Exercise sheet 9

Numerical Analysis 2022

1 PS.9

Given pairwise different $\{x_k\}_{k=0}^n \subset \mathbb{R}$ and values $\{y_k\}_{k=0}^n \subset \mathbb{R}$, identify a polynomial p_n of degree n such that

$$p_n(x_k) = y_k, \qquad k = 0, \dots, n. \tag{1}$$

For each of the following data sets,

$$\begin{aligned}
\{(x_i, y_i)\}_{i=0}^4 &= \{(0,1),(1,1),(2,1),(3,1),(4,1)\},\\
\{(x_i, y_i)\}_{i=0}^4 &= \{(0,0),(1,0),(2,1),(3,0),(4,0)\},\\
\{(x_i, y_i)\}_{i=0}^4 &= \{(0,1),(1,1),(2,2),(3,1),(4,1)\},\\
\{(x_i, y_i)\}_{i=0}^4 &= \{(0,1),(1,2),(2,5),(3,10),(4,17)\},
\end{aligned}$$

determine the corresponding interpolating polynomial $p_4 \in \mathbb{P}_4$.

2 PS.9

Implement the computation of the interpolating polynomial based on the following basis polynomials:

a) monomials,

$$p_n(x) = \sum_{k=0}^n c_k x^k, \qquad \underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}}_{V_n} \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}. \tag{2}$$

b) Lagrange polynomials,

$$p_n := \sum_{k=0}^n y_k L_k, \qquad L_k(x) = \prod_{\substack{i=0\\ i\neq k}}^n \frac{x - x_i}{x_k - x_i}, \quad k = 0, \dots, n. \tag{3}$$

c) Newton polynomials,

$$\begin{aligned}
p_n = \sum_{k=0}^n y[0,\dots,k] N_k, \qquad N_k(x) = \prod_{i=0}^{k-1}(x - x_i), \qquad y[k] = y_k, \quad k = 0, \dots, n\\
y[j,\dots,j+k] = \frac{y[j+1,\dots,j+k] - y[j,\dots,j+k-1]}{x_{j+k} - x_j}, \quad k = 1, \dots, n-j, \qquad .
\end{aligned}$$

Solve now the 4 interpolation problems above with each of your implementations.

3 PS.9

Given nonzero pairwise distinct nodes $x_0, \ldots, x_n \in \mathbb{R}$, compute the n-th divided differences

$$y[0, \ldots, k], \qquad k = 0, \ldots, n,$$

for the function $f(x) = \frac{1}{x}$. (Hence, we derive $y_j = \frac{1}{x_j}$, for $j = 0, \ldots, n$).

4 PS.9

Consider Runge's function

$$f(x) = \frac{1}{1 + 25x^2}.$$

a) Plot f and its interpolating polynomial of minimal degree for the equidistant nodes

$$x_j = -1 + j\frac{2}{n}, \qquad j = 0, \ldots, n.$$

in the interval $[-1, 1]$ for $n = 10$.

b) Produce the analogous plot for Chebychev nodes

$$x_j = \cos\left(\frac{(2j+1)}{2(n+1)}\pi\right), \qquad j = 0, \ldots, n.$$

c) Plot an approximation of Λ_n for equidistant and for Chebyshev nodes with $n = 2, 4, \ldots, 34$. The following provides the approximation:

Let us endow $\mathbb{R}^{n+1}$ with the maximum norm and fix $x_0, \ldots, x_n \in [-1, 1]$. The norm of the interpolation operator

$$Y_n : \mathbb{R}^{n+1} \to \mathbb{P}_n|_{[-1,1]}$$

with $Y_n y = p_n$ and $p_n(x_j) = y_j$, for $j = 0, \ldots, n$, is the Lebesgue constant

$$\Lambda_n = \|Y_n\|_{op} = \sup_{\|y\|_\infty = 1} \|Y_n y\|_{[-1,1]} = \sup_{\|y\|=1} \sup_{s \in [-1,1]} |p_n(s)|. \tag{4}$$

Thus, for sufficiently dense $s_1, \ldots, s_m \in [-1, 1]$, we obtain

$$\Lambda_n \approx \sup_{\|y\|_\infty = 1} \max_{i=1,\ldots,m} |p_n(s_i)|.$$

This is the operator norm of the sampling operator

$$A_n : \mathbb{R}^{n+1} \to \mathbb{R}^m, \qquad y \mapsto \begin{pmatrix} p_n(s_1) \\ \vdots \\ p_n(s_m) \end{pmatrix}, \tag{5}$$

so that $\|A\_n\|_{op} \approx \Lambda\_n$.