Introductory Electricity

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July 6th, 2014

HSSP Summer 2014

Lecture 1

Welcome to Introductory Electricity!

→ Introductory remarks

→ Instructors: Christian, David, and Phong

→ Coure Structure:

- → 7 class sessions: 5.5 lectures and 1.5 project classes
- → 5 problem sets (optional but highly encouraged)
- → Solutions to problem-sets are posted on the Thursday following the corresponding lectures.
- → Classes 2-6 will end with a 15-minute quiz.
- → There will be an optional final exam released at the end of the 6th class.
- → All assignments and assessments are open-notes in this course

→ Course Outline:

- → Basic mathematics
- → Basic electrostatics
- → Circuitry
- → Electric fields in matter
- → Project

We will cover ~~an~~ an exciting application each lecture class to show you the importance of what you have just learned.

→ All course documents will be posted at

web.mit.edu/vophong/www/Introductory_Electricity.htm

→ If you have any questions about the course, please do not hesitate to reach out to us!

Introduction to Vectors:

Before we dive into the exciting physics of electricity, we first want to become acquainted with some important mathematical concepts that are relevant to our study. The first of our mathematical asides will be on vector analysis!

Many quantities can be purely described by a number (with or without units). These quantities are called scalars. Some examples include:

Temperature	36°C
Speed	28 mph
Time	15 minutes
Intensity	60%

However, there are many physical quantities which cannot be entirely described by just a number. Often, we need to give these quantities ~~a~~ directions as well. These are vectors:

Vectors: objects which are specified by both a magnitude (a number) and a direction.

Some examples of vectors are:

Velocity	38 mph 45° NW
	6 m/s to the right
Force	7 N to the left
Acceleration	96 m/s^2 60° from the diagonal

Just like with a scalar, we denote a vector by giving it a variable name (a letter which can be Greek or Roman). However to distinguish it from scalars, we put an arrow over its head. We denote the magnitude of a vector by placing the variable name around double bars.

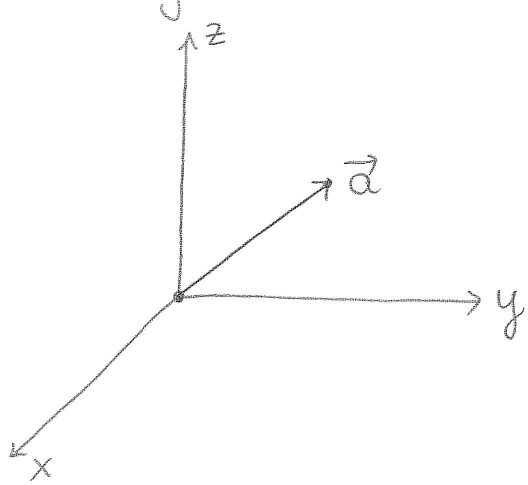
For example,

object	name	magnitude
Force	$\vec{F}$	$\|\vec{F}\|$
Acceleration	$\vec{a}$	$\|\vec{a}\|$
Velocity	$\vec{v}$	$\|\vec{v}\|$

So, how do we represent a vector?

Geometrical:

magnitude + arrow head



specify an arrow head and a magnitude.

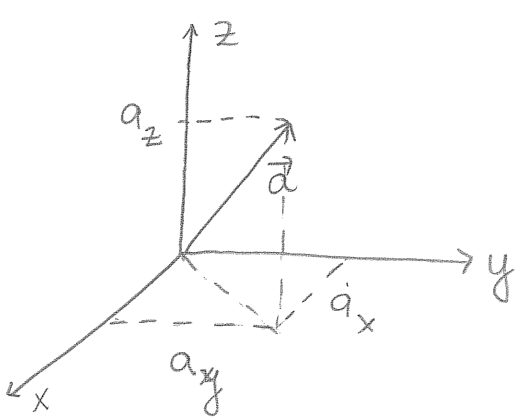
Compass-heading:

magnitude + compass reading (with respect to some reference)

72 m NW

68 m/s NE 30°

Component-form (most widely used):



unit vectors: vectors with magnitude one.

