Invariant Set and Controller Synthesis

FEANICSES 2018 Workshop

Danylo Malyuta, Dylan Janak, Behçet Açıkmeşe

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Autonomous Controls Laboratory, University of Washington

Overview

Robust Controlled Invariant Set

We consider Discrete Linear Time Invariant (DLTI) system:

$$x^+ = Ax + Bu + Dp,$$

where $x \in \mathbf{R}^n$, $p \in \mathcal{P} = \{p \in \mathbf{R}^d : Rp \leq r\}$ and $u \in \mathcal{U} = \{u \in \mathbf{R}^m : Hu \leq h\}$ are "specification" polytopes.

Controlled Robust Positively Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Robust Invariant Set

Robust Controlled Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Now consider that some control law exists and the system reduces to an autonomous one:

$$x^+ = Ax + Dp.$$

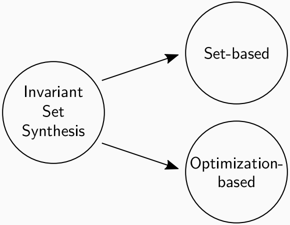
Robust Invariant Set

A set $\mathcal{X}$ is called *robust positively invariant* (RPI) if:

$$Ax + Dp \in \mathcal{X}, \quad \forall x \in \mathcal{X},\ p \in \mathcal{P}.$$

Goal: find an RPI $\mathcal{X}$.

Two Ways to Synthesize an Invariant Set



- Optimization-based methods rely on an explicit optimization problem (LP, LMI, etc.) to find $\mathcal{X}$
- Set-based methods rely on polytopic operations[1], i.e. computational geometry.

[1]These operations may implicitly involve an optimization, but what differentiates set-based methods is that people don't "talk" about it – they just assume that one can compute e.g. the Pontryagin difference.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Equivalent RPI Condition

$$\mathcal{X}(g) = \{x : Gx \leq g\} \text{ RPI} \Leftrightarrow \sigma(G_i \mid A\mathcal{X}(g)) + \sigma(G_i \mid D\mathcal{P}) \leq \sigma(G_i \mid \mathcal{X}(g)),$$

where $g \in \mathbf{R}^{n_g}$ and $\sigma(z \mid \mathcal{S}) \triangleq \sup\{y^T z : y \in \mathcal{S}\}$ is the support function of (some) set $\mathcal{S}$.

Note: $\sigma(G_i \mid \mathcal{X}(g)) \leq g_i$ with $<$ $\Leftrightarrow$ facet i is redundant.

One-Step Minimal RPI Computation

Trodden, “A One-Step Approach to Computing a Polytopic Robust Invariant Set”, 2016. [1]

Existence of an RPI Set

Fix G in $\mathcal{X}(g) = \{x : Gx \leq g\}$ (i.e. pick a “template”). Assumptions:

A1. $\mathcal{P}$ contains the origin

A2. $\lambda < 0 \; \forall \lambda \in \text{spec}(A)$

A3. The interior of $\mathcal{X}$ contains the origin

A4. For the chosen G, a g exists such that $\mathcal{X}(g)$ is RPI

Then there exists a g^* such that

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, \ldots, n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

Fixed-Point Solution Uniqueness

Given assumptions A1-A4, the g^* in the above statement is unique.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Existence of an RPI Set

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, ..., n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

g^* can be computed iteratively:

Algorithm 1 Iterative computation of g^*.

```
Set g ← 0
while True do
    g*_i ← σ(G_i | A𝒳(g)) + σ(G_i | D𝒫)  i = 1, ..., n_g
    if ‖g − g*‖_∞ < ε_tol then
        return g*
    g ← g*
```

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

- g^* can also be computed as a one-shot LP (main contribution of [1])
- Let $c_i(g) = \sigma(G_i \mid A\mathcal{X}(g))$, $d_i = \sigma(G_i \mid D\mathcal{P})$, $b_i(g) = \sigma(G_i \mid \mathcal{X}(g))$. Core realization (thanks to uniqueness of g^*):

$$g^* = \arg\min_g \{\|g\|_1 : c(g)+d = b(g)\} = \arg\max_g \{\|g\|_1 : c(g)+d = b(g)\}$$

