Invariant Set and Controller Synthesis

FEANICSES 2018 Workshop

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Overview

Robust Controlled Invariant Set

We consider Discrete Linear Time Invariant (DLTI) system:

$$x^{+} = Ax + Bu + Dp,$$

where $x \in \mathbf{R}^{n}$, $p \in \mathcal{P} = \{p \in \mathbf{R}^{d} : Rp \leq r\}$ and $u \in \mathcal{U} = \{u \in \mathbf{R}^{m} : Hu \leq h\}$ are "specification" polytopes.

Controlled Robust Positively Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^{n} : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Robust Invariant Set

Robust Controlled Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Now consider that some control law exists and the system reduces to an autonomous one:

$$x^+ = Ax + Dp.$$

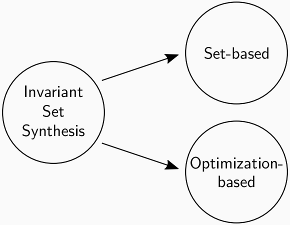
Robust Invariant Set

A set $\mathcal{X}$ is called *robust positively invariant* (RPI) if:

$$Ax + Dp \in \mathcal{X}, \quad \forall x \in \mathcal{X},\ p \in \mathcal{P}.$$

Goal: find an RPI $\mathcal{X}$.

Two Ways to Synthesize an Invariant Set



- Optimization-based methods rely on an explicit optimization problem (LP, LMI, etc.) to find $\mathcal{X}$
- Set-based methods rely on polytopic operations[1], i.e. computational geometry.

[1]These operations may implicitly involve an optimization, but what differentiates set-based methods is that people don't "talk" about it – they just assume that one can compute e.g. the Pontryagin difference.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Equivalent RPI Condition

$$\mathcal{X}(g) = \{x : Gx \leq g\} \text{ RPI} \Leftrightarrow \sigma(G_i \mid A\mathcal{X}(g)) + \sigma(G_i \mid D\mathcal{P}) \leq \sigma(G_i \mid \mathcal{X}(g)),$$

where $g \in \mathbf{R}^{n_g}$ and $\sigma(z \mid \mathcal{S}) \triangleq \sup\{y^T z : y \in \mathcal{S}\}$ is the support function of (some) set $\mathcal{S}$.

Note: $\sigma(G_i \mid \mathcal{X}(g)) \leq g_i$ with $<$ $\Leftrightarrow$ facet i is redundant.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Existence of an RPI Set

Fix G in $\mathcal{X}(g) = \{x : Gx \leq g\}$ (i.e. pick a "template"). Assumptions:

A1. $\mathcal{P}$ contains the origin

A2. $\lambda < 0 \; \forall \lambda \in \mathrm{spec}(A)$

A3. The interior of $\mathcal{X}$ contains the origin

A4. For the chosen G, a g exists such that $\mathcal{X}(g)$ is RPI

Then there exists a g^* such that

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, \ldots, n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

Fixed-Point Solution Uniqueness

Given assumptions A1-A4, the g^* in the above statement is unique.

One-Step Minimal RPI Computation

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Existence of an RPI Set

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, \ldots, n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

g^* can be computed iteratively:

Algorithm 1 Iterative computation of g^*.

```
Set g ← 0
while True do
    g*_i ← σ(G_i | AX(g)) + σ(G_i | DP)  i = 1, ..., n_g
    if ‖g − g*‖_∞ < ε_tol then
        return g*
    g ← g*
```

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

- g^* can also be computed as a one-shot LP (main contribution of [1])
- Let $c_i(g) = \sigma(G_i \mid A\mathcal{X}(g))$, $d_i = \sigma(G_i \mid D\mathcal{P})$, $b_i(g) = \sigma(G_i \mid \mathcal{X}(g))$.
  Core realization (thanks to uniqueness of g^*):

$$g^* = \arg\min_g \{\|g\|_1 : c(g)+d = b(g)\} = \arg\max_g \{\|g\|_1 : c(g)+d = b(g)\}$$

- Recalling that $b(g) \le g$, the above is readily converted to an LP:

$$g^* = c^* + d^*, \text{ where } (c^*, d^*) = \begin{array}{ll} \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\ldots,n_g\}}}{\arg\ \text{maximize}} & \displaystyle\sum_{i=1}^{n_g} c_i + d_i \\ \text{subject to} & c_i \le G_i A \xi^i \\ & G\xi^i \le c + d \\ & d_i \le G_i D \omega^i \\ & F\omega^i \le g. \end{array}$$

