Introductory Electricity

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Problem Set 5

This problem set is due **never** if you would like to receive feedback on your work. The exercises below vary in difficulty. Some are straightforward computational exercises intended to horn essential skills accompanying concepts in the previous lectures. Others may require more critical thinking to develop a deeper appreciation for the topics introduced. If you are unable to make progress on any particular problem and would like to obtain some hints before the solutions are released, please feel free to email us with your request.

1 Lecture Summary:

In lecture on August 17th, 2014, we discussed electric fields in matter.

1. The electric dipole moment $\mathbf{p}$ with separation distance $\mathbf{d}$ and charge q is given by

$$\mathbf{p} = -q\mathbf{d}.$$

   The electric field far away from the dipole is appropriately

$$\mathbf{E} = \frac{\mathbf{p}}{4\pi\epsilon_0 r^3},$$

   where r is the distance from the dipole.

2. A neutral atom or molecule in an electric field $\mathbf{E}$ induces a dipole $\mathbf{p}$. In general, we have

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

   where p_i's are the components of the dipole vector, $\alpha_{i,j}$'s are the components of the polarizability tensor, E_i's are the components of the electric field vector, and i and j denote x, y or z.

3. The polarization $\mathbf{P}$ is defined as the dipole moment per unit volume.

4. Suppose we have a dielectric material with a polarization $\mathbf{P}$. This polarization inducces bound charges

$$\sigma_b = \mathbf{P} \cdot \mathbf{n},$$
$$\rho_b = -\nabla \cdot \mathbf{P},$$

   where σ_b is the surface bound charge density, $\mathbf{n}$ is the normal surface vector, ρ_b is the volume bound charge density, and $\nabla\cdot$ denotes the divergence of a vector field.

5. Suppose we have a vector field $\mathbf{F} = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\rangle$ in Cartesian coordinate. We have
$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$
Often, we work with problems which have spherical symmetry. In these cases, it is best to use spherical coordinates. Denote $\mathbf{F} = F_r(r,\theta,\phi)\hat{r} + F_\theta(r,\theta,\phi)\hat{\theta} + F_\phi(r,\theta,\phi)\hat{\phi}$. The divergence is
$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(F_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.$$

6. Denote the total charge Q_{tot} as the sum of the free charge Q_f and the bound charge Q_b. The electric displacement $\mathbf{D}$ is
$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P},$$
where $\mathbf{E}$ is the total electric field and $\mathbf{P}$ is the polarization. The electric displacement follows a form of Gauss's Law with the free charge
$$\oint_{\delta\Omega} \mathbf{D}\cdot d\mathbf{A} = Q_f,$$
where Ω is the volume integral and $d\mathbf{A}$ is the normal surface element.

7. For some materials, the polarization is linearly proportional to the total electric field by a scaling factor χ_e called the electric susceptibility
$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}.$$
With these materials, we have
$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E}.$$
We define the following quantities:
   (a) Permitivity: $\epsilon = \epsilon_0(1+\chi_e)$
   (b) Relativity permitivity or dielectric constant: $\epsilon_r = \epsilon/\epsilon_0 = 1+\chi_e$

2 References:

For the convenience of the readers, we provide links to third-party websites below. We are *not* responsible for any content contained in those websites. Please view them at your own discretion.

1. The material covered in lecture resembles closely the presentation done in chapter of *Introduction to Electrodynamics* by David J. Griffiths. A PowerPoint version of this chapter appear here
   (a) `http://www.physics.mcgill.ca/~gang/PHYS340/Wiseman-Phys340-lecture-notes.update.pdf`
   (b) `http://www.phys.nthu.edu.tw/~thschang/notes/EM04.pdf`
2. For slightly different presentations on the same material, please see

   (a) http://physics.mq.edu.au/current/undergraduate/units/PHYS202/Electric%20fields%20in%20materials.pdf
   (b) http://www.physics.usu.edu/Wheeler/EM3600/Notes09FieldsInMatter.pdf
   (c) http://web.mit.edu/viz/EM/visualizations/coursenotes/modules/guide05.pdf

3. To further explore dielectrics, please consult
   (a) http://www.magneticsgroup.com/pdf/p18-25%20Dielectr.pdf
   (b) http://ecee.colorado.edu/~ecen3400/Chapter%207%20-%20Dielectrics%20in%20the%20Electrostatic%20Field.pdf
   (c) http://bayes.wustl.edu/etj/articles/nonlinear.dielectric.pdf

