Introductory Electricity

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Problem Set 2

This problem set is due **July 20th, 2014** if you would like to receive feedback on your work. The exercises below vary in difficulty. Some are straightforward computational exercises intended to horn essential skills accompanying concepts in the previous lectures. Others may require more critical thinking to develop a deeper appreciation for the topics introduced. If you are unable to make progress on any particular problem and would like to obtain some hints before the solutions are released, please feel free to email us with your request.

1 Lecture Summary:

In lecture on July 13th, 2014, we discussed the electric field, the electric potential, and the motion of charges in a conductor.

1. The electric field of a single charge q is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the radial vector connecting the location of the charge and the point where the electric field is evaluated. Suppose we have established an electric field $\mathbf{E}$ somehow. Then we bring a charge Q into this electric field, this charge experiences a force

$$\mathbf{F} = Q\mathbf{E}.$$

Because the electric field is derived from Coulomb's Law, it also obeys the superposition principle. That is, given an ensemble of discrete charges $\{q_1, q_2, q_3, ..., q_N\}$ located at positions $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, ..., \mathbf{r}_N\}$, the electric field at a point $\mathbf{R}$ (which for simplicity, is assumed to not coincide with any of the charges) is given by

$$\mathbf{E}(\mathbf{R}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i(\mathbf{R}-\mathbf{r_i})}{\|\mathbf{R}-\mathbf{r_i}\|^3}.$$

If we have a continuous distribution of charges instead of a discrete one, we can replace the sum by an integral. The general form of such an expression is not important for the purpose of this class.

2. In the case where the charge configuration exhibits symmetries, the electric field due to a continuous distribution of charges can be computed easily using *Gauss's Law*. In words, Gauss's Law states that the electric *flux* through a closed surface is proportional to the total charge enclosed by that surface. Symbolically, we have

$$\Phi = \oint_{\partial\Omega} \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \int_{\Omega} \rho dV = \frac{Q_{\text{enclosed}}}{\epsilon_0},$$

where Ω is the domain of integration, Φ is the electric flux, ρ is the charge density, and Q_{enclosed} is the total enclosed charge. To find the electric field of a symmetric configuration of charge, we follow these steps:

   (a) Determine the symmetry of the problem,

   (b) Partition space into separate domains of interest, if any (this is often a domain with charge and a domain without charge),

   (c) Draw a Gaussian surface with the determined symmetry for each of the domains above,

   (d) Calculate the electric flux by multiplying the electric field by the enclosed surface area,

   (e) Calculate the enclosed charge,

   (f) Divide the right-hand side of Gauss's Law by the area to acquire the magnitude of the electric field,

   (g) Designate the appropriate direction of the electric field based on the symmetry of the problem to the magnitude calculated above to obtain the electric field.

3. The electric potential is defined by the line integral of the electric field. Roughly speaking, it is the energy per charge. We define the electric potential as

$$V(\mathbf{r}) = -\int_{\text{reference}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l},$$

where $\mathbf{r}$ is the location where the potential is evaluated. If we use infinity as our reference where the potential is zero, then for a point charge q, the electric potential is

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r},$$

where r is the distance from the charge. If we were to bring another charge Q from infinity to a distance r from q, the configuration would gain an electric potential energy

$$U = QV(\mathbf{r}) = \frac{Qq}{4\pi\epsilon_0 r}.$$

Often, if we are only interested in the potential difference between two points, then

$$\Delta V = V(\mathbf{r_1}) - V(\mathbf{r_2}),$$

for some $\mathbf{r_1}$ and $\mathbf{r_2}$. If we pick infinity to be one of our points, then our potential difference is equal to the potential of a point charge. To find the potential difference (with infinity as the reference) at a point due to the presence of many discrete charges, we simply find the potential due to each charge at that point and then add up the different contributions. In the continuum limit of charges, we integrate.

