FIXED POINT THEOREM IN METRIC SPACES AND ITS APPLICATION TO THE COLLATZ CONJECTURE

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Abstract. In this paper, we show the new fixed point theorem in metric spaces. Furthermore, for this fixed point theorem, we apply to the Collatz conjecture.

1. Introduction

Let $\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$ and let C be a mapping from $\mathbb{N}$ into itself defined by

$$Cx \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2}x, & \text{if } x \text{ is even,} \\ 3x + 1, & \text{if } x \text{ is odd.} \end{cases}$$

Then the Collatz conjecture is as follows: for any $x \in \mathbb{N}$, there exists $c(x) \in \mathbb{N}$ such that $C^{c(x)}x = 1$. Many researchers have attempted to prove this conjecture. A most famous paper is probably [4] by Tao. In his paper, define $\mathrm{Col}_{\min}(N) \stackrel{\text{def}}{=} \inf\{N, C(N), C^2(N), \ldots\}$ and he showed that

Theorem 1.1. *Let f be a function from $\mathbb{N}$ into $\mathbb{R}$ with $\lim_{N\to\infty} f(N) = \infty$. Then $\mathrm{Col}_{\min}(N) < f(N)$ for allmost all $N \in \mathbb{N}$ in the sence of logarithmic density.*

Using this theorem, we obtain $\mathrm{Col}_{\min}(N) < \log\log\log\log N$ for almost all $N \in \mathbb{N}$. Furthermore, see the papers referenced in his paper.

In this paper, in the section 2, we show the new fixed point theorem in metric spaces. This fixed point theorem is an extension of result showed in [1]. Although these results applies to Banach spaces, see also [2] and [3].

Define $d(x, y) \stackrel{\text{def}}{=} |x - y|$ for any $x, y \in \mathbb{N}$. Then $(\mathbb{N}, d)$ is a complete metric space. In the section 3, for this fixed point theorem, we apply to the Collatz conjecture.

2. Fixed point theorem

Let (X, d) be a metric space with a metric d. We consider mappings α, β, γ, δ, ε, and ζ from $X \times X$ into $\mathbb{R}$. A mapping T from X into itself is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-weighted generalized pseudocontraction if

$$\begin{aligned} &\alpha(x,y)d(Tx,Ty)^2 + \beta(x,y)d(x,Ty)^2 + \gamma(x,y)d(Tx,y)^2 + \delta(x,y)d(x,y)^2 \\ &\quad + \varepsilon(x,y)d(x,Tx)^2 + \zeta(x,y)d(y,Ty)^2 \\ &\le 0 \end{aligned}$$

holds for any $x, y \in X$.

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Lemma 2.1. *Let (X,d) be a metric space. Then*

$$2\min\{\theta,0\}(d(x,z)^2+d(z,y)^2)\le\theta d(x,y)^2$$

holds for any $\theta\in\mathbb{R}$ and for any $x,y,z\in X$.

Proof. By the triangle inequality

$$|d(x,z)-d(z,y)|\le d(x,y)\le d(x,z)+d(z,y)$$

holds for any $x,y,z\in X$. Squaring each side, we obtain

$$d(x,z)^2-2d(x,z)d(z,y)+d(z,y)^2\le d(x,y)^2\le d(x,z)^2+2d(x,z)d(z,y)+d(z,y)^2.$$

Therefore, we obtain

$$\theta d(x,z)^2-2|\theta|d(x,z)d(z,y)+\theta d(z,y)^2\le\theta d(x,y)^2$$

for any $\theta\in\mathbb{R}$. Since

$$\begin{aligned}&\theta d(x,z)^2-2|\theta|d(x,z)d(z,y)+\theta d(z,y)^2\\&=|\theta|(d(x,z)-d(z,y))^2+(\theta-|\theta|)(d(x,z)^2+d(z,y)^2)\\&\ge(\theta-|\theta|)(d(x,z)^2+d(z,y)^2)\\&=2\min\{\theta,0\}(d(x,z)^2+d(z,y)^2),\end{aligned}$$

we obtain the desired result. □

Lemma 2.2. *Let (X,d) be a metric space, let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-weighted generalized pseudocontraction, let λ be a mapping from $X\times X$ into $[0,1]$, and let*

$$\begin{aligned}\alpha_\lambda(x,y)&\overset{def}{=}(1-\lambda(x,y))\alpha(x,y)+\lambda(x,y)\alpha(y,x);\\\beta_\lambda(x,y)&\overset{def}{=}(1-\lambda(x,y))\beta(x,y)+\lambda(x,y)\gamma(y,x);\\\gamma_\lambda(x,y)&\overset{def}{=}(1-\lambda(x,y))\gamma(x,y)+\lambda(x,y)\beta(y,x);\\\delta_\lambda(x,y)&\overset{def}{=}(1-\lambda(x,y))\delta(x,y)+\lambda(x,y)\delta(y,x);\\\varepsilon_\lambda(x,y)&\overset{def}{=}(1-\lambda(x,y))\varepsilon(x,y)+\lambda(x,y)\zeta(y,x);\\\zeta_\lambda(x,y)&\overset{def}{=}(1-\lambda(x,y))\zeta(x,y)+\lambda(x,y)\varepsilon(y,x).\end{aligned}$$

Then T is an $(\alpha_\lambda,\beta_\lambda,\gamma_\lambda,\delta_\lambda,\varepsilon_\lambda,\zeta_\lambda)$-weighted generalized pseudocontraction.