- Recalling that $b(g) \leq g$, the above is readily converted to an LP:

$$\begin{aligned} g^* = c^* + d^*, \text{ where } (c^*, d^*) = \ & \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\ldots,n_g\}}}{\arg\ \text{maximize}} && \sum_{i=1}^{n_g} c_i + d_i \\ & \text{subject to} && c_i \leq G_i A \xi^i \\ & && G\xi^i \leq c + d \\ & && d_i \leq G_i D \omega^i \\ & && F\omega^i \leq g. \end{aligned}$$

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

$$\boxed{\text{Let } c_i(g) = \sigma(G_i \mid A\mathcal{X}(g)),\ d_i = \sigma(G_i \mid D\mathcal{P}),\ b_i(g) = \sigma(G_i \mid \mathcal{X}(g))}$$

$$
\begin{aligned}
g^* = c^* + d^*,\ \text{where } (c^*, d^*) = \arg \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\dots,n_g\}}}{\text{maximize}} \quad & \sum_{i=1}^{n_g} c_i + d_i \\
\text{subject to} \quad & c_i \leq G_i A \xi^i \\
& G\xi^i \leq c + d \\
& d_i \leq G_i D \omega^i \\
& F\omega^i \leq g.
\end{aligned}
$$

The first two constraints evaluate $c_i(g)$ and the last two evaluate d_i. The first constraint holds with equality at optimality, since we want to maximize c_i. The RHS of the second constraint $= g^*$ at optimality, therefore the second constraint enforces $P\xi^i \leq g^*$, i.e. the definition of $b(g^*)$.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

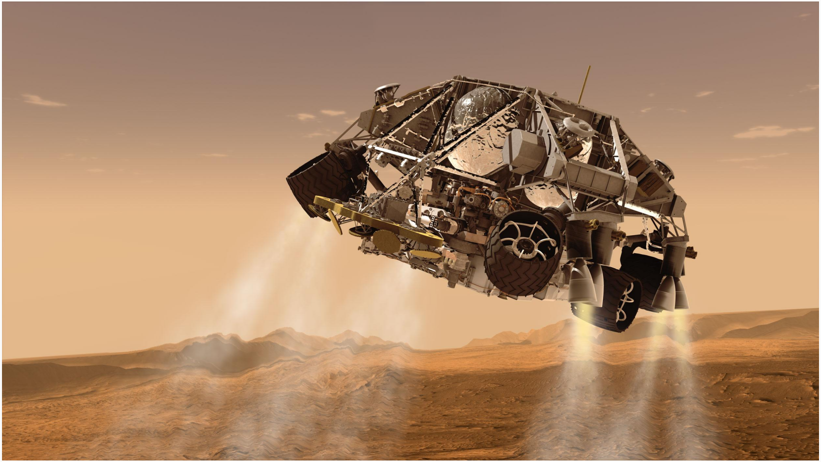


Image credit: NASA/JPL-Caltech

One-Step Minimal RPI Computation

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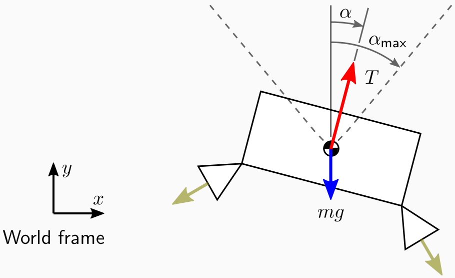


Parameters [2]:

$$
\begin{aligned}
m_{\text{wet}} &= 1905 \text{ kg} \\
g &= -3.7114 \text{ m/s}^2 \\
g_{\text{e}} &= 9.81 \text{ m/s}^2 \\
I_{\text{sp}} &= 225 \text{ s} \quad T_{\max} = 3.1 \text{ kN} \\
\phi &= 27 \text{ deg} \quad n = 6
\end{aligned}
$$

