Introductory Electricity

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Problem Set 4

This problem set is due **August 3rd, 2014** if you would like to receive feedback on your work. The exercises below vary in difficulty. Some are straightforward computational exercises intended to horn essential skills accompanying concepts in the previous lectures. Others may require more critical thinking to develop a deeper appreciation for the topics introduced. If you are unable to make progress on any particular problem and would like to obtain some hints before the solutions are released, please feel free to email us with your request.

1 Lecture Summary:

In lecture on July 27th, 2014, we discussed electric field energy, capacitance, and R-C circuits.

1. Electric fields store energy, which is proportional to the field strength squared. The electric field energy density (energy per volume) u of an electric field $\mathbf{E}$ is given by

$$u = \frac{\epsilon_0}{2}\|\mathbf{E}\|^2.$$

   If the electric field is contained in a volume Ω, then the total energy U of the electric field is

$$U = \int_\Omega u dV = \frac{\epsilon_0}{2}\int_\Omega \|\mathbf{E}\|^2 dV.$$

   The above equation simply states that to find the total energy, we add together all the energy contributions of at each point where the field is non-zero. To find the energy of electric fields in the entire universe, we can imagine integrating over all of space $\mathbb{R}^3$

$$U = \int_{\mathbb{R}^3} u dV = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} \|\mathbf{E}\|^2 dV$$

2. A related, but distinct, concept to energy is *power*. It is the rate of change of energy over time. A resistor connected to a power source of voltage V with a current I dissipates energy at a rate P

$$P = IV.$$

3. One way to store energy in an electric field that we can later extract is by using capacitors. We can construct a capacitor by using two *conducting* bodies of charges with opposite polarity with any geometrical configuration whatsoever. The field which is

established between these two bodies of charges stores energy. The most common idealized capacitors are parallel-plate capacitors where two plates are placed one on top of the other separated by a distance d. One of the plates has charge Q and the plate has $-Q$ (please note that the plates do not have to contain the same amount of charges). The capacitance C of a capacitor is a measure of the change in voltage between the two plates when we add an extra bit of charge defined as

$$C = \frac{Q}{\Delta V},$$

where V is the voltage and Q is the charge.

4. It turns out that capacitance is a purely geometrical quantity, dependent only on parameters which specify the shapes of the bodies of charges. For instance, the capacitance of two parallel plates with area A separated by a distance d is

$$C = \frac{\epsilon_0 A}{d}.$$

For other geometries, one has to appeal to the definition of capacitance to find an analytic form, if it exists.

5. The energy stored in a capacitor with a potential difference V, charge Q, and capacitance C is

$$U = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2.$$

6. Just like resistors, we can use capacitors as a circuit element. When a capacitor is connected to a voltage source V, the quantities of interest are charge Q (analogous to current I in a resistors circuit) and capacitance C (analogous to resistance R in a resistor circuit).

7. We can connect to capacitors $C_1, C_2, ..., C_n$ to a power source V via a parallel or series configuration. For capacitors in parallel,

   (a) The voltage across each capacitor is the same as that of the power source.

   (b) The total charge Q is the sum of charge, $Q_1, Q_2, ..., Q_n$, of each capacitor.

   (c) The equivalent capacitance C_{eq} is

$$C_{\text{eq}} = C_1 + C_2 + ... + C_n.$$

   For capacitors in series,

   (a) The voltage across each capacitor adds up to equal V.

   (b) The charge of each capacitor is the same as the charge of the others, if the capacitors are all initial uncharged.

   (c) The equivalent capacitance C_{eq} is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_n}.$$

8. We can connect a resistor R and a capacitor C to a voltage source V in series. Denote the time $t = 0$ at the moment we complete the circuit. The current through the circuit is

$$I(t) = \frac{V}{R} e^{-t/RC},$$

and the charge on the plates is

$$q(t) = VC(1 - e^{-t/RC}).$$

2 References:

For the convenience of the readers, we provide links to third-party websites below. We are *not* responsible for any content contained in those websites. Please view them at your own discretion.

