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Rudin # 1,2,3,4,5,6,8,9,12,13,15,20

1.1 If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Suppose, by way of contradiction, that $r + x$ is rational. Then $r + x$ can be written as a fraction, say $\frac{m}{n}$. Since $r \in \mathbb{R}$, we are guaranteed to have the additive inverse, so

$$\begin{aligned} r + x &= \frac{m}{n} \\ r + x + (-r) &= \frac{m}{n} + (-r) \\ 0 + x &= \frac{m}{n} - \frac{r}{1} \\ x &= \frac{m - rn}{n} \end{aligned}$$

Since $m - rn, n \in \mathbb{Z}$, this implies that x is rational which is a contradiction. It must be that $r + x$ is irrational.

Suppose, by way of contradiction, that rx is rational. Then we can write rx as a fraction, say $\frac{m}{n}$. Since $r \in \mathbb{R}$, we are guaranteed to have a multiplicative inverse (property $M5$):

$$\begin{aligned} \frac{1}{r}(rx) &= \frac{1}{r}\frac{m}{n} \\ 1x &= \frac{m}{rn} \end{aligned}$$

Since $m, rn \in \mathbb{Z}$, this means x is rational, which is a contradiction. It must be that rx is irrational. $\blacksquare$

1.2 Prove that there is no rational number whose square is 12.

Suppose, by way of contradiction, that $(\frac{m}{n})^2 = 12$ for $m, n \in \mathbb{Z}$ and m and n share no common factor.

$$\begin{aligned} \left(\frac{m}{n}\right)^2 &= \frac{m^2}{n^2} \\ &= 12 \\ m^2 &= 12n^2 \end{aligned}$$

So m^2 is divisible by 3 and since 3 is prime, m is divisible by 3. This allows us to write

$$\begin{aligned}\left(\frac{m}{n}\right)^2 &= \left(\frac{3k}{n}\right)^2 \text{ (for some } k \in \mathbb{Z}) \\ &= \frac{9k^2}{n^2}\end{aligned}$$

So $3k^2 = 4n^2$. This means that n^2 must be divisible by 3 (since 4 is certainly not). Once again, this implies that n is divisible by 3. But m and n share no common factor, so we reach a contradiction. Thus, there is no rational number whose square is 12. ■

1.3 Prove Proposition 1.15(a): If $x \neq 0$ and $xy = xz$ then $y = z$.

We know that $x, y, z \in F$. Since $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that $x(1/x) = 1$ (Property M5). Then,

$$\begin{aligned} xy &= xz \\ (1/x)xy &= (1/x)xz \\ 1y &= 1z \end{aligned}$$

Hence, $y = z$. ■

1.4 Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

By definition of of lower bound, for every element x in E, $\alpha \leq x$. By definition of upper bound, for every element x in E, $x \leq \beta$. In the case that $\alpha < x$ and $x < \beta$, we use Definition 1.5(ii) to conclude that $\alpha < \beta$ (note that we can use 1.5(ii) because E is a subset of an *ordered* set). For the special case $\alpha = x$ and $x < \beta$, we have $\alpha < \beta$ (substitution). For the special case $\alpha < x$ and $x = \beta$, we have $\alpha < \beta$. Finally, for the special case $\alpha = x = \beta$, we have $\alpha = \beta$ (since $=$ is an equivalence relation). Thus, $\alpha \leq \beta$. ■

1.5 Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Note that A is a subset of $\mathbb{R}$, A is non-empty, and A is bounded below so $inf(A)$ exists in $\mathbb{R}$ (by definition 1.10). We want to show that

1. $inf(A) \leq -sup(-A)$
2. $inf(A) \geq -sup(-A)$

By definition, for all $x \in A$, $x \geq inf(A)$. Then,

$$\begin{aligned}
x &\geq inf(A) \\
x + -x &\geq inf(A) + -x \text{ (additive inverse exists since } A \subseteq \mathbb{R}) \\
0 &\geq inf(A) - x \\
-inf(A) + 0 &\geq -inf(A) + inf A - x \\
-inf(A) &\geq -x
\end{aligned}$$

Note that $-x \in -A$. So for all $-x \in -A$, we have $-x \leq -inf(A)$. That is, $-inf(A)$ is *an* upper bound for $-A$. Since $sup(-A)$ is the *least* upper bound of $-A$, we have $sup(-A) \leq -inf(A)$ or, equivalently, $inf(A) \leq -sup(-A)$ (otherwise, supremum wouldn't be the smallest upper bound).

