Efficient, Truthful, Approximately Optimal Mechanisms

From S. Dughmi, T. Roughgarden, and Q. Yan

COS 597F — Open Problems in Algorithmic Game Theory
Prof. Matt Weinberg
Final Project Report

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1 Introduction

In the following report we examine *Dughmi, Roughgarden, and Yan's* paper *From Convex Optimization to Randomized Mechanisms: Toward Optimal Combinatorial Auctions* along with relevant supplemental materials [2]. We seek to provide value to someone reading the original paper in several ways:

- Provide a more clear outline of their approach and the components of their method
- Parse and give intuition and detail for their proofs
- Provide some relevant information which was later clarified in a follow-up paper by the same authors

To accomplish this, we first give a succinct revised background, followed by a visualization of the paper outline, an intuitive explanation of the results, and finally a presentation of the more pertinent proofs, providing detail and explanation beyond what is given in the original paper to make them more understandable to the unacquainted reader.

Disclaimer.[1] All the results in this paper are from [2], [1], or one of their references. We claim no ownership over any of the content.

2 Background

In the following sections we provide relevant definitions for the problem setup and properties we wish to examine. We then set up the modelling framework used, including the valuation function classes examined, and the oracle models considered.

2.1 The Problem Setup

The problem we consider is *welfare maximization* in *combinatorial auctions*. Here we define a **combinatorial auction** to have a set of m items and n bidders. The *mechanism* must provide a **feasible allocation** $x = [S_1, ..., S_n] \subset [\{0,1\}^m]^n := \mathcal{S}$ of the items to the bidders such that $S_1, ...S_n$ are a partition of the items, as well as a payment rule that specifies how much each bidder should be charged. Given that bidder i has value $v_i(S_i)$ for an allocated subset of items, we define the **welfare** to be $w(x) = \sum_i v_i(S_i)$. Thus the base problem we consider is

$$\begin{aligned} &\text{maximize } w(x) \\ &\text{subject to } x \in \mathcal{S}. \end{aligned} \tag{1}$$

After providing an allocation, the *seller* will provide prices that the bidders must pay in order to get the items. The seller does not initially know the valuations of the bidders, but must obtain this information in some way (through a bid, or queries of some kind). However, in the setting we consider, since the bidders are trying to maximize their utility, they may be incentivized to lie. Thus we consider only cases where the mechanism is *truthful* - that is to say, the bidders are incentivized to give their true valuations when queried.

[1] Honor code is scary.

The goal of this paper is to design a mechanism which is efficient, truthful, and which provides welfare comparable to that which would be obtained if the seller knew the valuations up-front.

In this paper, the notion of **efficient** will be for everything to run in time polynomial in n and m.

The notion of truthfulness will be *truthful in expectation*. Here we first define $(1-\epsilon)$-**approximately truthful in expectation** ($(1-\epsilon)$-**TIE**) as follows:
Definition 1 ([1]). *A mechanism with allocation rule A and payment rule p is $(1-\epsilon)$-approximately truthful-in-expectation ($(1-\epsilon)$-TIE) if, for every bidder i, true valuation function v_i, reported valuation function v_i', and reported valuation functions v_{-i} of the other bidders:*

$$\begin{aligned} &\mathbb{E}[v_i(A(v_i, v_{-i})) - p_i(v_i, v_{-i})] \\ &\geq (1-\epsilon)\mathbb{E}[v_i(A(v_i', v_{-i})) - p_i(v_i', v_{-i})]. \end{aligned} \tag{2}$$

Where standard truthfulness in expectation (TIE) results from setting $\epsilon = 0$. We note that this notion of approximate truthfulness in expectation corresponds to the situation where telling the truth costs a bidder at most an ϵ-fraction of their optimal utility. Intuitively this notion may make sense to use if bidders don't care beyond this ϵ-fraction enough to go out of their way to exploit the system, but this notion also results naturally when using a particular oracle access model that we will discuss later.

