Junior Seminar on Geometry Final Paper: Hyperbolic Geometry and Knot Complements

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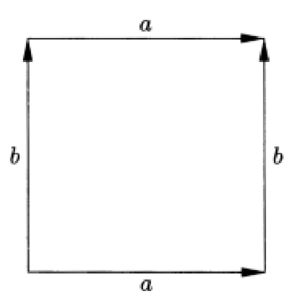
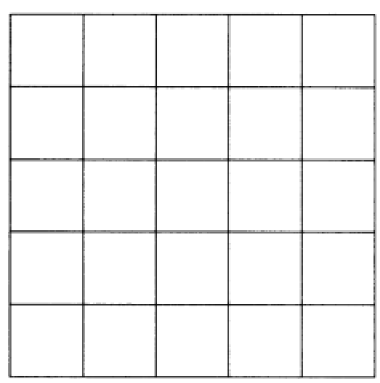
1 Introduction

1.1 Hyperbolic Geometry

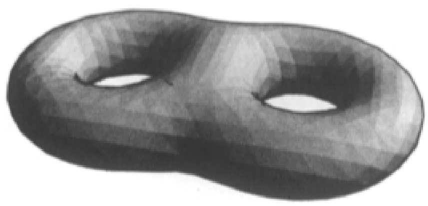
In the study of geometry, a very naturally arising question when looking at different manifolds is to ask: which ones can be given certain metrics? In the first chapter of his famous book *Three-Dimensional Geometry and Topology, Volume 1*, Thurston motivates this by showing that the *gluing pattern* [1] for a torus can be fit together to tessellate $\mathbb{R}^2$ (note that the edges/arrows line up correctly), thus it is easy to see that we can give the torus a standard Euclidean metric; it is just inherited from $\mathbb{R}^2$.

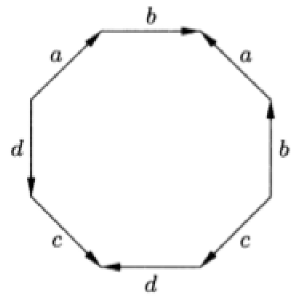




[1] A gluing pattern is a way to represent manifolds where one identified points along labelled edges with points labelled identically. I.e. the points alone edge a below match to the identical points on the other side; where the arrow indicates orientation.

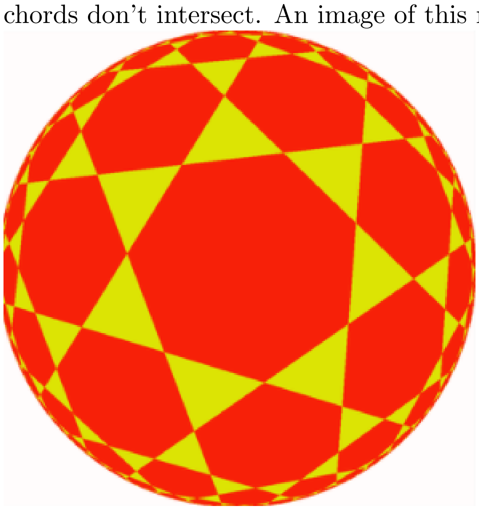
He continues by showing that the gluing pattern for a genus two surface