Proof. By assumption

$$\begin{aligned}&\alpha(x,y)d(Tx,Ty)^2+\beta(x,y)d(x,Ty)^2+\gamma(x,y)d(Tx,y)^2+\delta(x,y)d(x,y)^2\\&\quad+\varepsilon(x,y)d(x,Tx)^2+\zeta(x,y)d(y,Ty)^2\\&\le 0\end{aligned}$$

holds for any $x,y\in X$. Exchanging x and y, we obtain

$$\begin{aligned}&\alpha(y,x)d(Tx,Ty)^2+\gamma(y,x)d(x,Ty)^2+\beta(y,x)d(Tx,y)^2+\delta(y,x)d(x,y)^2\\&\quad+\zeta(y,x)d(x,Tx)^2+\varepsilon(y,x)d(y,Ty)^2\\&\le 0\end{aligned}$$

for any $x,y\in X$. By multiplying the upper inequality by $1-\lambda(x,y)$, multiplying the lower inequality by $\lambda(x,y)$, and adding them together, we obtain the desired result. □

Theorem 2.1. *Let (X,d) be a metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-weighted generalized pseudocontraction. Suppose that there exists a mapping λ from $X\times X$ into $[0,1]$ such that one of the following holds:*

(1) $\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ *and* $\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}\geq 0$ *for any* $x,y\in X$*;*

(2) $\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}\geq 0$ *and* $\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ *for any* $x,y\in X$*;*

(3) $\alpha_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\}>0$ *and* $\delta_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\}\geq 0$ *for any* $x,y\in X$*;*

(4) $\alpha_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\}\geq 0$ *and* $\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ *for any* $x,y\in X$*;*

(5) *there exists* $A\in(0,1)$ *such that for any* $x,y\in X$, $\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ *and*
$$-\frac{\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}{\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}\leq A,$$
or $\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}>0$ *and*
$$-\frac{\delta_\lambda(y,x)+\zeta_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}{\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}\leq A.$$

Then $\{T^nx\mid n\in\mathbb{N}\}$ *is Cauchy for any* $x\in X$*.*

Proof. By Lemma 2.2 T is an $(\alpha_\lambda,\beta_\lambda,\gamma_\lambda,\delta_\lambda,\varepsilon_\lambda,\zeta_\lambda)$-weighted generalized pseudocontraction. By Lemma 2.1 we obtain
$$\begin{aligned}&\alpha_\lambda(x,y)d(Tx,Ty)^2+\gamma_\lambda(x,y)d(Tx,y)^2+(\delta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\})d(x,y)^2\\&\quad+\varepsilon_\lambda(x,y)d(x,Tx)^2+(\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\})d(y,Ty)^2\\&\leq 0,\\&\alpha_\lambda(x,y)d(Tx,Ty)^2+\beta_\lambda(x,y)d(x,Ty)^2+(\delta_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\})d(x,y)^2\\&\quad+(\varepsilon_\lambda(x,y)+2\min\{\gamma_\lambda d(x,y),0\})d(x,Tx)^2+\zeta_\lambda(x,y)d(y,Ty)^2\\&\leq 0.\end{aligned}$$
Replacing x with $T^{n-1}x$ and y with T^nx in the inequality above, we obtain
$$\begin{aligned}&(\alpha_\lambda(T^{n-1}x,T^nx)+\zeta_\lambda(T^{n-1}x,T^nx)+2\min\{\beta_\lambda(T^{n-1}x,T^nx),0\})\\&\qquad\times d(T^nx,T^{n+1}x)^2\\&\quad+(\delta_\lambda(T^{n-1}x,T^nx)+\varepsilon_\lambda(T^{n-1}x,T^nx)+2\min\{\beta_\lambda(T^{n-1}x,T^nx),0\})\\&\qquad\times d(T^{n-1}x,T^nx)^2\\&\leq 0.\end{aligned}$$
In the case of (1), since
$$d(T^nx,T^{n+1}x)=0$$
holds for any $n\in\mathbb{N}$, $\{T^nx\mid n\in\mathbb{N}\}$ is Cauchy. In the case of (2), since
$$d(T^{n-1}x,T^nx)=0$$
holds for any $n\in\mathbb{N}$, $\{T^nx\mid n\in\mathbb{N}\}$ is Cauchy. Replacing x with T^nx and y with $T^{n-1}x$ in the inequality below, we obtain
$$\begin{aligned}&(\alpha_\lambda(T^nx,T^{n-1}x)+\varepsilon_\lambda(T^nx,T^{n-1}x)+2\min\{\gamma_\lambda(T^nx,T^{n-1}x),0\})\\&\qquad\times d(T^{n+1}x,T^nx)^2\end{aligned}$$

$$
\begin{aligned}
&+(\delta_\lambda(T^n x, T^{n-1}x) + \zeta_\lambda(T^n x, T^{n-1}x) + 2\min\{\gamma_\lambda(T^n x, T^{n-1}x), 0\})) \\
&\qquad \times d(T^n x, T^{n-1}x)^2 \\
&\leq 0.
\end{aligned}
$$

In the case of (3), since

$$d(T^{n+1}x, T^n x) = 0$$

holds for any $n \in \mathbb{N}$, $\{T^n x \mid n \in \mathbb{N}\}$ is Cauchy. In the case of (4), since

$$d(T^n x, T^{n-1}x) = 0$$

holds for any $n \in \mathbb{N}$, $\{T^n x \mid n \in \mathbb{N}\}$ is Cauchy. In the case of (5), since

$$
\begin{aligned}
&d(T^n x, T^{n+1}x)^2 \\
&\leq -\frac{\delta_\lambda(T^{n-1}x, T^n x) + \varepsilon_\lambda(T^{n-1}x, T^n x) + 2\min\{\beta_\lambda(T^{n-1}x, T^n x), 0\}}{\alpha_\lambda(T^{n-1}x, T^n x) + \zeta_\lambda(T^{n-1}x, T^n x) + 2\min\{\beta_\lambda(T^{n-1}x, T^n x), 0\}} \\
&\qquad \times d(T^{n-1}x, T^n x)^2 \\
&\leq A d(T^{n-1}x, T^n x)^2 \\
&\leq A^n d(x, Tx)^2
\end{aligned}
$$

or

$$
\begin{aligned}
&d(T^{n+1}x, T^n x)^2 \\
&\leq -\frac{\delta_\lambda(T^n x, T^{n-1}x) + \zeta_\lambda(T^n x, T^{n-1}x) + 2\min\{\gamma_\lambda(T^n x, T^{n-1}x), 0\}}{\alpha_\lambda(T^n x, T^{n-1}x) + \varepsilon_\lambda(T^n x, T^{n-1}x) + 2\min\{\gamma_\lambda(T^n x, T^{n-1}x), 0\}} \\
&\qquad \times d(T^n x, T^{n-1}x)^2 \\
&\leq A d(T^{n-1}x, T^n x)^2 \\
&\leq A^n d(x, Tx)^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
d(T^n x, T^m x) &\leq \sum_{k=n}^{m-1} d(T^k x, T^{k+1}x) \\
&\leq \sum_{k=n}^{m-1} A^{\frac{k}{2}} d(x, Tx) \\
&\leq \frac{A^{\frac{n}{2}}}{1 - A^{\frac{1}{2}}} d(x, Tx)
\end{aligned}
$$

for any $m > n$. Therefore, $\{T^n x \mid n \in \mathbb{N}\}$ is Cauchy. □

By Theorem 2.1 we obtain the following directly.