For optimality of welfare maximization, we will define an **approximately optimal mechanism** as follows:
Definition 2. *Let the optimal achievable social welfare be OPT. Then a mechanism is an α-approximation for welfare maximization if it achieves at least $\alpha \cdot OPT$ welfare in expectation (over randomness in the allocation rule).*

2.2 Basic Approach

The best-known approach to obtain truthfulness for randomized mechanisms (as the authors examine here) is using *Vickrey-Clarke-Groves (VCG)*-like payment schemes for algorithms which are *maximal-in-distributional-range*. We recall that VCG-like payment schemes work by aligning the utility of the bidders with the welfare; thus when bidders maximize their own utility, they maximize the welfare as a byproduct. For deterministic algorithms, **maximal-in-range** allocation algorithms (for which standard VCG payment schemes can be used) select the allocation which maximizes welfare from among a set of feasible allocations, and then VCG payments make it so that it is incentive compatible for the bidders to be truthful. For randomized algorithms, **maximal-in-distribution-range (MIDR)** allocation algorithms select a *distribution* D_x from a precommitted class of distributions $\mathcal{D}$ over feasible allocations (the *distributional range*) which maximizes the expected (reported) welfare [4].

Thus the approach of this paper is as follows:

- Recall that the welfare-maximization problem can be written as an (integer) optimization problem
- Since this is NP-hard, relax the problem to make it tractable, and then map the relaxed solution back to an integer solution via a *rounding scheme*

- Make sure that the allocation algorithm produced is MIDR, so that then we can use VCG-like payments to make the mechanism truthful

Note that this rounding approach **must** cost us *something*; the hardness of the original problem necessitates this. In this particular paper, the cost manifests itself as obtaining a constant-factor approximation of the optimal welfare, and also as some cost to efficiency and/or truthfulness. There is some optional trade-off in the cost to truthfulness or tractability that will be discussed in the next section.

2.3 Relaxation Model and Motivation

We recall from above the welfare-optimization problem P that we're considering:

$$\begin{aligned} &\text{maximize } w(x) \\ &\text{subject to } x \in \mathcal{S}, \end{aligned}$$

where we want to maximize the welfare function over the domain of feasible allocations.

The standard relaxation of this problem gives a P' of the form:

$$\begin{aligned} &\text{maximize } w_{\mathcal{R}}(x) \\ &\text{subject to } x \in \mathcal{R}. \end{aligned} \tag{3}$$

where we extend the set of feasible allocations, and a natural extension of the welfare function to this relaxed set (for example, this could be fractional allocations with a linearly extended welfare function).

The key motivation and insight of this paper was that, up until this point, existing literature optimized on the outcome of the relaxation and then rounded. The authors wanted to instead consider rounding the relaxed solution *first* and then optimizing on this output.

More specifically, we let $r(x)$ denote a randomized rounding scheme, applied to x, a fractional allocation output of the relaxed optimization problem. Then we compute such an x that maximizes the expected welfare $\mathbb{E}_{y\sim r(x)}[w(y)]$. Formally, the new relaxed optimization problem is:

$$\begin{aligned} &\text{maximize } \mathbb{E}_{y\sim r(x)}[w(y)] \\ &\text{subject to } x \in \mathcal{R} \end{aligned} \tag{4}$$

We see that this formulation has an immediate advantage: Any solution will be MIDR since we are restricting to the class of distributions induced by the randomized rounding function (as applied to elements of the relaxed allocation space $\mathcal{R}$), and then maximizing over this distributional range.

Unfortunately, since this approach requires computing the expectations in (4), value queries are not sufficient unless we are willing to incur a large computational cost. We can get around this by utilizing a more powerful oracle to the reported valuation functions, called a *Lottery-Value Oracle (LVO)* [2].

[2]Lottery-value queries are #P-hard to answer for the class of *matroid rank functions*, the class we examine later. Motivated by this, the authors released a follow-up paper which uses the more standard *value oracle* model, at a small cost for efficiency, truthfulness, and optimality. We discuss this in further detail in the conclusion.