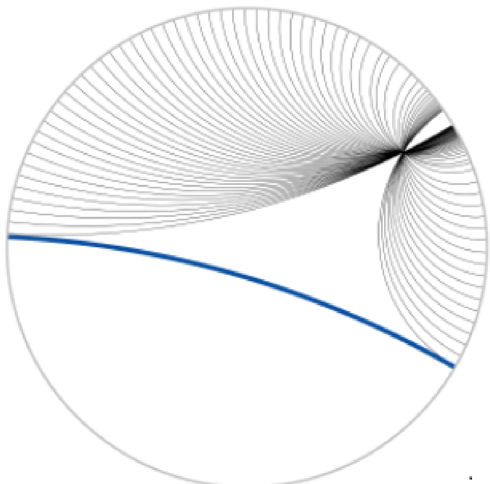


can not (at first glance) be fit together to tessellate $\mathbb{R}^2$ (Thurston). In the world of Euclidean geometry, this is quite clear since the gluing pattern requires 8 angles (which must sum to 1080 degrees), far beyond the 360 degrees required to tessellate. However, this limitation on sum of the interior angles of an octagon is only implied by Euclid's parallel postulate - since this forces a triangle's interior angles to sum to 180 degrees, and thus the necessary angles for other polygons can easily be deduced - and despite how intuitive this postulate is, by the early 1800s it was independently discovered by several mathematicians (including Bolyai, Gauss, and Lobachevsky) that one could construct another self-consistent geometry without this postulate, now better known as hyperbolic geometry.

We briefly introduce several representations of the hyperbolic plane. The first is the *Klein-Beltrami* (or simply Klein) model. This model is represented as an open disk in the Euclidean plane such that chords of the disk correspond to hyperbolic lines. Lines are parallel in this model when their chords don't intersect. An image of this model tiled by triangles and heptagons is given below.

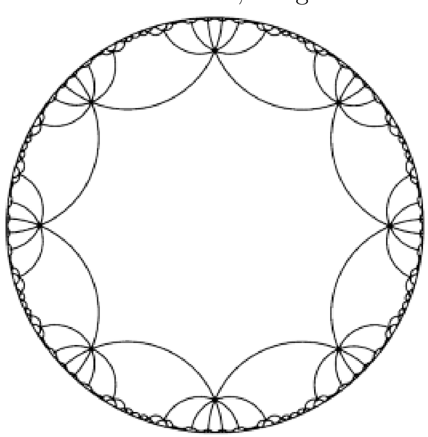


Another representation of the hyperbolic plane is the Poincaré hyperbolic disk, also called the conformal disk, since angles in this model are preserved. This model is a disk, where straight lines are arcs of circles that are perpendicular to the boundaries of the disk. The image below shows lines parallel to the blue line, as represented in this model.



Of course there exist other useful models of hyperbolic space. These, as well as a more in-depth discussion of the models mentioned here, are given in Chapter 2 of Martelli's *An Introduction to Geometric Topology.*

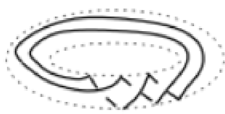
Going back to our problem of tiling two dimensional space with the gluing pattern for a genus two surface, we see that with the addition of hyperbolic metrics, we may resolve our above conundrum. We can give the genus two surface's gluing-pattern-octagon eight 45-degree angles, which may then tessellate $\mathbb{R}^2$ with the hyperbolic metric, as shown below in the hyperbolic plane via the Poincaré disk model, image from Chapter 1 of Thurston's referenced book.[2]

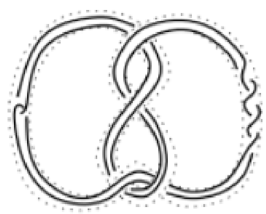


[2]There is a story about Thurston given in an AMS article that in his graduate oral exam, when asked for an example of the universal cover of a space, "Bill chose the surface of genus two and started drawing awkward octagons with many (eight) coming together at each vertex. This exposition quickly became an unconvincing mess on the blackboard. I think Bill was the only one in that exam room who had ever thought about such a nontrivial universal cover." Presumably he made this example the motivation for his textbook ironically (Gabai).