Dynamics:

$$
\begin{aligned}
(\dot{x}, \dot{y}) &= (v_x, v_y) \\
(\dot{v}_x, \dot{v}_y) &= (T_x, T_y)/m + g \\
\dot{m} &= -\frac{\|(T_x, T_y)\|_2}{I_{\text{sp}} g_{\text{e}} \cos\phi}
\end{aligned}
$$

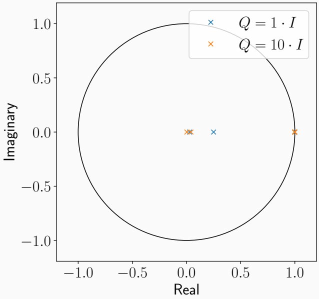
Letting $T \leftarrow T + mg$ be the gravity compensated control, the system is linearized about $(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y, \bar{m}) = (0, 0, 0, 0, m_{\text{wet}})$ and $(\bar{T}_x, \bar{T}_y) = (0, 0)$.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Synthesize an LQR stabilizing controller:

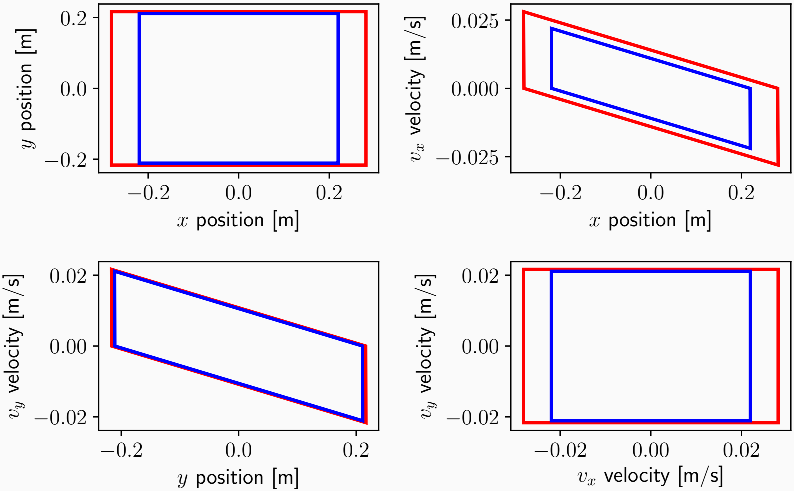
- State scaling: $D_x = \begin{bmatrix} 1 & 1 & 0.05 & 0.05 & 0.1 \end{bmatrix}$
- Input scaling: $D_u = \begin{bmatrix} nT_{\max} \cos\phi \sin\alpha_{\max} & nT_{\max} \cos\phi \end{bmatrix}$
- State penalty $Q = D_x^{-1} \hat{Q} D_x$ with $\hat{Q} \in \{I_5, 10I_5\}$
- Input penalty $R = D_x^{-1} \hat{R} D_x$ with $\hat{R} = I_2$



One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

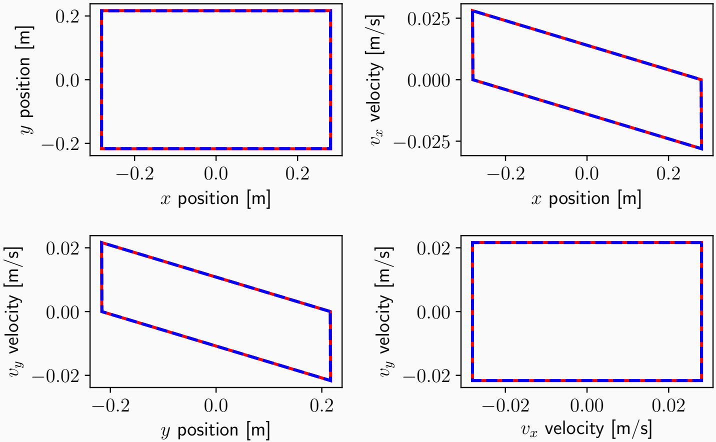
Direct application of LP on slide 10 ($\hat{Q} = I_5$, $\hat{Q} = 10I_5$):



One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

The one-shot LP of slide 10 and the iterative algorithm of slide 8 are identical...



... but iterative takes ≈ 315 s while one-shot takes ≈ 0.2 s!