Theorem 2.2. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-weighted generalized pseudocontraction. Suppose that there exists a mapping λ from $X \times X$ into $[0, 1]$ such that one of the following holds:*

(1) $\alpha_\lambda(x, y) + \zeta_\lambda(x, y) + 2\min\{\beta_\lambda(x, y), 0\} > 0$ *and* $\delta_\lambda(x, y) + \varepsilon_\lambda(x, y) + 2\min\{\beta_\lambda(x, y), 0\} \geq 0$ *for any* $x, y \in X$*;*
(2) $\alpha_\lambda(x, y) + \zeta_\lambda(x, y) + 2\min\{\beta_\lambda(x, y), 0\} \geq 0$ *and* $\delta_\lambda(x, y) + \varepsilon_\lambda(x, y) + 2\min\{\beta_\lambda(x, y), 0\} > 0$ *for any* $x, y \in X$*;*
(3) $\alpha_\lambda(x, y) + \varepsilon_\lambda(x, y) + 2\min\{\gamma_\lambda(x, y), 0\} > 0$ *and* $\delta_\lambda(x, y) + \zeta_\lambda(x, y) + 2\min\{\gamma_\lambda(x, y), 0\} \geq 0$ *for any* $x, y \in X$*;*

(4) *$\alpha_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\}\geq 0$ and $\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ for any $x,y\in X$;*

(5) *there exists $A\in(0,1)$ such that for any $x,y\in X$, $\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ and*

$$-\frac{\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}{\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}\leq A,$$

or $\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}>0$ and

$$-\frac{\delta_\lambda(y,x)+\zeta_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}{\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}\leq A.$$

Then $\{T^nx\mid n\in\mathbb{N}\}$ is convergent to a point in X for any $x\in X$.

By Theorem 2.2 we obtain the following.

Theorem 2.3. *Let (X,d) be a complete metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-weighted generalized pseudocontraction. Suppose that there exists a mapping λ from $X\times X$ into $[0,1]$ such that one of the following holds:*

(1) *$\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ and $\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}\geq 0$ for any $x,y\in X$;*

(2) *$\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}\geq 0$ and $\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ for any $x,y\in X$;*

(3) *$\alpha_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\}>0$ and $\delta_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\}\geq 0$ for any $x,y\in X$;*

(4) *$\alpha_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\gamma_\lambda(x,y),0\}\geq 0$ and $\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$ for any $x,y\in X$;*

(5) *there exist $A\in(0,1)$ and $B\in(0,\infty)$ such that for any $x,y\in X$, $\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$,*

$$-\frac{\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}{\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}\leq A,$$

and $\alpha_\lambda(x,y)+\beta_\lambda(x,y)+\zeta_\lambda(x,y)\geq B$, or $\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}>0$,

$$-\frac{\delta_\lambda(y,x)+\zeta_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}{\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}\leq A,$$

and $\alpha_\lambda(y,x)+\gamma_\lambda(y,x)+\varepsilon_\lambda(y,x)\geq B$. Furthermore, there exissts $M\in(0,\infty)$ such that $|\alpha(x,y)|\leq M$, $|\beta(x,y)|\leq M$, $|\gamma(x,y)|\leq M$, $|\delta(x,y)|\leq M$, $|\varepsilon(x,y)|\leq M$, and $|\zeta(x,y)|\leq M$ for any $x,y\in X$.

Then T has a fixed point u, where $u=\lim_{n\to\infty}T^nx$ for any $x\in X$.

Proof. In the cases of (1) and (3), the set of all fixed points of T is equal to $T(X)$ and $\lim_{n\to\infty}T^nx=Tx$. In the cases of (2) and (4), the set of all fixed points of T is equal to X and $\lim_{n\to\infty}T^nx=x$.

We show in the case of (5). By Theorem 2.2 $\{T^nx\mid n\in\mathbb{N}\}$ is convergent to a point u in X for any $x\in\mathbb{N}$. Replacing x with T^nx and y with u, and replacing x with u and y with T^nx, we obtain

$$\begin{aligned}&\alpha_\lambda(T^nx,u)d(T^{n+1}x,Tu)^2+\beta_\lambda(T^nx,u)d(T^nx,Tu)^2\\&+\gamma_\lambda(T^nx,u)d(T^{n+1}x,u)^2+\delta_\lambda(T^nx,u)d(T^nx,u)^2\\&+\varepsilon_\lambda(T^nx,u)d(T^nx,T^{n+1}x)^2+\zeta_\lambda(T^nx,u)d(u,Tu)^2\end{aligned}$$

$$
\begin{aligned}
&\leq 0,\\
&\alpha_\lambda(u,T^nx)d(Tu,T^{n+1}x)^2+\beta_\lambda(u,T^nx)d(u,T^{n+1}x)^2\\
&\quad+\gamma_\lambda(u,T^nx)d(Tu,T^nx)^2+\delta_\lambda(u,T^nx)d(u,T^nx)^2\\
&\quad+\varepsilon_\lambda(u,T^nx)d(u,Tu)^2+\zeta_\lambda(u,T^nx)d(T^nx,T^{n+1}x)^2\\
&\leq 0.
\end{aligned}
$$

