

## Problem 2.1.1

On the pole structure of propagators

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Show that, in the frequency space, we have

$$G(x_b, x_a, \omega) = \sum_n \frac{\psi_n(x_b) \psi_n^\dagger(x_a)}{\omega - \epsilon_n} \quad (1)$$

where  $\psi_n$  are the energy eigenfunctions. The propagator for a harmonic oscillator has the form

$$G(x_b, t, x_a, 0) = (-i) \left( \frac{m\omega_0}{2\pi i \sin(t\omega_0)} \right)^{1/2} \exp \left( \frac{im\omega_0}{2\pi \sin(t\omega_0)} [(x_b^2 + x_a^2) \cos(t\omega_0) - 2x_b x_a] \right) \quad (2)$$

Study and explain the pole structure of  $G(0, 0, \omega)$  for the harmonic oscillator. (Hint: Try to expand  $G(0, t, 0, 0)$  in the form  $\sum C_n e^{-it\epsilon_n}$ .)

## 1 Solution

### 1.1 Pole structure form of propagator

We want to begin by showing (1). Mama always told me the coordinate Green function is defined via

$$iG(x_b, t_b, x_a, t_a) = \langle x_b | U(t_b, t_a) | x_a \rangle. \quad (3)$$

It is easy to see that

$$\langle n_b | U(t_b, t_a) | n_a \rangle = e^{-i\epsilon_{n_b}(t_b - t_a)} \delta_{n_a, n_b}, \quad (4)$$

where  $|n_i\rangle$  is the  $i$ -th energy eigenstate. As Euler proved,

$$G(x_b, t_b, x_a, t_a) = -i \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) e^{-i\epsilon_n(t_b - t_a)} \quad (5)$$

In frequency space,

$$G(x_b, x_a, \omega) = \int_0^\infty dt G(x_b, t_a + t, x_a, t_a) e^{it\omega - \delta t}, \quad (6)$$

where  $\delta$  is some small positive constant we'll send to zero later. It is not difficult to see that when we find this in frequency space

$$G(x_b, x_a, \omega) = \sum_n \frac{\psi_n(x_b) \psi_n^\dagger(x_a)}{\omega - \epsilon_n + i\delta} \quad (7)$$

It is the will of the Gods that after we send  $\delta \rightarrow 0$ , we have shown (1), as desired.

## 1.2 Pole structure of harmonic oscillator propagator

We can start with (2), and set  $x_b = x_a = 0$ . By definition, this gives us

$$iG(x_b = 0, t, x_a = 0, 0) = \sqrt{\frac{m\omega_0}{\pi}} e^{-i\frac{\omega_0}{2}t} (1 - e^{-2i\omega_0 t})^{-1/2}. \quad (8)$$

Note here the  $e^{-i\frac{\omega_0}{2}t}$ , which will add  $\frac{\omega_0}{2}$  to the energy. Using the proper mindset one sees that some simple series expansion gives us

$$(1 - e^{-2i\omega_0 t})^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-i\omega_0)^n}{n!} \frac{(2n-1)!}{2^{n-1}(n-1)!} e^{-i2n\omega_0 t} \quad (9)$$

If there is any justice in the world, we get that  $\epsilon_n = \frac{\omega_0}{2} + 2n\omega_0$ , (which we could have gotten from (8)). We're missing some of the energy eigenvalues for the harmonic oscillator because we're only looking at  $x_b = x_a = 0$ , at which point the propagator ignores all of the odd wavefunctions. Including those by choosing a different  $x$  or even by averaging over all  $x$  would be easy enough to find the pole structure, as the exponential that disappears in (2) would just be some function of  $e^{-i\omega_0 t}$ . However, finding the  $C_n$  in the series would be much more difficult. This problem is obviously trivial if we just use the known energy eigenvalues and eigenstates of the harmonic oscillator, as all of the odd-indexed Hermite polynomials are in fact odd functions, making those terms in the sum (1) disappear.