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

We consider Discrete Linear Time Invariant (DLTI) system:

$$x^+ = Ax + Bu + Dp,$$

where $x \in \mathbf{R}^n$, $p \in \mathcal{P} = \{p \in \mathbf{R}^d : Rp \leq r\}$ and $u \in \mathcal{U} = \{u \in \mathbf{R}^m : Hu \leq h\}$ are "specification" polytopes.

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Maximal CRPI Set

A set $\mathcal{X}_\infty \subseteq \mathcal{X}$ is called *maximal CRPI* (maxCRPI) if it is CRPI and contains all other CRPI sets in $\mathcal{X}$, i.e. $\mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}_\infty\ \forall \mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}$ RCPI [3].

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

maxCRPI Set Convexity

Given the system $x^+ = Ax + Bu + Dp$ where $p \in \mathcal{P}$, $u \in \mathcal{U}$, consider $\mathcal{X}$ the set of "safe" states. If $\mathcal{X}, \mathcal{P}, \mathcal{U}$ are convex then the associated maxCRPI set $\mathcal{X}_\infty$ is convex.

Recall the maxCRPI set definition:

$$\mathcal{X}_\infty = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X}_\infty,\ \forall p \in \mathcal{P}\}.$$

The definition is recursive ($\mathcal{X}_\infty$ on both sides) $\Rightarrow$ compute $\mathcal{X}_\infty$ *iteratively*.

Core step: *preimage set* computation.

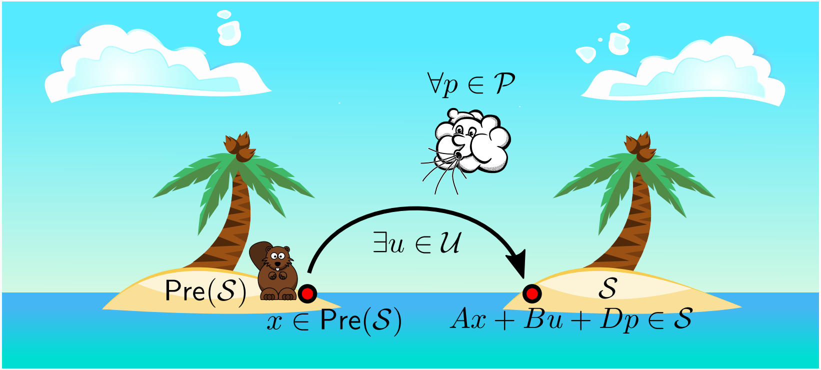
Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Preimage Set

$$\mathrm{Pre}(\mathcal{S}) \triangleq \{x \mid \exists u \in \mathcal{U},\ Ax + Bu + Dp \in \mathcal{S}\ \forall p \in \mathcal{P}\}$$

Remark: $\mathcal{S}$ CRPI $\Leftrightarrow \mathcal{S} \subseteq \mathrm{Pre}(\mathcal{S})$.



Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

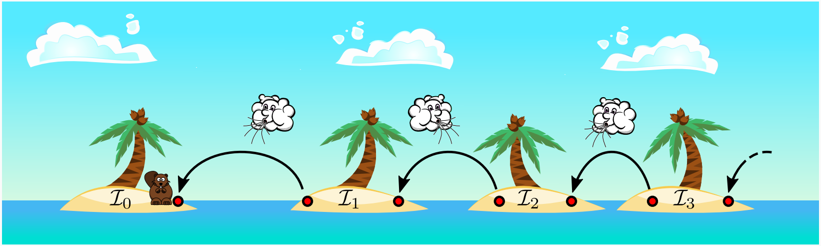
maxCRPI Iterative Computation

Execute the following dynamic programming-type algorithm:

$$\mathcal{I}_0 = \mathcal{X}$$

$$\mathcal{I}_{k+1} = \mathrm{Pre}(\mathcal{I}_k) \cap \mathcal{I}_k \quad k = 0, 1, 2, \ldots$$

STOP if $\mathcal{I}_{k+1} = \mathcal{I}_k$. Then, $\mathcal{I}_k = \mathcal{I}_\infty$ is the maxCRPI set.