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

$$\boxed{\text{Let } c_i(g) = \sigma(G_i \mid A\mathcal{X}(g)),\ d_i = \sigma(G_i \mid D\mathcal{P}),\ b_i(g) = \sigma(G_i \mid \mathcal{X}(g))}$$

$$\begin{aligned} g^* = c^* + d^*,\ \text{where } (c^*, d^*) = \ & \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\dots,n_g\}}}{\text{arg maximize}} && \sum_{i=1}^{n_g} c_i + d_i \\ & \text{subject to} && c_i \leq G_i A \xi^i \\ & && G\xi^i \leq c + d \\ & && d_i \leq G_i D \omega^i \\ & && F\omega^i \leq g. \end{aligned}$$

The first two constraints evaluate $c_i(g)$ and the last two evaluate d_i. The first constraint holds with equality at optimality, since we want to maximize c_i. The RHS of the second constraint $= g^*$ at optimality, therefore the second constraint enforces $P\xi^i \leq g^*$, i.e. the definition of $b(g^*)$.

One-Step Minimal RPI Computation

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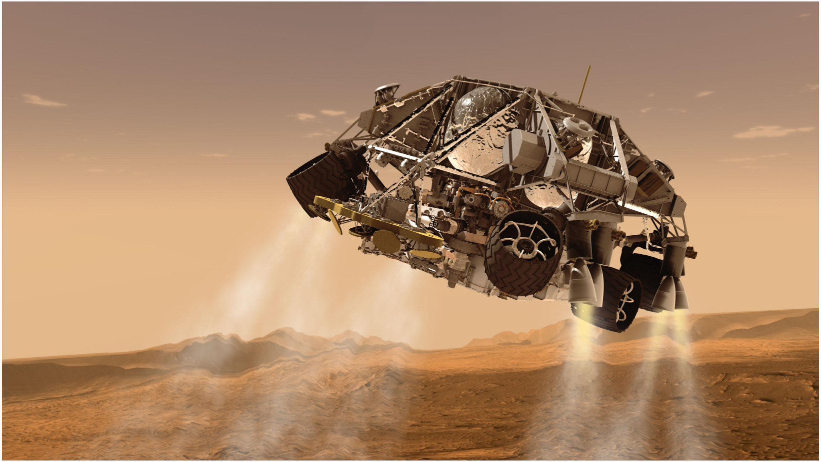

Image credit: NASA/JPL-Caltech

One-Step Minimal RPI Computation

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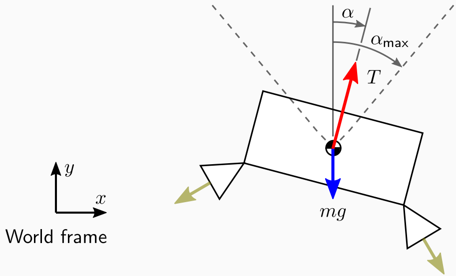


Parameters [2]:

$$m_{\text{wet}} = 1905 \text{ kg}$$
$$g = -3.7114 \text{ m/s}^2$$
$$g_e = 9.81 \text{ m/s}^2$$
$$I_{sp} = 225 \text{ s} \quad T_{\max} = 3.1 \text{ kN}$$
$$\phi = 27 \text{ deg} \quad n = 6$$

Dynamics:

$$(\dot{x}, \dot{y}) = (v_x, v_y)$$
$$(\dot{v}_x, \dot{v}_y) = (T_x, T_y)/m + g$$
$$\dot{m} = -\frac{\|(T_x, T_y)\|_2}{I_{sp} g_e \cos\phi}$$

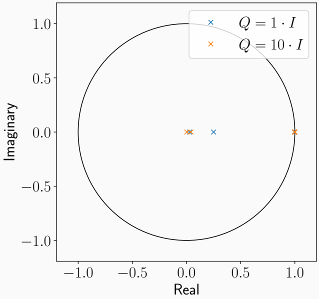
Letting $T \leftarrow T + mg$ be the gravity compensated control, the system is linearized about $(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y, \bar{m}) = (0, 0, 0, 0, m_{\text{wet}})$ and $(\bar{T}_x, \bar{T}_y) = (0, 0)$.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Synthesize an LQR stabilizing controller:

- State scaling: $D_x = \begin{bmatrix} 1 & 1 & 0.05 & 0.05 & 0.1 \end{bmatrix}$
- Input scaling: $D_u = \begin{bmatrix} nT_{\max} \cos\phi \sin\alpha_{\max} & nT_{\max} \cos\phi \end{bmatrix}$
- State penalty $Q = D_x^{-1} \hat{Q} D_x$ with $\hat{Q} \in \{I_5, 10 I_5\}$
- Input penalty $R = D_x^{-1} \hat{R} D_x$ with $\hat{R} = I_2$



