The semantics of propositional logic

Readings: Sections 1.3 and 1.4 of Huth and Ryan.

In this module, we will nail down the formal definition of a logical formula, and describe the semantics of propositional logic, which will be familiar to you from Math 135 and CS 251. Our final goal is to prove the soundness and completeness of propositional logic, but to do that, we need to be precise about induction, which we review carefully.

The formal language of formulas (1.3)

Our formulas can be viewed as strings over an alphabet composed of our atoms ($p, q, r, p_1, p_2, \ldots$), plus the symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, and the open and close parentheses, (and). ($\bot$ is a convenience in our proofs, and shouldn't appear in formulas.)

But not just any string over this alphabet is possible; $p \wedge \wedge \rightarrow\rightarrow ($ is not a well-formed formula.

We will describe well-formed formulas by a recursive definition. You have seen recursive definitions of data structures in CS 134 or CS 135.

A well-formed formula (wff) is either an atom, or it is of one of the forms $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, or $(\phi \rightarrow \psi)$, where ϕ and ψ are well-formed formulas (wffs).

Another (equivalent) way of describing wffs is by a grammar, as is used in CS 135 and CS 241.

$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \psi) \mid (\phi \vee \psi) \mid (\phi \rightarrow \psi)$$

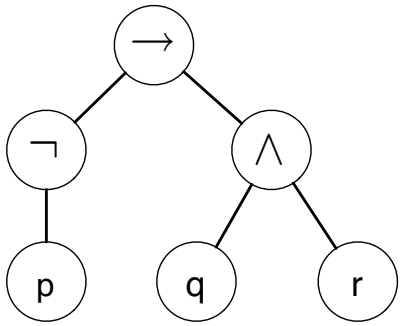
Here p stands for any atom.

Note that both our recursive definition and our grammar define fully parenthesized formulas; the abbreviated formulas using binding priorities, such as $\neg p \rightarrow q \wedge r$, do not satisfy either definition. But $((\neg p) \rightarrow (q \wedge r))$, which is really what we mean when we write the abbreviated version, does meet both definitions.

CS 241 covers the process of extending a grammar like this to handle operator precedence; equivalently, we could replace the recursive definition with a set of mutually recursive definitions.

Since our goal in this module is to prove things about all wffs, we need not worry about how convenient it is to write one down, and so we will not pursue these modifications.

Using notions from CS 135 and CS 241, we can consider the parse tree (a rooted, ordered tree of fanout at most two) for the formula $((\neg p) \rightarrow (q \wedge r))$.



It's tempting to try to make this the definition of a formula, especially as you have seen how to prove things about trees in CS 134 and CS 136. But a tree also has a recursive definition, and the drawing is just a visualization of it. The tree is really representing the process of showing that the formula meets our first definition:

p,q,r are wffs because they are atoms.
$(q \wedge r)$ is a wff because q and r are wffs.
$(\neg p)$ is a wff because p is.
$((\neg p) \rightarrow (q \wedge r))$ is a wff because $(\neg p)$ and $(q \wedge r)$ are.

The formulas occurring in this process are **subformulas** of $((\neg p) \rightarrow (q \wedge r))$. They correspond to complete subtrees rooted at nodes of the parse tree.

Apart from reordering some of the lines in the justification on the previous slide, there is no other way to show that our formula meets the definition. Another way of saying this is that the parse tree is unique.

This is true for every formula that satisfies the definition. This will be important for our semantics. The textbook calls this the **inversion principle**, meaning it is possible to invert the process of building formulas.

The textbook asks you to take this on faith, but for those of little faith, we will prove it in a little while, since we'll have the tools to do so.

The ideas of a formal definition of a logical formula and a separate semantics for assigning a truth value to such a formula come from the work of Gottlob Frege in 1879, though the notation he used was quite different.

Many mathematicians and philosophers built on Frege's ideas, including Bertrand Russell, Alfred North Whitehead, Ludwig Wittgenstein, and Emil Post. Throughout their work, the notation changed and evolved.