In the case where $\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}>0$,

$$
-\frac{\delta_\lambda(x,y)+\varepsilon_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}{\alpha_\lambda(x,y)+\zeta_\lambda(x,y)+2\min\{\beta_\lambda(x,y),0\}}\leq A,
$$

and $\alpha_\lambda(x,y)+\beta_\lambda(x,y)+\zeta_\lambda(x,y)\geq B$, for any $\rho\in(0,\infty)$, there exists $N\in\mathbb{N}$ such that $|d(T^nx,Tu)^2-d(u,Tu)^2|<\rho$, $d(T^nx,u)^2<\rho$, and $d(T^nx,T^{n+1}x)^2<\rho$ for any $n>N$. Therefore,

$$
\begin{aligned}
&\alpha_\lambda(T^nx,u)d(T^{n+1}x,Tu)^2+\beta_\lambda(T^nx,u)d(T^nx,Tu)^2\\
&\quad+\gamma_\lambda(T^nx,u)d(T^{n+1}x,u)^2+\delta_\lambda(T^nx,u)d(T^nx,u)^2\\
&\quad+\varepsilon_\lambda(T^nx,u)d(T^nx,T^{n+1}x)^2+\zeta_\lambda(T^nx,u)d(u,Tu)^2\\
&>\alpha_\lambda(T^nx,u)d(u,Tu)^2-M\rho+\beta_\lambda(T^nx,u)d(u,Tu)^2-M\rho\\
&\quad-M\rho-M\rho\\
&\quad-M\rho+\zeta_\lambda(T^nx,u)d(u,Tu)^2\\
&\geq Bd(u,Tu)^2-5M\rho.
\end{aligned}
$$

Since ρ is arbitrary, $d(u,Tu)\leq 0$, that is, u is a fixed point of T. In the case where $\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}>0$,

$$
-\frac{\delta_\lambda(y,x)+\zeta_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}{\alpha_\lambda(y,x)+\varepsilon_\lambda(y,x)+2\min\{\gamma_\lambda(y,x),0\}}\leq A,
$$

and $\alpha_\lambda(y,x)+\gamma_\lambda(y,x)+\varepsilon_\lambda(y,x)\geq B$, for any $\rho\in(0,\infty)$, there exists $N\in\mathbb{N}$ such that $|d(Tu,T^nx)^2-d(Tu,u)^2|<\rho$, $d(u,T^nx)^2<\rho$, and $d(T^{n+1}x,T^nx)^2<\rho$ for any $n>N$. Therefore,

$$
\begin{aligned}
&\alpha_\lambda(u,T^nx)d(Tu,T^{n+1}x)^2+\beta_\lambda(u,T^nx)d(u,T^{n+1}x)^2\\
&\quad+\gamma_\lambda(u,T^nx)d(Tu,T^nx)^2+\delta_\lambda(u,T^nx)d(u,T^nx)^2\\
&\quad+\varepsilon_\lambda(u,T^nx)d(u,Tu)^2+\zeta_\lambda(u,T^nx)d(T^nx,T^{n+1}x)^2\\
&>\alpha_\lambda(u,T^nu)d(Tu,u)^2-M\rho-M\rho\\
&\quad+\gamma_\lambda(u,T^nx)d(Tu,u)^2-M\rho-M\rho\\
&\quad+\varepsilon_\lambda(u,T^nx)d(u,Tu)^2-M\rho\\
&\geq Bd(Tu,u)^2-5M\rho.
\end{aligned}
$$

Since ρ is arbitrary, $d(Tu,u)\leq 0$, that is, u is a fixed point of T. □

3. Applying to the Collatz conjecture

Let C be a mapping from $\mathbb{N}$ into itself defined by

$$
Cx\overset{\mathrm{def}}{=}\begin{cases}\frac{1}{2}x, & \text{if } x \text{ is even},\\ 3x+1, & \text{if } x \text{ is odd}.\end{cases}
$$

Then the Collatz conjecture is as follows: for any $x\in\mathbb{N}$, there exists $c(x)\in\mathbb{N}$ such that $C^{c(x)}x=1$.

Clearly, $c(1) = 3$. Since $3x + 1$ is even whenever x is odd, $C^2x = \frac{1}{2}(3x + 1)$ for any odd number x. Let T be a mapping from $\mathbb{N}$ into itself defined by

$$Tx \stackrel{\text{def}}{=} \begin{cases} C^3x = 1, & \text{if } x = 1, \\ Cx = \frac{1}{2}x, & \text{if } x \text{ is even}, \\ C^2x = \frac{1}{2}(3x+1), & \text{if } x \text{ is odd and } x \geq 3. \end{cases}$$

Then, if we can prove that for any $x \in \mathbb{N}$, there exists $t(x) \in \mathbb{N}$ such that $T^{t(x)}x = 1$, then the Collatz conjecture is true.

Define $d(x, y) \stackrel{\text{def}}{=} |x-y|$ for any $x, y \in \mathbb{N}$. Then $(\mathbb{N}, d)$ is a complete metric space.

Theorem 3.1. *Let*

$$\alpha(x,y) \quad \stackrel{def}{=} \quad \begin{cases} 1, & \textit{if } x = 1 \textit{ and } y = 1, \\ 1, & \textit{if } x = 1 \textit{ and } y \textit{ is even}, \\ 0, & \textit{if } x = 1, y \textit{ is odd, and } y \geq 3, \\ 1, & \textit{if } x \textit{ is even and } y = 1, \\ 1, & \textit{if } x \textit{ is even and } y \textit{ is even}, \\ 0, & \textit{if } x \textit{ is even}, y \textit{ is odd, and } y \geq 3, \\ 1, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y = 1, \\ 0, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y \textit{ is even}, \\ 2, & \textit{if } x \textit{ is odd}, x \geq 3, y \textit{ is odd, and } y \geq 3; \end{cases}$$

$$\beta(x,y) \quad \stackrel{def}{=} \quad \begin{cases} 0, & \textit{if } x = 1 \textit{ and } y = 1, \\ 0, & \textit{if } x = 1 \textit{ and } y \textit{ is even}, \\ 0, & \textit{if } x = 1, y \textit{ is odd, and } y \geq 3, \\ 0, & \textit{if } x \textit{ is even and } y = 1, \\ 0, & \textit{if } x \textit{ is even and } y \textit{ is even}, \\ 0, & \textit{if } x \textit{ is even}, y \textit{ is odd, and } y \geq 3, \\ 0, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y = 1, \\ -2, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y \textit{ is even}, \\ \beta_0(x,y), & \textit{if } x \textit{ is odd}, x \geq 3, y \textit{ is odd, and } y \geq 3; \end{cases}$$