4. A conductor is an object that allows the consistent flow of electrons subject to a potential source. We idealize a conductor as an object with an infinite number of electrons unbounded to their atoms, free to move around as they choose. In the classical theory of conduction, we imagine a conductor as a lattice of atoms fixed in place with a sea of electrons in motion. When we apply an electric field across this conductor, these electrons feel a force in one direction and thus move in unison, on average, in that direction to create a current. Define the current density $\mathbf{J}$ as the number of charges moving past a given point per second per unit cross sectional area

$$\mathbf{J} = ne\mathbf{v}_{\text{drift}},$$

where n is the number density of electrons (number of electrons per unit volume), e is the elementary charge quantum, and $\mathbf{v}_{\text{drift}}$ is the average velocity of the electrons. According to the Drude model of conductivity, the current density is proportional to the applied electric field $\mathbf{E}$ because the electrons bounce off of the atomic obstacles

$$\mathbf{J} = \frac{ne^2\tau}{m}\mathbf{E},$$

where τ is the scattering time and m is the mass of the electron. We define the conductivity σ as

$$\sigma = \frac{ne^2\tau}{m}.$$

This constant is a property of the specific conductor, which is measured experimentally. An associated constant, called the resistivity $\rho = \frac{1}{\sigma}$ (please do not confuse this with the charge density), is a measure of the resistance to flow.

5. The current I is defined by

$$I = \mathbf{J} \cdot \mathbf{A},$$

where $\mathbf{A}$ is the cross-sectional area (assuming a uniform, planar surface).

2 References:

For the convenience of the readers, we provide links to third-party websites below. We are *not* responsible for any content contained in those websites. Please view them at your own discretion.

1. To learn more about electric fields and electric potential, please see sections 17.2, 18.1, and 18.2 of *Physics* by Raymond A. Serway and Jerry S. Faughn or sections 2.1 and 2.3 of *Introduction to Electrodynamics* by David J. Griffiths.

2. For some online resources on electric fields and electric potential, please consult:
   (a) `http://physics.bu.edu/~duffy/py106/Electricfield.html`
   (b) `http://web.mit.edu/8.02t/www/802TEAL3D/visualizations/coursenotes/modules/guide03.pdf`
   (c) `http://web.mit.edu/8.02t/www/802TEAL3D/visualizations/coursenotes/modules/guide04.pdf`

3. For simulations of electric fields, please see `http://web.mit.edu/8.02t/www/802TEAL3D/visualizations/electrostatics/index.htm`.

4. For more information on the Drude model of conductivity, please visit the following sites:
   (a) `http://web.mit.edu/8.02t/www/802TEAL3D/visualizations/coursenotes/modules/guide06.pdf`
   (b) `http://people.seas.harvard.edu/~jones/es154/lectures/lecture_2/drude_model/drude_model.html`
   (c) `http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-763-applied-superconductivity-fall-2005/lecture-notes/lecture4.pdf`
   (d) `http://www.phys.utk.edu/courses/fall%202009/physics671/chapter3a.pdf`
5. To explore semiconductors further, please see
   (a) `http://hyperphysics.phy-astr.gsu.edu/hbase/solids/semcn.html`
   (b) `http://matse1.matse.illinois.edu/sc/sc.html`
   (c) `http://ecee.colorado.edu/~bart/book/book/contents.htm`

3 Exercises:

Exercise 1 – The Electric Field of an Ensemble of Charges

Consider a configuration of charge where q is placed at $(d, 0)$ and another $2q$ is placed at $(-d, 0)$.

1. Draw the electric field lines of this charge configuration.
2. Find the set of all points where $\mathbf{E} = \mathbf{0}$.
3. Find the electric field, direction and magnitude, at the point $(0, 2)$.
4. Find the electric field, direction and magnitude, at the point $(1, 2)$.
5. A charge $-q$ is brought to the point $(0, 2)$. Assuming that the two original charges are pinned fixed in placed, find the electric force on the negative charge.