1. To learn more about electric energy density, please visit
   (a) `http://web.mit.edu/sahughes/www/8.022/lec03.pdf`
   (b) `http://www2.warwick.ac.uk/fac/sci/physics/current/teach/module_home/px263/lectures2011/energy_density.pdf`
2. To learn more about capacitors, please consult
   (a) `http://web.mit.edu/viz/EM/visualizations/coursenotes/modules/guide05.pdf`
   (b) `http://farside.ph.utexas.edu/teaching/302l/lectures/node47.html`
   (c) `https://www.khanacademy.org/test-prep/mcat/physical-processes/capacitors/v/capacitors-and-capacitance`
   (d) `http://micro.magnet.fsu.edu/electromag/electricity/capacitance.html`
3. For a book reference regarding R-C circuits, please read section 28.4 of *Physics for Scientists and Engineers* by Raymond A. Serway.
4. To review R-C circuits online, please see
   (a) `http://farside.ph.utexas.edu/teaching/302l/lectures/node46.html`
   (b) `http://hyperphysics.phy-astr.gsu.edu/hbase/electric/capchg.html`
   (c) `http://www.pa.msu.edu/courses/2000fall/PHY232/lectures/rccircuits/rc.html`
5. To learn more about complex impedance, please consider the following sites
   (a) `http://hyperphysics.phy-astr.gsu.edu/hbase/electric/impcom.html`
   (b) `http://www2.physics.umd.edu/~jacobson/273c/impedance.pdf`
6. To explore more about multiple-battery circuits and Kirchoff's Laws, please consult
   (a) `http://physics.bu.edu/~duffy/py106/Kirchoff.html`
   (b) `http://www.niu.edu/~mfortner/labelec/lect/Le1_012.pdf`
   (c) `http://www.niu.edu/iteams/documents/ueet602/KVL%20and%20KCL.pdf`
   (d) `http://people.clarkson.edu/~jsvoboda/Syllabi/ES250/ckts/KandOLaws_foc_ac.pdf`

3 Exercises:

Exercise 1 – Energy from Fields

Consider two concentric spherical shells (i.e. a shell within a shell with a common center) of non-zero radii a and b with $a < b$. The smaller shell has surface charge density σ, and the larger shell has surface charge density $-\sigma$.

1. Use Gauss's Law to find the electric field every where. *Hint:* The electric field is spherically symmetric (it only depends on the radius r from the center of the shells) because of the symmetry of the problem. Additionally, the field is only non-zero for $a \leq r \leq b$.
2. Find the electric field *energy density* as a function of r.
3. Find the total energy of this field configuration. *Hint:* You have found u in the previous part. This part simply asks you to find

$$U = \int_{\Omega} u dV.$$

In spherical coordinates, $dV = r^2 \sin\theta d\theta d\phi dr$. Because of spherical symmetry, all the angles integrate to 4π. So we only have

$$U = 4\pi \int_a^b u r^2 dr.$$

Exercise 2 – A Spherical Capacitor

Consider the geometry in Exercise 1.

1. Calculate the potential between the two shells. *Hint:* This requires you to integrate the electric field along the radial direction with $a \leq r \leq b$.
2. Find the capacitance C of this configuration using the definition and verify that it is a purely geometrical quantity.
3. Fix a and draw a qualitative plot of C as a function of b.
4. Fix b and draw a qualitative plot of C as a function of a.
5. Now draw similar plots for a parallel plate capacitor with A and d varying each the other one fixed.
6. Now assume that $a \approx b = R >> 1$. This means that the radii are really large, but the distance between them is very small. So the configuration effectively looks like a parallel-plate capacitor. Verify this result. *Hint:* you might find the approximation

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} \approx \frac{b-a}{R^2}$$

useful. Note also that the surface area of a sphere is $4\pi R^2$.

Exercise 3 – Finding the Capacitance

As noted in class, we can make a capacitor from any arbitrary geometry. So it is not easy to find the capacitance analytically in general. Fortunately, we can find the capacitance of a capacitor using many experimental methods. One such methods is using the energy characterization of capacitance. Capacitance is related to the stored energy via the voltage difference between the two conductors of the capacitor. The voltage difference can be easily measured using a voltmeter. Now, we just need to find the stored energy somehow. We can use a resistor and approximate its dissipated energy to accomplish feat. Suppose that you are given a 60-W light bulb, a stopwatch, a voltage meter, electrical wires, a capacitor of unknown capacitance, and a home-made sandwich. Propose an experiment, along with all of its assumptions, to approximate the capacitance.

Exercise 4 – Combination Capacitors

Consider the circuit in Fig. 1.