$$\gamma(x,y) \quad \stackrel{def}{=} \quad \begin{cases} 0, & \textit{if } x = 1 \textit{ and } y = 1, \\ 0, & \textit{if } x = 1 \textit{ and } y \textit{ is even}, \\ 0, & \textit{if } x = 1, y \textit{ is odd, and } y \geq 3, \\ 1, & \textit{if } x \textit{ is even and } y = 1, \\ -1, & \textit{if } x \textit{ is even and } y \textit{ is even}, \\ -2, & \textit{if } x \textit{ is even}, y \textit{ is odd, and } y \geq 3, \\ -1, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y = 1, \\ 0, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y \textit{ is even}, \\ -\beta_0(x,y), & \textit{if } x \textit{ is odd}, x \geq 3, y \textit{ is odd, and } y \geq 3; \end{cases}$$

$$\delta(x,y) \quad \stackrel{def}{=} \quad \begin{cases} 0, & \textit{if } x = 1 \textit{ and } y = 1, \\ -1, & \textit{if } x = 1 \textit{ and } y \textit{ is even}, \\ -2, & \textit{if } x = 1, y \textit{ is odd, and } y \geq 3, \\ -1, & \textit{if } x \textit{ is even and } y = 1, \\ 0, & \textit{if } x \textit{ is even and } y \textit{ is even}, \\ 1, & \textit{if } x \textit{ is even}, y \textit{ is odd, and } y \geq 3, \\ -1, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y = 1, \\ 1, & \textit{if } x \textit{ is odd, } x \geq 3, \textit{ and } y \textit{ is even}, \\ \delta_0(x,y), & \textit{if } x \textit{ is odd}, x \geq 3, y \textit{ is odd, and } y \geq 3; \end{cases}$$

$$\varepsilon(x,y) \overset{def}{=} \begin{cases} -1, & \textit{if } x=1 \textit{ and } y=1, \\ 0, & \textit{if } x=1 \textit{ and } y \textit{ is even}, \\ 1, & \textit{if } x=1, y \textit{ is odd, and } y\geq 3, \\ 0, & \textit{if } x \textit{ is even and } y=1, \\ -1, & \textit{if } x \textit{ is even and } y \textit{ is even}, \\ -2, & \textit{if } x \textit{ is even}, y \textit{ is odd, and } y\geq 3, \\ 0, & \textit{if } x \textit{ is odd, } x\geq 3, \textit{ and } y=1, \\ 2, & \textit{if } x \textit{ is odd, } x\geq 3, \textit{ and } y \textit{ is even}, \\ \varepsilon_0(x,y), & \textit{if } x \textit{ is odd}, x\geq 3, y \textit{ is odd, and } y\geq 3; \end{cases}$$

$$\zeta(x,y) \overset{def}{=} \begin{cases} 1, & \textit{if } x=1 \textit{ and } y=1, \\ 1, & \textit{if } x=1 \textit{ and } y \textit{ is even}, \\ 2, & \textit{if } x=1, y \textit{ is odd, and } y\geq 3, \\ 1, & \textit{if } x \textit{ is even and } y=1, \\ 1, & \textit{if } x \textit{ is even and } y \textit{ is even}, \\ 2, & \textit{if } x \textit{ is even}, y \textit{ is odd, and } y\geq 3, \\ 1, & \textit{if } x \textit{ is odd, } x\geq 3, \textit{ and } y=1, \\ -2, & \textit{if } x \textit{ is odd, } x\geq 3, \textit{ and } y \textit{ is even}, \\ \zeta_0(x,y) & \textit{if } x \textit{ is odd}, x\geq 3, y \textit{ is odd, and } y\geq 3; \end{cases}$$

for any $x, y \in \mathbb{N}$, where

$$\beta_0(x,y) \overset{def}{=} \begin{cases} -2, & \textit{if } k-\ell \leq -2, \\ k-\ell, & \textit{if } |k-\ell| \leq 1, \\ 2, & \textit{if } k-\ell \geq 2; \end{cases}$$

$$\delta_0(x,y) \overset{def}{=} \begin{cases} -2, & \textit{if } \begin{array}{l} k-\ell\leq -2 \textit{ and } 11k-10\ell+1\leq 0, \\ \textit{or} \\ k-\ell\geq 2 \textit{ and } -10k+11\ell+1\leq 0, \end{array} \\ -1, & \textit{otherwise}; \end{cases}$$

$$\varepsilon_0(x,y) \overset{def}{=} \begin{cases} 2, & \textit{if } k-\ell\leq -2 \textit{ and } 11k-10\ell+1\leq 0, \\ 0, & \textit{otherwise}; \end{cases}$$

$$\zeta_0(x,y) \overset{def}{=} \begin{cases} 2, & \textit{if } k-\ell\geq 2 \textit{ and } -10k+11\ell+1\leq 0, \\ 0, & \textit{otherwise}; \end{cases}$$

for any $x = 2k+1$ ($k \in \mathbb{N}$) and for any $y = 2\ell+1$ ($\ell \in \mathbb{N}$). Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-weighted generalized pseudocontraction.

Proof. In the case where $x = 1$ and $y = 1$: since $Tx = Ty = 1$, we obtain

$$\begin{aligned} &\alpha(1,1)d(1,1)^2 + \beta(1,1)d(1,1)^2 + \gamma(1,1)d(1,1)^2 + \delta(1,1)d(1,1)^2 \\ &\quad + \varepsilon(1,1)d(1,1)^2 + \zeta(1,1)d(1,1)^2 \\ &= 1\times 0^2 + 0\times 0^2 + 0\times 0^2 + 0\times 0^2 \\ &\quad + (-1)\times 0^2 + 1\times 0^2 \\ &= 0. \end{aligned}$$

