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Dummit & Foote 1.4.10.

1.4.10 Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R},\ a \neq 0,\ c \neq 0 \right\}$.

(a) Compute the product of $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ to show that G is closed under matrix multiplication.

(b) Find the matrix inverse of $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and deduce that G is closed under inverses.

(c) Deduce that G is a subgroup of $GL_2(\mathbb{R})$ (cf. Exercise 26, Section 1).

(d) Prove that the set of elements of G whose two diagonal entries are equal (i.e., $a = c$) is also a subgroup of $GL_2(\mathbb{R})$.

(a) The resulting matrix is $\begin{pmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix}$ and since the entries are all in $\mathbb{R}$, G is closed under multiplication.

(b) Using definition of inverse, we have $\begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix}$. All entries are in $\mathbb{R}$ and the form is preserved. In other words, G is closed under inverses.

(c) The only condition that needs to be satisfied is that for any matrix $M \in G$, $\det(M) \neq 0$. The determinant is explicitly ac, which is nonzero by the constraints on a, c.

(d) Such a group is a subgroup of G (because under multiplication, we have a^2, c^2 in top left, bottom right respectively). It follows that it's a subgroup of $GL_2(\mathbb{R})$ by part c.

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