Basis unit vectors; $\hat{x}, \hat{y}, \hat{z}$

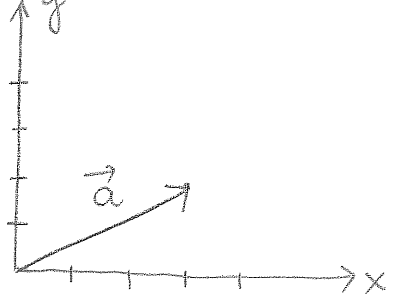
$\hat{i}, \hat{j}, \hat{k}$

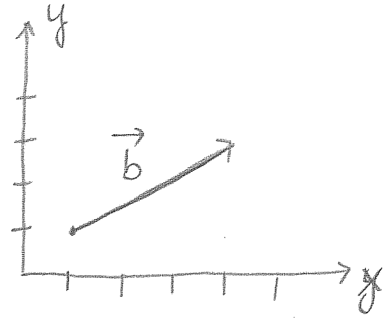
$$\vec{a} = \langle a_x, a_y, a_z \rangle = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

The components a_x, a_y, and a_z are scalars.

Now, we use a 2-D Cartesian coordinate system to simplify problems (4)

The components of the vectors do not uniquely determine the starting and ending points.



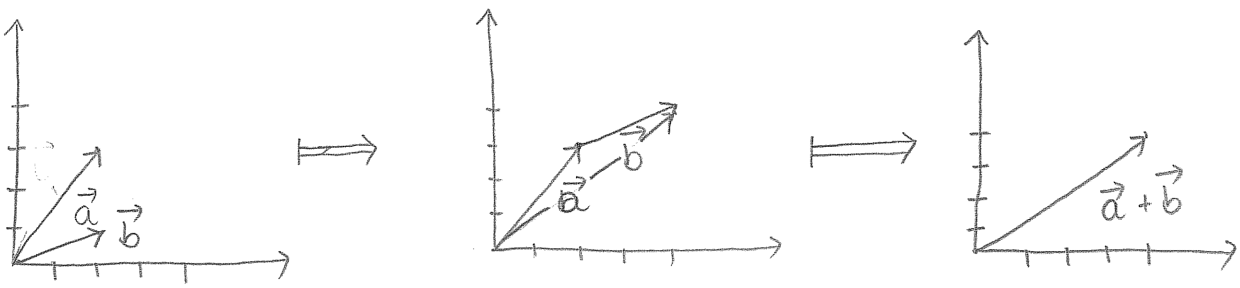


$\vec{a} = \langle 3, 2\rangle = \vec{b}$. They are the same vectors.

Now that we have vectors and know how to represent them, what can we do with them?

Addition: add tip-to-tail geometrically
component-wise analytically

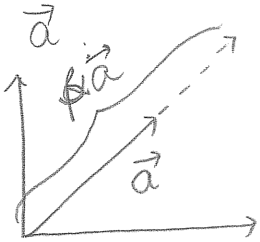
Suppose $\vec{a} = \langle 2, 3\rangle$ and $\vec{b} = \langle 2, 1\rangle$



$$\vec{a}+\vec{b} = \langle 2,3\rangle + \langle 2,1\rangle = \langle 2+2,\ 3+1\rangle = \langle 4, 4\rangle.$$

Scalar multiplication: changing the magnitude of a vector without affecting its direction.

Let $\vec{a}$ be a vector and β be a number; the vector $\beta\vec{a}$ is in the same direction as $\vec{a}$ but with a different magnitude than $\vec{a}$



$$\vec{a} = \langle a_x, a_y, a_z\rangle$$

$$\beta\vec{a} = \beta\langle a_x, a_y, a_z\rangle = \langle \beta a_x, \beta a_y, a_z\rangle$$

To subtract two vectors $\vec{a}$ and $\vec{b}$, we simply multiply one of them by $\beta = -1$ and then add $\vec{a} + \beta\vec{b} = \vec{a} - \vec{b}$.

Norm (finding the magnitude):

$$\vec{a} = \langle a_x, a_y, a_z \rangle \qquad \|\vec{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

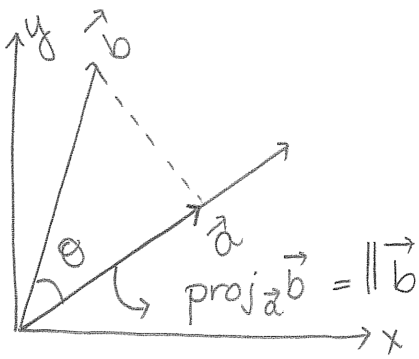
Unit vector: $\hat{a} = \dfrac{\vec{a}}{\|\vec{a}\|}$