What we have presented here is now standard terminology, though you will still see variations on it in other books (for example, the use of $\supset$ in place of $\rightarrow$).

The meaning of logical connectives

The set of truth values contains two elements, true (represented by T) and false (represented by F).

A **valuation** or **model** of a formula ϕ is a mapping of each propositional atom of ϕ onto a truth value.

Our example formula $((\neg p) \rightarrow (q \wedge r))$ has three propositional atoms. If we assign T to p, T to q, and F to r, that is one valuation. There are clearly $2^3 = 8$ different valuations for this formula.

This should be quite familiar, but we are doing it carefully to be able to build on it. (Though we'll mostly use "valuation" in this section, remember that word "model"; we'll see it a lot later on.)

We extend the idea of a valuation assigning a truth value to an entire formula by using the recursive definition of wff.

If a formula is of the form $(\phi \wedge \psi)$, then it is assigned T if both ϕ and ψ are; otherwise it is assigned F.

If a formula is of the form $(\phi \vee \psi)$, then it is assigned F if both ϕ and ψ are; otherwise it is assigned T.

If a formula is of the form $(\phi \rightarrow \psi)$, then it is assigned F if ϕ is assigned T and ψ is assigned F; otherwise, it is assigned T.

If a formula is of the form $(\neg\phi)$, then it is assigned F if ϕ is assigned T; otherwise, it is assigned T.

In Math 135, you saw the idea of using truth tables to do the assignment of truth values to a formula.

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	$\neg\phi$
T	F
F	T

These tables are just another way of describing the assignments on the previous slide.

We can also build a truth table for any formula – for example, $((\neg p) \rightarrow (q \wedge r))$.

p	q	r	$(\neg p)$	$(q \wedge r)$	$((\neg p) \rightarrow (q \wedge r))$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	F	F

When we proved a sequent $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$, the idea was that eventually we should be able to show that if all of $\phi_1, \phi_2, \ldots \phi_n$ were true, then ψ would also be true. Now we have a way of describing a circumstance (a valuation) under which we can evaluate the truth value of formulas.

This suggests that what we want to show is that any valuation which makes $\phi_1, \phi_2, \ldots \phi_n$ true should make ψ true.

If this is in fact the case, we write $\phi_1, \phi_2, \ldots \phi_n \models \psi$. (The symbol $\models$ is read "models"). Though we hoped for this for a particular sequent, the notion is actually independent of our previous notion of proof.

We define $\phi_1, \phi_2, \ldots \phi_n \models \psi$ to mean that any valuation that makes $\phi_1, \phi_2, \ldots \phi_n$ true also makes ψ true. $\models$ is also called the **semantic entailment** relation.

Our goal is to prove that this is equivalent to our previous notion of valid sequents.

The property that if $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots \phi_n \models \psi$ is called **soundness**.

The property that if $\phi_1, \phi_2, \ldots \phi_n \models \psi$, then $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$ is called **completeness**.

Soundness states that any valid sequent says something about the relationship between the formulas involved, under all valuations.

Completeness states that given a relationship between the valuations of formulas, there must exist a proof of validity of the corresponding sequent.

It helps to think about the case where n (the number of formulas on the left-hand side) is zero. Soundness states that any formula that is a theorem is true under all valuations. Completeness says that any formula that is true under all valuations is a theorem.

We are going to prove these two properties for our system of natural deduction and our system of valuations.

The completeness and soundness of propositional logic (not the particular proof system we are studying, but an equivalent one) was shown by Emil Post in his PhD thesis in 1920. (Post went on to make contributions to early theoretical computer science, some of which you may encounter in CS 360 and CS 365.)

We will follow the proofs of completeness and soundness given in the textbook. These use the mathematical tool of induction, which was introduced in Math 135 and discussed in CS 134 and CS 136, but which deserves careful review.

Induction

Induction is used to prove some property M that depends on some integer n; that is, $M(n)$ is a statement about n. For example, $M(n)$ might be "Every even number greater than 2 but less than n is the sum of two primes".