1.2 Knot Theory

We now make a jump to an initially seemingly unrelated field of mathematics: knot theory. As Jessica Purcell writes in her notes on Hyperbolic Knot Theory: while knots have been studied since the early 1800s, their link to hyperbolic geometry was not deeply investigated until the work of William Thurston in the late 1900s. The main topic of study related to knots then become investigating their complements in S^3 (the one-point compactification of $\mathbb{R}^3$ - i.e. with a point at infinity). Thurston famously proved that every knot complement can be categorized into one of three types: Seifert fibered, toroidal, or hyperbolic, which correspond to torus, satellite, and hyperbolic knots, respectively. Torus knots are the knots which can be drawn on the surface on the torus, satellite knots can be drawn "inside the complement of a (possibly knotted) solid torus" (Thurston), and all others are hyperbolic. Images of a torus and satellite knot example are given below, respectively, from Purcell's notes.





It is worth noting that these classifications all relate to his famous geometrization conjecture - which attempted to classify the decomposition of 3-manifolds - that Grigori Perelman famously proved in 2003.

Here we will focus on hyperbolic knots, which make up a significant majority of knots. As stated by Purcell, we know from the research of Hoste, Thistlethwaite, and Weeks that 1,701,903 of the 1,701,936 knots with 16 crossings are hyperbolic. Through a sequence of results by Mostow, Prasad, Gordon, and Luecke, we know that the hyperbolic structure on a knot complement is a complete invariant of the knot, thus the investigation of hyperbolic structures on knot complements leads to classification of the largest subdivision of knots in accordance with Thurston's work.

2 Procedure and Intuition

The remainder of the paper will give an in-depth overview on the techniques used to construct hyperbolic metrics on complements of knots, and then run through some examples of the construction for the figure-eight knot, Whitehead link, and Borromean rings.

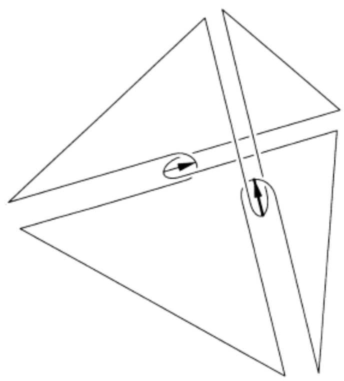
2.1 Decomposition into Ideal Polyhedra

The first and perhaps most significant part to the construction is to characterize the complement of the knot. We do this by dividing the complement into a union of *ideal polyhedra*.[3] By "dividing" it into a "union" of these polyhedra, I mean that the polyhedra we find can have their faces and edges labelled in such a way that they can be to each other to reconstruct the complement of the knot. One thing that is important to keep in mind when imagining this decomposition, is that we

[3] An ideal polyhedron is a polyhedron with the vertices removed. We will see the reason for this removal later.

are working in S^3. This means that, given a set of faces bounding a polyhedron in three dimensional space, this set of faces actually bounds two polyhedra: the "inside" and the "outside." It is this mental image that we will take advantage of when finding our decomposition.

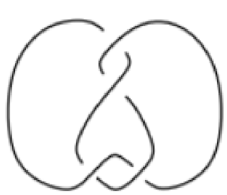
But how can we find a particular set of polyhedra corresponding to the complement of our particular knot? The answer is that we find a two-complex that spans the knot, and look at the regions it bounds. Unfortunately, this answer doesn't provide much insight into what we are actually doing, and in what way all of these things - the two-complex, the polyhedral decomposition, and the knot itself - relate. Fortunately, a particularly informative image is given below taken from page 41 of Thurston's aforementioned book.



We see here the example of the figure-eight knot - one to which we will later return - positioned in three-space such that the complex used is clear, and it is also clear that the complex bounds a tetrahedron. As stated above, this actually shows that it bounds two tetrahedra, thus giving the decomposition of the complement of the figure-eight to be two tetrahedra, glued at the faces of the two-complex. This image also highlights another important idea to keep in mind while going through the constructions: the knots should be thought about as being in three dimensions, and accordingly, the edges we create in our decomposition should be thought of in three dimensions (even if we are stuck with a 2 dimensional representation for the purposes of this paper).