In the case where $x = 1$ and y is even: let $y = 2\ell$ ($\ell \in \mathbb{N}$). Since $Tx = 1$ and $Ty = \ell$, we obtain

$$\begin{aligned} &\alpha(1,2\ell)d(1,\ell)^2 + \beta(1,2\ell)d(1,\ell)^2 + \gamma(1,2\ell)d(1,2\ell)^2 + \delta(1,2\ell)d(1,2\ell)^2 \\ &\quad + \varepsilon(1,2\ell)d(1,1)^2 + \zeta(1,2\ell)d(2\ell,\ell)^2 \end{aligned}$$

$$
\begin{aligned}
&= 1 \times (1-\ell)^2 + 0 \times (1-\ell)^2 + 0 \times (1-2\ell)^2 + (-1) \times (1-2\ell)^2 \\
&\quad + 0 \times 0^2 + 1 \times \ell^2 \\
&= -2\ell^2 + 2\ell \\
&\leq 0.
\end{aligned}
$$

In the case where x is even and $y = 1$: let $x = 2k$ ($k \in \mathbb{N}$). Since $Tx = k$ and $Ty = 1$, we obtain

$$
\begin{aligned}
&\alpha(2k,1)d(k,1)^2 + \beta(2k,1)d(2k,1)^2 + \gamma(2k,1)d(k,1)^2 + \delta(2k,1)d(2k,1)^2 \\
&\quad + \varepsilon(2k,1)d(2k,k)^2 + \zeta(2k,1)d(1,1)^2 \\
&= 1 \times (k-1)^2 + 0 \times (2k-1)^2 + 1 \times (k-1)^2 + (-1) \times (2k-1)^2 \\
&\quad + 0 \times k^2 + 1 \times 0^2 \\
&= -2k^2 + 1 \\
&\leq -1.
\end{aligned}
$$

In the case where x is even and y is even: let $x = 2k$ ($k \in \mathbb{N}$) and let $y = 2\ell$ ($\ell \in \mathbb{N}$). Since $Tx = k$ and $Ty = \ell$, we obtain

$$
\begin{aligned}
&\alpha(2k,2\ell)d(k,\ell)^2 + \beta(2k,2\ell)d(2k,\ell)^2 + \gamma(2k,2\ell)d(k,2\ell)^2 + \delta(2k,2\ell)d(2k,2\ell)^2 \\
&\quad + \varepsilon(2k,2\ell)d(2k,k)^2 + \zeta(2k,2\ell)d(2\ell,\ell)^2 \\
&= 1 \times (k-\ell)^2 + 0 \times (2k-\ell)^2 + (-1) \times (k-2\ell)^2 + 0 \times (2k-2\ell)^2 \\
&\quad + (-1) \times k^2 + 1 \times \ell^2 \\
&= -k^2 + 2k\ell - 2\ell^2 \\
&\leq -1.
\end{aligned}
$$

In the case where $x = 1$, y is odd, and $y \geq 3$: let $y = 2\ell + 1$ ($\ell \in \mathbb{N}$). Since $Tx = 1$ and $Ty = 3\ell + 2$, we obtain

$$
\begin{aligned}
&\alpha(1,2\ell+1)d(1,3\ell+2)^2 + \beta(1,2\ell+1)d(1,3\ell+2)^2 + \gamma(1,2\ell+1)d(1,2\ell+1)^2 \\
&\quad + \delta(1,2\ell+1)d(1,2\ell+1)^2 \\
&\quad + \varepsilon(1,2\ell+1)d(1,1)^2 + \zeta(1,2\ell+1)d(2\ell+1,3\ell+2)^2 \\
&= 0 \times (3\ell+1)^2 + 0 \times (3\ell+1)^2 + 0 \times (2\ell)^2 + (-2) \times (2\ell)^2 \\
&\quad + 1 \times 0^2 + 2 \times (\ell+1)^2 \\
&= -6\ell^2 + 4\ell + 2 \\
&\leq 0.
\end{aligned}
$$

In the case where x is odd, $x \geq 3$, and $y = 1$: let $x = 2k + 1$ ($k \in \mathbb{N}$). Since $Tx = 3k + 2$ and $Ty = 1$, we obtain

$$
\begin{aligned}
&\alpha(2k+1,1)d(3k+2,1)^2 + \beta(2k+1,1)d(2k+1,1)^2 + \gamma(2k+1,1)d(3k+2,1)^2 \\
&\quad + \delta(2k+1,1)d(2k+1,1)^2 \\
&\quad + \varepsilon(2k+1,1)d(2k+1,3k+2)^2 + \zeta(2k+1,1)d(1,1)^2 \\
&= 1 \times (3k+1)^2 + 0 \times (2k)^2 + (-1) \times (3k+1)^2 + (-1) \times (2k)^2 \\
&\quad + 0 \times (k+1)^2 + 1 \times 0^2 \\
&= -4k^2 \\
&\leq -4.
\end{aligned}
$$

In the case where x is even, y is odd, and $y \geq 3$: let $x = 2k$ ($k \in \mathbb{N}$) and let $y = 2\ell + 1$ ($\ell \in \mathbb{N}$). Since $Tx = k$ and $Ty = 3\ell + 2$, we obtain

$$\begin{aligned}
&\alpha(2k, 2\ell+1)d(k, 3\ell+2)^2 + \beta(2k, 2\ell+1)d(2k, 3\ell+2)^2 \\
&\quad + \gamma(2k, 2\ell+1)d(k, 2\ell+1)^2 + \delta(2k, 2\ell+1)d(2k, 2\ell+1)^2 \\
&\quad + \varepsilon(2k, 2\ell+1)d(2k, k)^2 + \zeta(2k, 2\ell+1)d(2\ell+1, 3\ell+2)^2 \\
&= 0 \times (k - 3\ell - 2)^2 + 0 \times (2k - 3\ell - 2)^2 + (-2) \times (k - 2\ell - 1)^2 \\
&\quad + 1 \times (2k - 2\ell - 1)^2 + (-2) \times k^2 + 2 \times (\ell + 1)^2 \\
&= -2\ell^2 + 1 \\
&\leq -1.
\end{aligned}$$

