Problem 2.1.1

On the pole structure of propagators: gory version

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Show that, in the frequency space, we have

$$G(x_b, x_a, \omega) = \sum_n \frac{\psi_n(x_b)\psi^\dagger(x_a)}{\omega - \epsilon_n} \tag{1}$$

where ψ_n are the energy eigenfunctions. The propagator for a harmonic oscillator has the form

$$G(x_b, t, x_a, 0) = (-i)\left(\frac{m\omega_0}{2\pi i \sin(t\omega_0)}\right)^{1/2} \exp\left(\frac{im\omega_0}{2\pi \sin(t\omega_0)}\left[(x_b^2 + x_a^2)\cos(t\omega_0) - 2x_b x_a\right]\right) \tag{2}$$

Study and explain the pole structure of $G(0, 0, \omega)$ for the harmonic oscillator. (Hint: Try to expand $G(0, t, 0, 0,)$ in the form $\sum C_n e^{-it\epsilon_n}$.)

1 Solution

1.1 Pole structure form of propagator

We want to begin by showing (1). The coordinate Green function is defined via

$$iG(x_b, t_b, x_a, t_a) = \langle x_b| U(t_b, t_a) |x_a\rangle \tag{3}$$

We can insert the identity twice:

$$G(x_b, t_b, x_a, t_a) = -i \langle x_b| U(t_b, t_a) |x_a\rangle \tag{4}$$

$$G(x_b, t_b, x_a, t_a) = -i \sum_{n_b, n_a} \langle x_b|n_b\rangle \langle n_b| U(t_b, t_a) |n_a\rangle \langle n_a|x_a\rangle \tag{5}$$

where $|n_i\rangle$ is the $i-$th energy eigenstate. Mama always told me

$$\langle n_b| U(t_b, t_a) |n_a\rangle = e^{-i\epsilon_{n_b}(t_b - t_a)}\delta_{n_a, n_b}, \tag{6}$$

Ergo,

$$
\begin{aligned}
G(x_b, t_b, x_a, t_a) &= -i \sum_{n_b, n_a} \langle x_b | n_b \rangle \langle n_b | U(t_b, t_a) | n_a \rangle \langle n_a | x_a \rangle && (7) \\
&= -i \sum_{n_b, n_a} \langle x_b | n_b \rangle e^{-i\epsilon_{n_b}(t_b - t_a)} \delta_{n_a, n_b} \langle n_a | x_a \rangle && (8) \\
&= -i \sum_{n_b} \langle x_b | n_b \rangle e^{-i\epsilon_{n_b}(t_b - t_a)} \langle n_b | x_a \rangle && (9) \\
&= -i \sum_{n} \langle x_b | n \rangle e^{-i\epsilon_{n}(t_b - t_a)} \langle n | x_a \rangle && (10) \\
&= -i \sum_{n} \psi_n(x_b) e^{-i\epsilon_{n}(t_b - t_a)} \psi_n^\dagger(x_a) && (11) \\
&= -i \sum_{n} \psi_n(x_b) \psi_n^\dagger(x_a) e^{-i\epsilon_{n}(t_b - t_a)} && (12)
\end{aligned}
$$

In frequency space,

$$
G(x_b, x_a, \omega) = \int_0^\infty \mathrm{d}t\, G(x_b, t_a + t, x_a, t_a) e^{it\omega - \delta t}, \tag{13}
$$

where δ is some small positive constant we'll send to zero later. Plugging this in,

$$
\begin{aligned}
G(x_b, x_a, \omega) &= \int_0^\infty \mathrm{d}t \left(-i \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) e^{-i\epsilon_n t} \right) e^{it\omega - \delta t} && (14) \\
G(x_b, x_a, \omega) &= -i \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) \int_0^\infty \mathrm{d}t\, e^{-i\epsilon_n t} e^{it\omega - \delta t} && (15)
\end{aligned}
$$

It was Lebesgue who first asked whether

$$
\begin{aligned}
G(x_b, x_a, \omega) &= -i \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) \int_0^\infty \mathrm{d}t\, e^{-i\epsilon_n t} e^{it\omega - \delta t} && (16) \\
G(x_b, x_a, \omega) &= -i \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) \int_0^\infty \mathrm{d}t\, e^{t(-\delta + i(\omega - \epsilon_n))} && (17) \\
G(x_b, x_a, \omega) &= -i \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) \frac{1}{(-\delta + i(\omega - \epsilon_n))} \left. e^{-\delta t} \right|_{t=0}^{\infty} && (18) \\
G(x_b, x_a, \omega) &= -i \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) \frac{-1}{(-\delta + i(\omega - \epsilon_n))} && (19) \\
G(x_b, x_a, \omega) &= \sum_n \psi_n(x_b) \psi_n^\dagger(x_a) \frac{i}{(-\delta + i(\omega - \epsilon_n))} && (20) \\
G(x_b, x_a, \omega) &= \sum_n \frac{\psi_n(x_b) \psi_n^\dagger(x_a)}{\omega - \epsilon_n + i\delta} && (21)
\end{aligned}
$$

It can be shown that after we send $\delta \to 0$, we have shown (1), as desired.