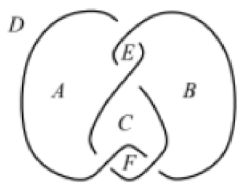
This construction of the decomposition of the figure-eight knot may be shown very nicely by Thurston, but for most knots, coming up with such a nice geometric construction is impractical, thus we want a more systematic approach. Such a device is given by Purcell, and it has the following steps (with several images from Purcell's notes):

1. Begin with a *knot diagram* of the knot. [4]



[4]This is simply a projection of the knot into two dimensions, with the over/under-crossings indicated above.

2. Label the regions bounded by the knot diagram (including the outermost region, extending to infinity). These represent a superset of the faces of our to-be-constructed ideal polyhedra.



3. Add a vertical edge at each crossing. Here we must imagine the knot in three dimensions, where the new edge runs vertically between the part of the knot crossing over/under. It may help to label these new edges in a way that distinguishes them; Purcell does this with different numbers of tick-marks on arrows representing the edges.

4. We now begin to think of the two ideal polyhedra that our faces will be bounding as the space above and the space below the knot. While each extends to infinity, this "extending to infinity" part is bounded by region "bounded" by the knot diagram that extends to infinity. We now see why we are using ideal polyhedra, since in this step we "shrink" the knot to ideal vertices of the "top" polyhedron. We can do this since we are looking for the complement, so the knot isn't a part of the manifold. The edges we added in step 3 connect to the knot on both sides, thus both of the vertices for these edges are ideal. Since this is the case for all edges of our new polyhedra, the polyhedra are themselves ideal. A useful way to think about this may be to imagine a tube of air representing our knot. Then in step 3 we add tubing with a supply of water corresponding to our knew edges. We'll pretend that the original air has some adhesive property that keeps it lumped together, and that there's no gravity causing it to rise or anything. As we fill our tube with water, the original air is compressed down to a few bubbles, while most of the original tubing is eventually filled with water. The water-filled parts are our new edges, the air is "missing space" representing our ideal vertices, and the faces exist as described in step 2. Essentially we just think about where our added edges can expand to, while contracting our original edges to vertices.

5. Finally we remove *nugatory* crossings, and *bigons*, shapes with only one or two edges, respectively. This leaves the final set of faces, edges, and vertices of our ideal polyhedra, as desired.

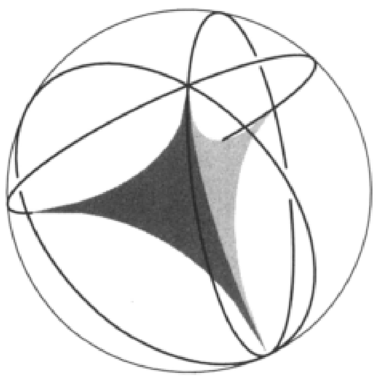
2.2 Realization in Hyperbolic Space

We now attempt to represent these polyhedra as hyperbolic and in such a way that they can be glued together, thus forming a hyperbolic manifold. This is normally done by making our n-hedron into an ideal regular n-hedron in hyperbolic space (appearing as a regular n-hedron in the Klein model). Then we must determine the *dihedral* angles of the polyhedra [5] (Recall our motivating example giving a tessellation of hyperbolic 2 dimensional space with the two-dimensional gluing

[5]The angles between adjacent faces.

pattern for a genus two surface, where the regular octagons had 45 degree angles. We do essentially the same thing here but with dihedral angles and three dimensional space.) This can be determined by looking at the polyhedron in the Poincaré model, then looking at the angles between the circles determined by the faces, since the angle between two planes in hyperbolic space is the same as the angle of their bounding circles on the sphere at infinity in the Poincaré model (this is given by Thurston in his book on page 128).

We note here that for a tetrahedron, this is 60 degrees since, in the image below, we see that there are 6 symmetric subdivisions of the sphere at infinity, thus $360/6 = 60$ is the measure of each angle.



Similarly, for an octahedron, there are two intersecting circles at infinity, thus four subdivisions, thus the angles are each $360/4 = 90$ degrees each. These two values will be used in our later examples.