4. Alternative energy is a major field of research with much depth and breadth. For general overviews, please see
   (a) http://energy.gov/science-innovation/energy-sources/renewable-energy
   (b) http://www.nrel.gov/

5. To learn more about solar cells, please consult
   (a) http://www.fsec.ucf.edu/en/consumer/solar_electricity/basics/how_pv_cells_work.htm
   (b) http://www.nrel.gov/learning/re_photovoltaics.html
   (c) http://www.nrel.gov/docs/legosti/old/16319.pdf
   (d) http://web.mit.edu/taalebi/www/scitech/pvtutorial.pdf
   (e) http://userwww.sfsu.edu/ciotola/solar/pv.pdf
   (f) http://www.camse.org/scienceonthemove/documents/DSSC_manual.pdf

6. To explore fuel cells further, please see
   (a) http://www.transportation.anl.gov/pdfs/FC/521.PDF
   (b) http://energy.gov/eere/fuelcells/fuel-cells-basics
   (c) http://micro.magnet.fsu.edu/primer/java/fuelcell/
   (d) http://www.mrl.ucsb.edu/~seshadri/2012-SummerSchool-Talks/UCSB-2012-Barnett.pdf
   (e) http://www3.nd.edu/~msen/Teaching/DirStudies/FuelCells.pdf
   (f) http://physics.nist.gov/MajResFac/NIF/pemFuelCells.html

7. To delve deeper into wind energy, please visit
   (a) http://windeis.anl.gov/guide/basics/
   (b) http://web.mit.edu/windenergy/windweek/Presentations/Wind%20Energy%20101.pdf
   (c) http://www.ewea.org/wind-energy-basics/how-a-wind-turbine-works/
   (d) http://www.eia.gov/kids/energy.cfm?page=wind_home-basics
   (e) http://ro.uow.edu.au/cgi/viewcontent.cgi?article=1958&context=engpapers
   (f) http://dspace.mit.edu/bitstream/handle/1721.1/59899/676694167.pdf

3 Exercises:

Exercise 1 – The Electric Dipole

Verify that the electric field of a dipole with separation vector $\mathbf{d} = \langle 0, d, 0\rangle$ with the center placed at the origin (the configuration done in class) is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{\langle x, y - d/2, z\rangle}{(x^2 + [y - d/2]^2 + z^2)^{3/2}} - \frac{\langle x, y + d/2, z\rangle}{(x^2 + [y + d/2]^2 + z^2)^{3/2}}\right), \tag{1}$$

where q is the charge.

Suppose that $d = 1$ m and $\mathbf{r} = \langle 100, 100, 100\rangle$ m. Verify numerically that

$$\mathbf{E} \approx -\frac{q\mathbf{d}}{4\pi\epsilon_0 r^3} = \frac{\mathbf{p}}{4\pi\epsilon_0 r^3}. \tag{2}$$

That is, compute the electric field using Eq. 1 and Eq. 2 separately. Then compare the results and show that they roughly agree.

Exercise 2 – The Polarizability Tensor

For some material, the polarizability tensor is

$$\alpha = \begin{pmatrix} 6 & 7 & 0 \\ 7 & 2 & 6 \\ 0 & 6 & 3 \end{pmatrix},$$

measured in the appropriate units.

1. Find the units of polarizability using the definition.
2. Find the dipole moment with applied electric field $\mathbf{E} = \langle E_x, E_y, E_z\rangle$
3. Find an electric field for which the dipole moment is parallel to the field. *Hint:* there are many possible solutions to this part. You are only asked to find one. It may be helpful to recall the properties of the dot and cross products.

Exercise 3 – Bound Charges, Part I

In lecture, we discussed an insulating material with polarization $\mathbf{P} = \langle 3x^2, 6y + 2x, 3z + 6\rangle$. If this material is a cube with one corner at $(0, 0, 0)$ and another corner at $(1, 1, 1)$, find the bound charge surface density for all six faces of this cube. *Hint:* Be careful with the directions of the normal unit vectors.

Exercise 4 – Bound Charges, Part II

An infinite cylinder of radius b. It has a polarization $\mathbf{P} = \kappa s\hat{\mathbf{s}}$, where κ is a constant, $s \leq b$ is the distance from the center axis of the cylinder, and $\hat{\mathbf{s}}$ is the spherical radial unit vector. Of course, $\mathbf{P} = \mathbf{0}$ for $s > b$.