1.2 Pole structure of harmonic oscillator propagator

We can start with (2), and set $x_b = x_a = 0$. We will take as an axiom that this gives us

$$G(x_b = 0, t, x_a = 0, 0) = (-i)\left(\frac{m\omega_0}{2\pi i \sin(t\omega_0)}\right)^{1/2} \exp\left(\frac{im\omega_0}{2\pi \sin(t\omega_0)}[0]\right) \tag{22}$$

$$= (-i)\left(\frac{m\omega_0}{2\pi i \sin(t\omega_0)}\right)^{1/2} \tag{23}$$

$$iG(0,t,0,0) = \sqrt{\frac{m\omega_0}{\pi}}\frac{1}{(2i\sin(t\omega_0))^{1/2}} \tag{24}$$

$$iG(0,t,0,0) = \sqrt{\frac{m\omega_0}{\pi}}\frac{1}{(2i\sin(t\omega_0))^{1/2}} \tag{25}$$

$$iG(0,t,0,0) = \sqrt{\frac{m\omega_0}{\pi}}\frac{1}{\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)^{1/2}} \tag{26}$$

$$iG(0,t,0,0) = \sqrt{\frac{m\omega_0}{\pi}}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)^{-1/2} \tag{27}$$

$$iG(0,t,0,0) = \sqrt{\frac{m\omega_0}{\pi}}e^{-i\frac{\omega_0}{2}t}\left(1 - e^{-2i\omega_0 t}\right)^{-1/2} \tag{28}$$

We can now take a brief aside to investigate the series expansion of $(1 - x)^{-1/2}$. By definition, the nested derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-1/2} = \frac{1}{2}(1-x)^{-3/2} \tag{29}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1-x)^{-1/2} = \frac{1}{2}\cdot\frac{3}{2}(1-x)^{-5/2} \tag{30}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(1-x)^{-1/2} = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1-x)^{-7/2} \tag{31}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x)^{-1/2} = \frac{(2n-1)!!}{2^n}(1-x)^{-2n+1/2}. \tag{32}$$

And verily, we can write the series as

$$(1-x)^{-1/2} = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{(2n-1)!!}{2^n}x^n, \tag{33}$$

where we've worked out the constant term through careful application of inserting $x = 0$ into the series. Note that we get convergence issues when $x = 1$. This is only a mild annoyance, however. The voices insist that $(2n-1)!! = \frac{(2n-1)!}{2^{n-1}(n-1)!}$.

Accordingly,

$$(1-x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(2n-1)!}{2^{2n-1}(n-1)!} x^n \tag{34}$$

$$(1-e^{-2i\omega_0 t})^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(2n-1)!}{2^{2n-1}(n-1)!} (-2i\omega_0)^n \, e^{-i2n\omega_0 t} \tag{35}$$

$$(1-e^{-2i\omega_0 t})^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-i\omega_0)^n}{n!} \frac{(2n-1)!}{2^{n-1}(n-1)!} e^{-i2n\omega_0 t} \tag{36}$$

If we now set $x = e^{-2i\omega_0 t}$, we get a series we can use in (28), which we can relate to the form given in the hint (that aside about the series just fixes C_n). We get then that $\epsilon_n = \frac{\omega_0}{2} + 2n\omega_0$, (which again, we could have gotten from (28)). We're missing some of the energy eigenvalues for the harmonic oscillator because we're only looking at $x_b = x_a = 0$, at which point the propagator ignores all of the odd wavefunctions. Including those by choosing a different x or even by averaging over all x would be easy enough to find the pole structure, as the exponential that disappears in (2) would just be some function of $e^{-i\omega_0 t}$. However, finding the C_n in the series would be much more difficult. This problem is obviously trivial if we just use the known energy eigenvalues and eigenstates of the harmonic oscillator, as all of the odd-indexed Hermite polynomials are in fact odd functions, making those terms in the sum (1) disappear.

Todo list