1: Proofs and functions

By sequence $p \Rightarrow q$. Establish $p \Rightarrow p_1 \Rightarrow \ldots \Rightarrow p_n \Rightarrow q$.
By induction. Prove p_0 true; assume p_k true; deduce p_{k+1} true.
By contraposition $p \Rightarrow q$. Prove that $\neg q \Rightarrow \neg p$.
By contradiction $p \Rightarrow q$. Show that $p \Rightarrow \neg q$ gives contradiction.
If and only if. Must prove ($\Rightarrow$) and ($\Leftarrow$).
Set equality. Prove $A \subseteq B$ (e.g. $x \in A \Rightarrow x \in B$) & $B \subseteq A$.

$f : X \to Y$. X dom, Y codom, $\{y \in Y | \exists x \in X, f(x) = y\}$ range. ① **Injective** $\Leftrightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. ② **Surjective** $\Leftrightarrow \forall y \in Y \exists x \in X, y = f(x)$. ③ **Bijective** $\Leftrightarrow$ inj & surj.

① $g_L : Y \to X$ left inv $\Leftrightarrow g_L \circ f = 1_X$. ② $g_R : Y \to X$ right inv $f \Leftrightarrow f \circ g_R = 1_Y$. ③ $g : Y \to X$ inv $\Leftrightarrow$ both left and right inv. f invertible if $\exists$ inverse of f; then inverse is unique and left=right.

① f has left inv $\Leftrightarrow$ injective. ② f has right inv $\Leftrightarrow$ surjective. ③ f invertible $\Leftrightarrow$ bijective. ④ If f invertible, inverse is unique!

Func. compos. 1. Associative 2. Compos of injections $\Rightarrow$ injection; surj $\Rightarrow$ surj; bij $\Rightarrow$ bij 3. $(f \circ g)^{-1} = (g^{-1} \circ f^{-1})$.

2: Introduction to Algebra

Group $(G, *)$: ⓪ $*$ closed ① $*$ assoc: $a*(b*c) = (a*b)*c$ ② $\exists$ iden $e \in G, a*e = e*a = a$ ③ $\exists a^{-1} \in G, a*a^{-1} = a^{-1}*a = e$ (4.) **Abelian**: $*$ commutative: $a*b = b*a$

Ring. $(R, +, \cdot)$, bin ops $+ : R \times R \to R$ (**closed** addition), $\cdot : R \times R \to R$ (**closed** multiplic.) s.t.
1. Addition satisfies: • Assoc: $a + (b + c) = (a + b) + c$. • Commut: $a + b = b + a$ • $\exists$ iden $0 \in R$, $a + 0 = a$. • $\exists$ inv: $(-a) \in R$, $a + (-a) = (-a) + a = 0$.
2. Multiplication satisf: • Assoc: $\forall a, b, c \in R, a \cdot (b \cdot c) = (a \cdot b) \cdot c$. • $\exists$ iden $\exists 1 \in R, \forall a \in R, 1 \cdot a = a \cdot 1 = a$.
3. Multip. distributive wrt add.: $\forall a, b, c \in R, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$
(4) **Commutative ring** if $\forall a, b \in R, a \cdot b = b \cdot a$

$(R, +, \cdot)$ ring: 1. $a \cdot 0 = 0 \cdot a = 0$ 2. $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$

$\mathbb{R}[s]$: $\sum_0^n a_i s^i$; $\mathbb{R}(s)$: $\sum_0^m a_i s^i / \sum_0^n b_i s^i$; $\mathbb{R}_p(s)$: $\sum_0^n a_i s^i / \sum_0^n b_i s^i$

Field $(F, *, \cdot)$ commut ring & $\forall a \in F \backslash 0, \exists a^{-1} \in F, a \cdot a^{-1} = 1$

Linear space (LS) $(V, F, \oplus, \odot)$: set V of **vectors**, field $(F, +, \cdot)$ of **scalars**, w/ bin ops $\oplus : V \times V \to V$ and $\odot : F \times V \to V$ s.t.
1. $\oplus$ satisfies: • Assoc: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$. • Commut: $x \oplus y = y \oplus x$. • $\exists$ iden $\theta \in V$ s.t. $x \oplus \theta = x$. • $\exists$ inverse: $\forall x \in V, \exists (\ominus x) \in V, x \oplus (\ominus x) = \theta$.
2. $\odot$ satisfies: • Assoc: $a \odot (b \odot x) = (a \cdot b) \odot x$ • Mult by iden $1 \in F$ leaves elems unchanged: $\forall x \in V, 1 \odot x = x$.
3. $\odot$ distribut. wrt $\oplus$: $\forall a, b \in F, \forall x, y \in V, (a + b) \odot x = (a \odot x) \oplus (b \odot x)$ and $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$.

$(V, F, \oplus, \odot)$ LS: 1. $0 \odot x = \theta$ 2. $(-a) \odot x = \ominus(a \odot x) = a \odot (\ominus x)$

Product space $(V \times W, F, \oplus, \odot)$ is LS of pairs $(v, w) \in V \times W$ with $(v_1, w_1) \oplus (v_2, w_2) = (v_1 \oplus_V v_2, w_1 \oplus_W w_2)$ and $a \odot (v, w) = (a \odot_V v, a \odot_W w)$.

Def: $C^k([t_0, t_1], \mathbb{R}^n)$ LS of k-times diffbl funcs $f : [t_0, t_1] \to \mathbb{R}^n$

Linear subspace (LSS) $(W \subseteq V, F)$ of $V \Leftrightarrow$ itself LS, i.e. $\theta_V \in W$ and $\forall w_1, w_2 \in W, \forall a_1, a_2 \in F$, we have $a_1 w_1 + a_2 w_2 \in W$.

$\{(W_i, F)\}_{i=1}^n$ LSS's of (V, F): • $\bigcap_{i=1}^n (W_i, F)$ always LSS • $\bigcup_{i=1}^n (W_i, F)$ LSS only if W_i embedded ($\exists W_k$ encompassing all)

LSS of (V, F) gen. by S: $\text{Span}(S) = \left\{\sum_{i=1}^n a_i v_i | a_i \in F, v_i \in S\right\}$.

$\{v_i\}_{i=1}^n$ **lin indep** $\Leftrightarrow (\sum_{i=1}^n a_i v_i = 0 \Rightarrow a_i = 0, \forall i = 1, \ldots, n)$.

Set $S \subseteq V$ is **basis** of $(V, F) \Leftrightarrow$ it is lin. indep. and $\text{Span}(S) = V$.

All bases have same # of elements $= \dim(V)$ (**dimension** of V).

Representation $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in F^n$ of $x \in V$ wrt basis $\{b_i\}_{i=1}^n$ such that $x = \sum_{i=1}^n \xi_i b_i$, **unique** wrt a basis!

$\mathcal{A} : U \to V$ **linear** $\Leftrightarrow \mathcal{A}(a_1 u_1 + a_2 u_2) = a_1 \mathcal{A}(u_1) + a_2 \mathcal{A}(u_2)$.

$\mathcal{A} : U \to V$ • **null space** $\text{Null}(\mathcal{A}) = \{u \in U | \mathcal{A}(u) = \theta_V\} \subseteq U$ • **range space** $\text{Range}(\mathcal{A}) = \{v \in V | \exists u \in U : v = \mathcal{A}(u)\} \subseteq V$

$\mathcal{A} : U \to V$ **surjectv** $\Leftrightarrow \text{Range}(\mathcal{A}) = V$; **injectv** $\Leftrightarrow \text{Null}(\mathcal{A}) = \theta_U$

Eigenvalue $\lambda \in F$ of $\mathcal{A} : V \to V \Leftrightarrow \exists v \in V$ s.t. $v \neq \theta_V \wedge \mathcal{A}(v) = \lambda \cdot v$. Then v called **eigenvector** of $\mathcal{A}$ for the eigenvalue λ.

The set of eigenvectors $\{v \in V | \mathcal{A}(v) = \lambda v\}$ is a subspace of V.

Given field F, every matrix $A \in F^{m \times n}$ defines a linear map $\mathcal{A} : (F^n, F) \to (F^m, F)$ by matrix multiplication.

$\mathcal{A}(x) = Ax$: $\text{Rank}(A) := \dim\text{Range}(\mathcal{A})$, $\text{Nullity}(A) := \dim\text{Null}(\mathcal{A})$

$A \in F^{n \times m}$:
- $\text{Rank}(A) + \text{Nullity}(A) = m$
- $0 \leqslant \text{Rank}(A) \leqslant \min\{m, n\}$
- If $P \in F^{m \times m}$ and $Q \in F^{n \times n}$ invertible, then:

$$\begin{cases} \text{Rank}(A) = \text{Rank}(AP) = \text{Rank}(QA) = \text{Rank}(QAP) \\ \text{Nullity}(A) = \text{Nullity}(AP) = \text{Nullity}(QA) = \text{Nullity}(QAP) \end{cases}$$

Let $A \in F^{n \times n}$ matrix rep of $\mathcal{A} : F^n \to F^n$, $\mathcal{A}(x) = Ax$. Equiv: 1. A inv 2. $\mathcal{A}$ bij 3. $\mathcal{A}$ inj 4. $\mathcal{A}$ surj 5. $\text{Rank}(A) = n$ 6. $\text{Nullity}(A) = 0$ 7. A cols are lin indep 8. A rows are lin indep

$A \in F^{n \times n}$: • $\lambda \in \mathbb{C}$ eigval of $A \Leftrightarrow \text{Det}(\lambda I - A) = 0$ • $v \in \mathbb{C}^n \backslash 0, Av = \lambda v$ **r eigvec** • $\eta \in \mathbb{C}^n \backslash 0, \eta^T A = \lambda \eta^T$ **l eigvec**

Spectrum $\text{Spec}[A \in F^{n \times n}] = \{\lambda_1, \ldots, \lambda_n\}$; A inv $\Leftrightarrow \lambda_i \neq 0 \forall i$

Any lin. map $\mathcal{A} : F^n \to F^m$ between 2 finite-dim. spaces can be represented by a **unique** matrix $A \in F^{m \times n}$ (wrt fixed bases).