Example: $\vec{a} = \langle 6, 5, 3 \rangle$. Find $\hat{a}$:

$$\|\vec{a}\| = \sqrt{6^2 + 5^2 + 3^2} = \sqrt{36 + 25 + 9} = \sqrt{70}$$

$$\hat{a} = \frac{1}{\sqrt{70}} \langle 6, 5, 3 \rangle$$

Scalar probuct (dot product):



$$\text{proj}_{\vec{a}}\vec{b} = \|\vec{b}\| \cos\theta \, \hat{a} = \frac{\vec{a}\, \|\vec{b}\| \cos\theta}{\|\vec{a}\|} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2}$$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ a scalar!

$$\Rightarrow \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|} = \|\vec{b}\| \cos\theta$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

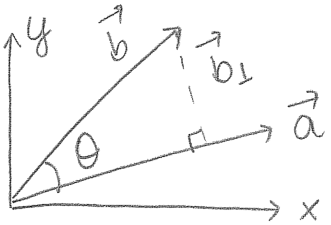
Two vectors are perpendicular or orthoganal if $\theta = 90°$

$$\Rightarrow \|\vec{a}\| \|\vec{b}\| \cos\theta = \vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$$

Cross product (vector product):

Similar to the dot product, but is produces a vector as a result.

This only works in 3 dimensions (for this course)



$$\|\vec{a}\times\vec{b}\| = \underbrace{\|\vec{a}\|\|\vec{b}\|\sin\theta}_{\text{area of the parallelogram formed by } \vec{a} \text{ and } \vec{b}}$$

The direction is given by the right-hand rule.

Notice that $\vec{a}\times\vec{b} \perp \vec{a}$ and $\vec{b}$.

Also, if $\vec{a} \parallel \vec{b}$, $\vec{a}\times\vec{b} = \vec{0}$ because $\theta = 0°$

Analytically, if $\vec{a} = \langle a_x, a_y, a_z \rangle$ and $\vec{b} = \langle b_x, b_y, b_z \rangle$, then

$$\vec{a}\times\vec{b} = \Big\langle a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x \Big\rangle$$

From the right-hand rule, it is clear that

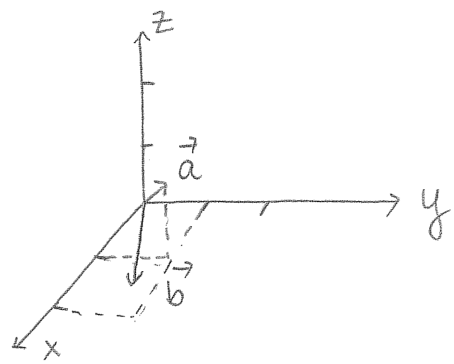
$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}.$$

Remark: A point on the plane is denoted by (a, b, c) whereas a vector on the same plane is $\vec{a} = \langle a_x, a_y, a_z \rangle$

Once we fix a coordinate system, a point has a unique representation on the plane, whereas a vector does not, unless specifically indicated.

Example: Given two vectors $\vec{a} = \langle 1, 1, 1 \rangle$ and $\vec{b} = \langle 2, 1, 1 \rangle$

(a) Draw them on a Cartesian coordinate system



(b) Find $\vec{a} + 6\vec{b}$, $\vec{a} \cdot 2\vec{b}$, $9\vec{a} - \vec{b}$, $\|\vec{a} + \vec{b}\|$, $\hat{a} + \vec{b}$,

$$\vec{a} + 6\vec{b} = \langle 1, 1, 1 \rangle + 6\langle 2, 1, 1 \rangle = \langle 1, 1, 1 \rangle + \langle 12, 6, 6 \rangle$$
$$= \langle 1+12, 1+6, 1+6 \rangle$$
$$= \langle 13, 7, 7 \rangle$$

$$\vec{a} \cdot 2\vec{b} = \langle 1, 1, 1 \rangle \cdot 2\langle 2, 1, 1 \rangle = \langle 1, 1, 1 \rangle \cdot \langle 4, 2, 2 \rangle$$
$$= 1 \times 4 + 1 \times 2 + 1 \times 2 = 4 + 2 + 2 = 8$$

$$9\vec{a} - \vec{b} = 9\langle 1, 1, 1 \rangle - \langle 2, 1, 1 \rangle = \langle 9, 9, 9 \rangle - \langle 2, 1, 1 \rangle$$
$$= \langle 9-2, 9-1, 9-1 \rangle$$
$$= \langle 7, 8, 8 \rangle$$

