Invariant Set and Controller Synthesis

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Overview

Robust Controlled Invariant Set

We consider Discrete Linear Time Invariant (DLTI) system:

$$x^+ = Ax + Bu + Dp,$$

where $x \in \mathbf{R}^n$, $p \in \mathcal{P} = \{p \in \mathbf{R}^d : Rp \leq r\}$ and $u \in \mathcal{U} = \{u \in \mathbf{R}^m : Hu \leq h\}$ are "specification" polytopes.

Controlled Robust Positively Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Robust Invariant Set

Robust Controlled Invariant Set

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$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Now consider that some control law exists and the system reduces to an autonomous one:

$$x^+ = Ax + Dp.$$

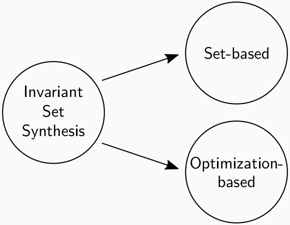
Robust Invariant Set

A set $\mathcal{X}$ is called *robust positively invariant* (RPI) if:

$$Ax + Dp \in \mathcal{X}, \quad \forall x \in \mathcal{X},\ p \in \mathcal{P}.$$

Goal: find an RPI $\mathcal{X}$.

Two Ways to Synthesize an Invariant Set



- Optimization-based methods rely on an explicit optimization problem (LP, LMI, etc.) to find $\mathcal{X}$
- Set-based methods rely on polytopic operations[1], i.e. computational geometry.

[1]These operations may implicitly involve an optimization, but what differentiates set-based methods is that people don't "talk" about it – they just assume that one can compute e.g. the Pontryagin difference.

Overview

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Equivalent RPI Condition

$$\mathcal{X}(g) = \{x : Gx \leq g\} \text{ RPI} \Leftrightarrow \sigma(G_i \mid A\mathcal{X}(g)) + \sigma(G_i \mid D\mathcal{P}) \leq \sigma(G_i \mid \mathcal{X}(g)),$$

where $g \in \mathbf{R}^{n_g}$ and $\sigma(z \mid \mathcal{S}) \triangleq \sup\{y^T z : y \in \mathcal{S}\}$ is the support function of (some) set $\mathcal{S}$.

Note: $\sigma(G_i \mid \mathcal{X}(g)) \leq g_i$ with $<$ $\Leftrightarrow$ facet i is redundant.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Existence of an RPI Set

Fix G in $\mathcal{X}(g) = \{x : Gx \leq g\}$ (i.e. pick a "template"). Assumptions:

A1. $\mathcal{P}$ contains the origin

A2. $\lambda < 0 \; \forall \lambda \in \mathrm{spec}(A)$

A3. The interior of $\mathcal{X}$ contains the origin

A4. For the chosen G, a g exists such that $\mathcal{X}(g)$ is RPI

Then there exists a g^* such that

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, \ldots, n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

Fixed-Point Solution Uniqueness

Given assumptions A1-A4, the g^* in the above statement is unique.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Existence of an RPI Set

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, \ldots, n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

g^* can be computed iteratively:

Algorithm 1 Iterative computation of g^*.

Set $g \leftarrow 0$
while True **do**
  $g_i^* \leftarrow \sigma(G_i \mid A\mathcal{X}(g)) + \sigma(G_i \mid D\mathcal{P})$ $i = 1, \ldots, n_g$
  if $\|g - g^*\|_\infty < \epsilon_{\text{tol}}$ **then**
    return g^*
  $g \leftarrow g^*$

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

- g^* can also be computed as a one-shot LP (main contribution of [1])
- Let $c_i(g) = \sigma(G_i \mid A\mathcal{X}(g))$, $d_i = \sigma(G_i \mid D\mathcal{P})$, $b_i(g) = \sigma(G_i \mid \mathcal{X}(g))$.
  Core realization (thanks to uniqueness of g^*):

$$g^* = \arg\min_g\{\|g\|_1 : c(g)+d = b(g)\} = \arg\max_g\{\|g\|_1 : c(g)+d = b(g)\}$$

- Recalling that $b(g) \leq g$, the above is readily converted to an LP:

$$g^* = c^* + d^*, \text{ where } (c^*, d^*) = \begin{array}{ll} \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\dots,n_g\}}}{\arg\ \text{maximize}} & \displaystyle\sum_{i=1}^{n_g} c_i + d_i \\ \text{subject to} & c_i \leq G_i A \xi^i \\ & G\xi^i \leq c + d \\ & d_i \leq G_i D \omega^i \\ & F\omega^i \leq g. \end{array}$$