Let $\mathcal{A} : (U, F) \to (V, F)$, $\{u_j\}_{j=1}^n \to \{v_i\}_{i=1}^m$ then:

$$y_j = \mathcal{A}(u_j) = \sum_{i=1}^m a_{ij} v_i \Rightarrow A := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in F^{m \times n}$$

Let $x = \sum_{j=1}^n \xi_j u_j \in U$, rep of x is $\xi = (\xi_1, \ldots, \xi_n)$. Then rep of $\mathcal{A}(x) \in V$ is $\eta = A\xi$ s.t. $\mathcal{A}(x) = \sum_{i=1}^m \eta_i v_i$, $\eta_i = \sum_{j=1}^n a_{ij} \xi_j$.

Let $\{u_j\}_{j=1}^n$ basis of (U, F) and $\mathcal{A} : (U, F) \to (V, F)$. Repr. A of $\mathcal{A}$ wrt $\{u_j\}_{j=1}^n$ is found col-by-col: $A_{\text{col},(i)} = \mathcal{A}(u_i) (= Au_i)$.

Easy to see with $\{e_j\}_{j=1}^n$ the canonical $(0, \ldots, 1, \ldots, 0)$ basis.

Rep of compos $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$ is $C = A \cdot B$ (matr mult).

A representation of $\mathcal{A} : V \to V$, then A^{-1} is the rep of $\mathcal{A}^{-1}$.

Change of Basis.

$$\begin{array}{ccc} (U, F) & \xrightarrow{\mathcal{A}} & (V, F) \\ \{u_j\}_{j=1}^n & \xrightarrow{A \in F^{m \times n}} & \{v_i\}_{i=1}^m \\ Q \in F^{n \times n} \quad \uparrow & & \downarrow \quad P \in F^{m \times m} \\ \{\tilde{u}_j\}_{j=1}^n & \xrightarrow{\tilde{A} \in F^{m \times n}} & \{\tilde{v}_i\}_{i=1}^m \end{array}$$

$$\Rightarrow \tilde{A} = PAQ$$

Find Q? $Q\tilde{u}_j = [\tilde{u}_j]_{\{u_j\}_{j=1}^n} = \alpha_1 u_1 + \cdots + \alpha_n u_n \Rightarrow Q_{\text{col}}^{(j)} = [\alpha_1 \cdots \alpha_n]^T$. Same for P ☺.

Change of basis matrices are **invertible**.

$A \in F^{m \times n}$ and $\tilde{A} \in F^{m \times n}$ are **equivalent** $\Leftrightarrow \exists Q \in F^{n \times n}, P \in F^{m \times m}$ s.t. $\tilde{A} = PAQ$.

2 matrices equivlnt $\Leftrightarrow$ they are representations of same lin map

When $\mathcal{A} : U \to U$ then $\tilde{A} = Q^{-1} A Q$ **change of basis** formula.

3: Introduction to Analysis

Norm on LS (V, F) is func $\| \cdot \| : V \to \mathbb{R}_+ \equiv \{x \in \mathbb{R} | x \geqslant 0\}$ s.t. ⓪ Maps to $\mathbb{R}_+$!!! ① $\forall v_1, v_2 \in V, \|v_1 + v_2\| \leqslant \|v_1\| + \|v_2\|$ ② $\forall v \in V, \forall a \in F, \|av\| = |a| \|v\|$ ③ $\|v\| = 0 \Leftrightarrow v = 0$

Normed linear space is $(V, F, \| \cdot \|)$.

$$\|x\|_1 = \sum_i |x_i| \quad \|x\|_2 = \sqrt{\sum_i |x_i|^2} \quad \|x\|_\infty = \max_i |x_i|$$

(Open) ball in $(V, F, \| \cdot \|)$: $B(v, r) = \{v' \in V | \|v - v'\| < r\}$. $B(0, 1)$ **unit ball**. $S \subseteq V$ **bounded** if $\exists r \in \mathbb{R}_+ | S \subseteq B(0, r)$.

Sequence $\{v_i\}_{i=0}^\infty$ in $(V, F, \| \cdot \|)$ **converges** to $\bar{v} \in V \Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} | \forall m \geqslant N, \|v_m - \bar{v}\| < \varepsilon$.

Set $K \subseteq V$ **closed** $\Leftrightarrow$ contains all its limit points $\Leftrightarrow \forall \{v_i\}_{i=0}^\infty \subseteq K$, if $v_i \to v \in V$ then $v \in K$. K is **open** $\Leftrightarrow$ complement $V \backslash K$ closed. K **compact** if both closed and bounded.

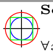
$f : U \to V$ **continuous at** $u \in U$ $\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0$ s.t. $\|u - u'\|_U < \delta \Rightarrow \|f(u) - f(u')\|_V < \varepsilon$. f **cont on** $U \Leftrightarrow$ cont $\forall u \in U$.

$\| \cdot \|_a$ & $\| \cdot \|_b$ **equiv** $\Leftrightarrow \exists m_u \geqslant m_l > 0, m_l \|v\|_a \leqslant \|v\|_b \leqslant m_u \|v\|_a$

$x \in V, B_a(0, 1) = \{\|x\|_a < 1\}$, $B_b(0, 1) = \{\|x\|_b < 1\}$: $m_l \|x\|_a \leqslant \|x\|_b \leqslant m_u \|x\|_a \Rightarrow B_b(0, m_l) \subseteq B_a(0, 1) \subseteq B_b(0, m_u)$

$x \in B_a(0, 1) \Rightarrow \|x\|_a < 1 \Rightarrow \|x\|_b < m_u \Rightarrow x \in B_b(0, m_u) \Rightarrow B_a(0, 1) \subseteq B_b(0, m_u)$. $y \in B_b(0, m_l) \Rightarrow m_l \|x\|_a \leqslant \|y\|_b < m_l \Rightarrow \|x\|_a < 1 \Rightarrow x \in B_a(0, 1) \Rightarrow B_b(0, m_l) \subseteq B_a(0, 1)$. ■

$$\begin{aligned} \|x\|_2 &\leqslant \|x\|_1 \leqslant \sqrt{n} \|x\|_2 \\ 1 \cdot \|x\|_\infty &\leqslant \|x\|_2 \leqslant \sqrt{n} \|x\|_\infty \\ \|x\|_\infty &\leqslant \|x\|_1 \leqslant n \|x\|_\infty \end{aligned}$$



Schwarz's Inequality: $\sum_{i=1}^n (|x_i| \cdot |y_i|) \leqslant \|x\|_2 \|y\|_2$ $\forall x, y \in F^n$.

$\| \cdot \|_a, \| \cdot \|_b$ equiv. $\{v_i\}_{i=0}^\infty$ CV in $(V, F, \| \cdot \|_a) \Leftrightarrow$ CV in $(V, F, \| \cdot \|_b)$.

Open/closed sets remain open/closed $\forall$ equivalent norms.

Any 2 norms on a finite-dim space are equivalent!

Weierstrass continuous $f : S \to \mathbb{R}$ defined on $S \subseteq \mathbb{R}^n$ that's compact in $(\mathbb{R}^n, \| \cdot \|_2)$ attains a (non-$(-\infty)$) **minimum** on S.

Infinite-dim linear spaces

$$\|f\|_1 = \int_{t_0}^{t_1} \|f(t)\|_2 dt, \; \|f\|_2 = \sqrt{\int_{t_0}^{t_1} \|f(t)\|_2^2 dt}, \; \|f\|_\infty = \max_{[t_0, t_1]} \|f(t)\|_2$$

All $\|f\|_i$ not equiv. Proof: family $f_n(t) = t^n \in C([0, 1], \mathbb{R})$, $n \in \mathbb{N}$.

Cauchy seq $\{v_i\}_{i=0}^\infty \Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N}, \forall m \geqslant N, \|v_m - v_N\| < \varepsilon$

Convergent sequence $\Rightarrow \not\Leftarrow$ Cauchy.

$(V, F, \| \cdot \|)$ **complete (Banach)** $\Leftrightarrow$ all Cauchy seq's in it converge

Every finite-dimensional lin. space (V, F) is Banach, for any $\| \cdot \|$.

$(\mathbb{R}, \mathbb{R}, \| \cdot \|)$ is Banach. $(C([t_0, t_1], \mathbb{R}^n), \mathbb{R}, \| \cdot \|_\infty)$ is Banach.

Let $f : (U, F, \| \cdot \|_U) \to (V, F, \| \cdot \|_V)$. **Induced norm** of f:

$$\|f\| = \sup_{u \neq 0} \frac{\|f(u)\|_V}{\|u\|_V}$$

For lin maps $\mathcal{A} : U \to V$, suffice: $\|\mathcal{A}\| = \sup_{\|u\|_U = 1} \|\mathcal{A}(u)\|_V$

Induced norms of $\mathcal{A} : F^n \to F^m$:
$$\begin{cases} \|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^m |a_{ij}| & \text{row sum} \\ \|A\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^n |a_{ij}| & \text{col sum} \\ \|A\|_2 = \max_{\lambda \in \text{Spec}[A^T A]} \sqrt{\lambda} & \text{max eig.} \end{cases}$$

Let *linear* map $\mathcal{A} : (U, F, \| \cdot \|_U) \to (V, F, \| \cdot \|_V)$. *Equivalent*: 1. $\mathcal{A}$ continuous. 2. $\mathcal{A}$ continuous at 0. 3. $\sup_{\|u\|_U = 1} \|\mathcal{A}(u)\|_V < \infty$ and induced norm $\|\mathcal{A}\|$ is well-defined.

All lin. funcs. between **finite-dim.** spaces are continuous.

