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Stewart: # 8.1, 8.4

8.1 Find the monomorphisms $\sigma_i : K \to \mathbb{C}$ for the following fields and determine the number s of the σ_i satisfying $\sigma_i(K) \subseteq \mathbb{R}$, and the number t of distinct conjugate pairs σ_i, σ_j such that $\overline{\sigma_i} = \sigma_j$:

(i) $\mathbb{Q}(\sqrt{5})$

(ii) $\mathbb{Q}(\sqrt{-5})$

(iii) $\mathbb{Q}(\sqrt[4]{5})$

(iv) $\mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/7}$

(v) $\mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/p}$ for a rational prime p.

(i) Note that the minimal polynomial is $x^2 - 5$ which has the roots $\pm\sqrt{5}$. The monomorphisms are

$$\sigma_1 : \sqrt{5} \to \sqrt{5} \qquad\qquad \sigma_2 : \sqrt{5} \to -\sqrt{5}$$

and both are real, so $s = 2$ and $t = 0$.

(ii) Note that the minimal polynomial is $x^2 + 5$ which has the roots $\pm i\sqrt{5}$, so the monomorphisms are

$$\sigma_1 : i\sqrt{5} \to i\sqrt{5} \qquad\qquad \sigma_2 : i\sqrt{5} \to -i\sqrt{5}$$

and both are imaginary, so $s = 0, t = 1$.

(iii) The minimal polynomial for $\sqrt[4]{5}$ is $x^4 - 5$ which has the roots $\sqrt[4]{5}, \omega\sqrt[4]{5}, \omega^2\sqrt[4]{5}, \omega^3\sqrt[4]{5}$ for $\omega = e^{\frac{2\pi i}{4}}$. Now note that $\omega = i, \omega^2 = -1$ and $\omega^3 = -i$. So the monomorphisms are

$$\sigma_1 : \sqrt[4]{5} \to \sqrt[4]{5}$$
$$\sigma_2 : \sqrt[4]{5} \to i\sqrt[4]{5}$$
$$\sigma_3 : \sqrt[4]{5} \to -\sqrt[4]{5}$$
$$\sigma_4 : \sqrt[4]{5} \to -i\sqrt[4]{5}$$

where σ_1, σ_3 are real and σ_2, σ_4 are complex. Thus, $s = 2, t = 1$.

(iv) The minimal polynomial for ζ is $x^7 - 1$ which has the roots ζ^k where $k = 0, 1, 2, 3, 4, 5, 6$ and the associated seven morphisms:

$$\sigma_{k+1} : \zeta \to \zeta^k$$

By Euler's formula, $\zeta^k = \cos(\frac{2\pi k}{7}) + i \sin(\frac{2\pi k}{7})$ so to find the number of morphisms that are real, it is sufficient to find values of k for which $\sin(\frac{2\pi k}{7}) = 0$. In other words, for $n \in \mathbb{Z}$, we require that

$$\begin{aligned} \frac{2\pi k}{7} &= n\pi \\ 2k &= 7n \\ &\Downarrow \\ 2k &\equiv 0 \mod 7 \\ &\Downarrow \\ k &\equiv 0 \mod 7 \text{ (because 7 doesn't divide 2)} \end{aligned}$$

The unique solution to this is $k = 0$ (out of $0, 1, 2, 3, 4, 5, 6$) and so σ_1 is the only real monomorphism. The other six monomorphisms have a nontrivial imaginary part. Thus, $s = 1, t = 3$.

(v) For $p = 2$, the minimal polynomial is $x^2 - 1$ so the roots are ± 1, both real so $s = 2$. Now suppose p is odd and consider the associated p monomorphisms:

$$\sigma_{k+1} : \zeta \to \zeta^k$$

for $k \in \{0, \cdots, p-1\}$. Using the same approach as above, we find all real monomorphisms by looking at values of k for which $\sin(\frac{2\pi k}{p}) = 0$. In other words, for any $n \in \mathbb{Z}$,

$$\begin{aligned} \frac{2\pi k}{p} &= n\pi \\ 2k &= np \\ &\Downarrow \\ 2k &\equiv 0 \mod p \\ k &\equiv 0 \mod p \text{ (because } p \text{ doesn't divide 2)} \end{aligned}$$

Since p is prime, all positive integers less than it are relative prime to it, so the unique solution to the above equality is $k = 0$, corrsesponding to $\sigma_1 : \zeta \to 1$. Thus, $s = 1$ and $t = \frac{p-1}{2}$.

