Math 117 / SyDe 111 Assignment #4 Fall 2006

due: October 13th

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1. Use the precise definition of the limit of a sequence (from page 8.3 of the lecture notes) to prove that $\frac{n}{n+1} \to 1$ as $n \to \infty$.

   (Your proof should begin with a statement like "Let ε be given." You'll then need to figure out how large n must be to satisfy the conditions of the definition.)

2. "Infinite Limits" require a slightly different definition: $a_n \to \infty$ as $n \to \infty$ if, for any $M > 0$, there exists a number $N > 0$ such that

   $a_n > M$ whenever $n > N$ (i.e. $n > N \Rightarrow a_n > M$).

   Use this to prove that $\frac{n^2}{n+1} \to \infty$ as $n \to \infty$.

3. Determine the limits of the following sequences, if they exist. You may need to perform some algebraic manipulations in order to apply the limit theorems.

   a) $\left\{ \frac{5n^2 - 2n}{3n^2 + n + 1} \right\}$ b) $\left\{ \frac{3n^2 + 1}{2n - 4} \right\}$ c) $\left\{ \cos\left(\frac{\pi n}{n+5} \right) \right\}$

   d) $\left\{ \sqrt{n^2 + 3n} - n \right\}$ e) $\left\{ \frac{2^n + 3^n}{3^n + 1} \right\}$

4. Many sequences arise naturally in "recursive" form; that is, rather than having an explicit formula for a_n as a function of n, we have only a rule for obtaining each term from the previous one. For example, we might know that

$$a_1 = 1, \quad \text{while} \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{11}{a_n}\right)$$

a) Calculate a_5.

b) From (a) it should appear that the sequence is convergent. Find the limit. (Hint: <u>assume</u> that it converges, and realize that if $\{a_n\}$ converges to L, then the shifted sequence $\{a_{n+1}\}$ must also converge to L.)

5. The same set of limit rules applies to functions as applies to sequences, so, for example, we can evaluate $\lim\limits_{x\to\infty} \frac{x}{\sqrt{2x+1}}$ in the same way as $\lim\limits_{n\to\infty} \frac{n}{\sqrt{2n+1}}$.

For functions, though, we can also consider limits as $x \to a$. If we know that f is <u>continuous</u> at a, then of course $\lim\limits_{x\to a} f(x) = f(a)$; that is, direct substitution of x with a is justified. Even if f is not continuous at a, the behaviour <u>may</u> be clear; eg. $\frac{1}{x} \to +\infty$ as $x \to 0^+$, while $\frac{1}{x} \to -\infty$ as $x \to 0^-$.

The difficulties arise when substitution results in the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, etc. In these cases we may be able to get around the problem using algebraic manipulation or identities. With this in mind, evaluate the following:

a) $\lim\limits_{x\to -1} \frac{x^2(1-x^3)}{2x^2+x+1}$ b) $\lim\limits_{x\to 4} \frac{x^2-16}{x-4}$ c) $\lim\limits_{x\to 0^+} \frac{\sqrt{1+x}-1}{\sqrt{x}}$

cont.

d) $\lim_{x \to 0} \dfrac{\sin 2x}{\sin x}$

e) $\lim_{x \to 0} \dfrac{\sin 3x}{x}$

6. a) Sketch the graph of $y = \dfrac{x^2 + 2x}{x^2 + 2x - 8}$, by considering the behaviour near any discontinuities and as $x \to \pm\infty$.

You may want to use a few specific points to improve the accuracy of your sketch; I'd suggest finding $y(-5)$, $y(-1)$, and $y(3)$.

b) Use the same principles to sketch the graph of $y = \dfrac{2x^3 + 3x^2}{x^2 + 2x + 1}$.

Note: this graph has an oblique asymptote; you can identify it by performing long division.