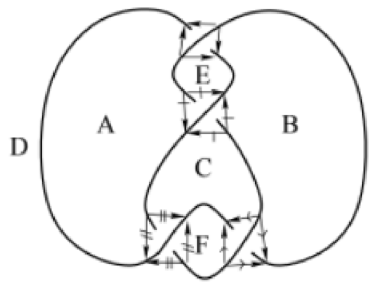
Finally, we can glue together our polyhedra (which, since we've given them representations in hyperbolic space, each have hyperbolic metrics) with hyperbolic isometries along their faces. We check that their edges can be identified such that the dihedral angles around each edges add to 360 degrees, and there's no vertex condition to check since the polyhedra are ideal. Thus such a gluing yields a hyperbolic manifold as desired.

An interesting note given by Thurston is that this sequence of steps is determined clearly enough that it has been repeatedly implemented on computers.

3 Examples

3.1 Figure-Eight Knot

While we have already shown the decomposition of the figure-eight knot via Thurston's geometric construction, we will finish the construction via Purcell's procedure as well, for clarity and completeness. The knot diagram and division shown above both correspond to the figure-eight knot. We continue by adding edges at the four crossings. Purcell's image below marks each edge by four edges (that could be glued back together) so that it's easier to see which edges are associated with each faces in our 2-dimensional model.



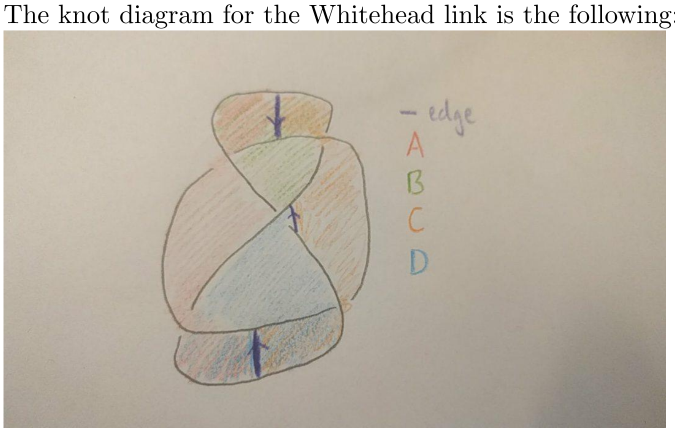
This completes step 3. Step 4 is mainly a mental one, showing that the construction works, the next real thing for us to do is to look at which faces are nugatory or bigonal. We see that faces E and F both have only two adjacent edges, thus they can be ignored. We also see that the remaining faces A, B, C, and D each have three edges adjacent to them, and they are all adjacent to each other. Thus each face is triangular, and the ideal polyhedron must be tetrahedronal, thus verifying the geometric construction given above by Thurston.

As we already reasoned above that the dihedral angles are all 60 degrees when the tetrahedra are regular and given a hyperbolic metric in hyperbolic space, we see that the tetrahedral decomposition can be reglued with hyperbolic isometries with six dihedral edges at a time, such that the angles add back to 360 degrees. More formally, the hyperbolic isometry we use is the unique orientation-preserving one taking identically labelled faces to each other (i.e. face A to A etc.). Then since these "tessellate" three dimensional hyperbolic space (more formally, since the hyperbolic space is a universal cover, but it is the same concept and intuition), we see that the quotient of the hyperbolic space by the group generated by the isometries is our manifold, and thus it has a hyperbolic metric.

3.2 Whitehead Link

We will construct the decomposition of the whitehead link slightly differently than using the above systematic method. We will still span our knot by a two-complex, but there is a more intuitive way to decompose the whitehead link than using Purcell's method. (Normally in step 2, when we label the faces, we are essentially exhibiting a spanning two-complex, but we can choose a different one if we so desire, as we will do here.)