The principle of mathematical induction says that if $M(1)$ is true (this is the base case), and in addition we can show that for any $n \geq 1$, if $M(n)$ is true, then $M(n+1)$ is true (this is the inductive step), then we can conclude that $M(n)$ is true for all natural numbers n. (We don't actually know how to do this for the specific statement $M(n)$ above, which is called "Goldbach's conjecture".)

Induction is a way of proving (in a mathematical sense) statements about an infinite number of situations or an infinite set.

We cannot prove the principle of mathematical induction; it is part of the definition of natural numbers. (We will make this idea more concrete later on in the course.)

The first proof by induction you saw in Math 135 was the proof that the sum of the first n natural numbers is $n(n+1)/2$. This is reproduced in our textbook (Theorem 1.31) in case you have lost or sold your Math 135 textbook. We will not bore you to tears by putting it into the slides.

In Math 135 you also saw that there were many different forms of induction, all equivalent to the basic form. We can vary the base case and we can vary the inductive step.

We could, for instance, declare the base case to be proving that $M(3)$ is true, and have the inductive step show that for $n \geq 3$, $M(n)$ being true implies $M(n+1)$ being true. The result proves that $M(n)$ is true for all $n \geq 3$.

Given such a proof, we can define the property P such that $P(n) = M(n+2)$, and then mechanically translate the proof involving M to a proof involving P. The base case of that proof would be proving $P(1)$ true, and the inductive step would show that for $n \geq 3$, if $P(n-2)$ is true, then $P(n-1)$ is true, claiming a conclusion that $P(n-2)$ is true for all $n \geq 3$.

Now the additional substitution $m = n - 2$ makes this into a proof that shows that $P(1)$ is true, and for $m \geq 1$, $P(m)$ being true implies that $P(m+1)$ is true. The conclusion becomes that $P(m)$ is true for all $m \geq 1$. This is the basic form of induction, and so this proof is valid. By reversing all our steps, we can show that the original proof is also valid.

A similar technique works to show that if we have a proof that keeps the base case of showing $M(1)$ true, but modifies the inductive step to show that for $n \geq 2$, $M(n-1)$ being true implies $M(n)$ true. Again, a suitable set of mechanical substitutions gets us to the basic form. This form might be preferable for certain properties M where the algebra or other reasoning is simplified going from $M(n-1)$ to $M(n)$.

There is a form called “simultaneous induction”, in which two (or more) statements are proved for all n.

As an example, we may have properties $M_1(n)$ and $M_2(n)$. The base case proves both $M_1(1)$ and $M_2(1)$. The inductive step assumes both $M_1(n)$ and $M_2(n)$, and uses these assumptions to prove both $M_1(n+1)$ and $M_2(n+1)$. The conclusion reached is that $M_1(n)$ is true for all n, and $M_2(n)$ is also true for all n.

By defining $P(n)$ to be the property “$M_1(n)$ is true and $M_2(n)$ is true”, we can mechanically reduce the above proof to the basic form.

The inductive step in basic induction is of the form “Assume $P(n)$ true and use it to prove $P(n+1)$ true.” From this, we conclude that $P(n)$ is true for all n. Sometimes, the induction is phrased in the following form.

“Assume $P(n)$ is not true for all n. Let n_0 be the smallest value of n for which $P(n)$ does not hold. Thus $P(n_0)$ does not hold, but $P(n_0 - 1)$ does. If, from these two facts, we can derive a contradiction, our assumption must be wrong, and $P(n)$ must hold for all n.”

This argument wraps basic induction in a proof by contradiction. It is sometimes called “proof by minimal counterexample”. It shows up, for instance, in CS 341, in the discussion of greedy algorithms.

Our basic form of induction is sometimes called “weak induction”, in contrast with a form called “strong induction”. (The adjectives refer to the strength of the hypothesis in the inductive step.)

In the version of strong induction done in Math 135, the base case shows that $M(1)$ is true, and the inductive step (for $n \geq 1$) assumes $M(1), M(2), \ldots M(n)$ are true, and uses that assumption to conclude $M(n+1)$ is true. Strong induction then allows the conclusion that $M(n)$ is true for all n.