Let $\mathcal{A}, \tilde{\mathcal{A}} : (V, F, \| \cdot \|_V) \to (W, F, \| \cdot \|_W)$ and $\mathcal{B} : (U, F, \| \cdot \|_U) \to (V, F, \| \cdot \|_V)$. Let $\| \cdot \|$ be the induced norms. ① $\forall v \in V, \|\mathcal{A}(v)\|_W \leqslant \|\mathcal{A}\| \cdot \|v\|_V$ ② $\forall a \in F, \|a\mathcal{A}\| = |a| \|\mathcal{A}\|$ ③ $\|\mathcal{A} + \tilde{\mathcal{A}}\| \leqslant \|\mathcal{A}\| + \|\tilde{\mathcal{A}}\|$ ④ $\|\mathcal{A}\| = 0 \Leftrightarrow \mathcal{A}(v) = 0 \forall v \in V$ (zero map) ⑤ $\|\mathcal{A} \circ \mathcal{B}\| \leqslant \|\mathcal{A}\| \|\mathcal{B}\|$

$\|\mathcal{A} \circ \mathcal{B}\| = \sup \|(\mathcal{A} \circ \mathcal{B})(u)\|_W \leqslant \|\mathcal{A}\| \sup \|\mathcal{B}(u)\|_V = \|\mathcal{A}\| \|\mathcal{B}\|$ ■

Ordinary Differential Equations

$u \in PC([t_0, t_1], \mathbb{R}^m)$ is **piecewise continuous (pwc)** $\Leftrightarrow$ cont at all $t \in \mathbb{R}$ except finite set of **discontinuity points** $D \subseteq \mathbb{R}$.

Consider ODE: $\dot{x}(t) = p(x(t), t) \in PC([t_0, t_1], \mathbb{R}^n)$ (i.e. pwc in t). $\phi : \mathbb{R} \to \mathbb{R}^n$ passing through $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ **solution** $\Leftrightarrow$ • $\phi(t_0) = x_0$ • $\forall t \in \mathbb{R} \backslash D$, ϕ differentiable at t & $\dot{\phi}(t) = p(\phi(t), t)$

$p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is **globally Lipschitz in** x (simply: "**Lipschitz**") $\Leftrightarrow \exists \, k : \mathbb{R} \to \mathbb{R}_+$ pwc s.t. $\forall x, x' \in \mathbb{R}^n, \forall t \in \mathbb{R}, \|p(x, t) - p(x', t)\| \leqslant k(t) \|x - x'\|$. $k(t)$ called **Lipschitz constant** of p at $t \in \mathbb{R}$.

Are globally Lipschitz: 1. All linear functions 2. All **differentiable** functions with **bounded derivatives**

Lipschitz $\Rightarrow \not\Leftarrow$ continuous. Lipschitz $\not\Rightarrow$ differentiable.

Differentiable func $f(x)$ is s.t. df/dx exists/is well-defined $\forall x$. Diffbl $\Rightarrow \not\Leftarrow$ cont $\forall x \in D$. Reason: ⋏, ∫, cont but not diffbl. Multivar func diffbl $\Leftrightarrow$ Jacobian well-defined $\forall x \in D$.

Methods to prove Lipschitzianity: M1 Show that derivative/Jac. bounded. M2 Show that function linear. M3 FTSOC, suppose $\exists k$ s.t. $|\sqrt{x} - \sqrt{y}| \leqslant k|x - y|$. Take $x = 1/n, y = 0 \Rightarrow (\sqrt{1/n} - \sqrt{0})/(1/n - 0) = \sqrt{n} \Rightarrow \sqrt{n} \leqslant k$. k const in n, letting $n \to \infty$ contradic.! $\sqrt{x}$ not Lipschitz. ■

Solution $\phi(t)$ to ODE $\dot{x}(t) = p(x(t), t)$ **exists** & is **unique** if p pwc wrt t and globally Lipschitz wrt x (*sufficient but not necessary!*). $\phi(t)$ is **pwc** e.g.

$\forall \| \cdot \|$ on $\mathbb{R}^n$, $\forall t_0, t_1 \in \mathbb{R}$: $\left\| \int_{t_0}^{t_1} f(t) dt \right\| \leqslant \left| \int_{t_0}^{t_1} \|f(t)\| dt \right|$

• $\forall m, k \in \mathbb{N}, (m + k)! \geqslant m! \cdot k!$ • $\forall c \in \mathbb{R}, \lim_{m \to \infty} [c^m / m!] = 0$

Fund thrm calculus. Let $g : \mathbb{R} \to \mathbb{R}$ pwc w/ disc set $D \subseteq \mathbb{R} \Rightarrow \forall t_0 \in \mathbb{R}, f(t) = \int_{t_0}^t g(\tau) d\tau$ cont and $\forall t \in \mathbb{R} \backslash D$, $\frac{d}{dt} f(t) = g(t)$.

Leibniz: $\frac{d}{dt} \left[\int_{a(t)}^{b(t)} f(t, \tau) d\tau \right] = \int_a^b \frac{\partial f(t, \tau)}{\partial t} d\tau + f(t, b)\dot{b} - f(t, a)\dot{a}$

Gronwall Lemma. Let $u(\cdot), k(\cdot) : \mathbb{R} \to \mathbb{R}_+$ pwc, $c_1 \geqslant 0, t_0 \in \mathbb{R}$. $\forall t, u(t) \leqslant c_1 + \left| \int_{t_0}^t k(\tau) u(\tau) d\tau \right| \Rightarrow \forall t, u(t) \leqslant c_1 \exp \left| \int_{t_0}^t k(\tau) d\tau \right|$

$-k\|s(t, t_0, x_0) - \hat{x}\| \leqslant \frac{d}{dt} \|x(t, t_0, x_0) - \hat{x}\| \leqslant k\|s(t, t_0, x_0) - \hat{x}\|$
$\Rightarrow \|x_0 - \hat{x}\| e^{-k(t - t_0)} \leqslant \|s(t, t_0, x_0) - \hat{x}\| \leqslant \|x_0 - \hat{x}\| e^{k(t - t_0)}$

4: Time varying linear systems: Solutions

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t) \tag{1}$$
$t \in \mathbb{R}, x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$.

Let $(x_o(t), u_o(t))$ **trajectory**. **Perturbations** around it: $x(t) = x_o(t) + \delta x(t)$, $u(t) = u_o(t) + \delta u(t)$. **Linearization about traj**:
$$d(\delta x(t))/dt = A(t)\delta x(t) + B(t)\delta u(t)$$
$A(t) := \partial f(x_o(t), u_o(t))/\partial x$; $B(t) := \partial f(x_o(t), u_o(t))/\partial u$.

Let $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ be pwc. Then:

$\forall u(\cdot) : \mathbb{R} \to \mathbb{R}^m$ pwc and $\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n \exists! x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ and $y(\cdot) : \mathbb{R} \to \mathbb{R}^p$ for system (1). This unique solution is:
$x(t) = s(t, t_0, x_0, u)$ **state transition map**
$y(t) = \rho(t, t_0, x_0, u) = C(t)s(t, t_o, x_o, u) + D(t)u(t)$

$D_x = \bigcup$ disc sets of A, B, u; $D_y = \bigcup$ of disc sets of C, D, u.
1. $\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n, u(\cdot) \in PC(\mathbb{R}, \mathbb{R}^m)$
   - $s(\cdot, t_0, x_0, u) \in C^1(t \in \mathbb{R} \backslash D_x, \mathbb{R}^n)$.
   - $\rho(\cdot, t_0, x_0, u) \in PC(\mathbb{R}, \mathbb{R}^p)$ w/ discont. set D_y.
2. $s(t, t_0, \cdot, u), \rho(t, t_0, \cdot, u) \in C(\mathbb{R}, \mathbb{R}^n$ and $\mathbb{R}^p$ respect.)
3. Below true for s **and** ρ:
$$s(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 s(t, t_0, x_{01}, u_1) + a_2 s(t, t_0, x_{02}, u_2)$$
4. Below true for s **and** ρ (change **trans.→resp.**):
$$\underbrace{s(t, t_0, x_0, u)}_{\textbf{state trans.}} = \underbrace{s(t, t_0, x_0, 0)}_{\textbf{0 inp. trans.}} + \underbrace{s(t, t_0, 0, u)}_{\textbf{0 state trans.}}$$

Rep. of $s(t, t_0, \cdot, 0)$ wrt to a basis is called **state transition matrix (STM)** $\Phi(t, t_0)$ s.t. $s(t, t_0, x_0, 0) = \Phi(t, t_0)x_0$. Properties:
1. $\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$ with $\Phi(t_0, t_0) = I$.
2. $\forall t_0, t_1 \in \mathbb{R}, \Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$.
3. $\forall t_1, t_0, \Phi(t_1, t_0)$ invtbl and $[\Phi(t_1, t_0)]^{-1} = \Phi(t_0, t_1)$.

Showing 1 & invoke exist& uniq enough to prove a func is STM.

$$\underbrace{s(t, t_0, x_0, u)}_{\textbf{state trans.}} = \underbrace{\Phi(t, t_0)x_0}_{\textbf{0 inp. trans.}} + \underbrace{\int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau}_{\textbf{0 state trans.}}$$

$$\underbrace{\rho(\cdot)}_{\textbf{out. resp.}} = \underbrace{C(t)\Phi(t, t_0)x_0}_{\textbf{0 inp. resp.}} + \underbrace{C(t)\int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)}_{\textbf{zero state response}}$$

To prove $L(t) = R(t)$, suff to show $L(t)$, $R(t)$ satisfy same linear DE for same IC, hence $L(t) = R(t)$ $\forall t$ by uniq of sol's. ■

5: LTI systems: solutions and transfer functions

$$\dot{x}(t) = Ax(t) + Bu(t) \quad y(t) = Cx(t) + Du(t) \tag{2}$$

Mat exp: $e^{At} = I + At + 1/(2!)A^2t^2 + \cdots = \sum_{k=0}^\infty 1/(k!)A^k t^k$

$\forall t, t_0 \in \mathbb{R}_+$, $\Phi(t, t_0) = e^{A(t - t_0)}$ (**for LTI, not LTV**)!