Conversely, $\forall -x \in -A$, we have $-x \leq sup(-A)$ or $x \geq -sup(-A)$ for all $x \in A$. That is, $-sup(-A)$ is *a* lower bound for A. Since $inf(A)$ is the *greatest* lower bound, then $inf(A) \geq -sup(-A)$. Thus, we've shown that $inf(A) = -sup(-A)$. ■

1.6 Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

(a) First, by equivalence relation we have $mq = pn$. Now let $a = (b^m)^{1/n}$. By Theorem 1.21, $a^n = b^m$ and

$$\begin{aligned}
(a^n)^q &= (b^m)^q \text{ (repeated multiplication)} \\
a^{nq} &= b^{mq} \\
&= b^{pn} \\
(a^q)^n &= (b^p)^n
\end{aligned}$$

Now by Theorem 1.21, a^q must equal b^p (uniqueness of the argument).But then by Theorem 1.21, $a = (b^p)^{1/q}$. This implies that

$$(b^m)^{1/n} = (b^p)^{1/q},$$

as desired.

(b) Let $r = \frac{p}{q}$ and $s = \frac{j}{k}$. Then,

$$\begin{aligned}
b^{r+s} &= b^{p/q+j/k} \\
&= b^{\frac{pk+jq}{qk}} \text{ (addition of rational numbers)} \\
&= (b^{pk+jq})^{1/qk} \text{ (part a)} \\
&= (b^{pk}b^{jq})^{1/qk} \\
&= b^{\frac{pk}{qk}}b^{\frac{jq}{qk}} \text{ (Corrolary, page 11)} \\
&= b^{\frac{p}{q}}b^{\frac{j}{k}} \\
&= b^r b^s
\end{aligned}$$

(c) Consider an arbitrary element $b^r \in B(x)$ for r rational such that $r \leq x$. We show that $b^r = supB(r)$ by showing the necessary properties of a supremum:

1. b^r is an upper bound of $B(r)$.
2. If $\gamma < b^r$ then γ is not an upper bound.

By definition, $B(r)$ is the set of all elements such that $t \leq r$. Let $t = \frac{m}{n}$ and $r = \frac{p}{q}$. By equivalence relation, $mq \leq pn$, and so $b^{mq} \leq b^{pn}$ (this is easy to verify directly). Now note that for any $x, y \in \mathbb{R}$, if $x \leq y$ then $x^{1/n} \leq y^{1/n}$ for integer $n > 0$ because if we let $x = a^n, y = b^n$ then we have $a^n \leq b^n$. Using induction, one can show that if $a > b$ then $a^n > b^n$. The contrapositive of that statement shows that $a \leq b$, which by Theorem 1.21 translates to $x^{1/n} \leq y^{1/n}$. Using this fact, we have

$$\begin{aligned}
b^{mq} &\leq b^{pn} \\
(b^{mq})^{\frac{1}{qn}} &\leq (b^{pn})^{\frac{1}{qn}} \\
(b^{mq/qn}) &\leq (b^{pn/qn}) \text{ (part a)} \\
b^{m/n} &\leq b^{p/q} \\
b^t &\leq b^r
\end{aligned}$$

Thus, b^r is an upper bound. Now note that $b^r \in B(r)$ so that $\gamma < b^r$ can't be an upper bound of $B(r)$. Thus, $b^r = sup(B(r))$. Note: It is not obvious why proving this for $r \in \mathbb{Q}$ implies it is true for all reals. At this point, I do not have an answer.

(d) We will show that $sup(B(x+y)) = sup(B(x))sup(B(y))$. It is easy to show that $sup(B(x)B(y)) = sup(B(x))sup(B(y))$ (use $\leq, \geq$). Taking this to be true, we want to show that $sup(B(x+y)) \leq sup(B(x)B(y))$ and $sup(B(x+y)) \geq sup(B(x)B(y))$. First,

$$B(x)B(y) = \{b^s b^t | s, t \in \mathbb{Q}, s \leq x, t \leq y\}$$

By part (b) of this problem, $b^s b^t = b^{s+t}$ so that

$$B(x)B(y) = \{b^{s+t} | s, t \in \mathbb{Q}, s + t \leq x + y\} \subseteq B(x+y)$$

This implies that $sup(B(x+y)) \geq sup(B(x)B(y))$.