The knot diagram for the Whitehead link is the following:



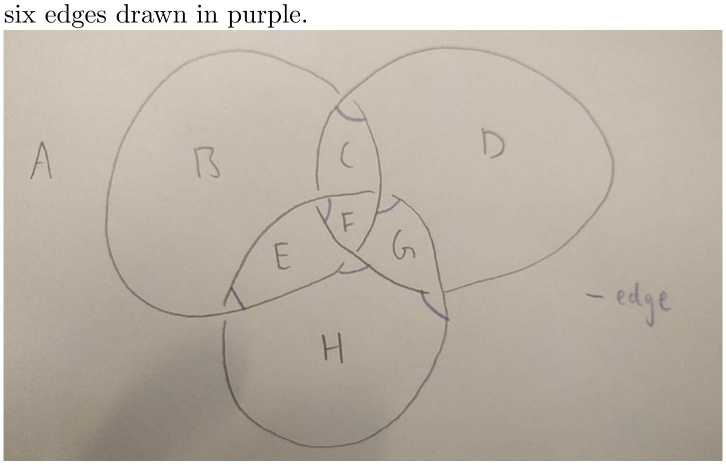
where the faces are labelled by colors corresponding to A, B, C, and D on the right, and the edges added in step 3 are the three in purple. Note that in two dimensions, it looks like some of the faces overlap, but this is because we are operating in three dimensions, so we can essentially look at the knot as a few overlapping slices, which we then choose to span by this particular two-complex. Imagining the whitehead link in three dimensions, one can verify that the added edges do indeed bound the given four faces once we expand the added edges so as to contract the original knot into the ideal vertices.

Now we must recall that we are operating in S^3, and since the two-complex given clearly doesn't separate three-dimensional space (it looks a bit like intersecting petals of a flower), this implies that our decomposition is in fact one octahedron, where each face here is actually two faces of the octahedron - one for each side of the two-complex sections here. (I.e. the octahedron contains the compactification point, the one "at infinity," thus we have to, in some sense, invert space if we wanted to see the octahedron we've constructed as the conventional representation in $\mathbb{R}^3$, but then the faces would all be an infinity instead.)

As we saw earlier that an ideal octahedron's bihedral angles are 90 degrees each, we see that the octahedral decomposition can be reglued four at a time (to itself, such that the faces in the two-complex that correspond to the two faces of the octahedron are reunited) with the unique orientation-preserving isometry taking these faces to each other, thus since these "tessellate" three dimensional hyperbolic space, we see that the quotient of the hyperbolic space by the group generated by these isometries is, as before, our manifold with an imbued hyperbolic metric.

3.3 Borromean Rings

Our final example will be that of the Borromean Rings. For this example we will return to the familiar technique given above. The knot diagram is given below, with the faces labelled, and the six edges drawn in purple.



Thus we see that there are 8 faces labelled A-H, andn that each face is bounded by 3 adjacent edges. Thus there are no nugatory nor bigonal edges, but there are in fact 8 triangles that together divide space into an upper and lower half. It can be trivially verified that the pattern these triangles glue together in is that of an octahedron (take an octahedron made of something malleable, remove one face, and then stretch the rest of it onto the plane and it can be deformed into this pattern, where the removed face is then the rest of the plane). Thus our decomposition is found to be two regular octahedra. But we need to be very careful how we glue them together. Thinking of the above shape as a two-complex in three dimensional space, we realize that in order to "match up" the corresponding faces when we glue/unglue the two octahedra, we need to twist them together by 120 degrees. Noting this, we've now reduced the rest of the problem to the same case as the whitehead link, where we use the same isometry/quotient, and still glue together edges 4 at a time since the angles are still 90 degree ones, thus giving the complement of the Borromean Rings a hyperbolic metric identically.

4 Works Cited

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I pledge my honour that this paper represents my own work in accordance with University regulations.

Dylan Mavrides 1/9/2018