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

$$\boxed{\text{Let } c_i(g) = \sigma(G_i \mid A\mathcal{X}(g)),\ d_i = \sigma(G_i \mid D\mathcal{P}),\ b_i(g) = \sigma(G_i \mid \mathcal{X}(g))}$$

$$
\begin{aligned}
g^* = c^* + d^*,\ \text{where } (c^*, d^*) = \ & \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\dots,n_g\}}}{\arg\text{maximize}} && \sum_{i=1}^{n_g} c_i + d_i \\
& \text{subject to} && c_i \leq G_i A \xi^i \\
& && G\xi^i \leq c + d \\
& && d_i \leq G_i D \omega^i \\
& && F\omega^i \leq g.
\end{aligned}
$$

The first two constraints evaluate $c_i(g)$ and the last two evaluate d_i. The first constraint holds with equality at optimality, since we want to maximize c_i. The RHS of the second constraint $= g^*$ at optimality, therefore the second constraint enforces $P\xi^i \leq g^*$, i.e. the definition of $b(g^*)$.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

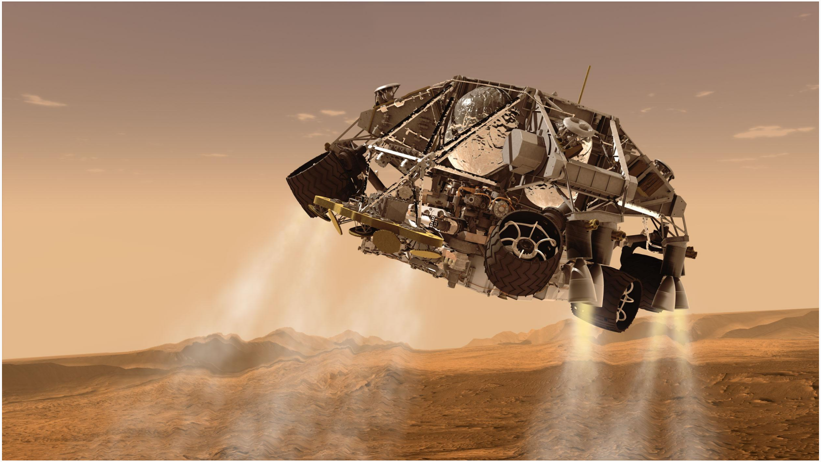


Image credit: NASA/JPL-Caltech

One-Step Minimal RPI Computation

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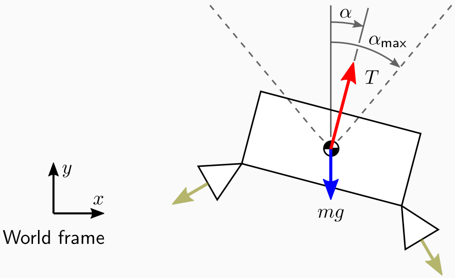


Parameters [2]:

$$
\begin{aligned}
m_{\text{wet}} &= 1905 \text{ kg} \\
g &= -3.7114 \text{ m/s}^2 \\
g_e &= 9.81 \text{ m/s}^2 \\
I_{\text{sp}} &= 225 \text{ s} \quad T_{\max} = 3.1 \text{ kN} \\
\phi &= 27 \text{ deg} \quad n = 6
\end{aligned}
$$

Dynamics:

$$
\begin{aligned}
(\dot{x}, \dot{y}) &= (v_x, v_y) \\
(\dot{v}_x, \dot{v}_y) &= (T_x, T_y)/m + g \\
\dot{m} &= -\frac{\|(T_x, T_y)\|_2}{I_{\text{sp}} g_e \cos\phi}
\end{aligned}
$$

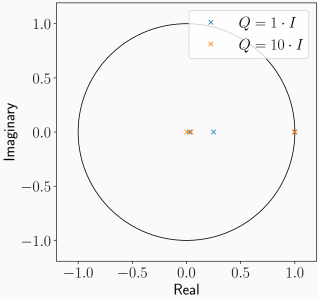
Letting $T \leftarrow T + mg$ be the gravity compensated control, the system is linearized about $(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y, \bar{m}) = (0, 0, 0, 0, m_{\text{wet}})$ and $(\bar{T}_x, \bar{T}_y) = (0, 0)$.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Synthesize an LQR stabilizing controller:

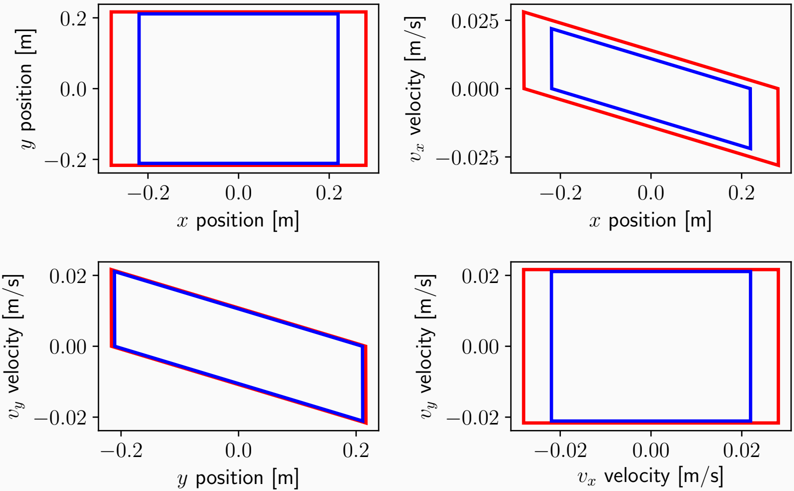
- State scaling: $D_x = \begin{bmatrix} 1 & 1 & 0.05 & 0.05 & 0.1 \end{bmatrix}$
- Input scaling: $D_u = \begin{bmatrix} nT_{\max}\cos\phi\sin\alpha_{\max} & nT_{\max}\cos\phi \end{bmatrix}$
- State penalty $Q = D_x^{-1}\hat{Q}D_x$ with $\hat{Q} \in \{I_5, 10I_5\}$
- Input penalty $R = D_x^{-1}\hat{R}D_x$ with $\hat{R} = I_2$



One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

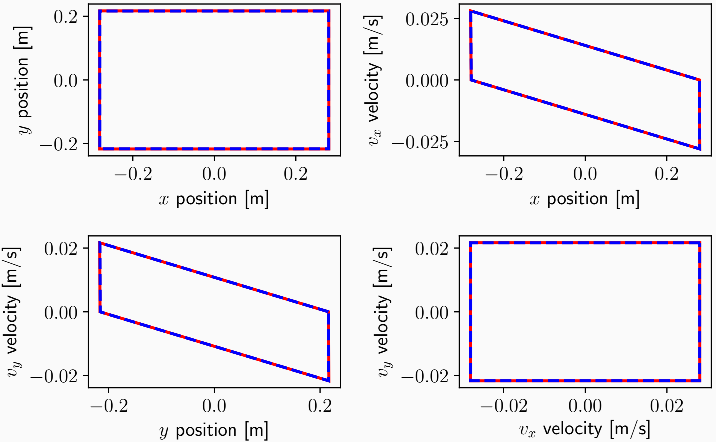
Direct application of LP on slide 11 ($\hat{Q} = I_5$, $\hat{Q} = 10I_5$):



One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

The one-shot LP of slide 11 and the iterative algorithm of slide 9 are identical...



... but iterative takes $\approx$ 315 s while one-shot takes $\approx$ 0.2 s!

Overview

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

We consider Discrete Linear Time Invariant (DLTI) system:

$$x^+ = Ax + Bu + Dp,$$

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Maximal CRPI Set

A set $\mathcal{X}_\infty \subseteq \mathcal{X}$ is called *maximal CRPI* (maxCRPI) if it is CRPI and contains all other CRPI sets in $\mathcal{X}$, i.e. $\mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}_\infty\ \forall \mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}$ RCPI [3].

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

maxCRPI Set Convexity

Given the system $x^+ = Ax + Bu + Dp$ where $p \in \mathcal{P}$, $u \in \mathcal{U}$, consider $\mathcal{X}$ the set of "safe" states. If $\mathcal{X}, \mathcal{P}, \mathcal{U}$ are convex then the associated maxCRPI set $\mathcal{X}_\infty$ is convex.

Recall the maxCRPI set definition:

$$\mathcal{X}_\infty = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X}_\infty,\ \forall p \in \mathcal{P}\}.$$

The definition is recursive ($\mathcal{X}_\infty$ on both sides) $\Rightarrow$ compute $\mathcal{X}_\infty$ *iteratively*.

Core step: *preimage set* computation.

