Exercise sheet 2

Numerical Analysis 2022

1 PS.2 (inner product is backwards stable)

Consider

$$f, \tilde{f} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = \langle x, y \rangle, \qquad \tilde{f}(x,y) = \bigoplus_{k=1}^{2} (x_k \odot y_k).$$

a) Verify that $\tilde{f}$ is partially backwards stable when considered as the map $\tilde{f} : \mathbb{R}^4 \to \mathbb{R}$, and recall that $x_k \odot y_k$ means $\mathrm{fl}(x_k) \odot \mathrm{fl}(y_k)$.

b) What is the partial conditioning of $f : \mathbb{R}_+^4 \to \mathbb{R}$, where $\mathbb{R}_+ = (0, \infty)$.

2 PS.2 (p-q-formula)

Consider the p-q-formula:

$$f(x) = x^2 - 2px + q, \qquad x_{1/2} = p \pm \sqrt{p^2 - q} \tag{1}$$

a) Explain why myPQnaive does not pass the tests below.

Vieta's formula yields

$$x^2 - 2px + q = (x - x_1)(x - x_2) \qquad \Rightarrow \qquad x_1 \cdot x_2 = q, \quad x_1 + x_2 = 2p$$

b) Write a function myPQ according to Vieta's formula. Check the testset to illustrate improvement upon myPQnaive.

3 PS.2 (two variance formulas)

If bad conditioned computations are not avoidable, then put them at the beginning, NOT at the end.

The sample variance of data $x_1, \dots, x_n \in \mathbb{R}$ is

$$\mathrm{Var}(x_1, \dots, x_n) = \frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x})^2, \qquad \bar{x} = \frac{1}{n} \sum_{k=1}^{n} x_k \tag{2}$$

This formula coincides with

$$
\begin{aligned}
\operatorname{Var}(x_1, \ldots, x_n) &= \frac{1}{n-1} \sum_{k=1}^{n} \left(x_k^2 - 2x_k\bar{x} + \bar{x}^2\right) && (3) \\
&= \frac{1}{n-1} \left(\sum_{k=1}^{n} x_k^2 - 2n\bar{x}^2 + n\bar{x}^2 \right) && (4) \\
&= \frac{1}{n-1} \left(\sum_{k=1}^{n} x_k^2 - n\bar{x}^2 \right) && (5) \\
&= \frac{1}{n-1} \left(\sum_{k=1}^{n} x_k^2 - \frac{1}{n} \left(\sum_{k=1}^{n} x_k \right)^2 \right) && (6)
\end{aligned}
$$

For small variances, the first formula is more stable than the second.

First formula:

$$
\operatorname{Var}(x_1, \ldots, x_n) = \frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x})^2, \qquad \bar{x} = \frac{1}{n} \sum_{k=1}^{n} x_k \tag{7}
$$

a) Implement the first formula for the sample variance.

Second formula:

$$
\operatorname{Var}(x_1, \ldots, x_n) = \frac{1}{n-1} \left(\sum_{k=1}^{n} x_k^2 - \frac{1}{n} \left(\sum_{k=1}^{n} x_k \right)^2 \right) \tag{8}
$$

b) Implement the second formula for the sample variance.

The mean of the data

$$
\begin{aligned}
x_1 &= 10,000,000.0 && (9) \\
x_2 &= 10,000,000.1 && (10) \\
x_3 &= 10,000,000.2 && (11)
\end{aligned}
$$

is $\bar{x} = x_2$ and the variance is

$$
\operatorname{Var}(x_1, x_2, x_3) = 2(0.1)^2/2 = 0.01. \tag{12}
$$

c) Decide which formula provides the more accurate result.

4 PS.2 (cancelations)

Consider the following functions

$$f_1(x) = x^3 - 6ax^2 + 12a^2x - 8a^3 = (x-2a)^3, \qquad a = \frac{1}{2}\cdot 10^7 - 1 = 4999999$$
$$f_2(x) = \frac{1}{x - \sqrt{x^2-1}}$$
$$f_3(x) = 1 - \sqrt{1-x^2}$$
$$f_4(x) = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512 = (x-2)^9$$
$$f_5(x) = \frac{1}{1+2x} - \frac{1-x}{1+x}.$$

a) Determine $\kappa_{f_1}(10^7)$. Which of the two versions in julia provides the more accurate value of $f_1(10^7)$?

b) Evaluate f_2 at $x = 10^7$ in julia. Reformulate f_2 to avoid cancelations and evaluate in julia again.

c) Determine $\lim_{x\to 0} \kappa_{f_3}(x)$. Reformulate f_3 to avoid cancelations when evaluating in julia at $x = 10^{-8}$.

d) Plot both versions of f_4 in julia in the interval $[1.925, 2.075]$.

e) Determine $\lim_{x\to 0} \kappa_{f_5}(0)$. Plot f_5 in julia in the interval $[-5\cdot 10^{-8}, 5\cdot 10^{-8}]$. Reformulate f_5 and obtain a more accurate plot.