1. Find the charge and potential difference across each capacitor.
2. Find the equivalent capacitance.

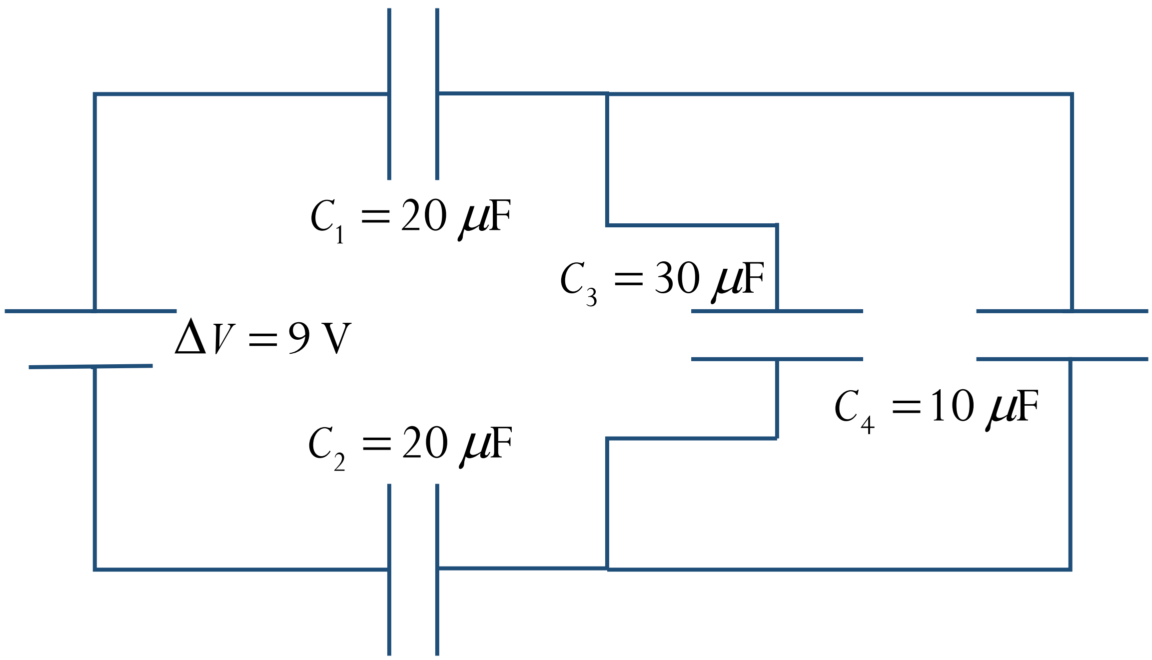

Figure 1: A combination of capacitors.

Exercise 5 – RC Circuit

Consider the circuit in Fig. 2. At time $t = 0$, the circuit is connected.

1. Verify that $\tau = RC$ has units of time using the definition of resistance and capacitance. This time τ is called the *characteristic time* of the system.
2. Find the current and charge on the capacitor as a function of time.
3. Review the graphical properties of the exponential function and plot current and charge as a function of time on the same sketch.
4. Explain in words what dynamics of the circuit for $t > 0$.
5. Calculate the complex impedance of this circuit.