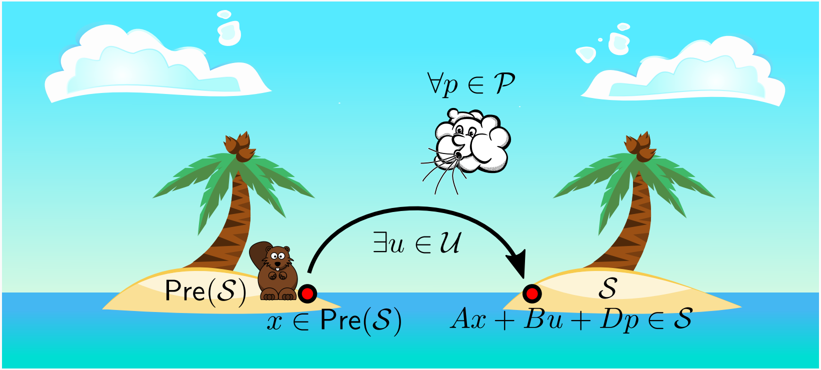
Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Preimage Set

$$\mathrm{Pre}(\mathcal{S}) \triangleq \{x \mid \exists u \in \mathcal{U},\ Ax + Bu + Dp \in \mathcal{S}\ \forall p \in \mathcal{P}\}$$

Remark: $\mathcal{S}$ CRPI $\Leftrightarrow \mathcal{S} \subseteq \mathrm{Pre}(\mathcal{S})$.



Maximal RCI Computation

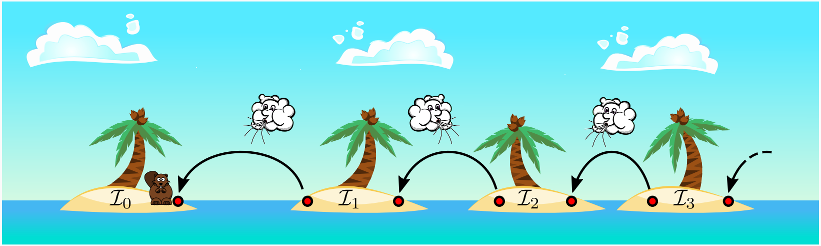
Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

maxCRPI Iterative Computation

Execute the following dynamic programming-type algorithm:

$$\mathcal{I}_0 = \mathcal{X}$$
$$\mathcal{I}_{k+1} = \mathrm{Pre}(\mathcal{I}_k) \cap \mathcal{I}_k \quad k = 0, 1, 2, \dots$$

STOP if $\mathcal{I}_{k+1} = \mathcal{I}_k$. Then, $\mathcal{I}_k = \mathcal{I}_\infty$ is the maxCRPI set.



(Proxy for convergence: distance between the islands.)

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Preimage Set Computation

$$\mathrm{Pre}(\mathcal{S}) = ((\mathcal{S} \ominus (D\mathcal{P})) \oplus (-B\mathcal{U}))A$$

where[2]:

- Minkowski sum: $\mathcal{A} \oplus \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$, $\mathcal{O}(c^n)$
- Pontryagin difference: $\mathcal{A} \ominus \mathcal{B} = \{a : a + b \in \mathcal{A}, \forall b \in \mathcal{B}\}$, $\mathcal{O}(n^c)$
- Direct mapping: $M\mathcal{A} = \{Ma : a \in \mathcal{A}\}$, $\mathcal{O}(c^n)$
- Inverse mapping: $\mathcal{A}M = \{a : Ma \in \mathcal{A}\}$, $\mathcal{O}(n^c)$

Minkowski sum is the most expensive operation (highest facet count, cannot be pre-computed).

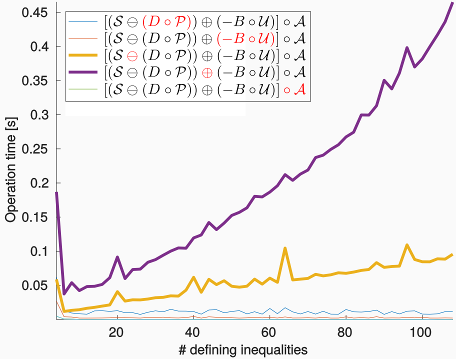
[2] n is the polytope facet count and c is a coefficient.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

$$\mathrm{Pre}(\mathcal{S}) = [(\mathcal{S} \ominus (D \circ \mathcal{P})) \oplus (-B \circ \mathcal{U})] \circ \mathcal{A}$$

For independent disturbances, Pontryagin difference ($\mathcal{O}(n^c)$) and especially Minkowski sum ($\mathcal{O}(c^n)$) are expensive[3].