(Proxy for convergence: distance between the islands.)

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Preimage Set Computation

$$\mathrm{Pre}(\mathcal{S}) = ((\mathcal{S} \ominus (D\mathcal{P})) \oplus (-B\mathcal{U}))A$$

where[2]:

- Minkowski sum: $\mathcal{A} \oplus \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$, $\mathcal{O}(c^n)$
- Pontryagin difference: $\mathcal{A} \ominus \mathcal{B} = \{a : a + b \in \mathcal{A}, \forall b \in \mathcal{B}\}$, $\mathcal{O}(n^c)$
- Direct mapping: $M\mathcal{A} = \{Ma : a \in \mathcal{A}\}$, $\mathcal{O}(c^n)$
- Inverse mapping: $\mathcal{A}M = \{a : Ma \in \mathcal{A}\}$, $\mathcal{O}(n^c)$

Minkowski sum is the most expensive operation (highest facet count, cannot be pre-computed).

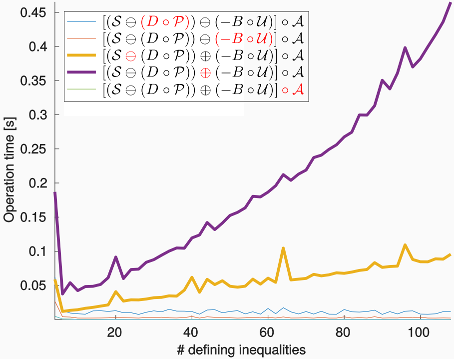
[2] n is the polytope facet count and c is a coefficient.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

$$\mathrm{Pre}(\mathcal{S}) = [(\mathcal{S} \ominus (D \circ \mathcal{P})) \oplus (-B \circ \mathcal{U})] \circ \mathcal{A}$$

For independent disturbances, Pontryagin difference ($\mathcal{O}(n^c)$) and especially Minkowski sum ($\mathcal{O}(c^n)$) are expensive[3].



[3] n is the polytope facet count and c is a coefficient.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

However, may wish to render invariant only *part* of the state. Examples:

- Some states do not make physical sense to render invariant (our case: skycrane mass)
- Some states may correspond to the controller (e.g. integrator)

In this case we want to render invariant the output $y = Cx$.

Controlled Robust Positively Output Invariant Set

The set $\mathcal{X}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{Y}\ \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Controlled Robust Positively Output Invariant Set

The set $\mathcal{Y}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(Ax + Bu + Dp) \in \mathcal{Y} \ \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Using $C^\dagger$ the pseudoinverse of C, we can write:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(A(C^\dagger y + \mathcal{N}(C)) + Bu + Dp) \subseteq \mathcal{Y} \ \forall p \in \mathcal{P}\},$$

where $\mathcal{N}(C)$ is the nullspace of C, i.e. $\mathcal{N}(C) = \{z : Cx = 0\}$ (which is a polytope!). The preimage set can be computed similarly to before:

$$\text{Pre}(\mathcal{Y}) = ((\mathcal{Y} \ominus (CD\mathcal{P} \oplus CA\mathcal{N}(C))) \oplus (-CB\mathcal{U}))CAC^\dagger$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

The following algorithm summarizes maxCRPOI set computation[4].

Algorithm 2 Iterative computation of maxCRPOI set $\mathcal{Y}_\infty$.

Set $\mathcal{Y}$ to the "safe outputs" specification

while True **do**

  $\text{Pre}(\mathcal{Y}) \leftarrow ((\mathcal{Y} \ominus (CD\mathcal{P} \oplus CA\mathcal{N}(C))) \oplus (-CB\mathcal{U}))CAC^\dagger$