One-Step Minimal RPI Computation

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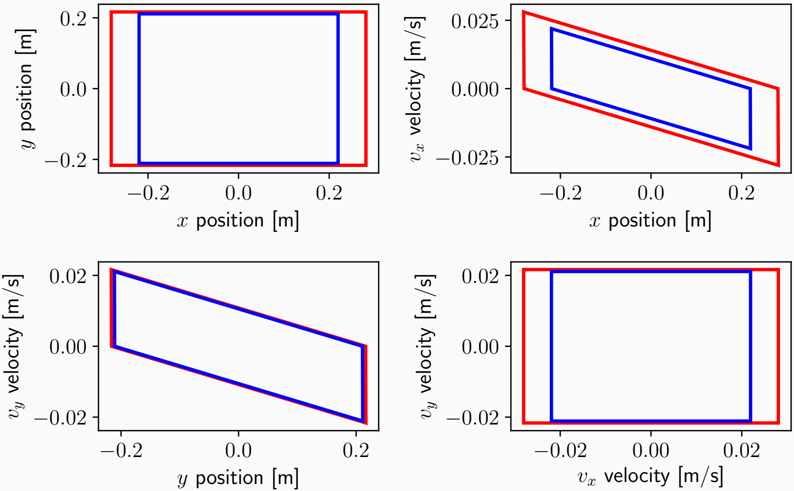
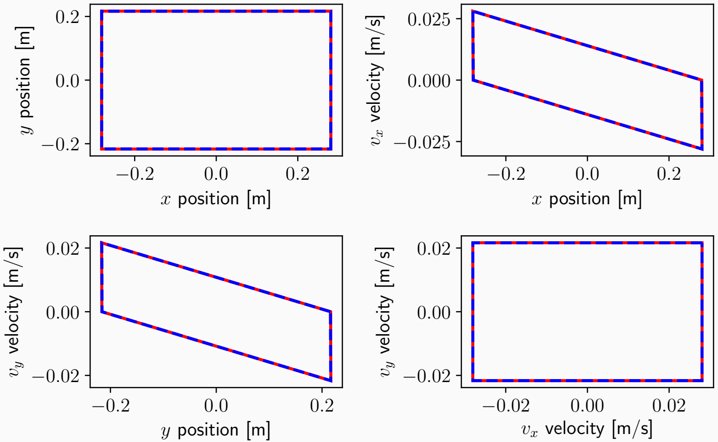
Direct application of LP on slide 10 ($\hat{Q} = I_5$, $\hat{Q} = 10I_5$):



One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

The one-shot LP of slide 10 and the iterative algorithm of slide 8 are identical...



... but iterative takes $\approx$ 315 s while one-shot takes $\approx$ 0.2 s!

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

We consider Discrete Linear Time Invariant (DLTI) system:

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Maximal CRPI Set

A set $\mathcal{X}_\infty \subseteq \mathcal{X}$ is called *maximal CRPI* (maxCRPI) if it is CRPI and contains all other CRPI sets in $\mathcal{X}$, i.e. $\mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}_\infty\ \forall \mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}$ RCPI [3].

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

maxCRPI Set Convexity

Given the system $x^+ = Ax + Bu + Dp$ where $p \in \mathcal{P}$, $u \in \mathcal{U}$, consider $\mathcal{X}$ the set of "safe" states. If $\mathcal{X}, \mathcal{P}, \mathcal{U}$ are convex then the associated maxCRPI set $\mathcal{X}_\infty$ is convex.

Recall the maxCRPI set definition:

$$\mathcal{X}_\infty = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X}_\infty,\ \forall p \in \mathcal{P}\}.$$

The definition is recursive ($\mathcal{X}_\infty$ on both sides) $\Rightarrow$ compute $\mathcal{X}_\infty$ *iteratively*.

Core step: *preimage set* computation.