Definition 3. *Given a valuation function v, and a probability distribution over items D_x, a* ***lottery-value oracle*** *outputs $F_v(x) = \mathbb{E}_{S \sim D_x}[v(S)]$.*

3 Summary and Exposition of Main Results

3.1 Framework Result

The foremost result of the paper is a blackbox-style framework which takes as input both a Randomized Rounding Scheme and class of valuation functions which together jointly satisfy certain conditions, namely

- The rounding algorithm runs in poly(n,m) time on possible inputs;
- Is α-*approximate* (defined below);
- And is a convex rounding scheme (defined below) on possible inputs;

and outputs a mechanism which satisfies the desired properties outlined above, reiterated:

- The protocol runs in poly(n,m) time;
- Gives an α-approximation to the optimal welfare;
- is truthful-in-expectation.

Definition 4. *A* ***rounding scheme*** *$r : \mathcal{R} \to \mathcal{S}$ is* ***convex*** *if $\mathbb{E}_{y \sim r(x)}[w(y)]$ is a concave function of $x \in \mathcal{R}$.*

Definition 5. *A rounding scheme is α-****approximate*** *for $\alpha \leq 1$, if for all $x \in S$:*

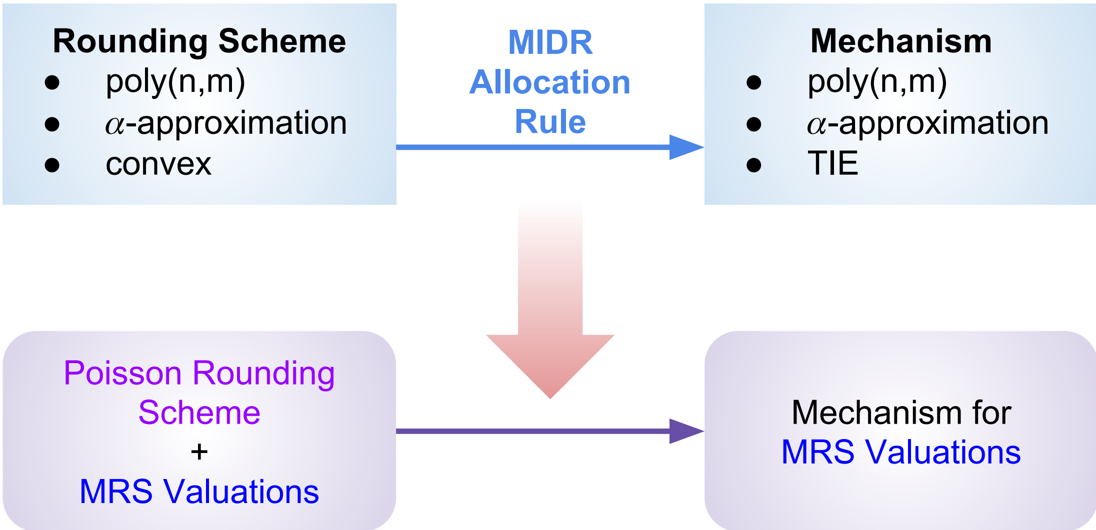
$$w(x) \geq \mathbb{E}_{y \sim r(x)}[w(y)] \geq \alpha \cdot w(x)$$

3.2 Motivating Application of Framework

Given this framework, the natural question is: so what? In particular, are there any polytime convex randomized rounding schemes for classes of valuation functions that people care about? It isn't immediately obvious that the above result is useful (in fact, there is an easy-to-see hardness result that essentially shows that any 'simple' rounding scheme can't give a useful approximation, due to the hardness of the original problem — if a rounding scheme rounds all feasible solutions to themselves, then any optimal solution to P' is a solution to P, and thus the relaxed problem must also be intractable assuming $P \neq NP$).

Considering this, the authors give an instance in which the framework is useful. Specifically, they show that, for *matroid rank sum (MRS) valuations*, the Poisson Rounding

Scheme fulfills the required properties (for $\alpha = 1 - 1/e$), and thus gives a mechanism with a $(1 - 1/e)$-approximation to the optimal welfare. [3]



We provide a series of definitions requisite to understanding this result below.