  $\mathcal{Y}^+ = \mathcal{Y} \cap \text{Pre}(\mathcal{Y})$

  if $\mathcal{Y} \subseteq \mathcal{Y}^+_{\epsilon_{\text{tol}}}$ and $\mathcal{Y}^+ \subseteq \mathcal{Y}_{\epsilon_{\text{tol}}}$ **then**

    return $\mathcal{Y}_\infty \leftarrow \mathcal{Y}^+$

  $\mathcal{Y} \leftarrow \mathcal{Y}^+$

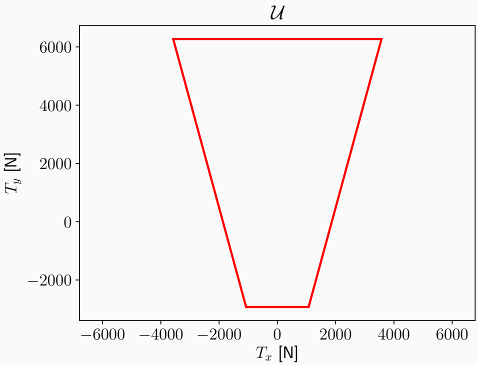
[4] If $\mathcal{S} = \{x : Px \leq p\}$, we denote $\mathcal{S}_{\epsilon_{\text{tol}}} = \{x : Px \leq p + \epsilon_{\text{tol}}\}$ the ϵ_{tol}-*dilation* of $\mathcal{S}$. In practical, dilation is a more robust stopping criterion than equality ($\mathcal{Y}^+ = \mathcal{Y}$) which is prone to numerical inaccuracy.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Going back to the skycrane example, consider the specifications:

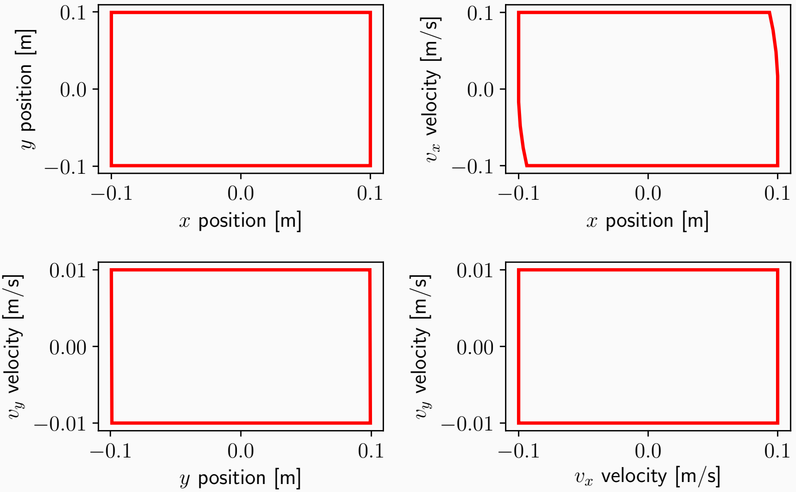
- ± 10 cm position error (in both x and y)
- ± 10 cm/s velocity error in x, ± 1 cm/s velocity error in y
- ± 400 N disturbance force (in both x and y)
- Input constraint set given by the rocket motor specs [2] (visualized below)



Maximal RCI Computation

Kvasnica et al., “Reachability Analysis and Control Synthesis for Uncertain Linear Systems...”, 2015. [3]

Direct application of algorithm on slide 23:



Maximal RCI Computation With Dependent Noise

Rakovic et al., "Reachability Analysis of Discrete-Time Systems With Disturbances", 2006. [4]

What happens if the disturbance is state and/or input dependent?

$$p \in \operatorname{Proj}_p \mathcal{P}(x, u) = \{\theta = (p, x, u) \in \mathbf{R}^{d+n+m} : R\theta \leq r\}$$

In this case $\operatorname{Pre}(\mathcal{X})$ can be computed in several steps:

$$\begin{aligned}
\mathcal{Z} &\triangleq \mathcal{X} \times \mathcal{U} \\
\mathcal{W} &\triangleq \{(x, u, p) : (x, u) \in \mathcal{Z}, p \in \mathcal{P}(x, u)\} \\
\Phi &\triangleq \{(x, u, p) : Ax + Bu + Dp \in \mathcal{S}\} \\
\Sigma &\triangleq \{(x, u) \in \mathcal{Z} \mid Ax + Bu + Dp \in \mathcal{S} \ \forall p \in \mathcal{P}(x, u)\} \\
&= \mathcal{Z} \setminus \operatorname{Proj}_{x,u}(\mathcal{W} \setminus \Phi) \\
\Rightarrow \operatorname{Pre}(\mathcal{S}) &= \operatorname{Proj}_x(\Sigma)
\end{aligned}$$