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{a}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{\vec{a}}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$\hat{a} + \vec{b} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle + \langle 2, 1, 1 \rangle$$

$$= \left\langle \frac{1}{\sqrt{3}} + 2, \frac{1}{\sqrt{3}} + 1, \frac{1}{\sqrt{3}} + 1 \right\rangle$$

(c) Find the angle between the two vectors

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta = \langle 1,1,1 \rangle \cdot \langle 2,1,1 \rangle = 2+1+1 = 4$

$\|\vec{a}\| = \sqrt{3}$

$\|\vec{b}\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

$\sqrt{3}\sqrt{6}\cos\theta = 8 \Rightarrow \sqrt{18}\cos\theta = 4$

$\cos\theta = \dfrac{4}{\sqrt{18}}$

$\theta = \cos^{-1}\left(\dfrac{4}{\sqrt{18}}\right) \approx 19.47°$

(d) Find the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$

$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

$= \sqrt{3}\sqrt{6} \sin 19.47° \approx 1.41$

(e) Find a vector that is ~~a~~ orthogonal to both $\vec{a}$ and $\vec{b}$

$\vec{a} \times \vec{b} = \langle 1\times1 - 1\times1,\ 1\times2 - 1\times1,\ 1\times1 - 1\times2 \rangle$

$= \langle 0, 1, -1 \rangle$

Check $(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle 0, 1, -1 \rangle \langle 1,1,1 \rangle$

$= 0\times1 + 1\times1 - 1\times1 = 0$ ✓

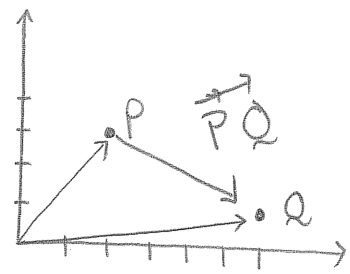
$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle 0,1,-1 \rangle \langle 2,1,1 \rangle$

$= 0\times1 + 1\times1 - 1\times1 = 0$ ✓

$\|\vec{a} \times \vec{b}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \approx 1.41$ ✓

Example: Given 2 points on the plane $P = (2,3,0)$ and $Q = (6,1,0)$

(a) Draw them



(b) Find the position vectors for P, Q

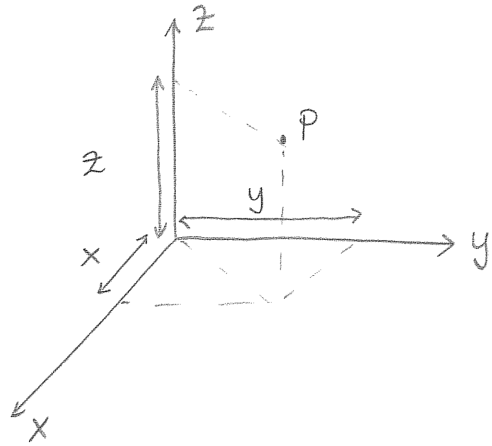
$$\vec{P} = \overrightarrow{OP} = \langle 2,3,0 \rangle$$
$$\vec{Q} = \overrightarrow{OQ} = \langle 6,1,0 \rangle$$

(c) Find a vector that goes from P to Q

$$\overrightarrow{PQ} = \vec{Q} - \vec{P} = \langle 6,1,0 \rangle - \langle 2,3,0 \rangle$$
$$= \langle 4,-2,0 \rangle$$