Given such a proof, we can convert it to the basic form by defining the property $P(n)$ to be “$M(m)$ is true for all $m \leq n$”. All of these forms are equivalent; we just choose the one that is most suitable for our purposes.

Finally, there is a variation called “structural induction”. You saw this in CS 134 or CS 136, where it was used to prove things about binary trees.

Recall that a binary tree is a collection of nodes that is either empty, or partitioned into three sets: a root, a left subtree, and a right subtree (the last two being binary trees).

You saw proofs of facts such as “in a non-empty binary tree, the number of leaves is one more than the number of nodes with exactly two children” with a base case of the empty tree, and an inductive step of assuming this was true of the left and right subtrees of a non-empty tree and proving it is true of the whole tree.

Structural induction, for a property of some structure defined by a recursive definition, is just strong induction on the number of times that the definition is invoked in order to justify a particular structure.

In the case of binary trees, a justification that a nonempty tree T satisfies the definition includes justifications that its left and right subtrees are also binary trees. Those justifications thus each use the definition fewer times than the justification for T.

We can do something similar to prove properties of formulas, because we have a recursive definition for them. When we use structural induction to prove a property M of a formula of the form $\phi \wedge \psi$, we assume M is true of ϕ and ψ and use that to prove M is true for $\phi \wedge \psi$. This is strong induction on the number of times the definition is used to show a formula is well-formed.

What we have called "strong induction", the textbook calls "course-of-values induction", and it also justifies structural induction by talking about the height of the parse tree, instead of the number of applications of the definition (which would correspond to the number of nodes in the tree).

However, these are relatively minor details. The point is that we have structural induction to use in order to prove properties of formulas.

Structural induction will prove useful in courses such as CS 341, CS 360, and CS 466. But all of these forms of induction are just different ways of expressing basic ("weak") induction. We choose the form that is most clear.

Proving the inversion principle

As a warmup to our proofs of soundness and completeness, we will prove the inversion principle, that there is exactly one way to define any wff. We start with an easy theorem. A wff is **balanced** if the number of open parentheses is equal to the number of close parentheses.

Thm (1.33): Every wff χ is balanced.

Proof: By structural induction on the wff χ.

The base case is if χ is an atom, for which there are no open and no close parentheses.

There are four inductive cases, for the four possible ways of building χ: $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, and $(\phi \rightarrow \psi)$.

We will just argue the last case, as the others are similar. (We can often, but not always, get away with this.)

If χ is of the form $(\phi \rightarrow \psi)$, then the inductive hypothesis lets us assume that ϕ is balanced, so is ψ. Then it is clear that $(\phi \rightarrow \psi)$ is balanced, just by adding up the numbers of open and close parentheses. □

If we scan any wff from left to right, we notice that the only time the number of open and close parentheses match is at the very beginning and at the very end. This property turns out to be important in proving the inversion principle (and, though we will not pursue this, leads to an algorithm for constructing the parse tree).

To define the property, we define a **strict prefix** of a wff, which is formed by deleting one or more symbols from the end (but not all symbols). A strict prefix is **open-heavy** if the number of open parentheses exceeds the number of close parentheses.

Thm: Every strict prefix of a wff χ is open-heavy.

Proof: By structural induction on the wff χ.

The base case is atoms, which have no strict prefixes, so the statement is vacuously true.

The inductive step has two cases: when χ is of the form $(\neg\phi)$, and when χ is of the form $(\phi \circ \psi)$, where $\circ$ is one of $\wedge, \vee, \rightarrow$.

When χ is of the form $(\neg\phi)$, a strict prefix of χ is of the form $($, $(\neg$, $(\neg\alpha$, where α is a strict prefix of ϕ, or $(\neg\phi$. The first and second subcases are clearly open-heavy. The fourth subcase is open-heavy because ϕ is balanced. In the third subcase, we invoke the inductive hypothesis to reason that α is open-heavy, thus so is $(\neg\alpha$.

