# Introductory Electricity

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HSSP Summer 2014

June 5, 2014

## Problem Set 1

This problem set is due **July 13th, 2014** if you would like to receive feedback on your work. The exercises below vary in difficulty. Some are straightforward computational exercises intended to horn essential skills accompanying concepts in the previous lectures. Others may require more critical thinking to develop a deeper appreciation for the topics introduced. If you are unable to make progress on any particular problem and would like to obtain some hints before the solutions are released, please feel free to email us with your request.

## 1 Lecture Summary:

In lecture on July 6th, 2014, we discussed the basics of *electrostatics* and some mathematical preliminaries.

1. Suppose we have a function $f(x)$ and $g(x)$ and constants $C$ and $D$, then the derivative of the sum of these two functions is
$$\frac{d[Cf(x) + Dg(x)]}{dx} = C\frac{df(x)}{dx} + D\frac{dg(x)}{dx}.$$
Likewise, the integral is
$$\int [Cf(x) + Dg(x)]dx = C\int f(x)dx + D\int g(x)dx.$$

2. If we have a function of the form $f(x) = x^n$ for some number $n$, then the derivative of this function is
$$\frac{d(x^n)}{dx} = nx^{n-1},$$
and its integral is
$$\int x^n dx = \frac{x^{n+1}}{n+1} + A,$$
for an arbitrary constant $A$ if $n \neq -1$, and is
$$\int x^{-1} dx = \ln|x| + A,$$
for the case $n = 1$.

3. Given two vectors in component-form, $\mathbf{v} = \langle v_x, v_y, v_z \rangle = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ and $\mathbf{w} = \langle w_x, w_y, w_z \rangle = w_x\hat{i} + w_y\hat{j} + w_z\hat{k}$, we can perform the following operations:
   (a) Addition: $\mathbf{v} + \mathbf{w} = \langle v_x + w_x, v_y + w_y, v_z + w_z \rangle$,
   (b) Subtraction:$\mathbf{v} - \mathbf{w} = \langle v_x - w_x, v_y - w_y, v_z - w_z \rangle$,
   (c) Scalar multiplication: $c\mathbf{v} = \langle cv_x, cv_y, cv_z \rangle$ for some scalar $c$,
   (d) Norm: $\|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$,
   (e) Inner (dot) product: $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$, where $\theta$ is the smallest angle between the vectors,
   (f) Vector (cross) product: $\mathbf{v} \times \mathbf{w} = \langle v_y w_z - v_z w_y, v_z w_x - v_z w_z, v_z w_y - v_y w_x \rangle$, and $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, where $\theta$ is the smallest angle between the vectors,
   (g) Unit vector: $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

4. There are two flavors of charges: plus (+) and minus (−). We take axiomatically that like charges repel and opposite charges attract. Charges are quantized to integral multiples of a third of the elementary charge $e = 1.60217657 \times 10^{-19}$ C.

5. Given two charges $q_1$ and $q_2$, the force between them is given by *Coulomb's Law*

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^3} \mathbf{r},$$

   where $\hat{\mathbf{r}}$ is the unit radial vector that joins the location of the two charges, $r$ is the distance between the two charges, and $\epsilon_0 = 8.85418782 \times 10^{-12}$ F/m is the vacuum permittivity.

6. Coulomb's Law obeys the *superposition principle*. That is, the force on a charge due to the presence of many charges is simply the sum of the forces between that charge and each of the other charges individually. Suppose that we have a collection of $N$ charges $\{q_1, q_2, q_3, ..., q_N\}$. Then the force on $q_1$ is simply

$$\mathbf{F} = \mathbf{F}_2 + \mathbf{F}_3 + ... + \mathbf{F}_N,$$

   where the subscript $i$ denotes the force between $q_1$ and $q_i$ for $i = 2, 3, 4, ..., N$.

## 2 References:


1. For a basic introduction to vectors (with some computational aids), please visit `http://hyperphysics.phy-astr.gsu.edu/hbase/vect.html`.

2. Those interested in learning more about differentiation and integration can consult any calculus textbook for more information. For instance, chapters 2 and 5 of *Calculus* by Gilbert Strang are relevant to our purpose. This textbook is free at `http://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/textbook/`.

3. For a nice conceptual coverage of Coulomb's Law, please see sections 17.1 and 17.2 of *Physics* by Raymond A. Serway and Jerry S. Faughn.