Definition 6. *A **matroid** $\mathcal{M}$ is a pair of a set E and a family of subsets of E, $\mathcal{I}$ called independent sets such that*

- $\emptyset \in \mathcal{I}$
- *Every subset of an independent set it also an independent set (i.e. it's downwards closed)*
- *The matroid exchange property: If A and B are two independent sets and $\#(A) > \#(B)$, then $\exists x \in A$ such that $B \cup \{x\} \in \mathcal{I}$.*

Definition 7. *The **rank function** $r(A)$ of a matroid has the following properties:*

- *The value of $r(A)$ is always a non-negative integer.*
- *For $A \subset E$, $r(A) \leq \#(A)$*
- *The rank is submodular*
- *For any set A and element x, $r(A) \leq r(A \cup \{x\}) \leq r(A) + 1$.*

Definition 8. *A **submodular** function is a set function $f : 2^{\Omega} \to \mathbb{R}$ (where Ω is a finite set), such that for every $X, Y \subseteq \Omega$, we have $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$.*

Definition 9. *A set function is a **matroid rank sum function** if there exists a family of matroid rank functions $u_1, ..., u_k$ and associated non-negative weights $w_1, ..., w_k$ such that $v(S) = \sum_{i=1}^{k} w_i u_i(S)$.*
Note that these are submodular.

Definition 10. *The **Poisson Rounding Scheme** is a randomized rounding scheme from the relaxed fractional solution space to the feasible solution space $r_{poiss} : \mathcal{R} \to \mathcal{S}$ by taking a fractional solution matrix $\boldsymbol{x}$ and assigning each item j to player i with probability $1 - e^{-x_{ij}}$ (and to no one otherwise).*

[3] A result from [5] shows that this is in fact tight, and thus a useful approximation factor.

4 Outline of Paper and Results

Now that we've given all of the necessary background and stated the main results, we give an illustration outlining how the authors go about proving the results. In particular, on the left side we give the framework in blue, with dependencies on the lemmas they prove and concepts they define. On the right side we give the main application they provide in orange, with dependencies on the corresponding lemmas.

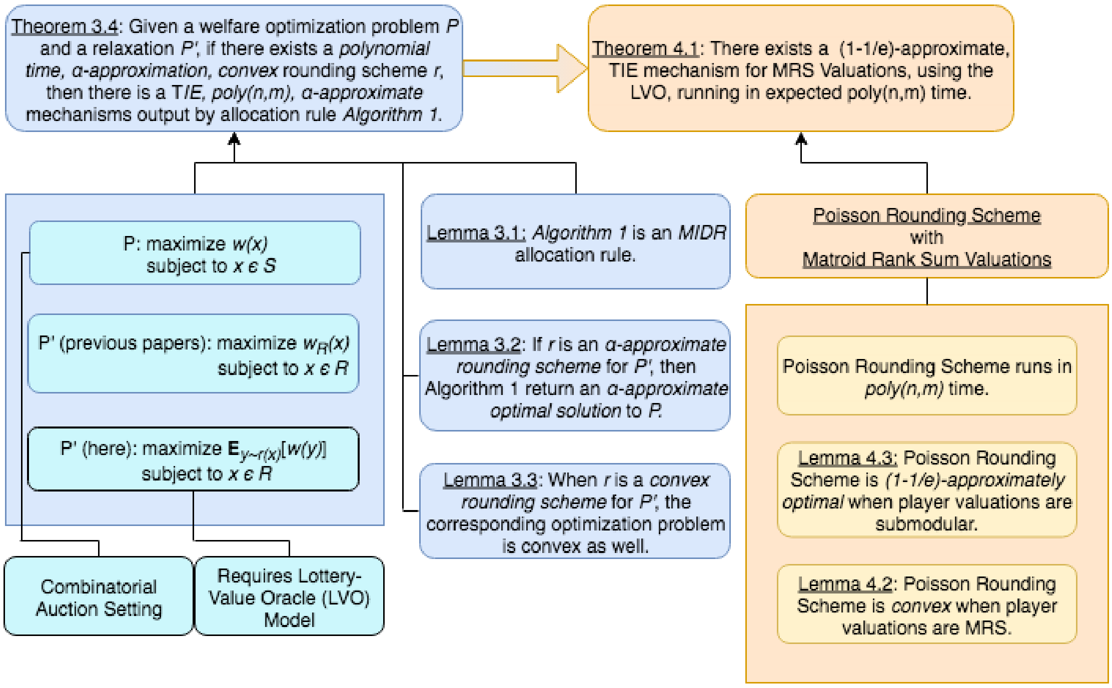


Figure 1: Outline of the structure and dependencies of [2]