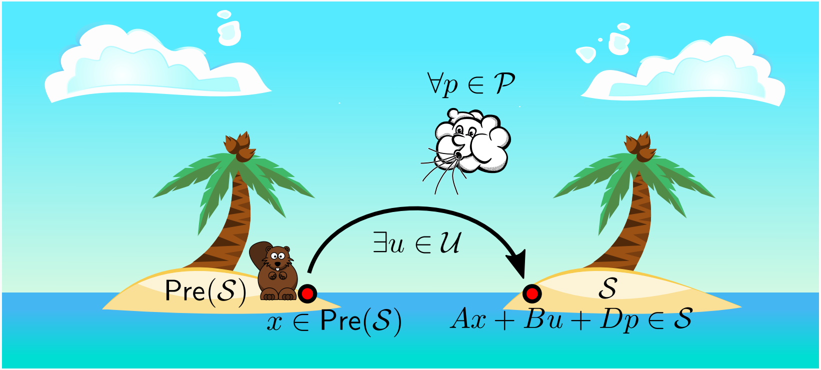
Maximal RCI Computation

Kvasnica et al., “Reachability Analysis and Control Synthesis for Uncertain Linear Systems...”, 2015. [3]

Preimage Set

$$\mathrm{Pre}(\mathcal{S}) \triangleq \{x \mid \exists u \in \mathcal{U},\ Ax + Bu + Dp \in \mathcal{S}\ \forall p \in \mathcal{P}\}$$

Remark: $\mathcal{S}$ CRPI $\Leftrightarrow \mathcal{S} \subseteq \mathrm{Pre}(\mathcal{S})$.



Maximal RCI Computation

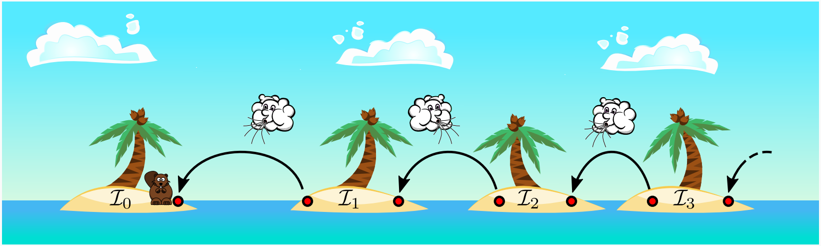
Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

maxCRPI Iterative Computation

Execute the following dynamic programming-type algorithm:

$$\mathcal{I}_0 = \mathcal{X}$$
$$\mathcal{I}_{k+1} = \mathrm{Pre}(\mathcal{I}_k) \quad k = 0, 1, 2, \ldots$$

STOP if $\mathcal{I}_{k+1} = \mathcal{I}_k$. Then, $\mathcal{I}_k = \mathcal{I}_\infty$ is the maxCRPI set.



(Proxy for convergence: distance between the islands.)

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Preimage Set Computation

$$\mathrm{Pre}(\mathcal{S}) = ((\mathcal{S} \ominus (D\mathcal{P})) \oplus (-B\mathcal{U}))A$$

where[2]:

- Minkowski sum: $\mathcal{A} \oplus \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$, $\mathcal{O}(c^n)$
- Pontryagin difference: $\mathcal{A} \ominus \mathcal{B} = \{a : a + b \in \mathcal{A}, \forall b \in \mathcal{B}\}$, $\mathcal{O}(n^c)$
- Direct mapping: $M\mathcal{A} = \{Ma : a \in \mathcal{A}\}$, $\mathcal{O}(c^n)$
- Inverse mapping: $\mathcal{A}M = \{a : Ma \in \mathcal{A}\}$, $\mathcal{O}(n^c)$

Minkowski sum is the most expensive operation (highest facet count, cannot be pre-computed).

[2] n is the polytope facet count and c is a coefficient

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

However, may wish to render invariant only *part* of the state. Examples:

- Some states do not make physical sense to render invariant (our case: skycrane mass)
- Some states may correspond to the controller (e.g. integrator)

In this case we want to render invariant the output $y = Cx$.

Controlled Robust Positively Output Invariant Set

The set $\mathcal{X}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{Y} \ \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Controlled Robust Positively Output Invariant Set

The set $\mathcal{Y}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(Ax + Bu + Dp) \in \mathcal{Y} \; \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Using $C^\dagger$ the pseudoinverse of C, we can write:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(A(C^\dagger y + \mathcal{N}(C)) + Bu + Dp) \subseteq \mathcal{Y} \; \forall p \in \mathcal{P}\},$$

where $\mathcal{N}(C)$ is the nullspace of C, i.e. $\mathcal{N}(C) = \{z : Cx = 0\}$ (which is a polytope!). The preimage set can be computed similarly to before:

$$\text{Pre}(\mathcal{Y}) = ((\mathcal{Y} \ominus (CD\mathcal{P} \oplus CA\mathcal{N}(C))) \oplus (-CB\mathcal{U}))CAC^\dagger$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

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The set $\mathcal{Y}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(A(C^\dagger y + \mathcal{N}(C)) + Bu + Dp) \subseteq \mathcal{Y} \ \forall p \in \mathcal{P}\}.$$