(***) Using above thm. and props of Φ, we have **for LTI only**:
1. $e^{At_1}e^{At_2} = e^{A(t_1 + t_2)}$ and $[e^{At}]^{-1} = e^{-At}$
2. $\Phi(t, t_0) = \Phi(t - t_0, 0)$
3. $s(t, t_0, x_0, u) = e^{A(t - t_0)}x_0 + \int_{t_0}^t e^{A(t - \tau)}Bu(\tau)d\tau$

If $AB = BA \Rightarrow e^{(A+B)t} = e^{At}e^{Bt}$, $Be^{At} = e^{At}B$
LTI: t_0 **doesn't matter, only** $t - t_0 \ldots$ WLOG, set $t_0 = 0$.

$A \in \mathbb{R}^{n \times n}$ **semi-simple** $\Leftrightarrow$ its right eigenvecs. $\{v_i\}_{i=1}^n \subseteq \mathbb{C}^n$ are lin. indep. in lin. space $(\mathbb{C}^n, \mathbb{C}) \Leftrightarrow \exists$ nonsingular matrix $T \in \mathbb{C}^{n \times n}$ and diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ s.t. $A = T^{-1}\Lambda T$.

If A semi-simp. $T^{-1} = [v_1 \, v_2 \ldots v_n]$ with v_i right eigenvec.'s:
$$e^{At} = T^{-1}e^{\Lambda t}T = T^{-1} \begin{bmatrix} e^{\lambda_1 t} \cdots & 0 \\ \vdots \quad \ddots & \vdots \\ 0 & \cdots e^{\lambda_n t} \end{bmatrix} T \tag{3}$$

$$\begin{array}{cccc} & \{e_i\}_{i=1}^n & \xrightarrow{e^{At}} & \{e_i\}_{i=1}^n \text{ basis of rep. of } \mathcal{A} \text{ by } A \\ T^{-1} & \uparrow & & \downarrow \; T \\ & \{v_i\}_{i=1}^n & \xrightarrow{e^{\Lambda t}} & \{v_i\}_{i=1}^n \text{ eigenvecetor basis} \end{array}$$

$A \in \mathbb{R}^{n \times n}$ **simple** $\Leftrightarrow$ its eigvals are distinct, i.e. $\lambda_i \neq \lambda_j \forall i \neq j$.

Simple matrix $\Rightarrow \not\Leftarrow$ semi-simple matrix.

$Av_i = \lambda_i v_i \Leftrightarrow (A - \lambda_i I)v_i = 0 \Leftrightarrow v_i \in \text{Null}[A - \lambda_i I]$.

Algebraic multiplicity of λ: # times λ appears in $\text{Spec}[A]$.
Geometric multiplicity of λ: $\dim \text{Null}[A - \lambda I]$.

When A non semi-simple?$\Rightarrow$ complete basis with Jordan chain:

Jordan chain length $\mu \in \mathbb{N}$ at eigval $\lambda \in \mathbb{C}$: family $\{v^j\}_{j=1}^\mu \subseteq \mathbb{C}^n$ s.t. ① $\{v^j\}_{j=1}^\mu$ lin ind ② $[A - \lambda I]v^1 = 0$; $[A - \lambda I]v^j = v^{j-1}$

$\{v^j\}_{j=1}^\mu$ called **generalized eigenvectors** at λ.

Jordan canonical form.
Let $A \in \mathbb{R}^{n \times n}$ have k lin. indep. eigenvecs $v_1, \ldots, v_k$ w/ max J chains $\{v_i^j\}_{j=1}^{\mu_i}$ at λ_i, $i = 1, \ldots, k$ w/ $\sum \mu_i = n$. Define:
$$T^{-1} = [v_1^1 \ldots v_1^{\mu_1} \ldots v_k^1 \ldots v_k^{\mu_k}] \in \mathbb{C}^{n \times n}$$

T invtbl; $J = TAT^{-1}$ is block-diag **Jordan canon form** of A:
$$J = \begin{bmatrix} J_1 \cdots & 0 \\ \vdots \quad \ddots & \vdots \\ 0 & \cdots J_k \end{bmatrix} \in \mathbb{C}^{n \times n}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \lambda_i & \ddots & \vdots \\ \vdots & \varnothing & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_i \end{bmatrix} \overset{\in \mathbb{C}^{\mu_i \times \mu_i}}{\textbf{Jordan blocks}}$$

Here's what makes computing e^{At} easy now:

$e^{At} = T^{-1}e^{Jt}T$ where
$$e^{Jt} = \begin{bmatrix} e^{J_1 t} \cdots & 0 \\ \vdots \quad \ddots & \vdots \\ 0 & \cdots e^{J_k t} \end{bmatrix}, \; e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{\mu_i - 1}e^{\lambda_i t}}{(\mu_i - 1)!} \\ 0 & e^{\lambda_i t} & \cdots & \frac{t^{\mu_i - 2}e^{\lambda_i t}}{(\mu_i - 2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_i t} \end{bmatrix}$$
NB: if $k = n$ we're back to (3).

Convolution product: $(f * g)(t) = \int_{-\infty}^t f(t - \tau)g(\tau)d\tau$, with $\int_0^t$ only if $f, g \in [0, \infty)$.

Laplace transform. Let $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n \times m}$. The Laplace transform of $f(\cdot)$: $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \in \mathbb{C}^{n \times m}$.
Let $A_1, A_2 \in \mathbb{R}^{p \times n}$ and $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}^{n \times m}$, properties of $\mathcal{L}$:
1. $\mathcal{L}\{A_1 f_1(t) + A_2 f_2(t)\} = A_1 \mathcal{L}\{f_1(t)\} + A_2 \mathcal{L}\{f_2(t)\}$
2. $\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$; $\mathcal{L}\{\ddot{f}(t)\} = s^2 F(s) - sf(0) - \dot{f}(0)$
3. $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$

Let $A \in \mathbb{R}^{n \times n}$, then $\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$. NB: $sI - A$ invertible $\forall s$ *except the eigvals of* A (there, $\text{Det}[sI - A] = 0$ by definition)!

$$(sI - A)^{-1} = \text{Adj}[sI - A]/\text{Det}[sI - A] = M(s)/\chi_A(s)$$
Denom is char poly of A: $\chi_A(s) = s^n + \chi_1 s^{n-1} + \cdots + \chi_n \in \mathbb{R}[s]$

Adjugate $\text{Adj}[A] \in \mathbb{R}^{n \times n}$: $\text{Adj}[A]_{ij} = (-1)^{i+j}\text{Det}[A_{j\bullet, \bullet i}]$.

Cayley-Hamilton. Every square matrix $A \in \mathbb{R}^{n \times n}$ satisfies its charac. poly: $\chi_A(A) = A^n + \chi_1 A^{n-1} + \ldots + \chi_n I = 0 \in \mathbb{R}^{n \times n}$

$A \in \mathbb{R}^{n \times n}, \forall k \in \mathbb{N}, A^k$ can be written as lin comb of $\{I, A, A^2, \ldots, A^{n-1}\}$ (**not a basis** since not nec lin indep!).

$A \in \mathbb{R}^{n \times n}$ **nilpotent** $\Leftrightarrow A^N = 0$ for some $N \in \mathbb{N}$.

Equiv: ① $A \in \mathbb{R}^{n \times n}$ nilpotent ② $A^n = 0$ ③ $\text{Spec}[A] = \{0, \ldots, 0\}$

$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$
This provides easy alt to 3 in (***) to compute sol to ODE:
$$Y(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}BU(s) + DU(s)$$

Transfer function of the system: $G(s) = C(sI - A)^{-1}B + D$
$\Rightarrow G(s) = C(M(s)/\chi_A(s))B + D = [CM(s)B + D\chi_A(s)]/\chi_A(s)$
Poles of system are roots of denom poly of $G(s)$, $\chi_A(s)$. So all poles of sys are eigvals of A, but not all eigvals of A are poles.

6: Stability

Let $\dot{x}(t) = p(x(t), t)$; $\hat{x} \in \mathbb{R}$ **equilibrium** $\Leftrightarrow p(\hat{x}, t) = 0 \forall t \in \mathbb{R}$.

If $\hat{x} \in \mathbb{R}$ equilib, then $s(t, t_0, \hat{x}) = \hat{x} \forall t, t_0 \in \mathbb{R}$ called **equilib sol.**

Stability defs: any norm will do as $\dim(\mathbb{R}^n) < \infty$, all norms equiv

Stable $\Leftrightarrow \forall t_0 \in \mathbb{R}, \forall \varepsilon > 0 \exists \delta > 0$ s.t.
$$\|x_0 - \hat{x}\| < \delta \Rightarrow \|s(t, t_0, x_0) - \hat{x}\| < \varepsilon \quad \forall t \geqslant t_0$$
Unstable $\Leftrightarrow$ not stable.
Uniformly stable $\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0$ s.t. $\forall t_0 \in \mathbb{R}$
$$\|x_0 - \hat{x}\| < \delta \Rightarrow \|s(t, t_0, x_0) - \hat{x}\| < \varepsilon \quad \forall t \geqslant t_0$$
Locally asymptot stable $\Leftrightarrow$ stable and $\forall t_0 \in \mathbb{R} \exists M > 0$ s.t.
$$\|x_0 - \hat{x}\| \leqslant M \Rightarrow \lim_{t \to \infty} \|s(t, t_0, x_0) - \hat{x}\| = 0$$
Globally asymptotically stable $\Leftrightarrow$ stable and $\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$
$$\lim_{t \to \infty} \|s(t, t_0, x_0) - \hat{x}\| = 0$$
Locally exp stable $\Leftrightarrow \forall t_0 \in \mathbb{R} \exists \alpha, m, M > 0$ s.t. $\forall t \geqslant t_0$:
$$\|x_0 - \hat{x}\| \leqslant M \Rightarrow \|s(t, t_0, x_0) - \hat{x}\| \leqslant m\|x_0 - \hat{x}\|e^{-\alpha(t - t_0)}$$
Glob exp stab $\Leftrightarrow \forall t_0 \in \mathbb{R} \exists \alpha, m > 0$ s.t. $\forall x_0 \in \mathbb{R}^n$ and all $t \geqslant t_0$
$$\|s(t, t_0, x_0) - \hat{x}\| \leqslant m\|x_0 - \hat{x}\|e^{-\alpha(t - t_0)}$$

uniformly stable	$\Rightarrow$	stable
local. asymp. stable	$\Rightarrow$	stable
glob. asymp. stable	$\Rightarrow$	local asymp. stable
loc. exp. stable	$\Rightarrow$	loc. asymp. stable
glob. exp. stable	$\Rightarrow$	glob. asymp. stable
glob. exp. stable	$\Rightarrow$	loc. exp. stable