Before moving on to the exact proof details, we first give an overview of the properties of r_{poiss} and MRS that are used in order to prove lemmas 4.2 and 4.3, since these are the most critical aspects to trying to extend these proofs to other classes of valuation functions.

Showing that r_{poiss} gives a $(1 - 1/e)$-approximation (Lemma 4.3 of [2]) uses basic algebraic manipulation combined with the submodularity of MRS. Note that this means that r_{poiss} may apply to other subclasses of submodular valuations, as long as they also are shown to be convex for those valuations. We will provide the proof in the next section.

Showing the convexity of r_{poiss} (Lemma 4.2 of [2]) is more complicated. To do so for MRS valuations, the authors first define a discrete version of the Hessian, and then they show that it's negative semi-definite for MRS valuations. The properties of MRS valuations that are used here are just the property of matroid rank functions that $v(T \cup \{j\}) \leq v(T) + 1$, and the matroid exchange property. Using these, they show that the negative discrete Hessian is symmetric, encodes a transitive relation, and is binary, thus it is a block diagonal matrix where each diagonal block is all-ones or all-zeros, which is known to be positive semi-definite, and thus the discrete Hessian is negative semi-definite. Then they use additional reasoning to conclude that this implies that r_{poiss} is convex for MRS valuation functions. Specific details and definitions for this proof are given in the following section.

5 Proofs

As stated previously, the authors provide an instance in which their framework is applicable. In particular, they show that the Poisson Rounding Scheme, coupled with MRS valuations satisfy the criteria of the framework. In this section, we provide lemmas justifying why each criterion is satisfied. These correspond to Lemma 4.2 and Lemma 4.3 in the original paper and were relatively more involved technically, so here we have tried to fill any gaps and provide more detailed explanations for each logical step.

It is easy to see that the Poisson Rounding Scheme is executable in poly(n, m) time. Showing the other two criteria — α-approximateness and convexity — is more involved and is outlined in the lemmas below.

Lemma 1. *The Poisson Rounding Scheme is* $1 - 1/\mathrm{e}$*-approximate with submodular valuations.*

Proof. We need to show that $w(x) \geqslant \mathbb{E}_{y\sim r(x)}[w(x)] \geqslant (1 - 1/\mathrm{e}) \cdot w(x)$ for every feasible allocation $x \in \mathcal{S}$. Fix any such allocation $\mathcal{S} = (S_1, \ldots, S_n)$ and the corresponding binary vector $x \in \{0, 1\}^{m\times n}$. Suppose it rounds to a different allocation $\mathcal{S}' = (S'_1, \ldots, S'_n) \sim r_{poiss}(x)$. We must show $\sum_{i=1}^{n} v_i(S_i) \geqslant \mathbb{E}_{y\sim r(x)}[\sum_{i=1}^{n} v_i(S'_i)] \geqslant (1 - 1/\mathrm{e}) \sum_{i=1}^{n} v_i(S_i)$.