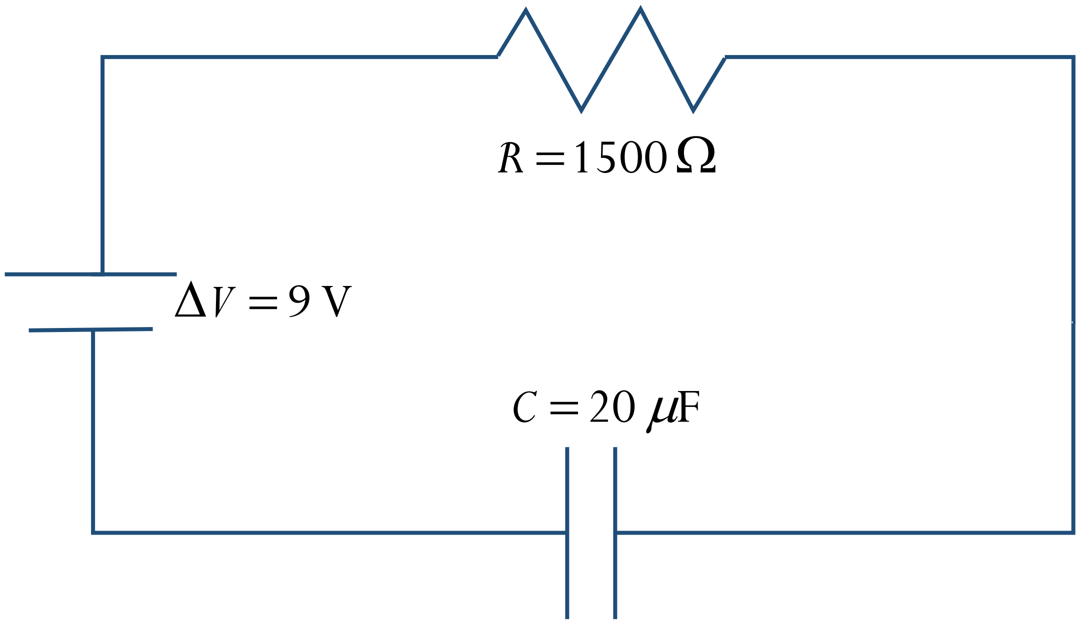

Figure 2: An RC circuit.

Exercise 6 – Household Appliances

Please complete problem 53 in chapter 19 of *Physics* by Serway and Faughn. In addition to using the bill provided in the book, please obtain an actual electric bill from your parents and answer the questions in the book using this bill.

Exercise 7 – Complex Circuits

Consider the circuit in Fig. 4.

1. Find the current and potential difference across each resistor.
2. Find the potential difference across points A and B.
3. Calculate the complex impedance of this circuit.

Hint: You may find Kirchhoff's Rules useful this problem.

References

[1] Serway, Raymond A., and Jerry S. Faughn. *Physics*. Boston: Holt, Rinehart, and Winston, 2002.

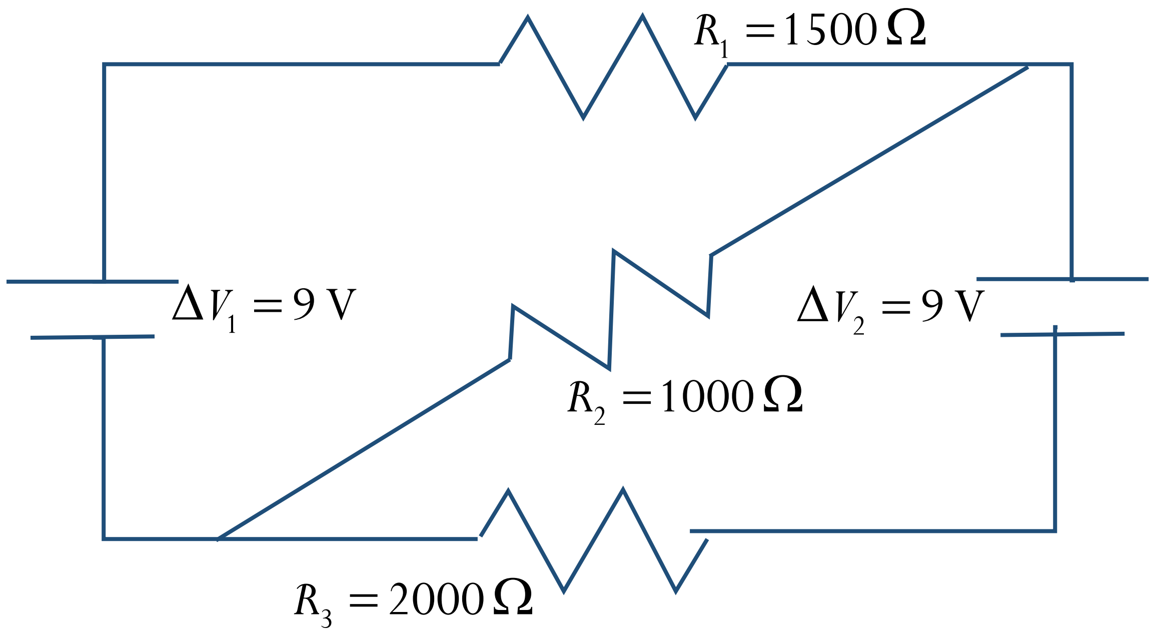

Figure 3: A complex circuit with multiple power sources. .