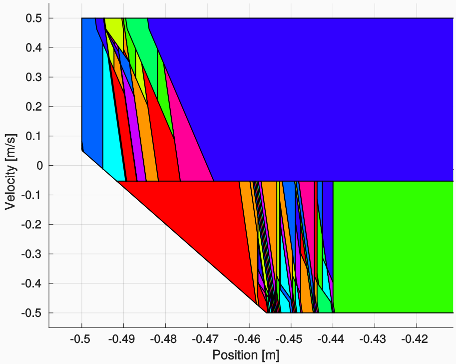
When sets are polytopes, all operations are possible via computational geometry.

Maximal RCI Computation With Dependent Noise

Rakovic et al., "Reachability Analysis of Discrete-Time Systems With Disturbances", 2006. [4]

$$\Sigma = \mathcal{Z} \setminus \mathrm{Proj}_{x,u}(\mathcal{W} \setminus \Phi).$$

Regiondiff operation (\) [5]) generates a union of polytopes, which suffers from severe "fracturing" of convex regions.



Furthermore, $\mathrm{Proj}_{x,u}$ is expensive when $\dim(\mathcal{W})$ is large!

Overview

The Control Problem for Independent Uncertainty

Consider a given DLTI system:

$$x^+ = Ax + Bu + Dp, \tag{DLTI}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $p \in \mathbf{R}^d$. Consider given polytopic sets ("specifications"):

$$\mathcal{X} \triangleq \{x \in \mathbf{R}^n \mid Gx \leq g\} \qquad \mathcal{U} \triangleq \{u \in \mathbf{R}^m \mid Hu \leq h\}$$
$$\mathcal{P} \triangleq \{p \in \mathbf{R}^d \mid Rp \leq r\}.$$

The control problem is to design a control policy $u = \mu(x)$ which ensures that $x^+ \in \mathcal{X}$ and $u \in \mathcal{U}$ for all $p \in \mathcal{P}$.

Linear Quadratic Regulator (LQR)

Solves an infinite-horizon deterministic optimal control problem:

$$\min_{u,x} \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k, \text{ s.t. } x_{k+1} = A x_k + B u_k,$$

where $Q \succeq 0$, $R \succ 0$. The control policy is given by:

$$u[k] = \mu_k(z[k]) \triangleq K z[k],$$

where:

$$\begin{aligned} K &= -(R + B^T P B)^{-1} B^T P A, \\ P &= Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A. \end{aligned} \tag{DARE}$$

- Easy to compute and to implement
- Does not handle uncertainty, so no guarantee of satisfying $x[k] \in \mathcal{X}\ \forall k$, nor $u[k] \in \mathcal{U}$ for that matter!

Linear Feedback Inducing $\mathcal{X}$ Invariance

- Consider a linear feedback control law $u[k] = \mu_k(z[k]) = Kz[k]$.
- K makes $\mathcal{X}$ robustly invariant if and only if:

$$\begin{aligned} \underset{z,p}{\text{maximize}} \quad & G_i((A+BK)z + Dp) - g_i \leq 0 \quad \forall i = 1, ..., n_g \\ \text{subject to} \quad & G(z - p_v) \leq g, \; Rp \leq r, \; HKz \leq h, \end{aligned}$$

where $p = (p_w, p_e, p_v)$ and $p_v = E_v p = \begin{bmatrix} 0 & 0 & I \end{bmatrix} p$ corresponds to estimation error. K can be found via the one-shot dual problem:

$$\begin{aligned} \underset{K,Y,M,S}{\text{minimize}} \quad & \|K\|_2 \text{ (or another norm or 0)} \\ \text{subject to} \quad & Yg + Mr \leq g \\ & YG = G(A+BK) \\ & MR = GD + YGE_v \\ & SG = HK, \quad Sg \leq h \\ & Y, M, S \geq 0. \end{aligned}$$

- K is neither guaranteed to exist nor to be fuel optimal!

