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Stewart: # 9.1, 9.5

9.1 Let $K = \mathbb{Q}(\sqrt{-5})$ and let P, Q, R be the ideals defined in Exercise 2 of Chapter 5 (page 124). Let H be the class group. Show that in H we have

$$[P]^2 = [O_K] \qquad [P][Q] = [O_K] \qquad [P][R] = [O_K]$$

and hence show that P, Q, R are equivalent.

Let $\mathcal{P}$ be the set of all fractional principal ideals and $\mathcal{F}$ be the group of fractional ideals with group operation defined as $[P][Q] = [PQ]$. We begin by noting that $O_K = \langle 1 \rangle$ so $O_K \in \mathcal{P}$. Furthermore, any principal ideal is in $\mathcal{P}$ (the constant 1 will trivially translate the ideal into O_K).

Note that $P^2 = \langle 2 \rangle$ so $P^2 \in \mathcal{P}$. Since $O_K \in \mathcal{P}$, O_K and P^2 are equivalent (definition on page 152) and so $[P^2] = [O_K]$. Using the defined group operation, we arrive at our desired result:

$$\begin{aligned}[O_K] &= [P^2] \\ &= [P][P] \\ &= [P]^2\end{aligned}$$

For PQ, we have that $PQ = \langle 1 + \sqrt{-5} \rangle$ so $PQ \in \mathcal{P}$. Thus, $[PQ] = [O_K]$ and so $[O_K] = [PQ] = [P][Q]$.

Finally, for $PR = \langle 1 - \sqrt{-5} \rangle$, we have $PR \in \mathcal{P}$ so $[PR] = [O_K]$. Thus, $[O_K] = [PR] = [P][R]$.

Finally, we show that P, Q, R are equivalent. From above, we have that $[P]^2 = [P][Q] = [P][R]$. Multiplying the whole expression by $[P]$, we have

$$\begin{aligned}&[P]([P]^2 = [P][Q] = [P][R]) \\ &[P][P]^2 = [P]^2[Q] = [P]^2[R] \\ &[P]^2[P] = [P]^2[Q] = [P]^2[R] \\ &[O_K][P] = [O_K][Q] = [O_K][R] \text{ (proven above)} \\ &[O_K P] = [O_K Q] = [O_K R] \text{ (multiplicative property of } \mathcal{H}\text{)}\end{aligned}$$

Now since P, Q, R are ideals in O_K, we know that $O_K P = P, O_K Q = Q$ and $O_K R = R$. Thus, the above equality reduces to

$$[P] = [Q] = [R]$$

which immediately implies that P, Q, R are equivalent. ∎

9.5 Find all squarefree integers d in $-10 < d < 10$ such that the class number of $\mathbb{Q}(\sqrt{d})$ is 1 (Hint: look up a few theorems!)

The squarefree integers between -10 and 10 are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 7$. By Theorem 9.1, $h = 1$ iff factorization in O_K is unique.

First, note that $-5, -6$ have non-unique factorization in O_K by Theorem 4.10. Thus, the class number corresponding to $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-6})$ is not 1.

Now by Theorem 4.17 and 4.19, O_K is Euclidean for $d = -1, \pm 2, \pm 3, 5, 6$ and ± 7. Since Euclidean domains are PID and PID are UFD (this we showed in class), the above values of d have a unique factorization. By Theorem 9.1, their class number is 1.

The remaining value to check is $d = 1$. This corresponds to $\mathbb{Q}$, whose ring of integers is $\mathbb{Z}$ (verified either by common sense or Theorem 3.2b). $\mathbb{Z}$ is clearly a UFD, so $d = 1$ corresponds $h = 1$ by Theorem 9.1. Thus, the squarefree integers d in $-10 < d < 10$ that have a class number of 1 are

$$d = \pm 1, \pm 2, \pm 3, 5, 6, \pm 7$$

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