Dummit & Foote Chapter 7.5: 4

7.5.4 Prove that any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

Let $F \subseteq \mathbb{R}$. Since F is a field, every element has an inverse, so we have $uu^{-1} = 1 \in F$. Taking advantage of the multiplicative structure on F, we see that $\langle 1 \rangle_{\mathbb{Z}} = \mathbb{Z} \subseteq F$ (that is, additive integer combinations of 1 generate $\mathbb{Z}$). Since every element has an inverse in F, we see that $pq^{-1} \in F$ for any $p, q \in \mathbb{Z}$. In other words, for any $\frac{p}{q} \in \mathbb{Q}$, we have that $\frac{p}{q} \in F$. It follows that $\mathbb{Q} \subseteq F$, as desired. $\blacksquare$