Conversely, $B(x+y) = \{b^r | r \in \mathbb{Q}, r \leq x+y\}$. Now choose $s \in \mathbb{Q}$ such that $s \leq x$. From this, we also obtain another rational: $t = r - s$. Note that $t = r - s \leq x + y - x = y$. Using part (b) again, we have

$$B(x+y) = \{b^{s+t} | s + t \in \mathbb{Q}, s \leq x, t \leq y\} = \{b^s b^t | s, t \in \mathbb{Q}, s \leq x, t \leq y\} \subseteq B(x)B(y)$$

Thus, $sup(B(x+y)) \leq sup(B(x)B(y))$. It follows that $b^{x+y} = sup(B(x+y)) = sup(B(x)B(y)) = b^x b^y$ for all real x and y (see statement of part (c)).

■

1.7 Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b*.)

(a) For any positive integer $n, b^n - 1 \geq n(b-1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$ then $b^{1/n} < t$.

(d) If w is such that $b^w < y$ then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = sup(A)$ satisfies $b^x = y$.

(g) Prove that this x is unique.

■

1.8 Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Suppose that $\mathbb{C}$ is an ordered field. By Proposition 1,18(d), if $x \neq 0$ then $x^2 > 0$. Clearly $i \neq 0$ so $i^2 > 0$. From this, we have:

$$\begin{aligned} i^2 &> 0 \\ -1 &> 0 \\ -1+1 &> 0+1 \\ 0 &> 1 \\ 0 &> 1^2 \text{ (Since 1 is a square of itself)} \end{aligned}$$

This implies that there is a square in $\mathbb{C}$ that is less than 0, a contradiction. Thus, no order can be defined in the complex field that turns it into an ordered field. ∎

1.9 Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Suppose $z, w, x \in \mathbb{C}$ such that $z = a + bi, w = c + di$ and $x = k + mi$ where $a, b, c, d, k, m \in \mathbb{R}$. To show that we have created an ordered set, we need to show that,

1. One and only one of the statements $z < w, z = w, w < z$ is true.
2. If $z < w$ and $w < x$ then $z < x$.

Consider the following cases for any two complex numbers $z = a + bi$ and $w = c + di$:

1. If $a < c$ then $z < w$ (by definition).
2. If $a > c$ then $z > w$
3. If $a = c$ and $b < d$, then $z < w$
4. If $a = c$ and $b > d$, then $z > w$
5. If $a = c$ and $b = d$, then $z = a + bi = c + di = w$.

Since $\mathbb{R}$ is an ordered set, these are the only possible cases and for each case one, and only one, condition held. Hence, one and only one of the statements $z < w, z = w, w < z$ is true.

We now show Part 2. Note that $z < w$ implies that $a < c$ or $a = c$ but $b < d$. Similarly, $w < x$ implies $c < k$ or $c = k$ but $d < m$:

1. If $z < w, w < x$ implies $a < c$ and $c < k$ then, since $\mathbb{R}$ is an ordered field, $a < k$ and so $z < x$.
2. If $z < w, w < x$ implies $a < c$ and $c = k$ but $d < m$ then $a < k$ and so $z < x$.

3. If $z < w, w < x$ implies $a = c$ but $b < d$ and $c < k$ then $a < k$ and so $z < x$.
4. If $z < w, w < x$ implies $a = c$ but $b < d$ and $c = k$ but $d < m$ then $a = k$ but $b < m$. Thus, $z < x$.

Since $\mathbb{R}$ is an ordered set, these are the only possible cases and they all imply that $z < x$. Therefore, if $z < w$ and $w < x$ then $z < x$. We have proven the necessary conditions for an ordered set.