When χ is of the form $(\phi \circ \psi)$, a strict prefix of χ is of the form $($, $(\alpha$ (where α is a strict prefix of ϕ), $(\phi$, $(\phi\circ$, $(\phi \circ \beta$ (where β is a strict prefix of ψ), or $(\phi \circ \psi$. Note that the inductive hypothesis applies to ϕ and ψ, so we can assume that α and β are open-heavy.

Of the possibilities, $($ is open-heavy, $(\alpha$ is open-heavy because α is, $(\phi$ and $(\phi\circ$ are open-heavy because ϕ is balanced, $(\phi \circ \beta$ is open-heavy because ϕ is balanced and β is open-heavy, and $(\phi \circ \psi$ is open-heavy because both ϕ and ψ are balanced. $\square$

The main reason to prove the previous theorem is that any strict prefix of a wff is not a wff (since it is open-heavy, not balanced). We're ready to prove the inversion principle.

Thm: For any wff χ, the justification of χ by means of the inductive definition of wffs is unique.

Pf: By structural induction on the wff χ.

If χ is an atom, the statement is true.

Suppose χ is of the form $(\phi \circ \psi)$, where ϕ, ψ are wffs and $\circ$ is a binary connective. We have to consider the possibilities for interpreting a non-atomic χ in two different ways using the inductive definition of wffs.

Can χ also be of the form $(\alpha \diamond \beta)$, where α, β are wffs, $\diamond$ is a binary connective, and $\alpha \neq \phi$? If it were, then one of α and ϕ would be a strict prefix of the other, and no strict prefix of a wff is a wff.

Can χ also be of the form $(\neg\alpha)$? Then ϕ would have to start with $\neg$, and the inductive definition does not allow any wff to start with $\neg$.

Thus there is only one way of interpreting χ as $(\phi \circ \psi)$ with ϕ and ψ wffs.

Since if χ is of the form $(\phi \circ \psi)$, there is only one way of interpreting it in that form, we now invoke the inductive hypothesis to assume that the justifications for ϕ and ψ are unique. So the justification for χ is formed by the justifications for ϕ and ψ plus the part of the recursive definition of a formula that permits the construction $(\phi \circ \psi)$.

The case where χ is of the form $(\neg\phi)$ is similar. We've already seen that interpretations in this form and also in the form $(\phi \circ \psi)$ cannot both happen, and another interpretation of χ as $(\neg\alpha)$ with $\alpha \neq \phi$ is impossible. Thus $\chi = (\neg\phi)$ has a unique interpretation; ϕ has a unique justification by the inductive hypothesis, and so does χ. $\square$

The inversion principle (also called unique readability) is important in computer science, because we want our programs to have unique interpretations.

CS 241 touches on this in the context of grammars; a grammar is **ambiguous** if there are two different ways to generate the same string (two different parse trees).

CS 360 discusses the concept further, and CS 462 gives a proof that a particular simple language is inherently ambiguous, that is, it has no unambiguous grammar.

Soundness

Recall our definition of soundness: if $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots \phi_n \models \psi$.

Soundness asserts that a valid sequent says something about valuations. A sequent is shown valid by means of a natural deduction proof. Our proof of soundness will proceed by induction on the length of this proof (the number of lines).

Note that our proofs are just flattened trees. We could have given a recursive definition of what a proof was (an application of a rule to one or more subproofs) in which case our proof by induction becomes structural induction. It might help to keep this in mind.

Thm: If $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots \phi_n \models \psi$.

Pf: By strong induction on the length k of a proof of the sequent $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$.

If the proof has one line, it must be of the form

1	ϕ	premise

So the sequent is $\phi \vdash \phi$, and clearly $\phi \models \phi$.

For the inductive step, we assume that soundness holds for sequents with proofs of length less than k, and prove that it holds for a sequent $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$ with a proof of length k.

The kth line of the proof must look like

k ψ justification

We have a number of cases depending on which rule is applied in this line. We'll do enough cases to get the general idea, since we have a lot of rules in natural deduction. (This proof is easier for other systems with fewer rules, but then using such systems is more difficult.)