Compute the maxCRPOI set:

TODO:

- Derive computation for output invariance
- Show example for maxCRPI for skycrane
- Show computationaly complexity (reuse graph I showed to JPL)
- Present Rakovic, but do not try to implement... show the fracturing problem (again reuse the slides)
- Then get started on the control section

First Way: Optimization

The control problem can be formulated as an optimization problem:

Control Policy Synthesis via Optimization

Let $\mathcal{I} \triangleq \{x \in \mathbf{R}^n \mid Gx \leq g\}$ be the maximal positively invariant set induced by the control policy $u[k] = \mu_k(z[k])$. Consider the following sequence of optimization problems (for $i = 1, \dots, n_\mathrm{p}$):

$$\begin{array}{lll} g_i^+ = & \underset{x,u,w,v,e,G,g,k}{\text{maximize}} & G_i(A[k]x + B[k](u+e) + E[k]w) \\ & \text{subject to} & x \in \mathcal{I},\ u \in \mathcal{U},\ w \in \mathcal{W}(x,u),\ v \in \mathcal{V}(x),\ e \in \mathcal{L}(u) \\ & & u = c_k(x+v) \\ & & \mathcal{I} \subseteq \mathcal{X}, \quad k \in \mathbf{Z}_+. \end{array}$$

The control policy solves the control problem if and only if $g^+ \leq g$.

Authors employing optimization solve this problem via clever tricks for the particular structure that they consider (yields an LP, an SDP, etc.).

Second Way: Set-Based Iterative

Predecessor Set

Given a set $\mathcal{R} \subseteq \mathcal{X}$, the *predecessor set* $\text{Pre}(\mathcal{R})$ is:

$$\text{Pre}(\mathcal{R}) \triangleq \{x \in \mathbf{R}^n \mid \exists u \in \mathcal{U} \text{ s.t. } A[k]x + B[k](u+v) + E[k]w \in \mathcal{R} \\ \forall v \in \mathcal{V}(x), w \in \mathcal{W}(x,u)\},$$

i.e. $\mathcal{R}$ is 1-step robustly *reachable* from $\text{Pre}(\mathcal{R})$.

Consider the algorithm:

$$\mathcal{I}_0 = \mathcal{X}, \quad \mathcal{I}_{k+1} = \text{Pre}(\mathcal{I}_k).$$

Then $\mathcal{I}_{k+1} \subseteq \mathcal{I}_k \; \forall i \in \mathbf{Z}_+$ and the *maximal robust controlled invariant* set in $\mathcal{X}$ is $\mathcal{I}_\infty \subseteq \bigcap_{i \in \mathbf{Z}_+} \mathcal{I}_i$ and $\mathcal{I}_\infty = \mathcal{I}_j$ for some $j \in \mathbf{Z}_+ \Leftrightarrow \mathcal{I}_{j+1} = \mathcal{I}_j$.

The resulting control policy is set valued and is obtained a posteriori:

$$c_k(x) = \{u \in \mathcal{U} \mid A[k]x + B[k](u+v) + E[k]w \in \mathcal{I}_\infty \; \forall v \in \mathcal{V}(x), w \in \mathcal{W}(x,u)\}.$$

Can then use e.g. dynamic programming to obtain some optimal point-valued policy.

Comparison of Set-Based versus Optimization-Based

- Optimization-based methods are faster and compute point-valued controllers directly
- Set-based methods are slower but can potentially accommodate more features and can be *anytime* (i.e. aborted at any point and yield a valid albeit imprecise answer anyway)
- Set-based methods compute set-valued controllers $\Rightarrow$ post-processing (e.g. dynamic programming) required to obtain point-valued controllers.

Whether one or the other will solve all our problems remains to be seen as we *actually try to solve all our problems*.

Overview

Invariant Set Synthesis

- Optimization-Based Methods
- Set-Theoretic Methods

Invariant Controller Synthesis

Overview

- Present "the control problem"
- LQR, Linear From Spec, Bertsekas, perhaps other new ones...

Bibliography

[1] P. Trodden, "A one-step approach to computing a polytopic robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 61, pp. 4100–4105, dec 2016.

[2] B. Acikmese and S. R. Ploen, "Convex programming approach to powered descent guidance for mars landing," *Journal of Guidance, Control, and Dynamics*, vol. 30, pp. 1353–1366, sep 2007.

[3] M. Kvasnica, B. Takács, J. Holaza, and D. Ingole, "Reachability analysis and control synthesis for uncertain linear systems in MPT," *IFAC-PapersOnLine*, vol. 48, no. 14, pp. 302–307, 2015.