On the other hand, this set does not have the l.u.b. property. Consider the subset $S = \{z \in \mathbb{C} | z = 1 + ki, k \in \mathbb{R}\} \subseteq \mathbb{C}$. Then, by above, S is an ordered set. Let $\alpha = a + bi$ be the l.u.b. Then for any $z \in S, z \leq \alpha$ and if β is any upper bound of S, then $\alpha \leq \beta$. However, let $\gamma = a + (b-1)i$ and note that $\gamma < \alpha$ because $b - 1 < b$ (if $b < 0$ then let $\gamma = a + (b+1)i$). Thus, α is not l.u.b., contradicting our initial assumption. ■

1.10 Suppose $z = a + bi, w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

First, note that $u^2 + v^2 = |w|^2$ so that $v = \pm\sqrt{|w|^2 - u^2}$. If $v \geq 0$ then $v = \sqrt{|w|^2 - u^2}$. We use this to get:

$$\begin{aligned}
w &= u + iv \\
&= (a^2 - b^2) + 2i\sqrt{\frac{|w|^2 - u^2}{4}} \\
&= a^2 - b^2 + 2i\sqrt{\frac{|w| + u}{2}}\sqrt{\frac{|w| - u}{2}} \\
&= a^2 + 2abi - b^2 \\
&= (a + bi)^2 \\
&= z^2
\end{aligned}$$

On the other hand, if $v \leq 0$ then $v = -sqrt|w|^2 - u^2$ and so

$$\begin{aligned}
w &= u + iv \\
&= (a^2 - b^2) - 2i\sqrt{\frac{|w|^2 - u^2}{4}} \\
&= a^2 - b^2 - 2i\sqrt{\frac{|w| + u}{2}}\sqrt{\frac{|w| - u}{2}} \\
&= a^2 - 2abi - b^2 \\
&= (a - bi)^2 \\
&= (\overline{z})^2
\end{aligned}$$

Thus, with the exception of 0, every complex number w has two square roots. ∎

1.11 If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Let $z = a + bi$. Choose r such that $a^2 + b^2 = r^2$ and let $w = \frac{a}{r} + \frac{b}{r}i$. Then $|w| = \frac{a^2}{r^2} + \frac{b^2}{r^2} = 1$ and $z = rw$. This also shows that w and r are not uniquely determined. ∎

1.12 If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|.$$

This is a more general triangle inequality. Since $|z|^2 = z\overline{z}$, we have that

$$\begin{aligned} |z_1 + z_2 + \cdots + z_n|^2 &= (z_1 + z_2 + \cdots + z_n)\overline{(z_1 + \cdots + z_n)} \\ &= (z_1 + z_2 + \cdots + z_n)(\overline{z_1} + \cdots + \overline{z_n}) \text{ (Property of conjugates)} \\ &= z_1\overline{z_1} + z_1\overline{z_2} + \cdots + z_2\overline{z_2} \cdots + z_n\overline{z_n} \\ &= |z_1|^2 + (z_1\overline{z_2} + \overline{z_1}z_2) + \cdots + (z_i\overline{z_j} + \overline{z_i}z_j) + \cdots + |z_n|^2 \end{aligned}$$

Now note that for any complex number, $z + \overline{z} = 2Re(z)$. Since $\overline{z_i}z_j = \overline{z_i\overline{z_j}}$, we get:

$$\begin{aligned} &= |z_1|^2 + 2Re(z_1\overline{z_2}) + 2Re(z_1\overline{z_3}) + \cdots + 2Re(z_i\overline{z_j}) + \cdots + |z_n|^2 \\ &\leq |z_1|^2 + 2|z_1\overline{z_2}| + \cdots + 2|z_i\overline{z_j}| + \cdots + |z_n|^2 \text{ (Theorem 1.33d)} \\ &= |z_1|^2 + 2|z_1||z_2| + \cdots + 2|z_i||z_j| + \cdots + |z_n|^2 \text{ (prop. of } |\cdot| \text{ and } |z_i| = |\overline{z_i}|) \\ &= (|z_1| + |z_2| + \cdots + |z_n|)^2 \end{aligned}$$