In the case where x is odd, $x \geq 3$, and y is even: let $x = 2k + 1$ ($k \in \mathbb{N}$) and $y = 2\ell$ ($\ell \in \mathbb{N}$). Since $Tx = 3k + 2$ and $Ty = \ell$, we obtain

$$\begin{aligned}
&\alpha(2k+1, 2\ell)d(3k+2, \ell)^2 + \beta(2k+1, 2\ell)d(2k+1, \ell)^2 \\
&\quad + \gamma(2k+1, 2\ell)d(3k+2, 2\ell)^2 + \delta(2k+1, 2\ell)d(2k+1, 2\ell)^2 \\
&\quad + \varepsilon(2k+1, 2\ell)d(2k+1, 3k+2)^2 + \zeta(2k+1, 2\ell)d(2\ell, \ell)^2 \\
&= 0 \times (3k - \ell + 2)^2 + (-2) \times (2k - \ell + 1)^2 + 0 \times (3k - 2\ell + 2)^2 \\
&\quad + 1 \times (2k - 2\ell + 1)^2 + 2 \times (k + 1)^2 + (-2) \times \ell^2 \\
&= -2k^2 + 1 \\
&\leq -1.
\end{aligned}$$

In the case where x is odd, $x \geq 3$, y is odd, and $y \geq 3$: let $x = 2k + 1$ ($k \in \mathbb{N}$) and $y = 2\ell + 1$ ($\ell \in \mathbb{N}$). Since $Tx = 3k + 2$ and $Ty = 3\ell + 2$, we obtain

$$\begin{aligned}
&\alpha(2k+1, 2\ell+1)d(3k+2, 3\ell+2)^2 + \beta(2k+1, 2\ell+1)d(2k+1, 3\ell+2)^2 \\
&\quad + \gamma(2k+1, 2\ell+1)d(3k+2, 2\ell+1)^2 + \delta(2k+1, 2\ell+1)d(2k+1, 2\ell+1)^2 \\
&\quad + \varepsilon(2k+1, 2\ell+1)d(2k+1, 3k+2)^2 + \zeta(2k+1, 2\ell+1)d(2\ell+1, 3\ell+2)^2 \\
&= 2 \times (3k - 3\ell)^2 + \beta_0(2k+1, 2\ell+1) \times (2k - 3\ell - 1)^2 \\
&\quad - \beta_0(2k+1, 2\ell+1) \times (3k - 2\ell + 1)^2 + \delta_0(2k+1, 2\ell+1) \times (2k - 2\ell)^2 \\
&\quad + \varepsilon_0(2k+1, 2\ell+1) \times (k+1)^2 + \zeta_0(2k+1, 2\ell+1) \times (\ell+1)^2 \\
&= (18 + 4\delta_0(2k+1, 2\ell+1))(k - \ell)^2 - 5\beta_0(2k+1, 2\ell+1)(k - \ell)(k + \ell + 2) \\
&\quad + \varepsilon_0(2k+1, 2\ell+1)(k+1)^2 + \zeta_0(2k+1, 2\ell+1)(\ell+1)^2.
\end{aligned}$$

In the case of $k - \ell \leq -2$ and $11k - 10\ell + 1 \leq 0$: since $\beta_0(2k+1, 2\ell+1) = -2$, $\delta_0(2k+1, 2\ell+1) = -2$, $\varepsilon_0(2k+1, 2\ell+1) = 2$ and $\zeta_0(2k+1, 2\ell+1) = 0$, we obtain

$$\begin{aligned}
&(18 + 4\delta_0(2k+1, 2\ell+1))(k - \ell)^2 - 5\beta_0(2k+1, 2\ell+1)(k - \ell)(k + \ell + 2) \\
&\quad + \varepsilon_0(2k+1, 2\ell+1)(k+1)^2 + \zeta_0(2k+1, 2\ell+1)(\ell+1)^2 \\
&= 10(k - \ell)^2 + 10(k - \ell)(k + \ell + 2) + 2(k+1)^2 \\
&= 2(k+1)(11k - 10\ell + 1) \\
&\leq 0.
\end{aligned}$$

In the case of $k - \ell \leq -2$ and $11k - 10\ell + 1 \geq 1$: since $\beta_0(2k+1, 2\ell+1) = -2$, $\delta_0(2k+1, 2\ell+1) = -1$, $\varepsilon_0(2k+1, 2\ell+1) = \zeta_0(2k+1, 2\ell+1) = 0$, we obtain

$$\begin{aligned}
&(18 + 4\delta_0(2k+1, 2\ell+1))(k - \ell)^2 - 5\beta_0(2k+1, 2\ell+1)(k - \ell)(k + \ell + 2) \\
&\quad + \varepsilon_0(2k+1, 2\ell+1)(k+1)^2 + \zeta_0(2k+1, 2\ell+1)(\ell+1)^2
\end{aligned}$$

$$
\begin{aligned}
&= 14(k-\ell)^2 + 10(k-\ell)(k+\ell+2) \\
&= 4(k-\ell)(6k-\ell+5).
\end{aligned}
$$

Since

$$5(-k+\ell-2)+(11k-10\ell) = 6k-5\ell-10 \geq 0$$

and $6k-\ell+5 > 6k-5\ell-10$, we obtain

$$4(k-\ell)(6k-\ell+5) \leq -8.$$

In the case of $k-\ell \geq 2$ and $-10k+11\ell+1 \leq 0$: since $\beta_0(2k+1,2\ell+1) = 2$, $\delta_0(2k+1,2\ell+1) = -2$, $\varepsilon_0(2k+1,2\ell+1) = 0$ and $\zeta_0(2k+1,2\ell+1) = 2$, we obtain

$$
\begin{aligned}
&(18+4\delta_0(2k+1,2\ell+1))(k-\ell)^2 - 5\beta_0(2k+1,2\ell+1)(k-\ell)(k+\ell+2) \\
&\quad +\varepsilon_0(2k+1,2\ell+1)(k+1)^2 + \zeta_0(2k+1,2\ell+1)(\ell+1)^2 \\
&= 10(k-\ell)^2 - 10(k-\ell)(k+\ell+2) + 2(\ell+1)^2 \\
&= 2(\ell+1)(-10k+11\ell+1) \\
&\leq 0.
\end{aligned}
$$

