1 Drude model σ and ϵ

The assumptions of the Drude model are simple: we have interaction-free electrons that occasionally undergo some scattering process during a time dt with probability $\frac{dt}{\tau}$, where τ is some phenomenological parameter. This scattering will randomise electron momentum.

Our ultimate goal will be to find the conductivity σ and the dielectric constant ϵ in the Drude model, with Drude relaxation time τ, electron density n and electron mass m. We'll find

$$\sigma_{\mathrm{DC}} = \frac{ne^2\tau}{m} \tag{1}$$

$$\sigma_{\mathrm{AC}} = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau} \qquad \text{For SI and Gaussian} \tag{2}$$

$$\epsilon_r = 1 + i\frac{4\pi\sigma}{\omega} \qquad \text{Gaussian} \tag{3a}$$

$$\epsilon_r = 1 + i\frac{\sigma}{\omega\epsilon_0} \qquad \text{SI} \tag{3b}$$

Our dielectric constant can be rewritten to plug in for σ, giving us

$$\begin{aligned}
\epsilon &= 1 + i\frac{4\pi\sigma}{\omega} && (4)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau} && (5)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1}{1-i\omega\tau}\frac{1+i\omega\tau}{1+i\omega\tau} && (6)\\
&= 1 + i\frac{4\pi\sigma_{\mathrm{DC}}}{\omega}\frac{1+i\omega\tau}{1+\omega^2\tau^2} && (7)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\omega\tau}{\omega\left(1+\omega^2\tau^2\right)}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) && (8)\\
&= \left(1 - \frac{4\pi\sigma_{\mathrm{DC}}\tau}{1+\omega^2\tau^2}\right) + i\left(\frac{4\pi\sigma_{\mathrm{DC}}}{\omega\left(1+\omega^2\tau^2\right)}\right) && (9)
\end{aligned}$$

This lets us write down the explicit real and imaginary of the Drude dielectric function.

1.1 Limiting forms of the Drude model

We can look at the large and small ω limits for the Drude dielectric function.

1.2 Derivations for Drude model

1.2.1 DC Conductivity

We can start unit-system independently, with the expression

$$\mathbf{j} = \sigma \mathbf{E}. \tag{10}$$

We can also relate our current to our average electron velocity: $\mathbf{j} = ne\mathbf{v}$. Imagine at time $t = 0$ our electron undergoes a Drude collision, and emerges with $\mathbf{v}_{t=0} = \mathbf{v_0}$. After a time t, the electron will accelerate with acceleration $-\frac{e\mathbf{E}}{m}$ (which fortunately remains unit independent). Because it will only accelerate for a time τ on average before a collision, it will end up with velocity $\mathbf{v} = -\frac{e\mathbf{E}}{m}\tau + \mathbf{v_0}$. The average velocity, and current, will be

$$\begin{aligned}
\langle \mathbf{v} \rangle &= -\frac{e\mathbf{E}}{m}\tau + \langle \mathbf{v_0} \rangle & (11)\\
&= -\frac{e\mathbf{E}}{m}\tau & (12)\\
\frac{\mathbf{j}}{ne} &= -\frac{e\mathbf{E}}{m}\tau & (13)\\
\mathbf{j} &= -\frac{ne^2\tau}{m}\mathbf{E}. & (14)
\end{aligned}$$

This of course gives us, unit-independently, our DC conductivity $\sigma_{\text{DC}} = \frac{ne^2\tau}{m}$.

The AC conductivity is also simple, but we want to be a bit more formal about it. We can write out the contributions to velocity in terms of probabilities. The velocity at a time dt will have probability dt/τ of being 0, and will otherwise be the original velocity minus $a\,dt$:

$$\begin{aligned}
\mathbf{v}(dt) &= \left(1 - \frac{dt}{\tau}\right)\left(\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\right) & (15)\\
&= \mathbf{v_0} - \frac{dt}{\tau}\mathbf{v_0} - \frac{e\mathbf{E}}{m}\,dt\,, & (16)
\end{aligned}$$

where we've invoked our inalienable right as physicists to ignore all terms $\mathcal{O}(dt^2)$. This reduces, using the definition of $d\mathbf{v} = \mathbf{v}(dt) - \mathbf{v_0}$, to

$$\begin{aligned}
d\mathbf{v} &= \frac{dt}{\tau}\mathbf{v} - \frac{e\mathbf{E}}{m}\,dt & (17)\\
\frac{d\mathbf{v}}{dt} &= \frac{\mathbf{v}}{\tau} - \frac{e\mathbf{E}}{m} & (18)
\end{aligned}$$

We can quickly Fourier transform this, using $\frac{d}{dt} \to -i\omega$, and we get (after surreptitiously dropping some vector signs)

$$-i\omega v(\omega) = -\frac{v(\omega)}{\tau} - \frac{eE(\omega)}{m} \tag{19}$$

$$v(\omega) = \frac{eE(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{20}$$

$$j(\omega) = \frac{ne^2E(\omega)}{m\left(\frac{1}{\tau} - i\omega\right)} \tag{21}$$

$$= \frac{ne^2\tau E(\omega)}{m\left(1 - i\omega\tau\right)}, \tag{22}$$

which gives us our AC conductivity in equation (2).

Now for our dielectric constant, we have to find some other defining relation on par with (10).