Exercise sheet 7

Numerical Analysis 2022

1 PS.7

Consider a sequence $(x_k)_{k\in\mathbb{N}}$ in a Banach space B such that $x_k \to x \in B$. We suppose that there are $c > 0$ and $r > 1$ such that

$$\|x_{k+1} - x\| \leq c\|x_k - x\|^r, \qquad \forall k \in \mathbb{N}.$$

Verify that

$$\|x_k - x\| \leq c^{-\frac{1}{r-1}} c^{\frac{r^k}{r-1}} \|x_0 - x\|^{r^k}, \qquad k = 1, 2, 3, \ldots$$

2 PS.7 (Horner's scheme)

Given $x_0 \in \mathbb{R}$ and a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n,$$

let us define $b_n = a_n$ and

$$b_{k-1} = a_{k-1} + b_k x_0, \qquad k = n, \ldots, 1.$$

Then the choice

$$q(x) = b_1 + b_2 x + \cdots b_n x^{n-1}$$

leads to

$$p(x) = (x - x_0)q(x) + b_0, \qquad p(x_0) = b_0, \qquad p'(x_0) = q(x_0).$$

To evaluate q at x_0, we also use a recursive scheme $c_n = b_n$ and

$$c_{k-1} = b_{k-1} + c_k x_0, \qquad k = n, \ldots, 2,$$

so that $q(x_0) = c_1$.

a) Implement Horner's scheme, so that you return $p(x_0)$, $p'(x_0)$, and the coefficients of q.

b) Write a function myNewton for polynomials that uses Horner's scheme for evaluation of the polynomial and its derivative.

3 PS.7 (Find all zeros of a polynomial)

Given a polynomial p, Newton's method provides an approximation x_0 with $p(x_0) \approx 0$. Horner's scheme yields

$$p(x) = (x - x_0)q(x) + p(x_0) \approx (x - x_0)q(x).$$

Thus, we may use q to compute the remaining zeros of p. We apply Newton's method to q (with initial value x_0) and obtain y with $q(y) \approx 0$. To increase accuracy, y is used as initialization for Newton's method applied to p:

0) Pick some initial value x_{start} and apply Newton's method to p. We obtain the first root x_0 of p, i.e.,

$$Newton(p, x_{start}) \to x_0$$

1) Apply Horner's scheme to remove the linear factor $(x - x_0)$ from $q_0 := p$. Then apply Newton's method to q_1 to obtain y_1. To increase accuracy, y_1 is used as initial value for Newton's method with the original p. This leads to the second root x_1 of p.

$$\begin{aligned} &Horner(p, x_0) \to q_1, \quad \text{such that} \quad p(x) = (x - x_0)q_1(x) \\ &Newton(q_1, x_0) \to y_1 \\ &Newton(p, y_1) \to x_1 \end{aligned}$$

2) Repeat the above step as follows:

$$\begin{aligned} &Horner(q_1, x_1) \to q_2, \quad \text{such that} \quad q_1(x) = (x - x_1)q_2(x) \\ &Newton(q_2, x_1) \to y_2 \\ &Newton(p, y_2) \to x_2 \end{aligned}$$

3) Repeat...

This process is repeated until all roots $x_0, x_1, x_2, \ldots$ are found.

a) Write a function findAllRoots that computes all roots of a polynomial via Newton's method and Horner's scheme. The input is supposed to be the coefficient vector a of the polynomial and some initial value x_0.

b) Apply your implementation to the polynomial $p(x) = (x - 1)(x - 5)^2(x - 7)(x - 9)$.

4 PS.7

Consider the system of nonlinear equations

$$\begin{aligned} ab + a - b - 1 &= 0 \\ ab &= 0, \end{aligned}$$

so that $a, b \in \mathbb{R}$ is to be solved for. Define a suitable function and apply one step of the associated bivariate Newton's method with initial vector $x_0 = (1, 1)^\top$.