Exercise 2 – Gauss's Law to the Rescue

1. Use Gauss's Law to find the electric field of a point charge q and confirm that this is indeed the same electric field as given by Coulomb's Law.
2. Consider a sphere of charge of radius R with charge density ρ. Find the electric field everywhere.
3. Consider an infinite sheet of charge with surface charge density σ. Find the electric field everywhere.
4. Consider an infinite cylinder of radius R with charge density ρ. Find the electric field everywhere.

Exercise 3 – The Electric Potential, Part I

Consider a configuration of charge where q is placed at $(d, 0)$ and another $-q$ is placed at $(-d, 0)$. This is known as the *electric dipole.*

1. Find the set of all points where the electric potential is zero. This is an example of an equipotential "surface." *Bonus:* Find all equipotential "surfaces." That is, find the set of all points where the potential is the same. Draw some of these "surfaces."

2. Find the electric potential at the point $(0, 2)$.
3. Find the electric potential at the point $(1, 2)$.
4. Find the electric potential *energy* of this configuration.

Exercise 4 – The Electric Potential, Part II

1. Recall that the potential is defined as the integral of the electric field. What is the electric field (magnitude and direction) in terms of the electric potential?
2. Consider a uniform electric field $\mathbf{E}$ in one dimension on the entire x-axis. What is the electric potential *difference* between two points a and b on the x-axis? Use the result from the previous part to verify that the electric potential you found here reproduces the correct electric field.

Exercise 5 – Comparing the Speed of Electrons

In this problem, we will establish the need for the Drude model of conductivity. Suppose you have two parallel plates connected to a potential source of potential difference $\Delta V = 120$ V. A uniform electric field is established between the plates. The distance d between the plates is 1 m. Suppose that the positive terminal is on the left and the negative terminal is on the right.

1. Use $V = Ed$ to calculate the electric field between the plates.
2. Now an electron is placed at the right terminal. It experiences a force and moves to left terminal. Find the acceleration that the electron experiences.
3. Use basic kinematics to find the velocity of the electron when it reaches the other side.
4. Now, we will compare this velocity with the drift velocity. Suppose we connect the two plates by a straight copper wire with resistivity of $\rho = 1.68 \times 10^{-8}\ \Omega$ m. Find the drift velocity of this electron according to the Drude model of conductivity.
5. Compare the drift velocity of the electron with its velocity in free space.
6. Repeat the previous five parts with $d = 10$ m. What do you notice?

Exercise 6 – Conceptual Questions about Conductors

In class, we stated the following properties of conductors:

1. The electric field inside a conductor is zero.
2. The electric field at the surface of a conductor is always perpendicular to it.
3. The charge density inside a conductor is zero.
4. All charges in a conductor reside at the surface.
5. The surface of a conductor is an equipotential surface.

Review your notes and provide a brief justification for each of the above statements.

Some of these properties seem to contradict the fact that we can drive a current through a metal wire (a conductor) by connecting it to a battery. If there is no electric field inside the conductor, how can charges flow inside the conductor, which we know they do? How would you reconcile this apparent paradox?

Exercise 7 – Practice with Band Gaps

Silicon is one of the most commonly-used semiconductors in the computer industry. It has a band gap energy of $\Delta E = 0.67$ eV, one of the smaller band gaps known.

1. Draw a band structure diagram for silicon showing the conduction band, the valence band, and the energy gap. Please label your energy axis.
2. Suppose that the valence band is completely filled and the conduction band is empty. To get an electron from the valence band to the conduction band, we need to excite it with some energy. We use light to provide this energy. Calculate the smallest frequency of light needed to excite the first electron.
3. Use the result from the previous part to draw an absorption diagram of silicon.
4. For whatever purpose for which we need silicon, we find that the band gap of $\Delta E = 0.67$ eV is too large. Discuss one way we can process silicon to narrow this band gap (at least for some electrons).