

1 General discussion of dielectric function

1.1 Fourier transform of dielectric components

We can start by looking at some basic relationships between \mathbf{D} , \mathbf{E} . In SI units, and writing ϵ instead of ϵ_r to avoid unnecessary subscripts,

$$\mathbf{D}_\alpha(r) = \int d^d r' \epsilon_{\alpha\beta}(r, r') \epsilon_0 \mathbf{E}_\beta(r'). \quad (1)$$

We can start by making the large assumption that the system is isotropic, and thus, the position dependence for ϵ must be of the form $\epsilon_{\alpha\beta}(r - r')$. This justifies the following:

$$\mathbf{D}_\alpha(r) = \int d^d r' \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') \quad (2)$$

$$= \int d^d r' \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') e^{ikr'} e^{-ikr'} \quad (3)$$

$$= \int d^d r' \epsilon_{\alpha\beta}(r - r') \epsilon_0 \mathbf{E}_\beta(r') e^{ikr'} e^{-ikr'} \quad (4)$$

$$= \int d^d r' \epsilon_{\alpha\beta}(r - r') e^{ikr'} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \quad (5)$$

$$\mathbf{D}_\alpha(r) e^{-ikr} = \int d^d r' \epsilon_{\alpha\beta}(r - r') e^{ikr'} e^{-ikr} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \quad (6)$$

$$\mathbf{D}_\alpha(r) e^{-ikr} = \int d^d r' \epsilon_{\alpha\beta}(r - r') e^{-ik(r-r')} \epsilon_0 \mathbf{E}_\beta(r') e^{-ikr'} \quad (7)$$

If we integrate this over r , and recognise our Fourier transforms, this becomes

$$D_\alpha(k) = \epsilon_{\alpha\beta}(k) \epsilon_0 E_\beta(k) \quad (8)$$

This is a fair result for the assumption of isotropy. We have an implicit sum over β here to look at. This serves as a useful working definition of $\epsilon(k)$.

One more assumption proves useful: for a given \mathbf{k} , we can look at the components of \mathbf{E} and \mathbf{D} parallel (longitudinal) and perpendicular (transverse) to \mathbf{k} . We may assume that only the parallel component of \mathbf{E} affects the parallel component of \mathbf{D} , and same for the perpendicular components. Specifically, this leads to

$$\epsilon_{\alpha\beta}(\mathbf{k}) = \epsilon_{\parallel}(\mathbf{k}) \hat{k}_\alpha \hat{k}_\beta + \epsilon_{\perp}(\mathbf{k}) (\delta_{\alpha\beta} - \hat{k}_\alpha \hat{k}_\beta). \quad (9)$$

A moment's reflection shows that this leads to two independent components of $\epsilon_{\alpha\beta}(\mathbf{k})$ (as can be seen by assuming \mathbf{k} points in the x direction). Also note that that for the transverse case, this explicitly only couples y to y and z to z ; there are no couplings between y and z .

why can we assume this?

include brief derivation of dielectric components

1.2 Electron gas densities and longitudinal eqs

To begin looking at an actual electron gas, with some external charge density ρ_{ext} imposed on it, we can start with Gauss's law:

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_{ext}(r) \quad (10)$$

This external charge density induces a charge density in the gas:

$$\rho(r) = \rho_{ind}(r) + \rho_{ext}(r) \quad (11)$$

We can define some scalar potentials Φ and Φ_{ext} , and therewith write

$$\mathbf{E}(r) = -\nabla\Phi(r) \quad (12)$$

$$\mathbf{D}(r) = -\epsilon_0 \nabla\Phi_{ext}(r) \quad (13)$$

Going Fourier with it leads to

$$\mathbf{E}(k) = -i\mathbf{k}\Phi(k) \quad (14)$$

$$\mathbf{D}(k) = -i\epsilon_0\mathbf{k}\Phi_{ext}(k) \quad (15)$$

These are the components of \mathbf{E} and \mathbf{D} parallel to \mathbf{k} , which means that these are really the *longitudinal* components.

Relating Gauss's laws to the scalar components, we get

$$\rho_{ext}(\mathbf{k}) = \nabla \cdot \mathbf{D}(\mathbf{k}) \quad (16)$$

$$= \nabla \cdot (-i\epsilon_0\mathbf{k}\Phi_{ext}(\mathbf{k})) \quad (17)$$

$$\rho_{ext}(\mathbf{k}) = \epsilon_0 k^2 \Phi_{ext}(\mathbf{k}) \quad (18)$$

Similarly, for \mathbf{E} and ρ , we get

$$\rho(\mathbf{k}) = \epsilon_0 k^2 \Phi(\mathbf{k}). \quad (19)$$

Because of (8), we should be able to write

$$\Phi_{ext}(\mathbf{k}) = \epsilon_{\parallel}(\mathbf{k})\Phi(\mathbf{k}) \quad (20)$$

Skipping over some details to fill in later (which involve defining potential energies $V = -e\Phi$ and using number densities defined by $\rho = -en$, we end up with

Fill in these details

$$\frac{1}{\epsilon_{\parallel}(k, \omega)} = 1 + \frac{4\pi e^2}{k^2} \Pi(k, \omega), \quad (21)$$

where $\Pi(k, \omega)$ is a response function satisfying

$$n_{ind}(k, \omega) = \Pi(k, \omega) V_f(k, \omega) \quad (22)$$

Here n_{ind} is the number density of induced electrons, and V_f is the voltage created by any free electrons in the metal (which isn't quite the same as an external voltage, but I think you might be able to ignore that difference).

There's no reason for $\Pi(\mathbf{k}, \omega)$ to be simple. In general, it describes the complete induced density response to any external potential.