

1 Reducing Lindhard to Drude

We want to see how we can reduce the Lindhard dielectric function

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)}, \quad (1)$$

where

$$f_l(x) = 1 - \frac{x}{2} \ln \frac{x+1}{x-1} \quad (2)$$

We can reduce things in the $k \rightarrow 0$ limit. The first half of (1) has the simple $\frac{1}{k^2}$ dependence, so we can look at how the rest of it behaves to start with.

1.1 f

$$f_l((\omega + i\nu)/kv_F) = 1 - \frac{(\omega + i\nu)/kv_F}{2} \ln \frac{(\omega + i\nu)/kv_F + 1}{(\omega + i\nu)/kv_F - 1} \quad (3)$$

Defining $\eta = \omega + i\nu$:

$$f_l((\omega + i\nu)/kv_f) = 1 - \frac{(\omega + i\nu)/kv_f}{2} \ln \frac{(\omega + i\nu)/kv_f + 1}{(\omega + i\nu)/kv_f - 1} \quad (4)$$

$$f_l(\eta/kv_F) = 1 - \frac{\eta}{2kv_F} \ln \frac{\eta + kv_F}{\eta - kv_F} \quad (5)$$

$$f_l = 1 - \frac{\eta}{2v_F} \frac{\ln \frac{\eta + kv_F}{\eta - kv_F}}{k} \quad (6)$$

$$\lim_{k \rightarrow 0} f_l = 1 - \frac{\eta}{2v_F} \frac{d \ln \frac{\eta + kv_F}{\eta - kv_F}}{dk} \quad (7)$$

$$= 1 - \frac{\eta}{2v_F} \frac{\eta - kv_F}{\eta + kv_F} \frac{v_F (\eta - kv_F) + v_F (\eta + kv_F)}{(\eta - kv_F)^2} \quad (8)$$

$$= 1 - \frac{\eta}{2} \frac{\eta - kv_F}{\eta + kv_F} \frac{(\eta - kv_F) + (\eta + kv_F)}{(\eta - kv_F)^2} \quad (9)$$

$$= 1 - \frac{\eta}{2} \frac{1}{\eta + kv_F} \frac{2\eta}{\eta - kv_F} \quad (10)$$

$$= 1 - \frac{\eta^2}{\eta^2 - k^2 v_F^2} \quad (11)$$

$$= \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2} \quad (12)$$

$$\lim_{k \rightarrow 0} f_l = 0 \quad (13)$$

Note that this goes to 0 for $k \rightarrow 0$.

1.2 Back to dielectric

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega + i\nu f_l((\omega + i\nu)/kv_f)} \quad (14)$$

In the denominator, we can note that ω should dominate $i\nu f$, because f goes to zero, so we can simplify that.

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega} \quad (15)$$

Using (12), we get

$$\epsilon_l = 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) f_l((\omega + i\nu)/kv_f)}{\omega} \quad (16)$$

$$= 1 + \frac{3\omega_p^2}{k^2 v_F^2} \frac{(\omega + i\nu) \frac{-k^2 v_F^2}{\eta^2 - k^2 v_F^2}}{\omega} \quad (17)$$

$$= 1 - 3\omega_p^2 \frac{(\omega + i\nu) \frac{1}{\eta^2}}{\omega} \quad (18)$$

$$= 1 - \frac{3\omega_p^2}{\omega(\omega + i\nu)} \quad (19)$$

This is the Drude limit.