For integral x specifically, each item in S_i also belongs to S'_i with probability $1 - 1/\mathrm{e}$ and no other item can belong to S'_i (so $S'_i \subseteq S_{'}$). The first inequality follows trivially from monotonicty and linearity of expectation:

$$\mathbb{E}_{y\sim r(x)}\left[\sum_{i=1}^{n} v_i(S'_i)\right] = \sum_{i=1}^{n} \mathbb{E}_{y\sim r(x)}[v_i(S'_i)] \leqslant \sum_{i=1}^{n} \mathbb{E}_{y\sim r(x)}\left[v_i(S_i)\right] = \sum_{i=1}^{n} v_i(S_i).$$

To show the second half, follows directly from a lemma due to Lee *et al* [6] (among others *e.g.* [3], who proved related results). Informally, this lemma states that given an allocation $S_1, \ldots, S_n$, and a second allocation $S'_1, \ldots, S'_n$ that is constructed by drawing every element of S_i into S'_i with some probability α. Then the expected welfare achieved by the S'_i is at least α times the expected welfare achieved by the S_i. □

Lemma 2. *The Poisson Rounding Scheme is convex with matroid rank sum valuations.*

Proof. Consider matroid rank sum valuation functions $\{v_i\}$. As before, let $\mathcal{S}$ be the set of feasible allocations, and $\mathcal{R}$ its relaxation for fractional allocations. Fix an $x \in \mathcal{R}$ and let $r_{poiss}(x) = (S_1, \ldots, S_n) \in \mathcal{S}$ be the corresponding allocation. We need to make the maximization problem 4 tractable by making the objective concave. In other words, we want to choose r such that $\mathbb{E}_{y\sim r(x)}[\sum_{i=1}^{n} v_i(S_i)]$ is concave. By linearity of expectation, and the fact that non-negative combinations of concave functions are concave, it suffices to show that $\mathbb{E}_{y\sim r(x)}[v_i(S_i)]$ is concave for each bidder i.

Part 1) Relationship between Hessian and discrete Hessian. In order to show concavity, we use a generalized notion of the Hessian matrix to set functions: Given a set function v, and a set $S \subseteq [m]$, the *discrete Hessian matrix* $\mathcal{H}_\mathcal{S}^v \in \mathbb{R}^{m\times m}$ of v at S is defined as

$$\mathcal{H}_\mathcal{S}^v(j,k) = v(\mathcal{S}\cup\{j,k\}) - v(\mathcal{S}\cup\{j\}) - v(\mathcal{S}\cup\{k\}) + v(\mathcal{S}).$$

The first claim is that the discrete Hessian works in roughly the same way as their continuous analogue in characterizing convexity: If all discrete Hessians of v are negative semi-definite, the objective function is concave.

To show this, we rewrite the expected welfare of bidder i as

$$G_v(x_1,\ldots,x_m) \coloneqq \mathbb{E}_{y\sim r(x)}[v(S)] = \sum_{S\subseteq[m]} v(S)\prod_{j\in S}(1-e^{-x_j})\prod_{j\notin S} e^{-x_j},$$

where we have dropped the i subscript in $v_i(\cdot)$, x_{ij}, and S_i for simplicity. We need to show that if $\mathcal{H}_\mathcal{S}^v$ is negative semi-definite for all $S \subseteq [m]$, then G_v is concave. Consider the mixed derivative of G_v with respect to x_j and x_k:

$$\begin{aligned}
&\frac{\partial^2 G_v(x)}{\partial x_j \partial x_k}\\
&= \sum_{S\subseteq[m]\setminus\{j,k\}} \prod_{l\in S}(1-e^{-x_l})\prod_{l\notin S} e^{-x_l}\Big(v(S) - v(S\cup\{j\}) - v(S\cup\{j\}) + v(S\cup\{j,k\})\Big)\\
&= \sum_{S\subseteq[m]} \prod_{l\in S}(1-e^{-x_l})\prod_{l\notin S} e^{-x_l}\Big(v(S) - v(S\cup\{j\}) - v(S\cup\{j\}) + v(S\cup\{j,k\})\Big)\\
&= \sum_{S\subseteq[m]} \prod_{l\in S}(1-e^{-x_l})\prod_{l\notin S} e^{-x_l}\mathcal{H}_S^v(j,k).
\end{aligned}$$