Rel to continuity. Equil $\hat{x}$ stable $\Leftrightarrow \forall t_0 \in \mathbb{R}$ func $s(\cdot, t_0, \odot) : (\mathbb{R}^n, \| \cdot \|) \to (C([t_0, \infty), \mathbb{R}^n), \| \cdot \|_{t_0, \infty})$ cont at $\hat{x}$ where:
$$\|s(\cdot, t_0, x_0)\|_{t_0, \infty} = \sup_{t \geqslant t_0} \|s(t, t_0, x_0)\|$$

Rate of CV. Let $\hat{x}$ equil of $\dot{x} = p(x(t), t)$. Assume $\exists k > 0$ s.t. $\forall x, x' \in \mathbb{R}^n, \|p(x, t) - p(x', t)\| \leqslant k\|x - x'\|$. Then
$$\|x_0 - \hat{x}\|e^{-k(t - t_0)} \leqslant \|s(t, t_0, x_0) - \hat{x}\| \leqslant \|x_0 - \hat{x}\|e^{k(t - t_0)}$$

Take-away: convergence can be at most exponential. If $k = k(t)$, then replace lower bound k above by $\inf(k)$ and upper bound by $\sup(k)$.

LTV systems

$$\dot{x}(t) = A(t)x(t) \qquad (4)$$

$s(t, t_0, x_0) = \Phi(t, t_0)x_0$ the solution linear wrt $x_0 \Rightarrow$ stability properties all depend on $\Phi(t, t_0)$. Equil. of (4) is $\hat{x} = 0$.

(★★) Let $\|\Phi(t, 0)\|$ ind norm of $\Phi(t, 0) \in \mathbb{R}^{n\times n}$ by $\|\cdot\|_2$. Equil $\hat{x} = 0$ of (4) is • stable⇔ $\forall t_0 \in \mathbb{R}\exists K \geqslant 0$ s.t. $\|\Phi(t, 0)\| \leqslant K \forall t \geqslant 0$ • loc asymp stable⇔ $\lim_{t\to\infty} \|\Phi(t, 0)\| = 0$.

$\hat{x} = 0$ of (4) is • Glob asymp stab ⇔ loc asymp stab • Glob exp stab ⇔ loc exp stab. ⇒ glob/loc equiv for LTV (∴LTI too!)

LTI systems

$$\dot{x}(t) = Ax(t) \Rightarrow \text{solution: } s(t, t_0, x_0) = e^{A(t-t_0)}x_0 \qquad (5)$$

Eq $\hat{x} = 0$ of (5) uniformly stable ⇔ stable.
Eq $\hat{x} = 0$ is asymp stab ⇔ exp stab ⇔ $\forall\lambda \in \text{SPEC}[A], \text{Re}[\lambda] < 0$.
Equil $\hat{x} = 0$ of LTI stable ⇔ 1 & 2 hold:
1. $\forall\lambda \in \text{SPEC}[A], \text{Re}[\lambda] \leqslant 0$
2. Algeb and geom mult $\forall\lambda \in \text{SPEC}[A]$ s.t. $\text{Re}[\lambda] = 0$ are equal

LTV syst may still be unstab even if all eigvals of $A(t)$ are $< 0\forall t$. To analyze LTV stability: use (★★).

$\forall\varepsilon > 0\exists m > 0$ s.t. $\forall t \in \mathbb{R}_+$, $\|e^{At}\| \leqslant me^{(\mu+\varepsilon)t}$ where $\|\cdot\|$ induced norm on $\mathbb{R}^{n\times n}$ and $\mu = \max\{\text{Re}[\lambda]|\lambda \in \text{Spec}[A]\}$. ∴ CV rate of resp of stable sys is determ by eigval w/ largest real part.

Systems with inputs/outputs. Consider (1) again.
BIBS: $\|s(\cdot, t_0, x_0, u)\|_{t_0,\infty}$ bounded if $\|u\|_{t_0,\infty}$ bounded.
BIBO: $\|\rho(\cdot, t_0, x_0, u)\|_{t_0,\infty}$ bounded if $\|u\|_{t_0,\infty}$ bounded.
NB: $\|f(\cdot)\|_{t_0,\infty} = \sup_{t\geqslant t_0} \|f(t)\|$

Lyapunov equation.
$P \in \mathbb{R}^{n\times n}$ symm pos def if: • $P = P^T$ • $x^T Px > 0 \ \forall x \neq 0$

Equiv: • Equil sol of $\dot{x}(t) = Ax(t)$ asymp stab • $\forall Q = Q^T > 0$, $\exists! P = P^T > 0$ satisf **Lyapunov eq** $A^T P + PA = -Q$.

Rayleigh quotient: $\lambda_{\min}(P)\cdot\|x\|^2 \leqslant x^T Px \leqslant \lambda_{\max}(P)\cdot\|x\|^2$ with $P = P^T$.

7: Inner product spaces

Let $F = \mathbb{R}$ or $\mathbb{C}$.
If $a(= a_1 + ja_2) \in \mathbb{C}$, $\bar{a} = a_1 - ja_2$ **complex conjugate** and $|a| = \sqrt{a_1^2 + a_2^2}$ **abs. value**. If $a \in \mathbb{R}$, $\bar{a} = a$.

Let (H, F) linear space. $\langle\cdot,\cdot\rangle : H \times H \to F$ **inner product**⇔ $\forall x, y, z \in H, \alpha \in F$:
1. $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$
   (a) Extra: $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
2. $\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$ (Extra: $\langle\alpha x, y\rangle = \bar{\alpha}\langle x, y\rangle$).
3. $\langle x, x\rangle \in \mathbb{R}_+ \forall x \neq 0$
4. $\langle x, y\rangle = \overline{\langle y, x\rangle}$

Then $(H, F, \langle\cdot,\cdot\rangle)$ called **inner product space** (:=IPS).

If $(H, F, \langle\cdot,\cdot\rangle)$ IPS, then $\|\cdot\| = \sqrt{\langle x, x\rangle} : H \to F$ is **norm def. by inner product** on (H, F). If $(H, F, \|\cdot\|)$ complete (Banach), then $(H, F, \langle\cdot,\cdot\rangle)$ **Hilbert space**.

Schwarz ineq. With $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$, we have: $|\langle x, y\rangle| \leqslant \|x\|\cdot\|y\|$

Consider lin. space (F^n, F); define $\langle\cdot,\cdot\rangle : F^n \times F^n \to F$:

$$\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i} y_i = \bar{x}^T \cdot y \quad \forall x, y \in F^n$$

The defined norm is $\|x\| = \sqrt{\sum_{i=1}^n |x_i|^2} = \|x\|_2$.

Space of square-integrable functions.
$L^2([t_0, t_1], F^n)$ **Hilbert space**: IPS of (Lebesgue) **square-integrable functions** $(f(\cdot) : [t_0, t_1] \to F^n$ s.t. $\int_{t_0}^{t_1} \|f(t)\|_2^2 dt < \infty)$ equipped with L^2 **inner product** $\langle f, g\rangle = \int_{t_0}^{t_1} \overline{f(t)}^T g(t)dt$ (induces the $\|f\|_2$ norm!). Function equivalence ⇔ $\int_{t_0}^{t_1} \|f_1(t) - f_2(t)\|_2^2 dt = 0$.
$x, y \in H$ **orthogonal**⇔ $\langle x, y\rangle = 0$.

Pythagoras theorem. Let $(H, F, \langle\cdot,\cdot\rangle)$ be ISP. $x, y \in H$ orthogonal⇒ $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Orthogonal complement of subspace M of IPS $(H, F, \langle\cdot,\cdot\rangle)$:

$$M^\perp = \{y \in H|\langle x, y\rangle = 0 \ \forall x \in M\}$$

$M^\perp$ is a closed subspace of H and $M \cap M^\perp = \{0\}$.

Let M, N subspaces of lin. space (H, F). **sum** of M and N is: $M + N = \{w|\exists u \in M, v \in N \text{ s.t. } w = u + v\}$
If $M \cap N = \{0\}$, then $M + N$ **direct sum** of M and N, $:= M \bigoplus N$.

NB: $M + N$ is a subspace of H.
$V = M \oplus N \Leftrightarrow \forall x \in V\exists! u \in M, v \in N$ s.t. $x = u + v$.

(■) Let M closed subspace of Hilbert space $(H, F, \langle\cdot,\cdot\rangle)$. Then:
1. $H = M \oplus M^\perp := M \overset{\perp}{\oplus} M^\perp$
2. $\forall x \in H\exists! y \in M$ s.t. $x - y \in M^\perp$, y called **orthogonal projection** of x onto M.
3. $\forall x \in H$, orthog. proj. $y \in M$ is the unique elem. of M achieving $\|x - y\| = \inf\{\|x - u\||u \in M\}$.