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8.4 Let $K = \mathbb{Q}(\theta)$ where $\theta \in \mathbb{R}$ and $\theta^3 = 3$. What is the map σ in this case? Pick a basis K and verify Theorem 8.1 for it.

Let $\theta = \sqrt[3]{3}$ and note that the three monomorphisms are

$$\sigma_1 : \theta \to \theta$$
$$\sigma_2 : \theta \to \omega\theta$$
$$\sigma_3 : \theta \to \omega^2\theta$$

where $\omega = e^{\frac{2\pi i}{3}}$. Since ω is imaginary, $s = 1$ and $t = 1$. By definition on Page 146, we have

$$\sigma : \mathbb{Q}(\theta) \to L^{1,1} \cong \mathbb{R}^1 \times \mathbb{C}^1 \cong \mathbb{R}^1 \times (\mathbb{R}^1 \times \mathbb{R}^1) \cong \mathbb{R}^3$$

defined by $\sigma = (\sigma_1, \sigma_2)$ or, equivalently, by $\sigma(\alpha) = (\sigma_1(\alpha), Re(\sigma_2(\alpha)), Im(\sigma_2(\alpha)))$. To simplify things, we compute the real and imaginary parts of ω^k for $k = 1, 2$:

$$Re(\omega) = -\frac{1}{2} \qquad Im(\omega) = \frac{\sqrt{3}}{2}$$
$$Re(\omega^2) = -\frac{1}{2} \qquad Im(\omega^2) = -\frac{\sqrt{3}}{2}$$

We now verify Theorem 8.1. Note that $\{1, \theta, \theta^2\}$ is a basis for $\mathbb{Q}(\theta)$. Then

$$\sigma(1) = \begin{bmatrix} \sigma_1(1) \\ Re(\sigma_2(1)) \\ Im(\sigma_2(1)) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\sigma(\theta) = \begin{bmatrix} \sigma_1(\theta) \\ Re(\sigma_2(\theta)) \\ Im(\sigma_2(\theta)) \end{bmatrix} = \begin{bmatrix} \theta \\ Re(\omega\theta) \\ Im(\omega\theta) \end{bmatrix} = \begin{bmatrix} \theta \\ -\frac{1}{2}\theta \\ \frac{\sqrt{3}}{2}\theta \end{bmatrix}$$
$$\sigma(\theta^2) = \begin{bmatrix} \sigma_1(\theta^2) \\ Re(\sigma_2(\theta^2)) \\ Im(\sigma_2(\theta^2)) \end{bmatrix} = \begin{bmatrix} \theta^2 \\ Re(\omega^2\theta^2) \\ Im(\omega^2\theta^2) \end{bmatrix} = \begin{bmatrix} \theta^2 \\ -\frac{1}{2}\theta^2 \\ -\frac{\sqrt{3}}{2}\theta^2 \end{bmatrix}$$

To show that they are linearly independent over $\mathbb{R}$, we need to show that if $\lambda_1\sigma(1) + \lambda_2\sigma(\theta) + \lambda_3(\sigma(\theta^2)) = 0$ then $\lambda_1 = \lambda_2 = \lambda_3 = 0$ for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Rewriting, we have

$$\begin{aligned} 0 = \lambda_1\sigma(1) + \lambda_2\sigma(\theta) + \lambda_3(\sigma(\theta^2)) &= \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} \theta \\ -\frac{\theta}{2} \\ \frac{\sqrt{3}}{2}\theta \end{bmatrix} + \lambda_3 \begin{bmatrix} \theta^2 \\ -\frac{\theta^2}{2} \\ -\frac{\sqrt{3}}{2}\theta^2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 + \lambda_2\theta + \lambda_3\theta^2 \\ \lambda_1 - \frac{\lambda_2}{2}\theta - \frac{\lambda_3}{2}\theta^2 \\ \frac{\sqrt{3}\lambda_2}{2}\theta - \frac{\sqrt{3}\lambda_3}{2}\theta^2 \end{bmatrix} \end{aligned}$$

The third entry implies $\lambda_2 = \theta\lambda_3$. Plugging this into the second entry yields $\lambda_1 = \lambda_3\theta^2$. Plugging these into the first entry yields $3\lambda_3\theta^2 = 0$. Since $\theta^2 \neq 0$, it must be that λ_3 and thus λ_1, λ_2 are zero. Thus, $\sigma(1), \sigma(\theta)$ and $\sigma(\theta^2)$ are linearly independent, as dictated by Theorem 8.1. ∎