The first equality holds by splitting the sum into 4 parts

- Sets $T \not\ni \{j,k\}$: The contribution of these terms to the coefficient of $v(T)$ is $a_1 \coloneqq \frac{\partial^2}{\partial x_j \partial x_k} e^{-x_j}e^{-x_k}C$, where C is a constant w.r.t x_j and x_j. This is equal to $(-e^{-x_j})(-e^{-x_j})C = e^{-x_j}e^{-x_k}C = a_1$,
- Sets T such that $j\in S$ and $k \notin S$: The contribution of these terms to the coefficient of $v(T)$ is $a_2 \coloneqq \frac{\partial^2}{\partial x_j \partial x_k} e^{-x_j}(1-e^{-x_k})C$, where C is a constant w.r.t x_j and x_j. This is equal to $(-e^{-x_j})(1-e^{-x_j})C = -a$. Furthermore, these sets can be written as $S\cup\{j\}$ where S varies over all subsets of $[m]\setminus\{j,k\}$.
- Sets T such that $k \in S$ and $j \notin S$: This is similar to the previous case.
- Sets T such that $\{j,k\} \in S$: The contribution of these terms to the coefficient of $v(T)$ is $a_4 \coloneqq \frac{\partial^2}{\partial x_j \partial x_k}(1-e^{-x_j})(1-e^{-x_k})C$, where C is a constant w.r.t x_j and x_j. This is equal to $(1-e^{-x_j})(1-e^{-x_j})C = a$. Furthermore, these sets can be written as $S\cup\{j,k\}$ where S varies over all subsets of $[m]\setminus\{j,k\}$.

The second equality holds since for matroid rank functions, $v(T\cup\{j\}) = v(T)$ if $j\in T$, and thus $v(S) - v(S\cup\{j\}) - v(S\cup\{j\}) + v(S\cup\{j,k\}) = 0$ when $j \in S$ or $k\in S$. The last equality was obtained via substituting the definition of the discrete Hessian $\mathcal{H}_\mathcal{S}^v$.

The above implies that the Hessian of G_v can be written as a non-negative weighted combination of discrete Hessian matrices. So any rounding function with negative semi-definite discrete Hessians is concave.

Part 2) Discrete Hessians are NSD for Poisson Rounding. First, we claim that it suffices to show that discrete Hessians are negative semi-definite for matroid rank *functions* (rather than sums). Observe that $\mathcal{H}_S^v$ is linear in $v(\cdot)$, and thus can be written as a non-negative weighted sum of $\mathcal{H}_S^{u_i}$, where u_i are matroid rank functions. Since non-negative sums of concave NSD matrices are NSD, it suffices to show $\mathcal{H}_S^v \preceq 0$ when v is a matroid rank function.

Let v be the rank function for some matroid M with ground set $[m]$. Fix some $S \subseteq [m]$. A simple case analysis implies

$$\mathcal{H}_S^v(j,k) = \begin{cases} -1 & \text{if } v(S)+1 = v(S \cup \{j\}) = v(S \cup \{k\}) = v(S \cup \{j,k\}) \\ 0 & \text{otherwise.} \end{cases}$$

Let M' be the contracted matroid M/S with independent set I' and rank function $v'(T) \coloneqq v(S \cup T) - v(S)$.

- $\{j\} \notin I'$: In this case,

  $$\begin{aligned} v(S \cup \{j\}) &= v(S) + v'(\{j\}) = v(S) \\ v(S \cup \{j,k\}) &= v(S) + v'(\{j,k\}) = v(S) + v'(\{k\}) = v(S \cup \{k\}), \end{aligned}$$

  where we have used the definition of the rank function of a contracted matroid, and the downwards closed property of independence. Therefore, $\mathcal{H}_S^v(j,k) = 0$.
- $\{k\} \notin I'$: This is similar to the previous case.
- $\{j,k\} \in I'$: By the downwards closed property, $\{j\}, \{k\} \in I'$. We have

  $$\begin{aligned} v(S \cup \{j,k\}) &= v(S) + v'(\{j,k\}) = v(S) + 2 \\ v(S \cup \{j\}) &= v(S) + v'(\{j\}) = v(S) + 1 = v(S) + v'(\{k\}) = v(S \cup \{k\}), \end{aligned}$$

  where again we have used the definition of the rank function of a contracted matroid, and the downwards closed property of independence. Overall, we again have that $\mathcal{H}_S^v(j,k) = 0$.
- $\{j\}, \{k\} \in I'$ and $\{j,k\} \notin I'$:

  $$\begin{aligned} v(S \cup \{j,k\}) &= v(S) + v'(\{j,k\}) = v(S) + 1 \\ v(S \cup \{j\}) &= v(S) + v'(\{j\}) = v(S) + 1 = v(S) + v'(\{k\}) = v(S \cup \{k\}), \end{aligned}$$

  In this case, we get $\mathcal{H}_S^v(j,k) = -1$.