If the kth line looks like

k ψ $\wedge$ i k_1, k_2

then ψ must be of the form $\psi_1 \wedge \psi_2$, where ψ_1 appears on line k_1, and ψ_2 appears on line k_2.

The sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ has a proof of length at most k_1 (the first k_1 lines of the proof we started with), so $\phi_1, \phi_2, \ldots, \phi_n \models \psi_1$. Similarly, $\phi_1, \phi_2, \ldots, \phi_n \models \psi_2$. Thus, by our definition of how to assign a truth value to a formula under a valuation, $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1 \wedge \psi_2$, as required.

This sort of reasoning works well for many other rules, such as $\wedge$e$_1$ and $\rightarrow$e. But it gets a little complicated if the last line in our proof closes off a box, as with $\rightarrow$i. Suppose we have such a proof of $\phi_1, \phi_2, \ldots, \phi_n \vdash \alpha \rightarrow \beta$.

1		
$\vdots$		
i	α	assumption
$\vdots$		
$k-1$	β	justification
k	$\alpha \rightarrow \beta$	$\rightarrow$ i i–$(k-1)$

If we delete the last line, this is not a complete proof.

But we can turn it into one by making the assumption of the open box into a premise.

1		
$\vdots$		
i	α	premise
$\vdots$		
$k-1$	β	justification

This is a proof of the sequent $\phi_1, \phi_2, \ldots, \phi_n, \alpha \vdash \beta$.

Since our proof of $\phi_1, \phi_2, \ldots, \phi_n, \alpha \vdash \beta$ has $k-1$ lines, we can apply the inductive hypothesis to conclude that $\phi_1, \phi_2, \ldots, \phi_n, \alpha \models \beta$.

We need to conclude that $\phi_1, \phi_2, \ldots, \phi_n \models \alpha \rightarrow \beta$. So consider a valuation that makes $\phi_1, \phi_2, \ldots, \phi_n$ true. If it makes α true, then we can apply the conclusion of the previous paragraph to say that it makes β true.

If it makes α false, then by our rules for assigning truth values to formulas, it must make $\alpha \rightarrow \beta$ true (this follows from the truth table for $\rightarrow$). So we have shown $\phi_1, \phi_2, \ldots, \phi_n \models \alpha \rightarrow \beta$.

This sort of reasoning works for the cases where the last line closes off a proof box.

The full proof involves a complete examination of each of the dozen or so rules of natural deduction, but we have already seen the two main ideas.

Soundness gives us a method of showing that a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ does **not** have a proof in natural deduction; we simply have to find a valuation that makes $\phi_1, \phi_2, \ldots, \phi_n$ true but ψ false.

Completeness

Thm: If $\phi_1, \phi_2, \ldots \phi_n \models \psi$, then $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$.

Proof: The proof is divided into three stages.

The first stage shows that if $\phi_1, \phi_2, \ldots \phi_n \models \psi$, then $\models \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$.

The second stage shows that if $\models \eta$, then $\vdash \eta$, for any formula η (including those looking like what the first stage produced). If $\models \eta$, then η is called a **tautology**.

The third stage shows that if $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$, then $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$.

The first and third stages are the easiest, so let's get them out of the way. We first need to show that if $\phi_1, \phi_2, \ldots \phi_n \models \psi$, then $\models \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$.

How can this last formula be assigned false by a valuation? ϕ_1 would have to be true and $\phi_2 \rightarrow (\phi_3 \rightarrow \ldots (\phi_n \rightarrow \psi) \ldots)$ would have to be false. We can continue to unravel the formula, and conclude that $\phi_1, \phi_2, \ldots \phi_n$ would all have to be true, and ψ false. But we know $\phi_1, \phi_2, \ldots \phi_n \models \psi$, so this is impossible.

For the third stage, we must take a proof of $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$, and from it construct a proof of $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$.