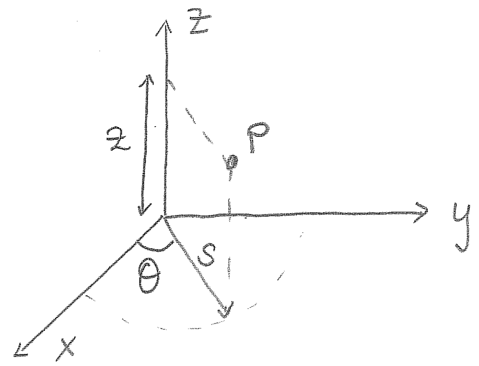
Coordinate System

Cartesian



$P = (x, y, z)$

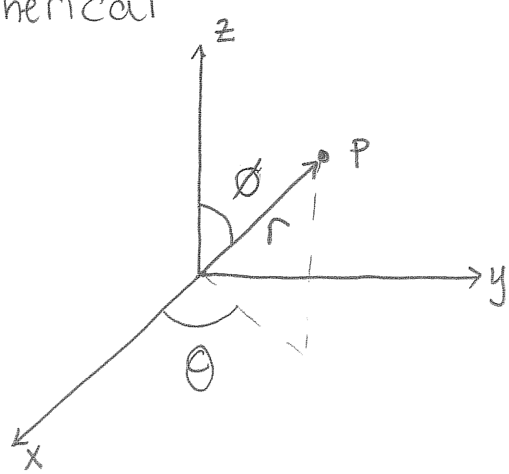
Cylindrical



$P = (s, \theta, z)$

$0 \leq \theta < 2\pi$ or 360°

Spherical



$P = (r, \theta, \phi)$

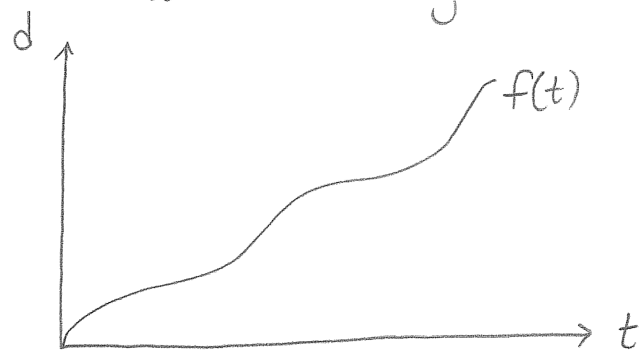
$0 \leq \theta < 2\pi$ or 360°

$0 \leq \phi < \pi$ or 180°

Differentiation and Integration:

Differentiation is a technique that allows one to find the instantaneous rate of change of a function.

Consider a car traveling with distance function f

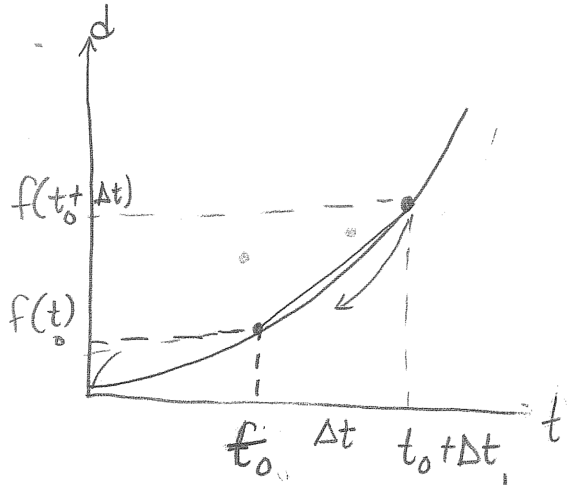


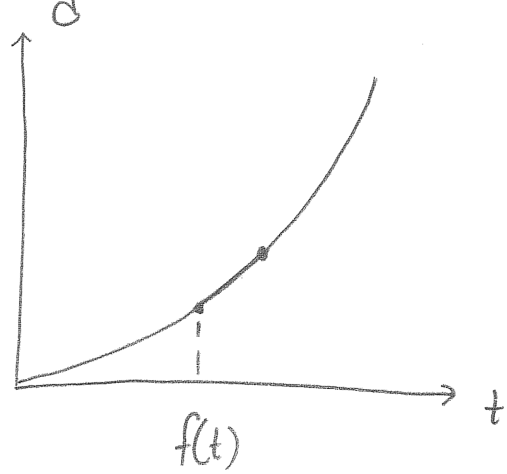
If you know the initial distance and time and the final distance and time, you can find the average speed

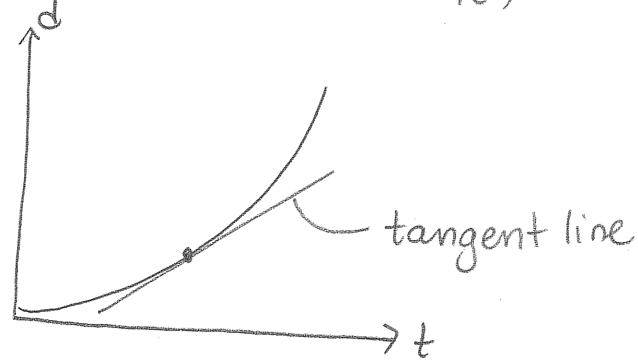
$$S_{ave} = \frac{f(t_f) - f(t_i)}{t_f - t_f}.$$

But the car does not necessarily travel at this average speed. Sometimes it speeds up and sometimes it slows. So what if we want to find the instantaneous speed?

