1 Drude model parameters

The assumptions of the Drude model are simple: we have interaction-free electrons that occasionally undergo some scattering process during a time dt with probability $\frac{dt}{\tau}$, where τ is some phenomenological parameter. This scattering will randomise electron momentum.

Our ultimate goal will be to find the conductivity σ and the dielectric constant ϵ in the Drude model, with Drude relaxation time τ, electron density n and electron mass m. We'll find

$$\sigma_{\mathrm{DC}} = \frac{ne^2\tau}{m} \tag{1}$$

$$\sigma_{\mathrm{AC}} = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau} \qquad \text{For SI and Gaussian} \tag{2}$$

$$\epsilon_r = 1 + i\frac{4\pi\sigma}{\omega} \qquad \text{Gaussian} \tag{3a}$$

$$\epsilon_r = 1 + i\frac{\sigma}{\omega\epsilon_0} \qquad \text{SI} \tag{3b}$$

Our dielectric constant can be rewritten to plug in for σ, giving us

$$\begin{aligned}
\epsilon &= 1 + i\frac{4\pi\sigma}{\omega} && (4)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau} && (5)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau}\frac{1+i\omega\tau}{1+i\omega\tau} && (6)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1+i\omega\tau}{1+\omega^2\tau^2} && (7)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\omega\tau}{\omega\left(1+\omega^2\tau^2\right)}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) && (8)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\tau}{1+\omega^2\tau^2}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) && (9)
\end{aligned}$$

This lets us write down the explicit real and imaginary of the Drude dielectric function.

1.1 Alternative forms of the Drude model

We can also rewrite the dielectric constant very slightly in terms of the plasma frequency. In Gaussian units:

$$\begin{aligned}
\epsilon &= 1 + i\frac{4\pi\sigma}{\omega} && (10)\\
&= 1 + i\frac{4\pi}{\omega}\frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau} && (11)\\
&= 1 + i\frac{4\pi}{\omega}\frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}\frac{i\nu}{i\nu} && (12)\\
&= 1 - \frac{4\pi}{\omega}\frac{ne^2}{m}\frac{1}{i\nu+\omega} && (13)
\end{aligned}$$

With $\omega_p^2 = \frac{4\pi ne^2}{m}$ in Gaussian units, this becomes

$$\epsilon = 1 - \frac{\omega_p^2}{\omega(\omega+i\nu)} \tag{14}$$

We'll see this again later.

1.2 Derivations for Drude model

1.2.1 DC Conductivity

We can start unit-system independently, with the expression

$$\mathbf{j} = \sigma\mathbf{E}. \tag{15}$$

We can also relate our current to our average electron velocity: $\mathbf{j} = ne\mathbf{v}$. Imagine at time $t = 0$ our electron undergoes a Drude collision, and emerges with $\mathbf{v}_{t=0} = \mathbf{v_0}$. After a time t, the electron will accelerate with acceleration $-\frac{e\mathbf{E}}{m}$ (which fortunately remains unit independent). Because it will only accelerate for a time τ on average before a collision, it will end up with velocity $\mathbf{v} = -\frac{e\mathbf{E}}{m}\tau + \mathbf{v_0}$. The average velocity, and current, will be

$$\begin{aligned}
\langle\mathbf{v}\rangle &= -\frac{e\mathbf{E}}{m}\tau + \langle\mathbf{v_0}\rangle && (16)\\
&= -\frac{e\mathbf{E}}{m}\tau && (17)\\
\frac{\mathbf{j}}{ne} &= -\frac{e\mathbf{E}}{m}\tau && (18)\\
\mathbf{j} &= -\frac{ne^2\tau}{m}\mathbf{E}. && (19)
\end{aligned}$$

This of course gives us, unit-independently, our DC conductivity $\sigma_{\mathrm{DC}} = \frac{ne^2\tau}{m}$.

1.2.2 AC Conductivity

The AC conductivity is also simple, but we want to be a bit more formal about it. We can write out the contributions to velocity in terms of probabilities. The velocity at a time dt will have probability dt/τ of being 0, and will otherwise be the original velocity minus $a\,dt$:

$$\mathbf{v}(dt) = \left(1 - \frac{dt}{\tau}\right)\left(\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\right) \tag{20}$$

$$= \mathbf{v_0} - \frac{dt}{\tau}\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\,, \tag{21}$$

where we've invoked our inalienable right as physicists to ignore all terms $\mathcal{O}(dt^2)$. This reduces, using the definition of $d\mathbf{v} = \mathbf{v}(dt) - \mathbf{v_0}$, to

$$d\mathbf{v} = \frac{dt}{\tau}\mathbf{v} - \frac{e\mathbf{E}}{m}\,dt \tag{22}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{v}}{\tau} - \frac{e\mathbf{E}}{m} \tag{23}$$

We can quickly Fourier transform this, using $\frac{d}{dt} \to -i\omega$, and we get (after surreptitiously dropping some vector signs)

$$-i\omega v(\omega) = -\frac{v(\omega)}{\tau} - \frac{eE(\omega)}{m} \tag{24}$$

$$v(\omega) = \frac{eE(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{25}$$

$$j(\omega) = \frac{ne^2E(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{26}$$

$$= \frac{ne^2\tau E(\omega)}{m\,(1 - i\omega\tau)}, \tag{27}$$

which gives us our AC conductivity in equation (2).

Now for our dielectric constant, we have to find some other defining relation on par with (15).