Adjoint of a linear map.
$(U, F, \langle\cdot,\cdot\rangle_U)$ and $(V, F, \langle\cdot,\cdot\rangle_V)$ Hilbert spaces. **Adjoint** $\mathcal{A}^* : V \to U$ of lin. map $\mathcal{A} : U \to V$ defined by:

$$\langle v, \mathcal{A}(u)\rangle_V = \langle\mathcal{A}^*(v), u\rangle_U \quad \forall u \in U, v \in V$$

Let $\mathcal{A} : U \to V$, $\mathcal{B} : U \to V$ and $\mathcal{C} : W \to U$ with U, V and W Hilbert spaces. Then: • $\mathcal{A}^*$ well defined, linear and continuous • $(\mathcal{A} + \mathcal{B})^* = \mathcal{A}^* + \mathcal{B}^*$ • $(a\mathcal{A})^* = \bar{a}\mathcal{A}^*$ • $(\mathcal{A} \circ \mathcal{C})^* = \mathcal{C}^* \circ \mathcal{A}^*$ • If $\mathcal{A}$ invertible, $(\mathcal{A}^{-1})^* = (\mathcal{A}^*)^{-1}$ • $\|\mathcal{A}^*\| = \|\mathcal{A}\|$ where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle_U}$ • $(\mathcal{A}^*)^* = \mathcal{A}$

Adjoint of linear map def. by mat. $A \in F^{m\times n}$ is $= \bar{A}^T (= [\overline{a_{ji}}]) \in F^{n\times m}$ (called **Hermitian transpose**). If $F = \mathbb{R}$, then $\text{ADJ}[A] = A^T$ simply.

Let $\mathcal{A}(u) = \int_{t_0}^{t_1} G(\tau)u(\tau)d\tau : L^2([t_0, t_1], F^m) \to F^n$. Adjoint $(\langle x, \mathcal{A}(u)\rangle = \langle(\mathcal{A}^*(x))(\cdot), u\rangle$: $(\mathcal{A}^*(x))(\cdot) = \overline{G(\cdot)}^T x$

$(H, F, \langle\cdot,\cdot\rangle)$ Hilbert space, $\mathcal{A} : H \to H$ linear and continuous. $\mathcal{A}$ **self-adjoint**⇔ $\mathcal{A}^* = \mathcal{A}$, ∴ $\forall x, y \in H$, $\langle x, \mathcal{A}(y)\rangle = \langle\mathcal{A}(x), y\rangle$.
In finite dim.: $\bar{A}^T = A$ (**A Hermitian**) if $F = \mathbb{C}$, or $A = A^T$ (**symmetric**) is $F = \mathbb{R}$.

$(H, F, \langle\cdot,\cdot\rangle)$ Hilbert space, $\mathcal{A} : H \to H$ linear, continuous and self-adjoint. Then: 1. All eigenvals of $\mathcal{A}$ real. 2. If λ_i, λ_j eigenvals with eigenvecs $v_i, v_j \in H$ and $\lambda_i \neq \lambda_j$, then $v_i \perp v_j$.

Finite Rank Lemma (FRL). $F = \mathbb{R}$ or $\mathbb{C}$, $(H, F, \langle\cdot,\cdot\rangle)$ and $(F^m, F, \langle\cdot,\cdot\rangle_{F^m})$ Hilbert (w/ latter finite-dim), let $\mathcal{A} : H \to F^m$ and $\mathcal{A}^* : F^m \to H$ its adjoint. Then:
1. $\mathcal{A} \circ \mathcal{A}^* : F^m \to F^m$, $\mathcal{A}^* \circ \mathcal{A} : H \to H$ lin., cont., self-adj.
2. $H = \text{RANGE}(\mathcal{A}^*) \overset{\perp}{\oplus} \text{NULL}(\mathcal{A})$, i.e. $\text{RANGE}(\mathcal{A}^*) \cap \text{NULL}(\mathcal{A}) = \{0\}$, $\text{RANGE}(\mathcal{A}^*) = (\text{NULL}(\mathcal{A}))^\perp$.
3. $F^m = \text{RANGE}(\mathcal{A}) \overset{\perp}{\oplus} \text{NULL}(\mathcal{A}^*)$.
4. *Restriction* of $\mathcal{A}$: $\mathcal{A}|_{\text{RANGE}(\mathcal{A}^*)} : \text{RANGE}(\mathcal{A}^*) \to \text{RANGE}(\mathcal{A})$ is a *bijection* "$\forall y \in \text{RANGE}(\mathcal{A})\exists!\tilde{x} \in \text{RANGE}(\mathcal{A}^*)$ s.t. $\mathcal{A}\tilde{x} = y$"
   $\tilde{x}$ unique in $\text{RANGE}(\mathcal{A}^*)$ but not in H (if $\text{NULL}(\mathcal{A}) \neq \{0\}$).
5. $\mathcal{A}^*|_{\text{RANGE}(\mathcal{A})} : \text{RANGE}(\mathcal{A}) \to \text{RANGE}(\mathcal{A}^*)$ is a *bijection*.
6. $\text{NULL}(\mathcal{A} \circ \mathcal{A}^*) = \text{NULL}(\mathcal{A}^*)$, $\text{RANGE}(\mathcal{A} \circ \mathcal{A}^*) = \text{RANGE}(\mathcal{A})$.
7. $\text{NULL}(\mathcal{A}^* \circ \mathcal{A}) = \text{NULL}(\mathcal{A})$, $\text{RANGE}(\mathcal{A}^* \circ \mathcal{A}) = \text{RANGE}(\mathcal{A}^*)$.

Matrix pseudo-inverse

$A \in \mathbb{R}^{m\times n}$ s.t. $\mathcal{A}(x) = Ax$. Let $b \in \mathbb{R}^m$. Assume $\text{RANGE}(\mathcal{A}) = \mathbb{R}^m$ (∴ A is **fat matrix**⇒ $m \leqslant n$, $\text{NULL}(\mathcal{A}^*) = \{0\}$). Then $\tilde{x} = A^T(AA^T)^{-1}b$ is:
- Unique elem. of $\text{RANGE}(\mathcal{A}^*)$ s.t. $A\tilde{x} = b$
- Orthog. proj. onto $\text{RANGE}(\mathcal{A}^*)$ of any $x \in \mathbb{R}^n$ s.t. $Ax = b$
- Unique minimizer of $\|x\|$ subject to $Ax = b$

$A^\dagger = A^T(AA^T)^{-1}$ **right pseudo-inverse** since $AA^\dagger = I$.

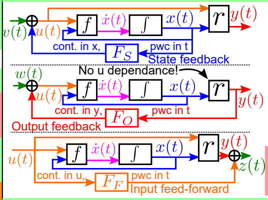
$A \in \mathbb{R}^{m\times n}$ s.t. $\mathcal{A}(x) = Ax$. Let $b \in \mathbb{R}^m$. Assume $\text{NULL}(\mathcal{A}) = \{0\}$ (∴ A is **tall matrix**⇒ $m \geqslant n$, $\text{RANGE}(\mathcal{A}^*) = \mathbb{R}^n$). Then $\tilde{x} = (A^TA)^{-1}A^Tb$ is:
- Unique elem. of $\mathbb{R}^n$ s.t. $A\tilde{x}$ orthog. proj. of b onto $\text{RANGE}(\mathcal{A})$
- Unique minimizer of $\{\|Ax - b\||x \in \mathbb{R}^n\}$

$A^\ddagger = (A^TA)^{-1}A^T$ **left pseudo-inverse** since $A^\ddagger A = I$.

8: Controllability and Observability

$$\text{Consider general non-linear system: } \begin{array}{l}\dot{x}(t) = f(x(t), u(t), t)\\ y(t) = r(x(t), u(t), t)\end{array} \qquad (6)$$

$u(\cdot) \in PC([t_0, t_1], \mathbb{R}^m)$ **steers** (x_0, t_0) to (x_1, t_1) ⇔ $s(t_1, t_0, x_0, u) = x_1$. System **controllable on** $[t_0, t_1]$⇔ $\forall x_0, x_1\exists u(\cdot) \in PC([t_0, t_1], \mathbb{R}^m)$ that steers (x_0, t_0) to (x_1, t_1)⇔ $\forall x_0$, $s(t_1, t_0, x_0, \cdot)$ is surjective.



- Open loop: $\dot{x} = f(x, u, t)$
- State feedback: $\dot{x} = f(x, v + F_S(x, t), t) = f_S(x, v, t)$
- Output feedback: $\dot{x} = f(x, w + F_O(y, t), t) = f_O(x, w, t)$
- Input feedforward: $z(t) = r(x, u, t) + F_F(u, t) = r_F(x, u, t)$

Controllable on $[t_0, t_1]$ open loop⇔state feedb.⇔output feedb.

System (6) **observable on** $[t_0, t_1]$⇔ $\forall x_0 \in \mathbb{R}^n, \forall u(\cdot) \in PC([t_0, t_1], \mathbb{R}^m)$, given $u(t \in [t_0, t_1])$ and corresp. $y(t \in [t_0, t_1])$, the val. of x_0 can be uniquely determined⇔ $\forall u(\cdot) \in PC([t_0, t_1], \mathbb{R}^m)$, $\rho(\cdot, t_0, \odot, u) : x_0 \mapsto \rho(\cdot, t_0, x_0, u) : [t_0, t_1] \to \mathbb{R}^p$ is injective.

NB: once x_0 known, $x(t) = s(t, t_0, x_0, u)$ uniquely established!
Observable on $[t_0, t_1]$ open loop⇔output feedb.⇔input feedf.

LTV Controllability (abbreviation: ctrb.)

$(A(\cdot), B(\cdot))$ **controllable on** $[t_0, t_1]$⇔ $\forall x_0, x_1 \in \mathbb{R}^n\exists u : [t_0, t_1] \to \mathbb{R}^m$ that steers (x_0, t_0) to (x_1, t_1): $x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$.