Taking the square root of both sides yields

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

∎

1.13 If x, y are complex, prove that

$$\big||x| - |y|\big| \leq |x - y|.$$

We use a similar approach as the one used on Page 15 to show this:

$$\begin{aligned} |x - y|^2 &= (x - y)\overline{(x - y)} \\ &= (x - y)(\overline{x} - \overline{y}) \\ &= x\overline{x} - x\overline{y} - y\overline{x} + y\overline{y} \\ &= |x|^2 - 2Re(x\overline{y}) + |y|^2 \text{ (Since } z + \overline{z} = 2Re(z)) \end{aligned}$$

Now by Theorem 1.33(d), $|Re(x)| \leq |z|$ which implies $-|Re(x)| \geq -|z|$ and so,

$$\begin{aligned}
|x|^2 - 2Re(x\overline{y}) + |y|^2 &\geq |x|^2 - 2|x\overline{y}| + |y|^2 \\
&= |x|^2 - 2|x||\overline{y}| + |y|^2 \text{ (By Theorem 1.33c)} \\
&= |x|^2 - 2|x||y| + |y|^2 \text{ (By Theorem 1.33b)} \\
&= (|x| - |y|)^2 \\
&= (|x| - |y|)(|x| - |y|) \\
&= (|x| - |y|)(|\overline{x}| - |\overline{y}|) \text{ (by Theorem 1.33b)} \\
&= (|x| - |y|)(\overline{|x| - |y|}) \text{ (by Theorem 1.31b)} \\
&= \big||x| - |y|\big|^2 \text{ (by Definition 1.32)}
\end{aligned}$$

Taking the square roots of both sides yields $||x| - |y|| \leq |x - y|$, as desired. ■

1.14 If z is a complex number such that $|z| = 1$, that is, such that $z\overline{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

By definition, $|1 + z|^2 = (1 + z)(1 + \overline{z})$ and $|1 - z|^2 = (1 - z)(1 - \overline{z})$. This yields

$$|1 + z|^2 + |1 - z|^2 = 1 + z + \overline{z} + z\overline{z} + 1 - z - \overline{z} + z\overline{z} = 4$$

■

1.15 Under what conditions does equality hold in the Schwarz inequality? Explain why.

Let $A = \sum_i |a_i|^2$, $B = \sum_i |b_i|^2$ and $C = \sum_i a_i\overline{b_i}$ (note that A and B are real). Under equality, the Schwarz inequality states that $AB = |C|^2$. From this, we have

$$
\begin{aligned}
AB &= |C|^2 \\
AB - |C|^2 &= 0 \\
&\Rightarrow B(AB - |C|^2) = B \cdot 0 = 0 \\
&\iff BAB - B|C|^2 = 0 \\
&\iff B^2 A - B|C|^2 = 0 \text{ (sums are elements of } \mathbb{C}\text{, which is commutative)} \\
&\iff B^2 A - 2B|C|^2 + B|C|^2 = 0 \\
&\iff B^2 A - BC\overline{C} - BC\overline{C} + B|C|^2 = 0 \text{ (since } C\overline{C} = |C|^2\text{)} \\
&\iff B^2 \left(\sum_i |a_i|^2 \right) - B \left(\sum_i a_i \overline{b_i} \right) \overline{C} - BC \overline{\left(\sum_i a_i \overline{b_i} \right)} + |C|^2 \left(\sum_i |b_i|^2 \right) = 0 \\
&\iff \sum_i \left(B^2 |a_i|^2 - B\overline{C} a_i \overline{b_i} - BC\overline{a_i} b_i + |C|^2 |b_i|^2 \right) = 0 \\
&\iff \sum_i (Ba_i - Cb_i)(B\overline{a}_i - \overline{Cb_i}) = 0 \text{ (since } z\overline{z} = |z|^2\text{)} \\
&\iff \sum_i (Ba_i - Cb_i)\overline{(Ba_i - Cb_i)} = 0 \text{ (note that } B \text{ is real)} \\
&\iff \sum_i |Ba_i - Cb_i|^2 = 0
\end{aligned}
$$

Since squares are zero iff they're trivial, we end up with:

$$
\begin{aligned}
&\iff Ba_i - Cb_i = 0 \ (\forall i) \\
&\iff Ba_i = Cb_i \\
&\iff a_i = \frac{C}{B} b_i
\end{aligned}
$$

Thus, equality holds whenever $\vec{a} = (a_1, a_2, \cdots, a_n)$ and $\vec{b} = (b_1, b_2, \cdots, b_m)$ are scalar multiples of each other.