In the case of $k-\ell \geq 2$ and $-10k+11\ell+1 \geq 1$: since $\beta_0(2k+1,2\ell+1) = 2$, $\delta_0(2k+1,2\ell+1) = -1$, $\varepsilon_0(2k+1,2\ell+1) = \zeta_0(2k+1,2\ell+1) = 0$, we obtain

$$
\begin{aligned}
&(18+4\delta_0(2k+1,2\ell+1))(k-\ell)^2 - 5\beta_0(2k+1,2\ell+1)(k-\ell)(k+\ell+2) \\
&\quad +\varepsilon_0(2k+1,2\ell+1)(k+1)^2 + \zeta_0(2k+1,2\ell+1)(\ell+1)^2 \\
&= 14(k-\ell)^2 - 10(k-\ell)(k+\ell+2) \\
&= 4(k-\ell)(k-6\ell-5).
\end{aligned}
$$

Since

$$5(-k+\ell-2)+(10k-11\ell) = 5k-6\ell-10 \leq 0$$

and $k-6\ell-5 < 5k-6\ell-10$, we obtain

$$4(k-\ell)(k-6\ell-5) \leq -8.$$

In other cases: since $\beta_0(2k+1,2\ell+1) = k-\ell$, $\delta_0(2k+1,2\ell+1) = -1$, $\varepsilon_0(2k+1,2\ell+1) = \zeta_0(2k+1,2\ell+1) = 0$, we obtain

$$
\begin{aligned}
&(18+4\delta_0(2k+1,2\ell+1))(k-\ell)^2 - 5\beta_0(2k+1,2\ell+1)(k-\ell)(k+\ell+2) \\
&\quad +\varepsilon_0(2k+1,2\ell+1)(k+1)^2 + \zeta_0(2k+1,2\ell+1)(\ell+1)^2 \\
&= 14(k-\ell)^2 - 5(k-\ell)^2(k+\ell+2) \\
&= (k-\ell)^2(4-5(k+\ell)) \\
&\leq 0.
\end{aligned}
$$

From the above, T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-weighted generalized pseudocontraction. □

Remark 3.1. By Theorem 3.1 T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-weighted generalized pseudocontraction. Let λ be a mapping from $X \times X$ into $[0,1]$ defined by $\lambda(x,y) = 1$, let $A = \frac{1}{2}$, let $B = 2$, and let $M = 2$.

Then, do α_λ, β_λ, γ_λ, δ_λ, ε_λ, and ζ_λ satisfy (5) in Theorem 2.3?

Clearly, $|\alpha_\lambda(x,y)| \leq M$, $|\beta_\lambda(x,y)| \leq M$, $|\gamma_\lambda(x,y)| \leq M$, $|\delta_\lambda(x,y)| \leq M$, $|\varepsilon_\lambda(x,y)| \leq M$, and $|\zeta_\lambda(x,y)| \leq M$.

In the cases where $x = 1$ and $y = 1$, $x = 1$ and y is even, x is even and $y = 1$, $x = 1$ and y is odd and $y \geq 3$, x is odd and $x \geq 3$ and $y = 1$, x is even and y is even, or x is even and y is odd and $y \geq 3$,

$$\begin{aligned}
\alpha_\lambda(x,y) + \zeta_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\} &= \alpha(x,y) + \zeta(x,y) + 2\min\{\beta(x,y),0\} \\
&= 2, \\
\delta_\lambda(x,y) + \varepsilon_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\} &= \delta(x,y) + \varepsilon(x,y) + 2\min\{\beta(x,y),0\} \\
&= -1, \\
\alpha_\lambda(x,y) + \beta_\lambda(x,y) + \zeta_\lambda(x,y) &= \alpha(x,y) + \beta(x,y) + \zeta(x,y) \\
&= 2 = B
\end{aligned}$$

and

$$-\frac{\delta_\lambda(x,y) + \varepsilon_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\}}{\alpha_\lambda(x,y) + \zeta_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\}} = \frac{1}{2} = A.$$

In the case where x is odd, $x \geq 3$, y is odd, and $y \geq 3$: if $x \geq y$, then since the possible combinations of $\alpha(x,y)$, $\beta(x,y)$, $\gamma(x,y)$, $\delta(x,y)$, $\varepsilon(x,y)$, and $\zeta(x,y)$ are $(2,2,-2,-2,0,2)$, $(2,2,-2,-1,0,0)$, $(2,1,-1,-1,0,0)$, and $(2,0,0,-1,0,0)$,

$$\begin{aligned}
&\alpha_\lambda(x,y) + \zeta_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\} \\
&= \alpha(x,y) + \zeta(x,y) + 2\min\{\beta(x,y),0\} \\
&= \begin{cases} 4, & \text{in the case of } (2,2,-2,-2,0,2), \\ 2, & \text{in the case of } (2,2,-2,-1,0,0), \\ 2, & \text{in the case of } (2,1,-1,-1,0,0), \\ 2, & \text{in the case of } (2,0,0,-1,0,0), \end{cases} \\
&\delta_\lambda(x,y) + \varepsilon_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\} \\
&= \delta(x,y) + \varepsilon(x,y) + 2\min\{\beta(x,y),0\} \\
&= \begin{cases} -2, & \text{in the case of } (2,2,-2,-2,0,2), \\ -1, & \text{in the case of } (2,2,-2,-1,0,0), \\ -1, & \text{in the case of } (2,1,-1,-1,0,0), \\ -1, & \text{in the case of } (2,0,0,-1,0,0), \end{cases} \\
&\alpha_\lambda(x,y) + \beta_\lambda(x,y) + \zeta_\lambda(x,y) \\
&= \alpha(x,y) + \beta(x,y) + \zeta(x,y) \\
&= \begin{cases} 6, & \text{in the case of } (2,2,-2,-2,0,2), \\ 4, & \text{in the case of } (2,2,-2,-1,0,0), \\ 3, & \text{in the case of } (2,1,-1,-1,0,0), \\ 2, & \text{in the case of } (2,0,0,-1,0,0) \end{cases} \\
&\geq B,
\end{aligned}$$

and

$$-\frac{\delta_\lambda(x,y) + \varepsilon_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\}}{\alpha_\lambda(x,y) + \zeta_\lambda(x,y) + 2\min\{\beta_\lambda(x,y),0\}} = \frac{1}{2} = A.$$

Unfortunately, the conditions of the theorem 2.3 are not satisfied in other cases.

Furthermore, since $\mathbb{N}$ is discrete, Theorem 2.2 is sufficient without using Theorem 2.3, however, the conditions of the theorem 2.2 are also not satisfied.

References

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