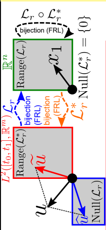
Equivalent: • $(A(\cdot), B(\cdot))$ controllable on $[t_0, t_1]$ • $\forall x_0\exists u$ steering (x_0, t_0) to $(0, t_1)$ (controllability to 0) • $\forall x_1\exists u$ steering $(0, t_0)$ to (x_1, t_1) (reachability from 0)

x_1 **reachable on** $[t_0, t_1]$⇔ $\exists u(\cdot) \in L^2$ steering $(0, t_0) \to (x_1, t_1)$. **Reachability map on** $[t_0, t_1]$ of $(A(\cdot), B(\cdot))$ is $\mathcal{L}_r(u) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau : L^2([t_0, t_1], \mathbb{R}^m) \to \mathbb{R}^n$, linear & continuous! $\text{RANGE}(\mathcal{L}_r)$:= **set of reachable states**! Since $\mathcal{L}_r : L^2([t_0, t_1], \mathbb{R}^m) \to \mathbb{R}^n$ Hilbert spaces, *can apply FRL*!
Controllability Gramian of $(A(\cdot), B(\cdot))$ on $[t_0, t_1]$: $W_r(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B(\tau)^T\Phi(t_1, \tau)^T d\tau \in \mathbb{R}^{n\times n}$, it's the matrix rep. of $\mathcal{L}_r \circ \mathcal{L}_r^* : \mathbb{R}^n \to \mathbb{R}^n$. *Use to analyze LTV ctrblty*!

$W_r(t_0, t_1)$ is symmetric, positive semidefinite and $\forall t_0' \leqslant t_0$, $W_r(t_0', t_1) \geqslant W_r(t_0, t_1)$ ("*less* effort req when more time available"), i.e. $x^T[W_r(t_0', t_1) - W_r(t_0, t_1)]x \geqslant 0\forall x \in \mathbb{R}^n$

$(A(\cdot), B(\cdot))$ controllable on $[t_0, t_1]$ ⇔ $\text{RANGE}(\mathcal{L}_r) = \mathbb{R}^n$ ⇔ $\text{RANGE}(\mathcal{L}_r \circ \mathcal{L}_r^*) = \mathbb{R}^n$ ⇔ $\text{DET}[W_r(t_0, t_1)] \neq 0$ (otherwise $= 0$).

LTV Minimum Energy Control



Idea: $\tilde{u} \in \text{RANGE}(\mathcal{L}_r^*)$ is the unique input 2-norm minimizer (cf. (■)).

$(A(\cdot), B(\cdot))$ controllable on $[t_0, t_1]$. Given $x_0, x_1 \in \mathbb{R}^n$, define:

$$\begin{aligned}\tilde{u}(t) &= \mathcal{L}_r^* \circ (\mathcal{L}_r \circ \mathcal{L}_r^*)^{-1}[x_1 - \Phi(t_1, t_0)x_0]\\ &= B(t)^T\Phi(t_1, t)^TW_r(t_0, t_1)^{-1}[x_1 - \Phi(t_1, t_0)x_0] \quad \forall t \in [t_0, t_1]\end{aligned}$$

1. $\tilde{u}$ steers $(x_0, t_0) \to (x_1, t_1)$
2. $\tilde{u}$ pwc w/ discont. set of $B(\cdot)$. $\tilde{u}$ cont.⇔ $B(\cdot)$ cont.
3. $\|\tilde{u}\|_2^2 = [x_1 - \Phi(t_1, t_0)x_0]^TW_r(t_0, t_1)^{-1}[x_1 - \Phi(t_1, t_0)x_0]$
4. If u steers $(x_0, t_0) \to (x_1, t_1) \Rightarrow \|u\|_2 \geqslant \|\tilde{u}\|_2$

From 3.: $\|\tilde{u}\|_2^2 \propto W_r(t_0, t_1)^{-1} \Rightarrow$ the "smaller" (i.e. more singular) $W_r(t_0, t_1)$, the "more energy" needed to steer to (x_1, t_1).

LTV Observability and Duality (abbreviation: obsvb.)

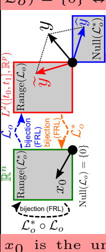
$(C(\cdot), A(\cdot))$ **observable on** $[t_0, t_1]$⇔ $\forall x_0 \in \mathbb{R}^n, \forall u : [t_0, t_1] \to \mathbb{R}^m$ can uniquely determine x_0 from $\{(u(t), y(t))|t \in [t_0, t_1]\}$.

x_0 **unobservable on** $[t_0, t_1]$⇔ $C(t)\Phi(t, t_0)x_0 = 0\forall t \in [t_0, t_1]$ ⇔ $x_0 \in \text{NULL}(\mathcal{L}_o)$ **observability map** $\mathcal{L}_o = C(t)\Phi(t, t_0)x_0 : \mathbb{R}^n \to L^2([t_0, t_1], \mathbb{R}^p)$ with $\text{RANGE}(\mathcal{L}_o) = PC([t_0, t_1], \mathbb{R}^p)$ w/ discont. set of $C(\cdot)$.

Consequence: $x_0 = 0$ is unobservable!

Observability Gramian of $(C(\cdot), A(\cdot))$ on $[t_0, t_1]$: $W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi(\tau, t_0)^TC(\tau)^TC(\tau)\Phi(\tau, t_0)d\tau \in \mathbb{R}^{n\times n}$, it's the matrix rep. of $\mathcal{L}_o^* \circ \mathcal{L}_o : \mathbb{R}^n \to \mathbb{R}^n$.

$(C(\cdot), A(\cdot))$ observable on $[t_0, t_1]$ ⇔ $\text{NULL}(\mathcal{L}_o) = \{0\}$ ⇔ $\text{NULL}(\mathcal{L}_o^* \circ \mathcal{L}_o) = \{0\}$ ⇔ $\text{DET}[W_o(t_0, t_1)] \neq 0$ (otherwise $= 0$).



Let ① $\begin{cases}\dot{x} = Ax + Bu\\ y = Cx + Du\end{cases}$ ② $\begin{cases}\dot{\bar{x}} = -A^T\bar{x} - C^T\bar{u}\\ \bar{y} = B^T\bar{x} + D^T\bar{u}\end{cases}$

① (w/ state. trans. mat. $\Phi(t, t_0)$) and ② (w/ state. trans. mat. $\Psi(t, t_0)$) (t ommitted) have:
- $\Psi(t, t_0) = \Phi(t_0, t)^T$ (solve $\dot{X}(t) = -A(t)^TX(t)$)
- ① ctrb. on $[t_0, t_1]$ ⇔ ② obsvb. on $[t_0, t_1]$.
- ① obsvb. on $[t_0, t_1]$ ⇔ ② ctrb. on $[t_0, t_1]$.

$(C(\cdot), A(\cdot))$ observable on $[t_0, t_1]$. Given $y \in L^2([t_0, t_1], \mathbb{R}^p)$, define:

$$\begin{aligned}x_0 &= (\mathcal{L}_o^* \circ \mathcal{L}_o)^{-1} \circ \mathcal{L}_o^*(y)\\ &= [W_o(t_0, t_1)]^{-1}\int_{t_0}^{t_1} \Phi(\tau, t_0)^TC(\tau)^Ty(\tau)d\tau\end{aligned}$$

x_0 is the unique minimizer of $\|y - \mathcal{L}_o(x)\|_2$ over $x \in \mathbb{R}^n$ w/ $\min_{x\in\mathbb{R}^n} \|y - \mathcal{L}_0(x)\|_2^2 = \|y\|_2^2 - x_0^TW_o(t_0, t_1)x_0$

LTI Observability

Observability matrix: $O = [C; CA; \cdots; CA^{n-1}] \in \mathbb{R}^{pn\times n}$.

• $\text{NULL}(O) = \text{NULL}(\mathcal{L}_o) = \{x_0 \in \mathbb{R}^n|\text{unobservable}\}$
• $\text{NULL}(O)$ is A invariant subspace,∴ $x \in \text{NULL}(O) \Rightarrow Ax \in \text{NULL}(O)$
$\forall[t_0, t_1]$, (C, A) observable on $[t_0, t_1]$ ⇔ $\text{RANK}(O) = n$ ⇔ $\forall\lambda \in \text{SPEC}[A]$, $\text{RANK}([\lambda I - A; C] \in \mathbb{R}^{(n+p)\times n}) = n$ (`MATLAB notation!`).

(C, A) observable on some $[t_0, t_1]$ ⇔ observable $\forall[t_0, t_1]$.

$\forall C \in \mathbb{R}^{p\times r}, A \in \mathbb{R}^{n\times n}\exists T \in \mathbb{R}^{n\times n}$, $\text{DET}[T] \neq 0$ s.t. in this basis the matrices decompose into:

$$\widehat{A} = TAT^{-1} = \begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix} \qquad \widehat{C} = CT^{-1} = [C_1 \quad 0]$$

and the pair of matrices (C_1, A_{11}) is **observable**!

FRL: $\mathbb{R}^n = \text{NULL}(O) \overset{\perp}{\oplus} \text{NULL}(O)^\perp = \text{NULL}(O) \overset{\perp}{\oplus} \text{RANGE}(O^T)$. So:
① New $\mathbb{R}^n$ basis $\{y_i\}_{i=1}^n = \{w_1, \ldots, w_{n-r}, v_1, \ldots, v_r\}$
 • $\{v_i\}_{i=1}^r$ basis of $\text{NULL}(O)$. • $\{w_i\}_{i=1}^{n-r}$ basis of $\text{RANGE}(O^T)$.
② T = transf mat $\{e_i\} \to \{y_i\}$ (where $\{e_i\}$ canonical basis).
 So: $T^{-1} = [w_1, \ldots, w_{n-r}, v_1, \ldots, v_r]$ simply!

NB: $\text{SPEC}[A] \equiv \text{SPEC}[\widehat{A}] = \text{SPEC}[A_{11}] \cup \text{SPEC}[A_{22}]$ where $\text{SPEC}[A_{11}]$ contains eigvals whose eigvecs obsvb. (**obsvb. modes**), $\text{SPEC}[A_{22}]$ eigevals whose eigvecs unobsvb. (**unobsvb. modes**).

Danger of unobservability: an unobservable mode may diverge $\to \infty$ if unstable, yet no indication at output!
If all unobservable modes are stable, system **detectable**.

