Exercise sheet 1

Numerical Analysis 2022

1 PS.1 (conditioning)

Consider subtraction and division as bivariate functions, i.e.,

$$
\begin{aligned}
f &: \mathbb{R}^2 \to \mathbb{R} & \qquad (x_1, x_2) &\mapsto x_1 - x_2 \\
g &: \mathbb{R} \times \mathbb{R}_* \to \mathbb{R} & \qquad (x_1, x_2) &\mapsto \frac{x_1}{x_2}
\end{aligned}
$$

a) Determine the conditioning of f with respect to the sum norm.

b) Determine the partial conditioning of f.

c) Is g partially well-conditioned?

2 PS.1 (evaluating polynomials)

We are given the polynomial

$$f(x, y) = 4x^4 - y^4 + 2y^2$$

a) Evaluate f at $(x, y) = (13860.0, 19601.0)$ in julia.

b) There could be rounding errors. What is the exact value of $f(13860.0, 19601.0)$?

c) Provide a lower bound on $\hat{\kappa}_f(13860, 19601)$.

3 PS.1 (ill-conditioned deblurring)

For $x \in \mathbb{R}^n$, you only observe

$$
y = Ax, \qquad A = \frac{1}{9}\begin{pmatrix}
3 & 2 & 1 & 0 & \cdots & 0 \\
2 & 3 & \ddots & \ddots & & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & 1 \\
\vdots & & \ddots & \ddots & 3 & 2 \\
0 & \cdots & 0 & 1 & 2 & 3
\end{pmatrix}.
$$

a) Write a function constr_A(n) that builds A provided that $n \geq 5$.

b) For fixed n and a suitable signal $x \in \mathbb{R}^n$, compute A and plot $y = Ax$.

c) Add independent, identically distributed Gaussian noise $\varepsilon \in \mathbb{R}^n$ to y, i.e.,

$$\tilde{y} = y + \varepsilon$$

and reconstruct $A^{-1}\tilde{y}$. Plot $A^{-1}y$ and $A^{-1}\tilde{y}$ for different noise intensities.

d) Plot $\|A^{-1}\|_1$ against n for $5 \leq n \leq 100$.

4 PS.1 (32-bit system)

Consider the following 32-bit System:

- sign: 1 bit
- exponent: 8 bits
- significant digits: 23 bits

a) Determine $\mathbb{F}_{32}$, $\mathbb{F}_{32,sub}$, r_{32}, and R_{32}.

b) Provide 3 positive integers that are smaller than R_{32} but not contained in $\mathbb{F}_{32}$.

c) Illustrate and verify your theoretical findings in a) and b) by julia.