}{x} + \frac{6g'}{x^3}\right) \tag{20}$$

$$= g' - g' + \frac{1}{6}x^2 g''' + g' \tag{21}$$

$$\frac{g(a+x)-g(a-x)}{x} = 2g' + \frac{1}{3}x^2 g''' + \mathcal{O}(x^4) \tag{22}$$

This type of result is to be expected: we are starting with an even function. For $g = \ln(\eta + kv_F)$, we have

$$g'(k=0) = \frac{v_F}{\eta} \tag{23}$$

$$g'''(k=0) = \frac{2v_F^3}{\eta^3} \tag{24}$$

Plugging these into (15) gives us:

$$\begin{aligned}
f_l(\eta/kv_F) &= 1 - \frac{\eta}{2v_F k}\ln\frac{\eta + kv_F}{\eta - kv_F} &(25)\\
&= 1 - \frac{\eta}{2v_F}\left(2\frac{v_F}{\eta} + \frac{1}{3}k^2\frac{2v_F^3}{\eta^3}\right) &(26)\\
&= 1 - 1 - \frac{1}{3}k^2\frac{v_F^2}{\eta^2} &(27)\\
&= -\frac{k^2 v_F^2}{3\eta^2} &(28)
\end{aligned}$$

This gives us a simple approximation for f_l in the long wavelength limit.

1.3 Back to dielectric

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2}\frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)} \tag{29}$$

In the denominator, we can note that ω should dominate $i\nu f$, because f goes to zero, so we can simplify that.

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2}\frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} \tag{30}$$

Using (28), we get

$$\begin{aligned}
\epsilon_l &= 1 + \frac{3\omega_p^2}{k^2 v_F^2}\frac{(\omega + i\nu)\, f_l((\omega + i\nu)/kv_f)}{\omega} &(31)\\
&= 1 + \frac{3\omega_p^2}{k^2 v_F^2}\frac{(\omega + i\nu)\,\frac{-k^2 v_F^2}{3\eta^2}}{\omega} &(32)\\
&= 1 - 3\omega_p^2\frac{\eta\frac{1}{3\eta^2}}{\omega} &(33)\\
&= 1 - \frac{\omega_p^2}{\omega\,(\omega + i\nu)} &(34)
\end{aligned}$$

This is the Drude limit, keeping in mind that $\omega_p^2 = \frac{4\pi n e^2}{m}$ in Gaussian units.