Exercise sheet 4

Numerical Analysis 2022

1 PS.4 (qr-decomposition)

For $A \in \mathbb{R}^{m \times n}$ with $rank(A) = n$, the solution of

$$\arg \min_{x \in \mathbb{R}^n} \|Ax - b\|_2.$$

via the qr-decomposition $A = QR$ is given by

$$R = \begin{pmatrix} R_0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = Q^\top b, \qquad R_0 x = c_0.$$

a) Write a function myLLS that takes A as input and returns the solution of the lls problem by calling myQR.

b) Check your implementation by running the testset.

2 PS.4 (qr-method for eigenvalues)

Given $A_1 := A$, compute

$$\begin{aligned} Q_k R_k &= A_k && \text{qr-decomposition of } A_k \\ A_{k+1} &= R_k Q_k && \text{recombine factors in reverse order} \end{aligned}$$

For $k \to \infty$ and A symmetric and regular, we have

$$A_k \to \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad Q_1 Q_2 \cdots Q_k \to V = (v_1, \dots, v_n), \qquad A v_j = \lambda_j v_j.$$

a) Write a julia function that computes eigenvalues and eigenvectors via the qr-algorithm.

We have

$$A v_j = \lambda_j v_j, \quad j = 1, \dots, n \quad \Leftrightarrow \quad AV = (Av_1, \dots, Av_n) = (\lambda_1 v_1, \dots, \lambda_n v_n) = V \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

b) Check your implementation with the following testset and then determine the eigenvalues and eigenvectors of the matrices A and B given below. Can you determine the exact values?

3 PS.4 (Gershgorin circles)

Consider $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and

$$C := \bigcup_{i=1}^{n} C_i, \qquad C_i = \Big\{\xi \in \mathbb{C} : |\xi - a_{ii}| \le \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\Big\}. \tag{1}$$

a) If λ is an eigenvalue of A, then $\lambda \in C$.

b) Illustrate Part a) for a few matrices of your choice by plotting the circles and the eigenvalues appropriately.

4 PS.4 (Convergence order)

For fixed $r, s > 1$, consider the sequence

$$x_k = s^{-r^k}, \qquad k = 0, 1, 2, \dots$$

a) What is the convergence order of $(x_k)_{k\in\mathbb{N}}$?

b) What is the convergence order of $y_k = s^{-k}$, $k = 0, 1, 2, \dots$? Plot $(y_k)_{k\in\mathbb{N}}$, for $s = 3, 4, 5$, in a logarithmic scale.