LTI Controllability

Controllability matrix: $P = [B, AB, \cdots, A^{n-1}B] \in \mathbb{R}^{n\times nm}$.

• $\text{RANGE}(P) = \text{RANGE}(\mathcal{L}_r) = \{x_1 \in \mathbb{R}^n|\text{reachable}\}$
• $\text{RANGE}(P)$ is an A invariant subspace
$\forall[t_0, t_1]$, (A, B) controllable on $[t_0, t_1]$ ⇔ $\text{RANK}(P) = n$ ⇔ $\forall\lambda \in \text{SPEC}[A]$, $\text{RANK}([\lambda I - A, B] \in \mathbb{R}^{n\times(n+m)}) = n$ (`MATLAB notation!`).
$\forall B \in \mathbb{R}^{n\times m}, A \in \mathbb{R}^{n\times n}\exists T \in \mathbb{R}^{n\times n}$, $\text{DET}[T] \neq 0$ s.t. in this basis:

$$\widehat{A} = TAT^{-1} = \begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix} \qquad \widehat{B} = TB = \begin{bmatrix}B_1\\ 0\end{bmatrix}$$

and the pair of matrices (A_{11}, B_1) is **controllable**!

FRL: $\mathbb{R}^n = \text{RANGE}(P) \overset{\perp}{\oplus} \text{RANGE}(P)^\perp = \text{RANGE}(P) \overset{\perp}{\oplus} \text{NULL}(P^T)$. So:
① New $\mathbb{R}^n$ basis $\{y_i\}_{i=1}^n = \{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$
 • $\{v_i\}_{i=1}^r$ basis of $\text{RANGE}(P)$. • $\{w_i\}_{i=1}^{n-r}$ basis of $\text{NULL}(P^T)$.
② T = transf mat $\{e_i\} \to \{y_i\}$ (where $\{e_i\}$ canonical basis).
 So: $T^{-1} = [v_1, \ldots, v_r, w_1, \ldots, w_{n-r}]$ simply!

NB: $\text{SPEC}[A] \equiv \text{SPEC}[\widehat{A}] = \text{SPEC}[A_{11}] \cup \text{SPEC}[A_{22}]$ where $\text{SPEC}[A_{11}]$ contains eigvals whose eigvecs reachable (**ctrb. modes**), $\text{SPEC}[A_{22}]$ eigvals whose eigvecs unreachable (**unctrb. modes**).

If all uncontrollable modes are stable, system **stabilizable**.

9: State Feedback and Observer Design (*LTI only*)

Consider system $\{A, B, C, D\}$ and change of basis $\tilde{x} = Tx\forall t \in \mathbb{R}_+$, $T \in \mathbb{R}^{n\times n}$ invertible. Then:
1. In new basis $\{\widetilde{A} = TAT^{-1}, \widetilde{B} = TB, \widetilde{C} = CT^{-1}, \widetilde{D} = D\}$
2. $\text{SPEC}[A] = \text{SPEC}[\widetilde{A}]$
3. $\widetilde{G}(s) = \widetilde{C}(sI - A)^{-1}\widetilde{B} + \widetilde{D} = C(sI - A)^{-1} + D = G(s)$
4. $(\widetilde{A}, \widetilde{B})$ controllable ⇔ (A, B) controllable.
5. $(\widetilde{C}, \widetilde{A})$ observable ⇔ (C, A) observable.

Linear state feedback for single-input (SI) systems

Let char. poly. $\text{DET}[\lambda I - A] = \lambda^n + \chi_1\lambda^{n-1} + \cdots + \chi_{n-1}\lambda + \chi_n$.
Define matrix S w/ $\{s_n = B, s_{n-k} = As_{n-k+1} + \chi_kB\}$:

$$S = [s_1 \cdots s_n] = \begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix}\begin{bmatrix}\chi_{n-1} & \chi_{n-2} & \cdots & \chi_1 & 1\\ \chi_{n-2} & \chi_{n-3} & \cdots & 1 & 0\\ \vdots & \vdots & ⋰ & \vdots & \vdots\\ \chi_1 & 1 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\end{bmatrix}$$

$As_1 + \chi_nB = 0$. NB: $S \in \mathbb{R}^{n\times n}$ invertible ⇔ (A, B) controllable.

Controllable canonical form. (A, B) controllable ⇔ $\exists T_c \in \mathbb{R}^{n\times n}$ invertible, $\tilde{x}(t) = T_cx(t)$, s.t. $\widetilde{A} = T_cAT_c^{-1}$, $\widetilde{B} = T_cB$ s.t.

$$\widetilde{A} = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -\chi_n & -\chi_{n-1} & -\chi_{n-2} & \cdots & -\chi_1\end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}, \quad T_c^{-1} \equiv S$$

LTI (single-input) state feedback: $u(t) = Kx(t) + r(t)$, $K \in \mathbb{R}^{m\times n}$ **gain matrix**, $r(t) \in \mathbb{R}$ "ext. input vec.". CL dynamics: $\dot{x}(t) = (A + BK)x(t) + Br(t)$.

(A, B) controllable ⇔ $\forall\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{C}\ \exists K \in \mathbb{R}^{m\times n}$ s.t. $\text{SPEC}[A + BK] = \{\lambda_1, \ldots, \lambda_n\}$ (the desired **CL poles**).

Pole placement. Find K that places CL poles at $\{\lambda_1, \ldots, \lambda_n\}$.
① Write $\text{DET}[\lambda I - A] = \lambda^n + \chi_1\lambda^{n-1} + \cdots + \chi_{n-1}\lambda + \chi_n$.
② Write $\text{DET}[\lambda I - (A + BK)] = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n) = \lambda^n + d_1\lambda^{n-1} + \cdots + d_{n-1}\lambda + d_n$.
③ Define $\widetilde{K} = [\tilde{k}_n \cdots \tilde{k}_1] \in \mathbb{R}^{1\times n}$. Compute $\tilde{k}_i = \chi_i - d_i\ \forall i$.
④ Obtain $K = \widetilde{K}T_c = \widetilde{K}S^{-1}$ ■
OR just solve ② with $K = [k_1 \cdots k_n]$ as vars ($n \times n$ lin. sys.).

LTI state observers for single-output (SO) systems

$C \in \mathbb{R}^{1\times n}, D \in \mathbb{R}^{1\times m}, u \in \mathbb{R}^m$, $L \in \mathbb{R}^{n\times 1}$ **observer gain matrix**. Then:

$$\textbf{Observer of } \{A, B, C, D\} : \begin{cases}\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})\\ \hat{y} = C\hat{x} + Du\end{cases}$$

$e := x - \hat{x}$ **estimation error**. Dynamics: $\dot{e}(t) = (A - LC)e(t)$.
Estimation works ⇔ $e(t) \to 0$ ⇔ $\text{Real}[\lambda] < 0\ \forall\lambda \in \text{SPEC}[A - LC]$.

Observable canonical form. (C, A) observable ⇔ $\exists T_o \in \mathbb{R}^{n\times n}$ invertible, $\tilde{x}(t) = T_ox(t)$, s.t. $\widetilde{A} = T_oAT_o^{-1}$, $\widetilde{B} = T_oB$ s.t.

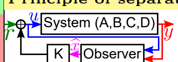
$$\widetilde{A} = \begin{bmatrix}0 & 0 & \cdots & 0 & -\chi_n\\ 1 & 0 & \cdots & 0 & -\chi_{n-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & -\chi_1\end{bmatrix}, \quad \widetilde{C} = [0 \quad \cdots \quad 0 \quad 1]$$

$$\text{where } T_o := \begin{bmatrix}\chi_{n-1} & \chi_{n-2} & \cdots & \chi_1 & 1\\ \chi_{n-2} & \chi_{n-3} & \cdots & 1 & 0\\ \vdots & \vdots & ⋰ & \vdots & \vdots\\ \chi_1 & 1 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix} \equiv S^T \text{(duality)}$$

Observer design. Find L that places error dynamics poles at $\{\lambda_1, \ldots, \lambda_n\}$.
① Write $\text{DET}[\lambda I - A] = \lambda^n + \chi_1\lambda^{n-1} + \cdots + \chi_{n-1}\lambda + \chi_n$.
② Write $\text{DET}[\lambda I - (A - LC)] = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n) = \lambda^n + d_1\lambda^{n-1} + \cdots + d_{n-1}\lambda + d_n$.
③ Define $\tilde{L} = [\tilde{l}_n; \cdots; \tilde{l}_1] \in \mathbb{R}^{n\times 1}$. Compute $\tilde{l}_i \overset{\text{order!}}{=} d_i - \chi_i\forall i$.
④ Obtain $L = T_o^{-1}\tilde{L}$. ■
OR just solve ② with $L = [l_1; \cdots; l_n]$ as vars ($n \times n$ lin. sys.).

(C, A) observable ⇔ $\forall\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{C}\ \exists L \in \mathbb{R}^{m\times n}$ s.t. $\text{SPEC}[A - LC] = \{\lambda_1, \ldots, \lambda_n\}$.

Principle of separation



$$\begin{bmatrix}\dot{x}\\ \dot{\hat{x}}\end{bmatrix} = \begin{bmatrix}A & BK\\ LC & A + BK - LC\end{bmatrix}\begin{bmatrix}x\\ \hat{x}\end{bmatrix} + \begin{bmatrix}B\\ B\end{bmatrix}r$$

Change basis: $[x; e] = [x; x - \hat{x}] = [I, 0; I, -I][x; \hat{x}] = T[x; \hat{x}]$.

$$\Rightarrow \begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}A + BK & -BK\\ 0 & A - LC\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}r = \overline{A}\begin{bmatrix}x\\ e\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}r$$

∴**Princip. of separ.**: $\text{SPEC}[\overline{A}] = \text{SPEC}[A + BK] \cup \text{SPEC}[A - LC]$ ⇒ CL ctrlr. + obsrvr. remains stable if ctrlr. & obsrvr. stable!