Exercise sheet 3

Numerical Analysis 2022

1 PS.3 (triangular solver)

$Ly = b$ with L lower triangular with 1 in the diagonal.

$$
\begin{aligned}
y_1 &= b_1, \\
y_2 &= b_2 - \ell_{2,1} y_1, \\
&\vdots \\
y_k &= b_k - \sum_{j=1}^{k-1} \ell_{k,j} y_j, \\
&\vdots \\
y_n &= b_n - \sum_{j=1}^{n-1} \ell_{n,j} y_j.
\end{aligned}
$$

a) Write a solver for $Ly = b$.

$Ux = y$ with U upper triangular and nonzero diagonal.

$$
\begin{aligned}
x_n &= \frac{y_n}{u_{n,n}} \\
x_{n-1} &= \frac{1}{u_{n-1,n-1}} \left(y_{n-1} - u_{n-1,n} x_n \right) \\
&\vdots \\
x_k &= \frac{1}{u_{k,k}} \left(y_k - \sum_{j=k+1}^{n} u_{k,j} x_j \right) \\
&\vdots \\
x_1 &= \frac{1}{u_{1,1}} \left(y_1 - \sum_{j=2}^{n} u_{1,j} x_j \right).
\end{aligned}
$$

b) Write a solver for $Ux = y$.

2 PS.3 (lu-decomposition without pivoting)

For $A \in \mathbb{R}^{n\times n}$, set $A^{(1)} := B^{(1)} := A$ and

$$\ell^{(k)} = \frac{1}{a_{k,k}^{(k)}} \begin{pmatrix} a_{k+1,k}^{(k)} \\ \vdots \\ a_{n,k}^{(k)} \end{pmatrix} \in \mathbb{R}^{n-k}, \qquad B^{(k+1)} = B^{(k)} - \ell^{(k)} \cdot \left(a_{k,k+1}^{(k)}, \cdots, a_{k,n}^{(k)}\right) \in \mathbb{R}^{(n-k)\times(n-k)} \tag{1}$$

$$A^{(k+1)} = L_k A^{(k)} = \begin{pmatrix} a_{1,1}^{(1)} & a_{1,2}^{1)} & \cdots & & \cdots & a_{1,n}^{(1)} \\ 0 & \ddots & & & & \\ \vdots & \ddots & a_{k,k}^{(k)} & & \cdots & a_{k,n}^{(k)} \\ 0 & & 0 & & & \\ \vdots & & \vdots & B^{(k+1)} & & \\ 0 & \cdots & 0 & & & \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ | & \ddots & \ddots & & & \vdots \\ | & & \ddots & \ddots & & \vdots \\ | & & | & \ddots & 0 & \vdots \\ | & & | & & 1 & 0 \\ \ell^{(1)} & \cdots & \ell^{(k)} & \cdots & \ell^{(n-1)} & 1 \end{pmatrix}, \qquad U = A^{(n)} \tag{2}$$

We may store $\ell^{(1)}, \dots, \ell^{(k)}$ in $A^{(k+1)}$ to get rid of L, i.e.,

$$\tilde{A}^{(k+1)} = \begin{pmatrix} a_{1,1}^{(1)} & a_{1,2}^{1)} & \cdots & & \cdots & a_{1,n}^{(1)} \\ | & \ddots & & & & \\ | & \ddots & a_{k,k}^{(k)} & & \cdots & a_{k,n}^{(k)} \\ | & & | & & & \\ | & & | & B^{(k+1)} & & \\ \ell^{(1)} & \cdots & \ell^{(k)} & & & \end{pmatrix} \tag{3}$$

a) Write a function with input A that returns L and U of the lu-decomposition.

b) Solve the system of linear equations $Ax = b$ for

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4}$$

by using myLU and myU, myL for $\varepsilon = 2$ and for $\varepsilon = 2^{-55}$. Are the results correct?

3 PS.3 (lu-decomposition with partial pivoting)

$$PA = LU, \qquad Ax = b \quad \Leftrightarrow \quad LUx = Pb \tag{5}$$

a) Solve the system of linear equations $Ax = b$ for

$$A = \begin{pmatrix} -2 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -8 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} -1 \\ 7 \\ -1 \end{pmatrix} \tag{6}$$

by using the lu-decomposition with partial pivoting by hand.

b) The function myLUpivot is provided below. Write a function myLinearSolver that solves $Ax = b$ based on myLUpivot. Solve part a) with your implementation. Run testset and, if necessary, adjust the error bounds.

4 PS.3 (Cholesky factorization)

For symmetric, positive definite $A = LL^\top$, the matrix $L = (l_{i,j})$ is given by

$$l_{j,j} = \sqrt{a_{j,j} - \sum_{k=1}^{j-1} l_{j,k}^2}, \qquad j = 1, \ldots, n, \tag{7}$$

$$l_{i,j} = \frac{1}{l_{j,j}} \left(a_{i,j} - \sum_{k=1}^{j-1} l_{i,k} l_{j,k} \right), \qquad i = j+1, \ldots, n. \tag{8}$$

a) Write a function with input A that returns L.

b) Complete the testset “myCholesky” and run the tests successfully.

5 PS.3 (lls via normal equations)

Given data $\alpha, b \in \mathbb{R}^m$, consider

$$\arg\min_{x \in \mathbb{R}^2} \sum_{k=1}^{m} |b_k - (x_1 + x_2 \alpha_k)|^2$$

The minimizer (x_1, x_2) defines an affine linear function $f(a) = x_1 + x_2 a$.

a) Write a julia function that takes input α, b and returns f.

b) Generate several data sets with illustrative plots.

6 PS.3 (svd)

For $m \geq n$, let $A \in \mathbb{R}^{m \times n}$ with $rank(A) = p$.

- The matrix $A^\top A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 \geq \ldots \geq \lambda_p > 0$ with pairwise orthonormal eigenvectors $v_1, \ldots, v_p \in \mathbb{R}^n$ that are collected into

$$V := (v_1, \ldots, v_p) \in \mathbb{R}^{n \times p}.$$

- The singular values $\sigma_k := \sqrt{\lambda_k}$, for $k = 1, \ldots, p$, are collected into

$$\Sigma := diag(\sigma_1, \ldots, \sigma_p) \in \mathbb{R}^{p \times p}.$$

- Define $u_k := \frac{1}{\sigma_k} A v_k$, for $k = 1, \ldots, p$, and build

$$U := (u_1, \ldots, u_p) \in \mathbb{R}^{m \times p}.$$

Then we obtain the decomposition

$$A = U \Sigma V^\top.$$

The pseudo inverse of A is $A^\# = V\Sigma^{-1}U^\top$.

a) Write a julia function that takes input A and returns U, $\sigma_1, \ldots, \sigma_p$, and V.

b) Write a julia function that takes input A and returns $A^\#$. Eventually use $A^\#$ to compute f in part a) of PS.3 (lls via normal equations).