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4.3.5 Let G be a group. If the center of G is of index n, prove that every conjugacy class has at most n elements.

Consider an arbitrary conjugacy class, say of $g \in G$. By Proposition 6,

$$|g^G| = |G : C_G(g)| = \frac{|G|}{|C_G(g)|}$$

where $C_G(g)$ is the centralizer of g. Furthermore, $Z(G) \subseteq C_G(g)$ as every element of $Z(G)$ will conjugate to equal g (thus be in g^G). This implies that $|Z(G)| \leq |C_G(g)$ and combining this with the above equality, we have

$$|g^G| = \frac{|G|}{|C_G(g)|} \leq \frac{|G|}{|Z(G)|}$$

By Lagrange's Theorem, $\frac{|G|}{|Z(G)|} = |G : Z(G)| = n$ so $|g^G| \leq n$, as desired. ∎