4. For some practice with Coulomb's Law, please see `http://dev.physicslab.org/Document.aspx?doctype=3&filename=Electrostatics_AdvancedCoulombsLawProblems.xml`

5. To review Thomson's discovery of the electron, please visit `http://www-outreach.phy.cam.ac.uk/camphy/electron/electron1_1.htm`.

6. Those interested in learning more about the different atomic models are encouraged to search online for more information. The models we explored in class include the plum-pudding model, Rutherford's planetary model, Bohr's atomic model, and the Schrodinger's electron cloud model.

# 3 Exercises:

**Exercise 1 – Differentiation and Integration**

Find the derivative and integral of the following functions with respect to the specified independent variable. All letters appearing in these functions, other than the independent variable, are assumed to be constants.

1. $f(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$
2. $f(x) = -kx$
3. $f(r) = \frac{l^2}{2mr^2} + \frac{GmM}{r}$

**Exercise 2 – Working with Vectors, Part I**

Given the following vectors, $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle 2, 4, 3\rangle$, find the following quantities:

1. $\|a\|$ and $\|b\|$
2. $\mathbf{a} + \mathbf{b}$
3. $\mathbf{a} - 6\mathbf{b}$
4. $\mathbf{a} \cdot 2\mathbf{b}$
5. $\mathbf{a} \times \mathbf{b}$
6. Find a vector which are both perpendicular to $\mathbf{a}$ and $\mathbf{b}$ by whatever means seems easiest.

**Exercise 3 – Working with Vectors, Part II**

Suppose we are working in the three-dimensional Cartesian coordinate system. Consider three points in the space: $P = (0, 0, 2)$, $Q = (0, 3, 0)$, and $R = (a, b, c)$, where $a, b$, and $c$ are unknown coordinates to be determined.

1. Find the position vectors associated with each given point. That is, denote the origin as $O = (0, 0, 0)$. We want to find the vectors $\mathbf{P} = \overrightarrow{OP}$, $\mathbf{Q} = \overrightarrow{OQ}$, and $\mathbf{R} = \overrightarrow{OR}$.
2. Find the vector $\overrightarrow{PQ}$ and its associated unit vector.
3. Now, we are interested in finding the vector $R$ subject to the following constraints: $\mathbf{P} \times \mathbf{R} = \mathbf{Q} \times \mathbf{R} = \mathbf{0}$ and $\|\mathbf{R}\| = 5$. Do these constraints produce a unique $R$? If not, what other constraints would you need to obtain uniqueness?
4. Now, we repeat the previous step with a different set of constraints: $\mathbf{P} \times \mathbf{R} = \mathbf{0}$, $\mathbf{Q} \cdot \mathbf{R} = 2$ and $\|\mathbf{R}\| = 1$. Do these constraints produce a unique $R$? If not, what other constraints would you need to obtain uniqueness?

**Exercise 4 – Coulomb's Law at Work**

Consider two charges $q_1 = -e$ and $q_2 = e$ with masses $m_1$ and $m_2$ respectively.

1. Find the acceleration on $m_1$ as a function of the separation distance $r$.
2. Suppose the two masses are initially separated with a distance $r_0 = 5.29 \times 10^{-11}$ m. Find the acceleration on $m_1$ here.
3. Now, imagine that $m_2$ is much much larger than $m_1$ that it remains essentially at rest throughout the entire interaction. If the acceleration on $m_1$ remains the same as that when it is at a distance $r_0$ from $m_2$, how much time does it take for $m_1$ to collide with $m_2$? Of course, this approximation is vastly inaccurate because the acceleration gets larger and larger as $r$ goes to zero. However, it does serve to provide an upper limit for the collision time.

**Exercise 5 – The Sum of Forces**

Let $q_1 = q_2 = e$ be placed at $(0, 2)$ and $(0, -2)$ respectively. These two charges are pinned in place. A third charge $q_3 = -2e$ is placed originally at $(x_0, 0)$ for some $x_0$.

1. Find the force, magnitude and direction, on $q_3$ as a function of the location of $q_3$.
2. Qualitatively describe the motion of $q_3$ if it were free to move according to Newton's laws.
3. Find the acceleration of $q_3$ as a function of its position using Newton's second law.
4. Solve for the position of $q_3$ as a function of time. *Hint:* The acceleration is the second derivative (i.e. the derivative of the derivative of) $x$ with respect to time. So, you should arrive at an expression

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

for some constant $\omega$ that you will determine. This *differential equation* has solutions

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$

for arbitrary constants $A$ and $B$ that are determined by initial conditions. In our problem, the initial conditions are that the charge start out at rest and is initially at $x_0$ on the $x$-axis. This reduces the solution to

$$x(t) = x_0 \cos(\omega t).$$

Does this solution confirm your intuition in part (b)?