This is the derivative!







Derivative: slope of the tangent line.

$$\boxed{\frac{d}{dt} f(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}}$$

In this course, we are only interested in the derivative of functions of the form $f(x) = x^n$ where n is a number.

$$\frac{d}{dx}(x^n) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

By the binominal expansion

$$(x+\Delta x)^n = \binom{n}{0} x^n \Delta x^0 + \binom{n}{1} x^{n-1} \Delta x + \binom{n}{2} x^{n-2} \Delta x^2 + \ldots$$

$$(x+\Delta x)^n - x^n = \binom{n}{1} x^{n-1} \Delta x + \ldots$$

$$\frac{(x+\Delta x)^n - x^n}{\Delta x} = \binom{n}{1} x^{n-1} + \theta(\Delta x)$$

Now, let $\Delta x \to 0$, we have

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

Properties of the derivative:

Let f and g be differentiable functions and α and β are numbers, then

$$\frac{d(\alpha f)}{dx} = \alpha \frac{df}{dx}$$

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d(\alpha f + \beta g)}{dx} = \alpha \frac{df}{dx} + \beta \frac{dg}{dx}$$

Let us now check to see if this makes sense.

$$f(x) = Cx = Cx^1$$

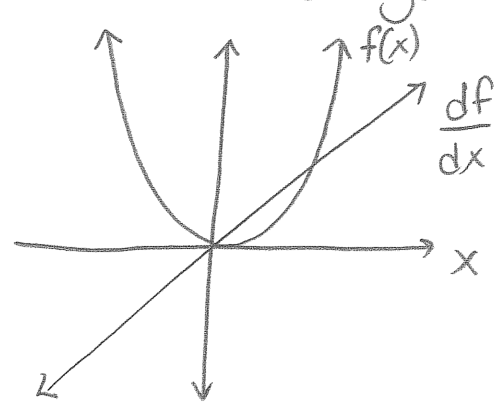
$$\frac{df}{dx} = C\frac{dx}{dx} = C \times 1\, x^{1-1} = C \Leftarrow$$ slope of linear function.

How about $f(x) = C = Cx^0$

$$\frac{df}{dx} = C \times 0\, x^{0-1} = 0 \Leftarrow$$ constant functions have zero slope

How about $f(x) = x^2$

$$\frac{df(x)}{dx} = 2x$$



Examples: Find the derivative of the following functions.

(a) $x^2 + x^3$ $\qquad 2x + 3x^3$

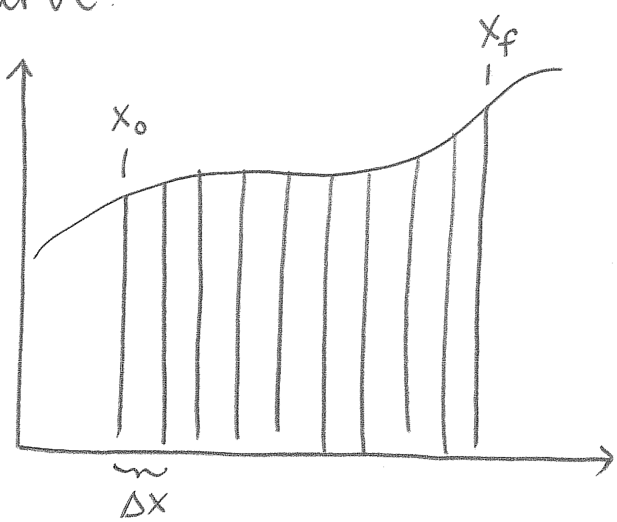
(b) $\frac{1}{x^2}$ $\qquad \frac{1}{x^2} = x^{-2} \Rightarrow -2x^{-3} = -\frac{2}{x^3}$

(c) $\frac{1}{x^2} + \frac{1}{x^3}$ $\qquad -\frac{2}{x^3} - \frac{3}{x^4}$

(d) $3x^{100} + 4x^{200}$ $\qquad 300x^{99} + 800x^{199}$

Integration:

Integration is the inverse operation of differentiation.
Geometrically, the integral of a function is the area under the curve.



$$A \approx \sum_i f(x_0 + i\Delta x)\Delta x$$

Let $i \to \infty$
$\Delta x \to 0$

$$A = \int_{x_0}^{x_f} f(x)\,dx$$

If we do not put bounds, then the integral
$F(x) = \int f(x)\,dx$ is called an indefinite integral

If we do put bounds, then

$F(x) = \int_a^b f(x)\,dx$ is called a definite integral.

