CS370 Assignment 3

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1. **Discrete Fourier by hand**

   i) $f[n] = (2, -2, 1, -1)$ $(n = 0, \ldots, 3; N = 4)$
   Recall that $F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-nk}$, and that $W = e^{\frac{2\pi i}{N}}$.
   This gives $W^0 = 1, W^{-1} = -i, W^{-2} = -1, W^{-3} = i, W^{-4} = W^0, W^{-5} = W^{-1}$, etc.
   $F_0 = \frac{1}{4}(f_0 + f_1 + f_2 + f_3) = 0$
   $F_1 = \frac{1}{4}(f_0 + f_1W^{-1} + f_2W^{-2} + f_3W^{-3}) = \frac{1}{4}(2 + 2i - 1 - i) = \frac{1+i}{4}$
   $F_2 = \frac{1}{4}(f_0 + f_1W^{-2} + f_2W^{-4} + f_3W^{-6}) = \frac{1}{4}(2 + 2 + 1 + 1) = \frac{3}{2}$
   $F_3 = \frac{1}{4}(f_0 + f_1W^{-3} + f_2W^{-6} + f_3W^{-9}) = \frac{1}{4}(2 - 2i - 1 + i) = \frac{1-i}{4}$

   ii) $f[n] = (1, 2, 4, 8)$ $(n = 0, \ldots, 3; N = 4)$
   Notice that the values for W^{-i} are the same as in (i) since N is the same.
   $F_0 = \frac{1}{4}(f_0 + f_1 + f_2 + f_3) = \frac{15}{4}$
   $F_1 = \frac{1}{4}(f_0 + f_1W^{-1} + f_2W^{-2} + f_3W^{-3}) = \frac{1}{4}(1 - 2i - 4 + 8i) = \frac{-3+6i}{4}$
   $F_2 = \frac{1}{4}(f_0 + f_1W^{-2} + f_2W^{-4} + f_3W^{-6}) = \frac{1}{4}(1 - 2 + 4 - 8) = \frac{-5}{4}$
   $F_3 = \frac{1}{4}(f_0 + f_1W^{-3} + f_2W^{-6} + f_3W^{-9}) = \frac{1}{4}(1 + 2i - 4 - 8i) = \frac{-3-6i}{4}$

2. **More Fourier**

   a) $f_n = W^{3n}$

$$
\begin{aligned}
F_k &= \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-nk} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} W^{3n}W^{-nk} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} W^{-n(k-3)} \\
&= \begin{cases} 1 \text{ if } k = 3 \\ 0 \text{ otherwise} \end{cases}
\end{aligned}
$$

   b) $f_n = \cos(\frac{4n\pi}{N})$
   First, we observe the following:

$$
\cos(\frac{4n\pi}{N}) = \frac{1}{2}(\cos(\frac{4n\pi}{N}) + \cos(\frac{4n\pi}{N}))
$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\left(\cos(\frac{4n\pi}{N}) + i\sin(\frac{4n\pi}{N})\right) + \left(\cos(\frac{4n\pi}{N}) - i\sin(\frac{4n\pi}{N})\right)\right] \\
&= \frac{1}{2}\left[\left(\cos(\frac{4n\pi}{N}) + i\sin(\frac{4n\pi}{N})\right) + \left(\cos(\frac{-4n\pi}{N}) + i\sin(\frac{-4n\pi}{N})\right)\right] \\
&= \frac{1}{2}\left(W^{2n} + W^{-2n}\right)
\end{aligned}
$$

So now,

$$
\begin{aligned}
F_k &= \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-nk} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \cos\left(\frac{4n\pi}{N}\right) W^{-nk} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} \left(W^{2n} + W^{-2n}\right) W^{-nk} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} \left(W^{-n(k-2)} + W^{-n(k+2)}\right) \\
&= \begin{cases} 1 \text{ if } k = 2 \textbf{ and } k = N-2 \text{ (only when k = 4)} \\ \frac{1}{2} \text{ if } k = 2 \textbf{ or } k = N-2 \text{ (but not both)} \\ 0 \text{ otherwise} \end{cases}
\end{aligned}
$$

c) $N = 4m$, $f_n = 0$ for $m \leq n < 3m$ and 1 otherwise.

3. **Fast Fourier Transform**

a) $g_n = \frac{1}{2}(f_n + f_{n+\frac{N}{2}})$ and $h_n = \frac{1}{2}(f_n - f_{n+\frac{N}{2}})W^{-n}$ where $W = e^{\frac{2\pi i}{N}}$.
Thus, $g = [\frac{-1+1}{2}, \frac{-2+2}{2}, \frac{-2+2}{2}, \frac{-1+1}{2}] = [0, 0, 0, 0]$
and $h = [\frac{-1-1}{2}W^0, \frac{-2-2}{2}W^{-1}, \frac{-2-2}{2}W^{-2}, \frac{-1-1}{2}W^{-3}]$
$= [-1, -2(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i), -2(-i), -1(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)]$
$= [-1, -\sqrt{2} + i\sqrt{2}, 2i, \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}]$