In other words, the matrix $-\mathcal{H}_S^v$ is a binary matrix, in which entry (j,k) is 1 if and only if the following two conditions are satisfied:

1. Both $\{j\}$ and $\{k\}$ are independent sets in M/S,
2. $\{j,k\}$ is a dependent set in M/S.

Finally, observe that $-\mathcal{H}_S^v$ is *symmetric*, *i.e.* $-\mathcal{H}_S^v(j,k) = -\mathcal{H}_S^v(k,j)$, since k and j are interchangeable. Furthermore, $-\mathcal{H}_S^v$ corresponds to a *transitive* relation, *i.e.* for all $j,k,l \in [m]$, $-\mathcal{H}_S^v(j,k) = -\mathcal{H}_S^v(k,l) = 1$ implies $-\mathcal{H}_S^v(j,l) = 1$. To see this, suppose $-\mathcal{H}_S^v(j,k) = -\mathcal{H}_S^v(k,l) = 1$. Similarly to above, let I' be the independent set of the contracted matroid M/S. Then, by the characterization above, we know that

1. $\{j\}, \{k\}, \{l\} \in I'$

2. $\{j,k\},\{k,l\} \notin I'$

Assume for contradiction that $-\mathcal{H}^v_S(j,l) = 0$. The only way for j and l to violate the characterizations is through $\{j,k\} \in I$. However, applying the exchange property to $\{k\}$ and $\{j,k\}$ implies that either $\{j,k\}$ or $\{k,l\}$ is in I', which is a contradiction.

We have shown that $-\mathcal{H}_S$ is a symmetric, transitive, binary matrix. Therefore, it must be a block diagonal matrix with all-ones and all-zeros blocks. One can check, for example via Sylvester's criterion, that such a matrix is positive semi-definite, so its negation is negative semi-definite, thus completing the proof. □

6 Concluding Remarks and Next Steps

There are several directions to take this paper. As noted in a footnote, the #P-hardness of LVO queries for MRS valuations means that this model may be unrealistic in general. The authors address this in their follow-up work [1]. In this paper, they use the more standard value query oracles (where a seller can ask simply for the value of a subset of the items $S \subseteq [m]$), and obtain a similar result, except that instead one obtains $(1-\epsilon)$-TIE, the mechanism runs in time poly$(n, m, 1/\epsilon)$, and obtains welfare $(1-1/e-\epsilon)\cdot OPT$, for $\epsilon \in 1/\text{poly}(m,n)$.

The other natural extension to their work would be to find other valuation function classes or rounding algorithms to apply their framework to. In order to use their results for the Poisson rounding scheme, the valuation class must be a proper subset of submodular valuations (they give an example of a submodular valuation function which makes the rounding scheme non-convex in their Appendix D.2), but as only a few properties of MRS functions are used in the relevant proofs, it may be possible to extend their result to a larger class.

It's also possible that, with small modifications or another rounding scheme, their framework could be extended to other valuation function classes. The basic idea of optimizing on the output of a rounding function while naturally obtaining a MIDR allocation rule may apply more generally. However, if we care about tractability, the rounding schemes cannot be 'too simple."

They note in Appendix D.1 that the mechanism they provide via VCG-like payments is individually rational and gives non-negative payments both in expectation. They leave open the question of whether or not it's possible to have truthfulness and non-negative payments ex-post, as opposed to in expectation as provided here [2].

Finally, while in this paper the authors wanted to have polytime efficiency, it's possible that their mechanism construction protocol could have implications if addressed in the context of communication/taxation complexity.

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