Math 119 Assignment #2 Winter 2007

due January 19th

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1. Suppose you wish to estimate the value of $\sqrt{70}$. You now have several options:

   a) Use linear interpolation, using the knowledge that $\sqrt{64} = 8$ and $\sqrt{81} = 9$.

   b) Use the concept of differentials (from Math 117), with $x_0 = 64$ as your starting point.

   c) Try a Taylor Polynomial with x_0 as its center. Part (b) is actually equivalent to using the first-order Taylor Polynomial, $P_{1,64}(x)$, so to improve on it, try $P_{3,64}(x)$ instead.

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2. a) Consider the sequence $\{y_n\}$ whose general term is $y_n = n^2 - n$, $n = 0, 1, 2, \ldots$. Construct the table of differences and show that $\Delta^2 y_n = 2$ and $\Delta^3 y_n = 0$ (all n).

   b) Generalization: Consider $\{y_n\}$, where $y_n = a_0 + a_1 n + a_2 n^2 + \ldots + a_k n^k$, where the a_k's are constant and k is an integer. Show that $\Delta^k y_n =$ constant and $\Delta^{k+1} y_n = 0$ (all n).

   Hint: Show that when Δ is applied to y_n it reduces the degree of the polynomial by one. The rest follows from there.
   (Note: you showed on Assignment #1 that Δ is a <u>linear</u> operator.)

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Some new material: <u>DIFFERENCE EQUATIONS</u>

(We won't be discussing these in class. We'll explain the basics within this assignment, and also refer you to MEM, Section 7.4, pp. 428–436. This <u>is</u> to be considered examinable material.)

A <u>difference equation</u> is a relation involving finite differences. For example,

(2.1) $\quad \Delta y_n = (a-1)y_n + b \qquad (a, b \text{ constants}, \ n = 0, 1, 2 \ldots)$

is a difference equation. In fact, since it involves only a first finite difference, it is a <u>first-order</u> difference equation.

3. a) Use the definition of the difference operator Δ to show that (2.1) may be expressed in the alternative form

(2.2) $\quad y_{n+1} = a y_n + b$

In this form the difference equation is also known as a <u>recurrence relation</u>, because it produces the next term of the sequence (i.e. y_{n+1}) from knowledge of the current term (i.e. y_n).

b) In MEM (p.428) it is shown that the solution to (2.2), subject to the initial condition $y_0 = c =$ constant, is given by

$$y_n = c a^n + \left(\frac{1-a^n}{1-a}\right) b \qquad (a \neq 1)$$

Hence determine the solution to the difference equation

$$\Delta y_n = 2y_n + 2, \quad \text{if} \quad y_0 = 1.$$

4\. a) Show that the second-order difference equation

(2.3) $\Delta^2 y_n - \Delta y_n = 0 \quad (n = 0, 1, 2, \ldots)$

can be written as the second-order recurrence relation

(2.4) $y_{n+2} - 3y_{n+1} + 2y_n = 0 \quad (n = 0, 1, 2, \ldots)$

b) Equations like (2.4) can be solved in the following manner. Since we know that the solution to the first-order recurrence relation (2.2) contains the exponential form a^n, we guess that (2.4) could have similar exponential solutions. Assume that $y_n = \lambda^n$, and determine if there are any values of λ which make this a solution to (2.4) (by plugging $y_n = \lambda^n$ directly into the equation).

c) In part (b) you should in fact have found two values of λ which work, and hence two different solutions to (2.4). The "principle of superposition" states that if $y_n^{(1)}$ and $y_n^{(2)}$ are two solutions to a linear equation, then the expression $y_n = C_1 y_n^{(1)} + C_2 y_n^{(2)}$ is also a solution, for any constants C_1 and C_2, and in fact for a 2nd-order equation all solutions must be of this form (we call it the "general solution"). Write out the general solution to (2.4), and then determine the values of C_1 and C_2 which must hold if $y_0 = 0$ and $y_1 = 1$.

Comment: Difference equations (or recurrence relations) are extremely important in the analysis of digital signals. You may see them again... for example in ECE 342.

5. Find the following Taylor polynomials:

(a) $P_{3,1}(x)$ for $f(x)=\sqrt[3]{x}$; (b) $P_{5,0}(x)$ for $g(x)=\sin(x)$;

(c) $P_{5,0}(x)$ for $h(x)=e^{x}$; (d) $P_{2,0}(x)$ for $u(x)=\dfrac{1}{1-x}$.

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6. (a) Find $P_{5,0}(x)$ for $\psi(x)=\cos(x)$; then take the derivative of the polynomial obtained in 5 (b) above, thus verifying that they are the same.

(b) In the polynomial obtained in 5 (d) above make the replacement $x \to (-x^2)$, and verify that what you get is in fact $P_{4,0}(x)$ for $v(x)=\dfrac{1}{1+x^2}$.

(c) In the polynomial obtained in part (b) above take the indefinite integral of both sides, and comment on the result.

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▨ Remark. As Probl. 6 shows, from a known Taylor polynomial one can get many others by making algebraic substitutions, taking derivatives, and integrating. This, of course, is a clever way of avoiding the (tedious) calculation of all those derivatives! ▨

7. Find the zeros of the function $f(x)=x^3-x-1$ correct to 4 d.p..

[Hint. First show that there is only one zero which is located between $x=1$ and $x=2$. Then use Newton's method in the modern version — i.e., Eq. (4.17) of L.N.]

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8. Use a rough sketch to show that the curves $y = e^{-x^2}$ and $y = x^2$ intersect twice. If your sketch is reasonably good, it should be clear that the rightmost intersection point lies at about $x_0 = 0.75$ Using this as your starting point, use Newton's Method to evaluate this root to 4 decimal places.

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9. Consider the equation $x \tan x = 2 - \cosh x$.

a) Show that for small values of x, $x \tan x \approx x^2$, while $2 - \cosh x \approx 1 - \frac{1}{2}x^2$. Draw a conclusion from this regarding the probable number and approximate locations of roots on the interval $[-1, 1]$.

b) Use Newton's Method to identify one of these roots to three decimal places of accuracy.