This can be done simply by augmenting the proof of $\vdash \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$ with n lines at the beginning introducing $\phi_1, \phi_2, \ldots \phi_n$ as premises, and n lines at the end, each applying $\to$e to peel off ϕ_1, then ϕ_2, and so on up to ϕ_n, which leaves ψ as the last line. The result is a proof of $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$.

Technically speaking, both of these stages should be proofs by induction on n, but they are so "iterative" or algorithmic in nature that the induction will only obscure what is going on. This is also true in the second stage, but there we also must do a structural induction on formulas, so we really need to keep things as clear as possible.

The second stage is more complicated than anything we have done so far, though it is not technically difficult. Our goal is to prove that for any formula η, if $\models \eta$, then $\vdash \eta$.

Suppose η has n distinct propositional atoms. (This n is different from the one in the first and third stages.) We are going to have to construct a proof knowing only that each of the 2^n valuations assign true to η. We do this by constructing 2^n subproofs, one for each valuation, and then putting them all together.

A valuation V assigns true or false to each of the atoms, which we'll call $p_1, p_2, \ldots, p_n$. (You can think of it as a line in a truth table for η.) Relative to V, we define $\hat{p}_i$ to be the formula p_i if V assigns true to p_i; otherwise $\hat{p}_i$ is the formula $\neg p_i$.

The induction on formulas is contained in the following lemma:

Lemma (1.38): Let ϕ be a formula with propositional atoms $p_1, p_2, \ldots p_n$, and V a valuation. If V makes ϕ true, then $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n \vdash \phi$; if V makes ϕ false, then $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n \vdash \neg\phi$.

Proof: By structural induction on the formula ϕ.

If ϕ is an atom p, the statement of the lemma says that $p \vdash p$ and $\neg p \vdash \neg p$, which have one-line proofs.

If ϕ is of the form $\neg\phi_1$, then we may apply the inductive hypothesis to ϕ_1. If V makes ϕ true, it must make ϕ_1 false, and by the inductive hypothesis, $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n \vdash \neg\phi_1$. But this just says $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n \vdash \phi$, as required. If V makes ϕ false, this reasoning yields $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n \vdash \phi_1$. Adding an application of $\neg\neg$i gives $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n \vdash \neg\neg\phi_1$, which is $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n \vdash \neg\phi$.

In the remaining cases, ϕ is of the form $\phi_1 \circ \phi_2$. These are all similar; we will do only one here, where ϕ is of the form $\phi_1 \to \phi_2$. (Unlike with the proof of soundness, the textbook treats all the cases.)

Let $q_1, \ldots, q_k$ be the propositional atoms of ϕ_1 and $r_1, \ldots r_\ell$ be the propositional atoms of ϕ_2. First, we treat the case where V makes ϕ false. Then V must make ϕ_1 true and ϕ_2 false.

Applying the inductive hypothesis, we have $\hat{q}_1, \ldots, \hat{q}_k \vdash \phi_1$, and $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg\phi_2$. Since both of the sets of mini-formulas on the left-hand sides of the sequents are subsets of $\hat{p}_1 \ldots \hat{p}_n$, we know that $\hat{p}_1, \ldots, \hat{p}_n \vdash \phi_1$, and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\phi_2$. We can put these proofs together with one application of $\wedge$i to show $\hat{p}_1, \ldots, \hat{p}_n \vdash \phi_1 \wedge \neg\phi_2$.

We've shown $\hat{p}_1, \ldots, \hat{p}_n \vdash \phi_1 \wedge \neg\phi_2$. But what we need to show is $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg(\phi_1 \rightarrow \phi_2)$. All we need is a proof of $\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \rightarrow \phi_2)$. The book leaves this as an exercise.

That was the subcase where V made $\phi = \phi_1 \rightarrow \phi_2$ false. If V makes ϕ true, there are three possibilities for what it does to ϕ_1 and ϕ_2: TT, FT, and FF. Each of these, using the same sort of reasoning, generates a sequent which is close to what we need; for example, TT generates $\hat{p}_1, \ldots, \hat{p}_n \vdash \phi_1 \wedge \phi_2$, and thus generates the exercise $\phi_1 \wedge \phi_2 \vdash \phi_1 \rightarrow \phi_2$.