**Exercise 6 – Balancing charges**

Consider a collection of four charges placed at the vertex of a square with sides of 2 m. Suppose that the charges are $q_1 = e$, $q_2 = -2e$, $q_3 = 3q$, and $q_4 = -4e$ arranged clockwise starting with the upper left vertex. Suppose that these four charges are somehow pinned down to where they are so that they cannot move by interacting with the other charges. We want to trap a fifth charge $q_5 = e$ inside the square. Where must $q_5$ be placed? *Hint:* This problem is not conceptually difficult (we believe), but it is computational cumbersome. So please do not become discouraged if you run into messy algebra. As a suggestion, it is crucial that you pay attention to the signs of the forces.

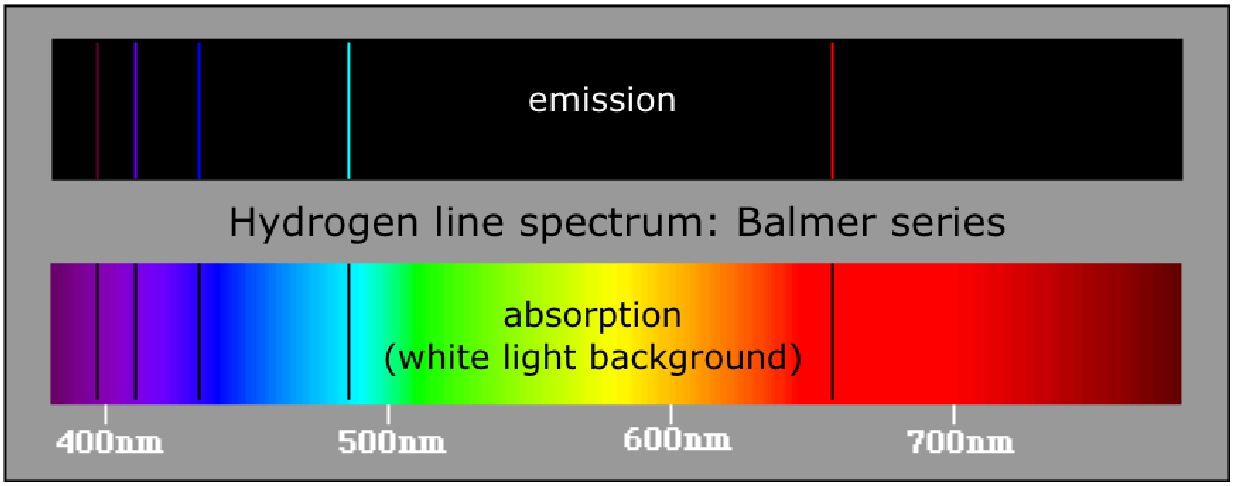


Figure 1: The Balmer series of the hydrogen spectrum. This image is taken from `http://chemwiki.ucdavis.edu/Under_Construction/chem1/Atoms_and_the_Periodic_Table/The_Bohr_Atom`.

**Exercise 7 – Spectrum of the Hydrogen**

In class, we stated that the energy levels of the hydrogen atom are quantized according to

$$E_n = \frac{-2.17896 \times 10^{-18}\ \mathrm{J}}{n^2}$$

for $n = 1, 2, 3, \ldots$. Thus, an electron that falls from an initial energy level with quantum number $n_i$ to a final energy level with quantum number $n_f$ releases a photon of light with energy corresponding to the difference in energy between the two levels (the energy that the electron lost). Recall that $E = h\frac{c}{\lambda}$, where $h = 6.626 \times 10^{-34}$ Js is Planck's constant, $c = 299,792,458\ \mathrm{m/s}^2$ and $\lambda$ is the wavelength of the emitted light. Calculate the wavelength of the emitted light for $n_f = 2$ and $n_i = 3, 4, \ldots, 7$. These spectral lines are known as the *Balmer Series*. Compare your result with the data in Fig. 1. Agreement of the Bohr's model with experimental spectroscopic data was one of the early triumphs of the old quantum model.