Ellipsoidal Linear Feedback

Bertsekas, "Infinite time reachability of state-space regions by using feedback control", 1972. [6]

Works on ellipsoidal sets, so reformulate $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{P}$ as maximal inscribed ellipsoids of their polytopic specifications[5]:

$$\mathcal{X} \triangleq \{x \in \mathbf{R}^n \mid x^T G x \leq 1\} \qquad \mathcal{U} \triangleq \{u \in \mathbf{R}^m \mid u^T H u \leq 1\}$$
$$\mathcal{P} \triangleq \{p \in \mathbf{R}^{d+n+m} \mid p^T R p \leq 1\}.$$

Linear Control Law Sufficient for Invariance [6]

A sufficient condition for $\mathcal{X}$ to be invariant is that $\exists \psi \succ 0$ and $\beta \in (0,1)$ such that

$$G = A^T(F^{-1} + BH^{-1}B^T)^{-1}A + \psi, \text{ where}$$
$$F = \left[(1-\beta)G^{-1} - \frac{1-\beta}{\beta}DR^{-1}D^T\right]^{-1} \succ 0.$$

A linear time-invariant control law achieves invariance:

$$u[k] = \mu_k(z[k]) = Kz[k] = -(H + B^T G B)^{-1} B^T F A z[k].$$

[5] G, H, R matrices here are different from their polytope counterparts.

Linear Feedback from Bertsekas (1972)

- The control law is asymptotically stable, so can be turned on outside $\mathcal{X}$ and will drive the system to inside $\mathcal{X}$, if possible
- If the system is stabilizable, Algorithm 3 finds a solution. At termination, satisfaction of original $\mathcal{X}$, $\mathcal{U}$ is not guaranteed!

Algorithm 3 Algorithm for determining X invariance-inducing control gain.

Choose $\rho \in (0, 1)$ relaxation factor
while $i <$ maximum number of relaxations **do**
  Initialize $\psi \leftarrow G$, $G_0 \leftarrow \psi$, $i \leftarrow 0$
  while $i <$ maximum number of inner iterations **do**
    $F_i \leftarrow \left[(1-\beta)G_i^{-1} - \frac{1-\beta}{\beta} D R^{-1} D^T\right]^{-1}$
    $G_{i+1} \leftarrow A^T (F_i^{-1} + B H^{-1} B^T)^{-1} A + \psi$
    $i \leftarrow i + 1$
    if $\|G_{i+1} - G_i\|_\infty <$ convergence tolerance **then**
      return $-(H + B^T G_{i+1} B)^{-1} B^T F_i A$ ▷ Invariance-sufficient control gain
  Relax $H \leftarrow \rho H$, $\psi \leftarrow \rho\psi$ ▷ Grows the state and input constraint sets

Thank You For Your Attention!

Appendix

Overview

Bibliography

[1] P. Trodden, "A one-step approach to computing a polytopic robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 61, pp. 4100–4105, dec 2016.

[2] B. Acikmese and S. R. Ploen, "Convex programming approach to powered descent guidance for mars landing," *Journal of Guidance, Control, and Dynamics*, vol. 30, pp. 1353–1366, sep 2007.

[3] M. Kvasnica, B. Takács, J. Holaza, and D. Ingole, "Reachability analysis and control synthesis for uncertain linear systems in MPT," *IFAC-PapersOnLine*, vol. 48, no. 14, pp. 302–307, 2015.

[4] S. Rakovic, E. Kerrigan, D. Mayne, and J. Lygeros, "Reachability analysis of discrete-time systems with disturbances," *IEEE Transactions on Automatic Control*, vol. 51, pp. 546–561, apr 2006.

[5] M. Baotić, "Polytopic Computations in Constrained Optimal Control," *Automatika, Journal for Control, Measurement, Electronics, Computing and Communications*, vol. 50, pp. 119–134, 2009.

[6] D. Bertsekas, "Infinite time reachability of state-space regions by using feedback control," *IEEE Transactions on Automatic Control*, vol. 17, pp. 604–613, oct 1972.