We then have to consider the cases where ϕ is of the form $\phi_1 \wedge \phi_2$ or $\phi_1 \vee \phi_2$. In total, eleven exercises are generated for future practice in natural deduction. That concludes the proof of the lemma, which is not difficult, but rather tedious.

Given the lemma, we can prove the main part of stage 2, which is to show that if $\models \eta$, then $\vdash \eta$. Since η is a tautology, all 2^n possible valuations make it true. Thus for every possible choice for $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$, we can use the lemma to get a proof of $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \eta$.

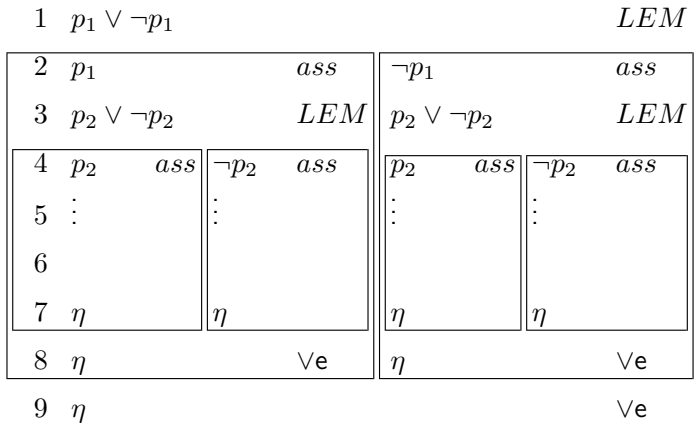
To get a proof of $\vdash \eta$, we use LEM to obtain $p_1 \vee \neg p_1$. Then we open two proof boxes with assumptions p_1 and $\neg p_1$, with the intention of using $\vee$e on them. (Though we have to do this sequentially, think of them as happening in parallel.)

Within each of those, we use LEM to obtain $p_2 \vee \neg p_2$, and do the same thing, nesting boxes. At the innermost level, we'll have 2^n boxes side by side, each one with assumptions accumulating one choice for each of $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$.

For each one of the 2^n innermost boxes, there are n enclosing proof boxes making assumptions that build up one choice of $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$.

Thus we can fill each one with one of the proofs provided by the lemma to conclude η within the innermost boxes. We then close the innermost proof boxes in pairs with $\vee$e, concluding η. Then we close the next matching pairs with $\vee$e, concluding η, and so on at each level, until all the proof boxes are closed.

The next slide illustrates this for the case $n = 2$.



The resulting proof is not one of which we should be particularly proud. But it is a valid proof of $\vdash \eta$.

This concludes the second stage, and thus the entire proof of the completeness of propositional logic. We have shown that if $\phi_1, \phi_2, \ldots \phi_n \models \psi$, then $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$.

Putting this together with soundness, we see that $\phi_1, \phi_2, \ldots \phi_n \models \psi$ if and only if $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$. Our two notions, proof by natural deduction (syntactic transformations) and semantics via truth-table valuations, have been shown to coincide. The next language for which we describe a proof system and semantics, predicate logic, is also sound and complete, but the proofs of those properties are beyond the scope of this course.

What's coming next?

Before we move to predicate logic, we will briefly discuss the parts of Chapter 1 that we are not covering in depth. We will also survey some alternate proof systems, and some implications of the material we have just covered, particularly as it relates to topics in other CS courses.

The definitions of both the system of natural deduction for predicate logic and the semantics associated with these more expressive formulas will prove to be considerably more complicated.

Goals of this module

You should understand the semantics presented for propositional logic in terms of valuations, and you should be very clear on the distinction between a formula being semantically true and a formula being syntactically provable.

You should be comfortable with the various forms of induction discussed, particularly structural induction, and be able to do such proofs yourself.

You should be able to fill in any of the holes left in the proofs of soundness and completeness, both in terms of understanding the big picture (what needs to go into the holes in order to complete the proof structure), and the specific details.