1. Find the bound surface charge density.
2. Find the bound volume charge density. *Hint:* The divergence in spherical coordinates of a vector field $\mathbf{F}$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{s}\frac{\partial(sF_s)}{\partial s} + \frac{1}{s}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

3. Since the system starts out as a neutral object, it should remain neutral as a whole after polarization. Verify that the sum of charges is, in fact, zero. *Hint:* Because this is an infinite cylinder, both the bound surface charge density and the bound volume charge density yield to infinite charges separately. However, for any finite portion of the cylinder (of length $z = L$ for any L), the total charge is zero. Hence, the total charge is zero for the system. For this part, you might need $dV = s ds d\theta dz$.

Exercise 5 – The Electric Displacement

Consider an infinite cylinder of radius b with uniform free charge volume density ρ_f.

1. If this conducting cylinder has no polarization, what is the electric field $\mathbf{E}$ everywhere? *Hint:* Use Gauss's Law.
2. Now, suppose we somehow fix a polarization $\mathbf{P} = \kappa s \hat{\mathbf{s}}$ as in the previous question, find the electric field everywhere in this case using two different methods as done in class.
   (a) Use Gauss's Law to find the electric field $\mathbf{E}$ directly from $\rho_{\text{tot}} = \rho_f + \rho_b$. Do not forget to include the bound surface charge.
   (b) Use Gauss's Law to find the electric displacement $\mathbf{D}$ from ρ_f and then use $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ to find the electric field $\mathbf{E}$.

Exercise 6 – Linear Dielectrics

We have a parallel-plate capacitor, with plate area A and gap distance d, whose gap is partially filled by a dielectric material with dielectric constant κ as shown in Fig. 1. Find the capacitance of this configuration.

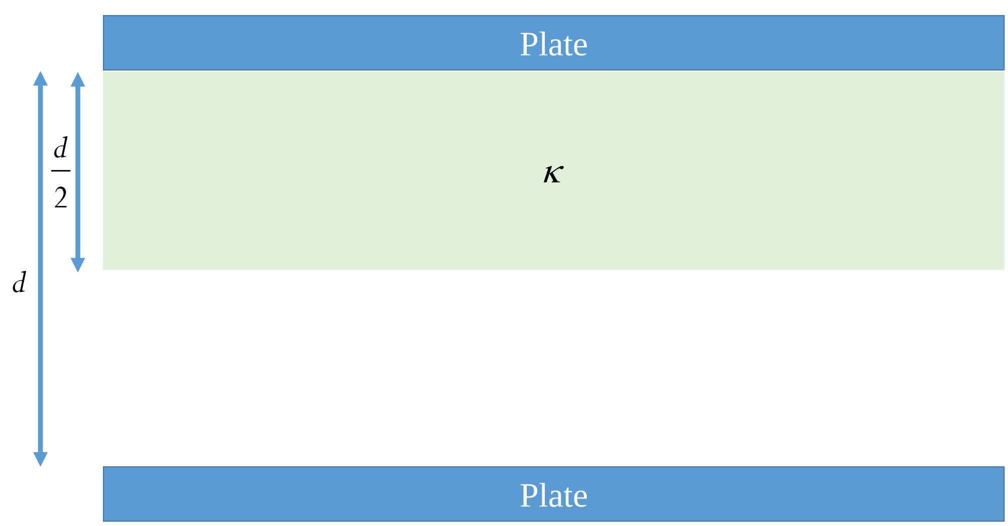

Figure 1: A parallel-plate capacitor.

Exercise 7 – The Future of Energy

In this exercise, we ask you to perform a few simple tasks to become more familiar with our potential future energy candidates. Please feel free to consult whatever sources you need to complete the following tasks.

1. What is the difference between a renewable and a non-renewable energy source?
2. List 3 examples of renewable energy and 3 examples of non-renewable energy.
3. Look up the energy consumption by source in the United States, and classify each source as either renewable or non-renewable.
4. Look up the energy consumption by source in the entire world, and classify each source as either renewable or non-renewable.
5. What is the percentage of consumption that is renewable in the world? How about non-renewable?
6. Of the 3 renewable sources that you listed, find 2 advantages and 2 disadvantages of 2 of those sources.
7. Propose several ways of how you as an individual can reduce your reliance on non-renewable energy.
8. Propose several ways of how the world can reduce its reliance on non-renewable energy.