Reflexive Polytopes and Differential Equations

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Introduction

In the past twenty years, much research in mathematical string theory has focused on mirror symmetry of Calabi-Yau varieties, geometric objects believed to encapsulate several extra dimensions of our Universe. These developments have caught the attention of mathematicians; a nice construction allows us to look at these manifolds through the geometric framework of lattice polytopes. We can also study Calabi-Yau varieties from an algebraic perspective by looking at hypersurfaces in toric varieties.

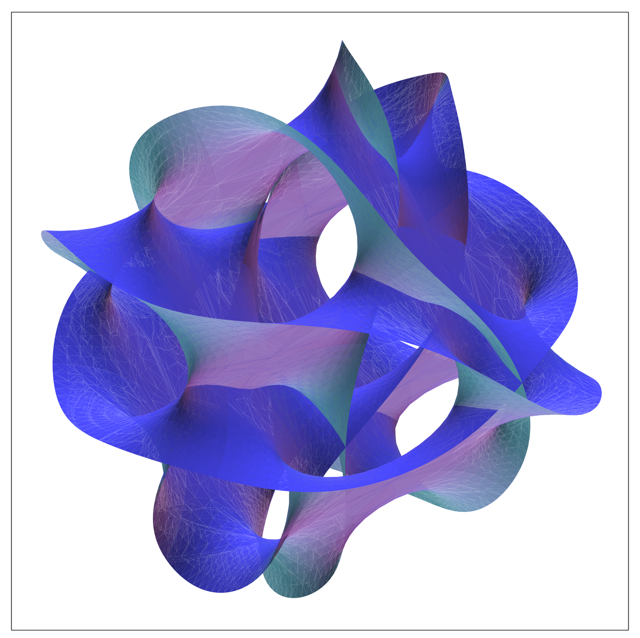

Figure 1: A Calabi-Yau manifold

Our research focused on 2-dimensional Calabi-Yau varieties, which are called K3 surfaces. We studied a family of K3 surfaces built from a polytope called the "skew octahedron," which is the last of the three-dimensional reflexive polytopes with symmetry group $\mathcal{S}_4$ to be studied. From this family of hypersurfaces, we constructed the Picard-Fuchs differential equation, which characterizes the family and the mirror symmetry associated with the family.

Reflexive Polytopes

A **polytope** is the convex hull of a finite set of points; an example is an octahedron. For any polytope $\Delta \subset \mathbb{R}^n$, we can construct its **dual polytope**

$$\Delta^\circ = \{v \in \mathbb{R}^n : \langle v, p \rangle \geq -1 \text{ for all } p \in \Delta\}$$

If every vertex of Δ is a lattice point, we say that Δ is a **lattice polytope**. If the dual of a lattice polytope Δ is also a lattice polytope, we say that Δ is **reflexive**.

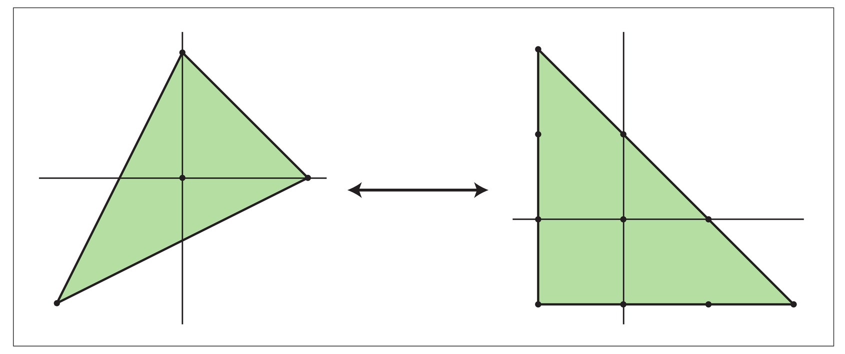

Figure 2: A reflexive polytope and its dual

Toric Varieties

We say that V is a **toric variety** if

- $(\mathbb{C}^*)^q$ is a dense, open subset of V, and
- the action of $(\mathbb{C}^*)^q$ on itself extends easily to an action on V.

Example: The complex projective line, $\mathbb{P}$, is a toric variety. By definition, $\mathbb{P} = (\mathbb{C}^*)^2/\sim$, where $\sim$ is the equivalence relation given by $(z_0, z_1) \sim (\lambda z_0, \lambda z_1)$ for some $\lambda \in \mathbb{C}^*$. Clearly $\mathbb{C}^* \subset \mathbb{P}$ and is dense and open; the action of $\mathbb{C}^*$ on itself is component-wise multiplication, which does extend nicely to an action on $\mathbb{P}$.

Let V be an irreducible n-dimensional variety and $W \subset V$. If W is a variety defined by a single polynomial, then we say that W is a **hypersurface** in V.

There exist straightforward algorithms to compute the toric variety of a polytope and generate families of hypersurfaces from polytopes [?]. Let Δ be a reflexive polytope, with dual polytope Δ°. Associate a variable z_i with each vertex of Δ. Define a hypersurface by

$$Q = \sum_{p_j \in \Delta^\circ} \alpha_j \prod_{v_i \in \Delta} z_i^{\langle v_i, p_j \rangle + 1}$$

for some coefficients $\alpha_j \in \mathbb{C}$. This is called the **anti-canonical hypersurface** of Δ; for a 3-dimensional polytope, the anti-canonical hypersurface is a K3 surface.