Properties:

Let f and g be integrable functions and α and β are numbers, then

$$\int \alpha f dx = \alpha \int f dx$$

$$\int (f+g) dx = \int f dx + \int g dx$$

$$\int (\alpha f + \beta g) dx = \alpha \int f dx + \beta \int g dx$$

The same is true for definitie integrals. Additionally,

$$\int_a^b f dx = - \int_b^a f dx.$$

Because differentiation and integration are inverse operations, we have

$$\frac{d}{dx}\left(\int f(x) dx \right) = f(x)$$

If $f(x) = x^n$ where n is a number

$$\frac{d}{dx}\left(\int f(x) dx \right) = \frac{d}{dx}\left(\int x^n dx \right) = x^n$$

$$= \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c \right) = x^n$$

any constant

$$\Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c \quad \text{for } n \neq -1$$

For $n = -1$

$$\int \frac{1}{x}\,dx = \ln x + C.$$

For definite integrals,

$$\int f(x)\,dx = F(x)$$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Examples: Integrate the following

$$\int (x^2 + x^3)\,dx = \frac{x^3}{3} + \frac{x^4}{4} + C$$

$$\int_2^3 x^6\,dx = \frac{x^7}{7} + C$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$\int_1^2 \left(\frac{6}{x^3} + \frac{1}{x}\right)dx = 6\int_1^2 \frac{1}{x^3}\,dx + \int_1^2 \frac{1}{x}\,dx$$

$$= 6\int_1^2 x^{-3}\,dx + \int_1^2 x^{-1}\,dx$$

$$= 6\left(\frac{x^{-2}}{-2}\right)\Big|_1^2 + \ln x\Big|_1^2$$

$$= -3\left(\frac{1}{x^2}\right)\Big|_1^2 + \ln x\Big|_1^2 = -3\left(\frac{1}{4} - 1\right) + \ln 2 - \ln 1$$

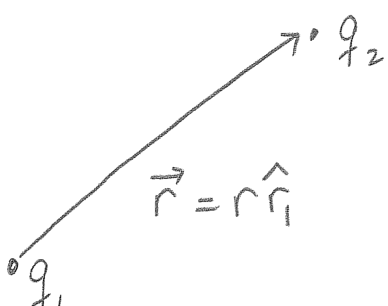
Electric Charge and Coulomb's Law:

The electric charge is a physical quantity that describes ~~the~~ a property of matter, like mass.

However, unlike mass, it comes in 2 flavors: plus (+) and minus (-)

Electric charge: Conserved quantity that facilitates the electric field between particles (a measure of the strength of a particle's interaction with an applied field). It is measured in Coulombs.

Coulomb's Law:

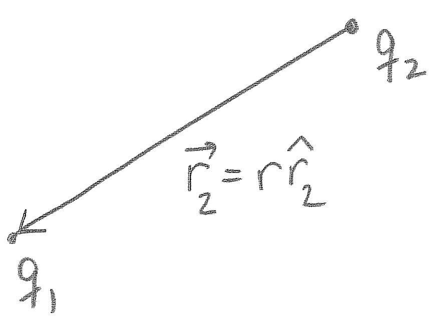


$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_1 \qquad \epsilon_0 = 8.854187 82 \times 10^{-12} \text{ F/m}$$

on ↗ ↖ due to

Notice that $\hat{r}$ is drawn to point away from the charge.

By Newton's third Law, $\vec{F}_{12} = -\vec{F}_{21}$



$$F_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_2$$

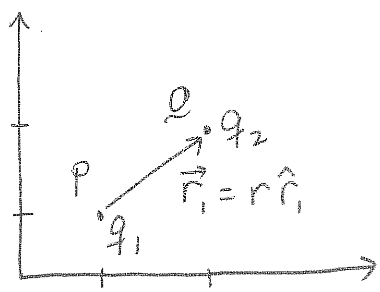
✓

$$\hat{r}_2 = -\hat{r}_1$$

Now, because the charge can be either positive or negative, the force can be attractive or repulsive.

Example:

Suppose q_1 is located at $(1,1)$ and q_2 at $(2,2)$, what is the force on q_1?