Conversely, given that $\vec{a}, \vec{b}$ are scalar multiples, we immediately have (using the above derivation) that

$$a_i = \frac{C}{B} b_i \iff B(AB - |C|^2) = 0$$

With the condition that $B \neq 0$, we end up with $AB - |C|^2$ or, equivalently, $AB = |C|^2$, the Schwartz equality. ∎

1.16 Suppose $k \geq 3$, $\vec{x}, \vec{y} \in R^k$, $|\vec{x}-\vec{y}| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\vec{z} \in R^k$ such that

$$|\vec{z}-\vec{x} = |\vec{z}-\vec{y}| = r$$

(b) If $2r = d$, there is exactly one such $\vec{z}$.

(c) If $2r < d$, there is no such $\vec{z}$.

■

1.17 Prove that

$$|\vec{x}+\vec{y}|^2 + |\vec{x}-\vec{y}|^2 = 2|\vec{x}|^2 + 2|\vec{y}|^2$$

if $\vec{x} \in R^k$ and $\vec{y} \in R^k$. Interpret this geometrically, as a statement about parallelograms.

■

1.18 If $k \geq 2$ and $\vec{x} \in R^k$, prove that there exists $\vec{y} \in R^k$ such that $\vec{y} \neq \vec{0}$ but $\vec{x} \cdot \vec{y} = 0$. Is this also true if $k = 1$.

■

1.19 Suppose $\vec{a} \in R^k, \vec{b} \in R^k$. Find $\vec{c} \in R^k$ and $r > 0$ such that

$$|\vec{x}-\vec{a}| = 2|\vec{x}-\vec{b}|$$

iff $|\vec{x}-\vec{c}| = r$ (*Solution*: $3\vec{c} = 4\vec{b}-\vec{a}, 3r = 2|\vec{b}-\vec{a}|$.)

■

1.20 With reference to the Appendix, suppose that property(III) were omitted from the defintion of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

As a reminder, elements of $\mathbb{R}$ are formed from cuts in $\mathbb{Q}$. Choose $r \in \mathbb{R}\ \mathbb{Q}$. Then a cut of r is defined as the set of all points less than r. Note that if it was defined as 'set of points greater than r', it would contradict (II) and if it was defined as 'set of points less than and greater than r', then it would contradict (I) since this definition implies the cut is all of $\mathbb{Q}$. Finally, our definition makes the cut strictly less than r for otherwise (III) would not be satisfied.

Let us ignore property (III). Define a cut with the following two properties:

1. α is not empty, and $\alpha \neq \mathbb{Q}$.

2. If $p \in \alpha, q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.

Let α, β, γ be cuts. Following page 17-19, steps 2 and 3 can be verified as written, excluding property (III). Thus, the resulting ordered set has the l.u.b. property.

Excluding property (III) is equivalently saying that the cut may contain the maximal element. That is, if $p \in \alpha$, we are not guaranteed to find a bigger element $r \in \alpha$, which implies that p may just as well be the biggest element. For this reason, we define 0^* to be the set of all negative rational numbers *and* 0.

We are now ready to show that the resulting ordered set satisfies A1 to A4 but not A5.

(A1) *If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.* This step is exactly the same as that of Step 4, with $<$ replaced with $\leq$.

(A2) *Addition is commutative.* See Step 4.

(A3) *Addition is associative.* See Step 4.

(A4) *F contains an element 0 such that $0 + x = x$ for every $x \in F$.* See Step 4, replacing strict inequality.

(A5) *To every $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.* Let α and β be cuts that satisfy properties I,II and III. Suppose, by way of contradiction, that $\alpha + \beta = 0^*$. Then for some $a \in \alpha$, $\exists b \in \beta$ such that $a + b = 0$. Since α satisfies III, we know that $\exists r \in \alpha$ such that $a < r$. Using this, we have

$$0 = a + b < r + b \in \alpha + \beta$$

Since this element is strictly greater than 0 and 0^* is the set of all elements less than, or equal to 0, then $r + b$ is an element of $\alpha + \beta$ that is not in 0^*. It follows that $\alpha + \beta \neq 0^*$.

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