[3] n is the polytope facet count and c is a coefficient.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

However, may wish to render invariant only *part* of the state. Examples:

- Some states do not make physical sense to render invariant (our case: skycrane mass)
- Some states may correspond to the controller (e.g. integrator)

In this case we want to render invariant the output $y = Cx$.

Controlled Robust Positively Output Invariant Set

The set $\mathcal{X}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{Y}\ \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Controlled Robust Positively Output Invariant Set

The set $\mathcal{Y}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(Ax + Bu + Dp) \in \mathcal{Y} \ \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Using $C^\dagger$ the pseudoinverse of C, we can write:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(A(C^\dagger y + \mathcal{N}(C)) + Bu + Dp) \subseteq \mathcal{Y} \ \forall p \in \mathcal{P}\},$$

where $\mathcal{N}(C)$ is the nullspace of C, i.e. $\mathcal{N}(C) = \{z : Cx = 0\}$ (which is a polytope!). The preimage set can be computed similarly to before:

$$\text{Pre}(\mathcal{Y}) = ((\mathcal{Y} \ominus (CD\mathcal{P} \oplus CA\mathcal{N}(C))) \oplus (-CB\mathcal{U}))CAC^\dagger$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

The following algorithm summarizes maxCRPOI set computation[4].

Algorithm 2 Iterative computation of maxCRPOI set $\mathcal{Y}_\infty$.

Set $\mathcal{Y}$ to the "safe outputs" specification

while True **do**

  $\text{Pre}(\mathcal{Y}) \leftarrow ((\mathcal{Y} \ominus (CD\mathcal{P} \oplus CA\mathcal{N}(C))) \oplus (-CB\mathcal{U}))CAC^\dagger$

  $\mathcal{Y}^+ = \mathcal{Y} \cap \text{Pre}(\mathcal{Y})$

  if $\mathcal{Y} \subseteq \mathcal{Y}^+_{\epsilon_{\text{tol}}}$ and $\mathcal{Y}^+ \subseteq \mathcal{Y}_{\epsilon_{\text{tol}}}$ **then**

    return $\mathcal{Y}_\infty \leftarrow \mathcal{Y}^+$

  $\mathcal{Y} \leftarrow \mathcal{Y}^+$

---

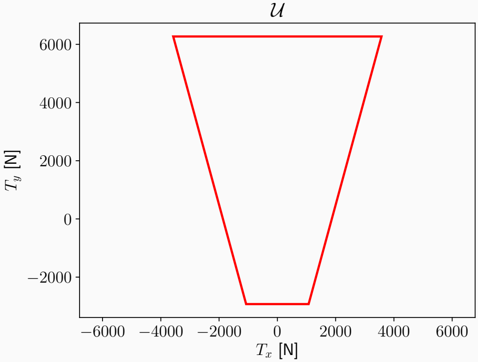
[4] If $\mathcal{S} = \{x : Px \leq p\}$, we denote $\mathcal{S}_{\epsilon_{\text{tol}}} = \{x : Px \leq p + \epsilon_{\text{tol}}\}$ the ϵ_{tol}-*dilation* of $\mathcal{S}$. In practical, dilation is a more robust stopping criterion than equality ($\mathcal{Y}^+ = \mathcal{Y}$) which is prone to numerical inaccuracy.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Going back to the skycrane example, consider the specifications:

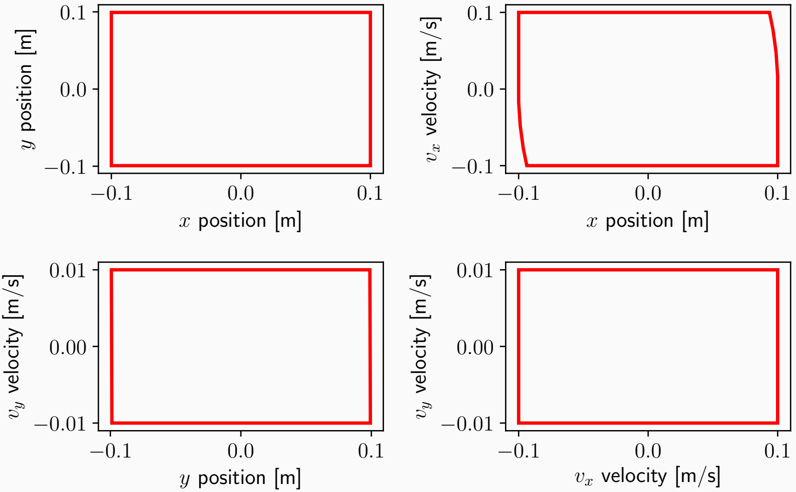
- ± 10 cm position error (in both x and y)
- ± 10 cm/s velocity error in x, ± 1 cm/s velocity error in y
- ± 400 N disturbance force (in both x and y)
- Input constraint set given by the rocket motor specs [2] (visualized below)



Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Direct application of algorithm on slide 25:



Overview

Robust Invariant Set Definitions

Minimal Robust Positively Invariant Set Computation

Maximal Controlled Robust Positively Invariant Set Computation (Independent Noise)

Maximal Controlled Robust Positively Invariant Set Computation (Dependent Noise)

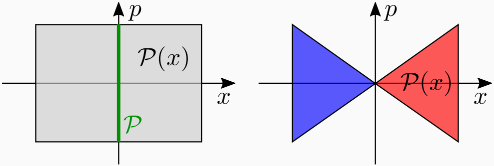
Conclusion

Maximal RCI Computation With Dependent Noise

Rakovic et al., "Reachability Analysis of Discrete-Time Systems With Disturbances", 2006. [4]

What happens if the disturbance is state and/or input dependent?

$$p \in \operatorname{Proj}_p \mathcal{P}(x, u) = \{\theta = (p, x, u) \in \mathbf{R}^{d+n+m} : R\theta \leq r\}$$



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In this case $\operatorname{Pre}(\mathcal{X})$ can be computed in several steps:

$$\begin{aligned}
\mathcal{Z} &\triangleq \mathcal{X} \times \mathcal{U} \\
\mathcal{W} &\triangleq \{(x,u,p) : (x,u) \in \mathcal{Z}, p \in \mathcal{P}(x,u)\} \\
\Phi &\triangleq \{(x,u,p) : Ax + Bu + Dp \in \mathcal{S}\} \\
\Sigma &\triangleq \{(x,u) \in \mathcal{Z} \mid Ax + Bu + Dp \in \mathcal{S}\ \forall p \in \mathcal{P}(x,u)\} \\
&= \mathcal{Z} \setminus \operatorname{Proj}_{x,u}(\mathcal{W} \setminus \Phi) \\
\Rightarrow \operatorname{Pre}(\mathcal{S}) &= \operatorname{Proj}_x(\Sigma)
\end{aligned}$$

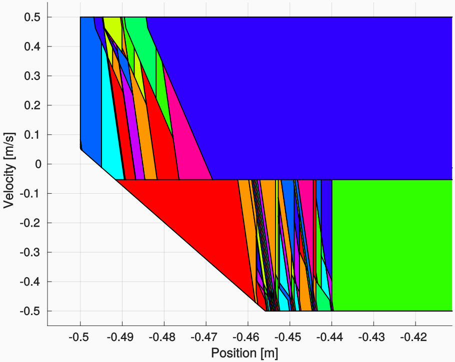
When sets are polytopes, all operations are possible via computational geometry.

Maximal RCI Computation With Dependent Noise

Rakovic et al., "Reachability Analysis of Discrete-Time Systems With Disturbances", 2006. [4]

$$\Sigma = \mathcal{Z} \setminus \mathrm{Proj}_{x,u}(\mathcal{W} \setminus \Phi).$$

Regiondiff operation (\) [5]) generates a union of polytopes, which suffers from severe "fracturing" of convex regions.



Furthermore, $\mathrm{Proj}_{x,u}$ is expensive when $\dim(\mathcal{W})$ is large!

Overview

Robust Invariant Set Definitions

Minimal Robust Positively Invariant Set Computation

Maximal Controlled Robust Positively Invariant Set Computation (Independent Noise)

Maximal Controlled Robust Positively Invariant Set Computation (Dependent Noise)

Conclusion

Conclusion

- Minimal RPI computation boils down to a one-shot LP [1]
- Maximal CRPI computation for generic polytopes has exponential complexity in set-based methods due to the Minkowski sum
- Maximal CRPI computation for dependent noise is computationally difficult due to non-convexity
- Further reading in [6, 7].

Thank You For Your Attention!

Appendix

Overview

Bibliography

[1] P. Trodden, "A one-step approach to computing a polytopic robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 61, pp. 4100–4105, dec 2016.

[2] B. Acikmese and S. R. Ploen, "Convex programming approach to powered descent guidance for mars landing," *Journal of Guidance, Control, and Dynamics*, vol. 30, pp. 1353–1366, sep 2007.

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[7] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Springer International Publishing, 2015.