$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_1$$

$$\hat{r}_1 = \frac{\vec{r}_1}{r}$$

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^3} \vec{r}_1$$

$$\vec{r}_1 = \overrightarrow{OQ} - \overrightarrow{OP} = \langle 2;2 \rangle - \langle 1,1 \rangle$$

$$= \langle 1,1 \rangle$$

$$\|\vec{r}_1\| = r = \sqrt{1^2+1^2} = \sqrt{2}$$

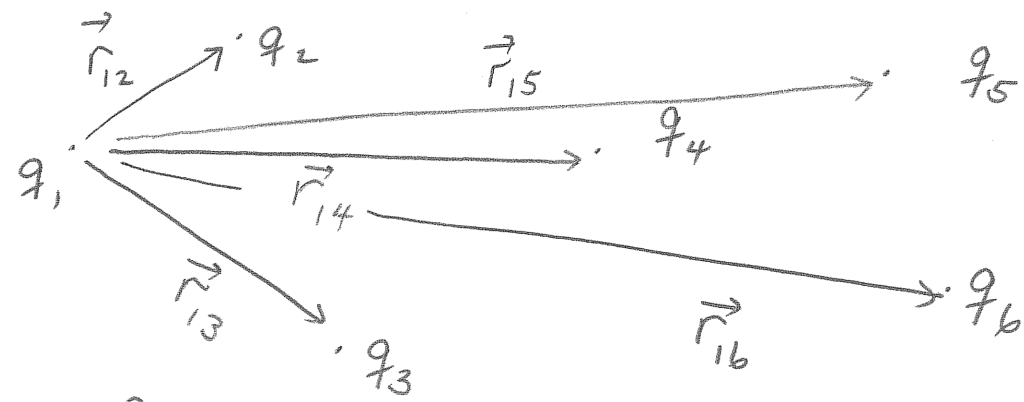
$$\boxed{\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{2^{3/2}} \langle 1,1 \rangle}$$

$$\vec{F}_{21} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{2^{3/2}} \langle 1,1 \rangle$$

Superposition Principle:

Suppose we have an esemble of charges

$\{ q_1, q_2, q_3, \ldots \}$



what is the force on q_1?

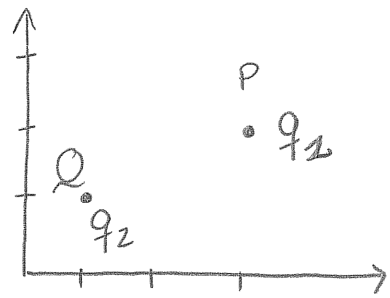
The superposition principle states that the force on ~~a~~ a test charge due to the presence of many charges is simply the vector sum of the forces of each charges with the test charge independently.

$$\vec{F}_1 = \frac{q_1}{4\pi\epsilon_0} \left(\frac{q_2}{\|r_{12}\|^3} \vec{r}_{12} + \frac{q_3}{\|r_{13}\|^3} \vec{r}_{13} + \ldots \right)$$

$$= \frac{q_1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\| r_{1i} \|^3} \vec{r}_{1i}$$

Example:

Let $q_1 = e$ and $q_2 = -2e$ be placed at $(3, 2)$ and $(1, 1)$ respectively. Where must $q_3 = -e$ be placed so that it experiences no force?



Let q_3 be placed at $(x, y) = R$

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32}$$

$$= \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^3}\vec{r}_1 + \frac{q_2}{r_2}\vec{r}_2\right)$$

$$\vec{r}_1 = \vec{OP} - \vec{OR} = \langle 3-x, 2-y \rangle$$

$$\vec{r}_2 = \vec{OQ} - \vec{OR} = \langle 1-x, 1-y \rangle.$$

$$\vec{F}_3 = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{\left(\sqrt{(3-x)^2 + (2-y)^2}\right)^3}\langle 3-x, 2-y \rangle + \frac{q_2}{\left(\sqrt{(1-x)^2 + (1-y)^2}\right)^3}\langle 1-x, 1-y \rangle\right) = \langle 0, 0 \rangle.$$

$$\Rightarrow \quad \frac{q_1(3-x)}{r_1^3} + \frac{q_2(1-x)}{r_2^3} = 0$$

$$\frac{q_1(2-y)}{r_1^3} + \frac{q_2(1-y)}{r_2^3} = 0$$

2 eq's, 2 unknowns, can be solved for x and y.

Applications:

Please see PowerPoint.