The Skew Octahedron

Our main result concerned the skew-octahedron, pictured in Figure **??**; we showed that the toric variety of this polytope is $V_\Sigma = \mathbb{P} \times \mathbb{P} \times \mathbb{P}/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

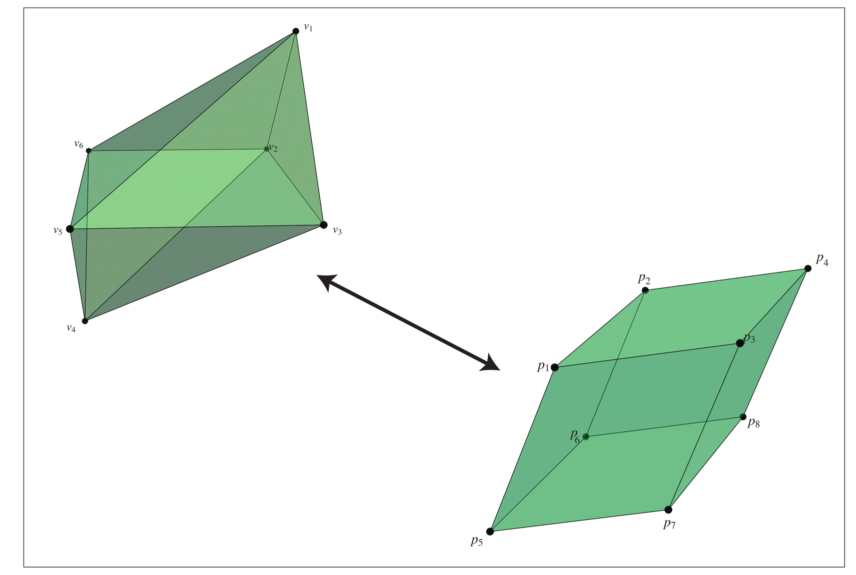

Figure 3: The skew octahedron and its dual

We looked at the family of hypersurfaces given by

$$Q = z_0^2 z_1^2 z_2^2 + z_0^2 z_1^2 z_3^2 + z_0^2 z_2^2 z_4^2 + z_0^2 z_3^2 z_4^2 + z_1^2 z_3^2 z_5^2 + z_2^2 z_4^2 z_5^2 + z_3^2 z_4^2 z_5^2 + z_1^2 z_2^2 z_5^2 + t z_0 z_1 z_2 z_3 z_4 z_5$$

We computed the Picard-Fuchs differential equation for this family of hypersurfaces using the **Griffiths-Dwork** method. Let Ω_0 be a holomorphic form on V_Σ and let $\omega = \int \frac{\Omega_0}{Q}$. The main idea is that successive derivatives of ω can be represented in terms of ω. Taking derivatives,

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial t} \int \frac{\Omega_0}{Q} = \int \frac{\partial}{\partial t} \frac{\Omega_0}{Q} = \int -(z_0 z_1 z_2 z_3 z_4 z_5) \frac{\Omega_0}{Q^2}$$

$$\frac{\partial^2 \omega}{\partial t^2} = \int 2(z_0 z_1 z_2 z_3 z_4 z_5)^2 \frac{\Omega_0}{Q^3}$$

$$\frac{\partial^3 \omega}{\partial t^3} = \int -6(z_0 z_1 z_2 z_3 z_4 z_5)^3 \frac{\Omega_0}{Q^4}$$

Now, define the Jacobian ideal $J = \left\langle \frac{\partial Q}{\partial z_i} \right\rangle$. It can be shown (as in [?]) that if $K = \sum_i A_i \frac{\partial Q}{\partial z_i}$ is in J, then

$$\frac{\Omega_0}{Q^{k+1}} \sum_i A_i \frac{\partial Q}{\partial z_i} = \frac{1}{k} \frac{\Omega_0}{Q^k} \sum_i \frac{\partial A_i}{\partial z_i} + \text{exact terms}$$

Doing this for the derivatives above gives a lower-order system of differential equations. We can eliminate terms to find a linear relationship between the derivatives, which is the Picard-Fuchs equation:

$$0 = \frac{\partial^3 \omega}{\partial t^3} + \frac{6(t^2 - 32)}{t(t^2 - 64)} \frac{\partial^2 \omega}{\partial t^2} + \frac{7t^2 - 64}{t^2(t^2 - 64)} \frac{\partial \omega}{\partial t} + \frac{1}{t(t^2 - 64)} \omega$$

There are several ways to verify that this is the correct form. It is straightforward to check that this differential equation is Fuchsian; furthermore, the Picard group for this polytope has rank 19, and from Hodge theory the order of the Picard-Fuchs equation should be $22 - 19 = 3$. We also showed that the differential equation has a symmetric square root, which is

$$(t^3 - 64t) \frac{\partial^2 \omega}{\partial t^2} + (2t^2 - 64) \frac{\partial \omega}{\partial t} + \frac{t}{4} \omega = 0$$

References

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Acknowledgments

We want to thank our advisor, Ursula Whitcher, for guiding us through this project. Additional thanks go to Andrey Novoseltsev, who implemented the reflexive polytope package in Sage, and to Dagan Karp, who taught us algebraic geometry in the beginning of summer. This research was performed in part using Sage and hex. Hex is a parallel-computing system operated by Harvey Mudd College's Department of Mathematics and the Quantitative Life Sciences Center; hex was made available through a grant from the W.M. Keck Foundation. We would also like to thank the University of Sydney for creating the Magma software package. We gratefully acknowledge support by the National Science Foundation under Grant DMS-083996.

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- The Sage, Mathematica, and Magma code we wrote is available at `http://www.cs.hmc.edu/~djmoore/research/`.

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