Math 119 Assignment # 1 Winter 2007

due January 12th

A warm-up problem (not to be submitted):

Consider the data

x	0	1	2	3	4
y	2	3	4	5	6

. Since there are 5 points, the Newton Forward Difference Formula for $n=4$ should be used. That is, we look for a 4th degree polynomial ... but what do we get? (answer at end)

Now the actual assignment:

1. At a certain location, temperature measurements are taken hourly, starting at midnight, as a storm moves through the area. The data result in the following table:

t_n	0	1	2	3	4	5
T_n	16	11	13	12	11	13

Find the Newton Interpolating Polynomial for these values, and use it to estimate the temperature at 4:30 am. If you try using it to <u>predict</u> the temperature at 6:00 am, what do you get? What about 7:00 am?

(You don't need to simplify the polynomial.)

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2. Find the Newton Interpolating Polynomial based on the nodes $x_n = 0, 1, 2, 3$ and the corresponding function values $y_n = 0, 1, 8, 27$. Simplify your result.

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3. Consider the function $y(x) = 2^x$. We know $y(1) = 2$, $y(2) = 4$, and $y(3) = 8$.

   a) Use these three pairs to construct a Newton Interpolating Polynomial, and use it to estimate the value of $2^{2.5}$

   b) We also know that $y(4) = 16$. By simply extending the difference table you set up in (a), find the new interpolating polynomial, and use it to find a new estimate for $2^{2.5}$.

   c) Compare your results to your calculator's value for $2^{2.5}$. What are the percentage errors? (Limit your calculations to 3 d.p.)

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4. The table below gives the values of the function $y = \sinh x$ at the given nodes, to 4 decimal places.

x_n	0.5	0.6	0.7	0.8
y_n	0.5211	0.6367	0.7586	0.8881

   a) Find the Newton Interpolating Polynomial (using the forward difference formula: 4.7′ of the L.N.) and use it to approximate $\sinh(0.56)$, also to 4 d.p.

   b) Use the Lagrange Linear Interpolation Formula to again approximate $\sinh(0.56)$

   c) Use your calculator to compare the errors in the two estimates.

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5. Consider the function $y(x) = x^3$

a) Calculate the second finite difference $\Delta^2 y_0$ using x_0, $x_0 + \Delta x$, and $x_0 + 2\Delta x$ as nodes (as done by Taylor).

b) Next, consider the ratio $\dfrac{\Delta^2 y_0}{\Delta x^2}$, and verify that (in this particular case at least) $\dfrac{\Delta^2 y_0}{\Delta x^2} \to y''(x_0)$ as $\Delta x \to 0$.

■ <u>Remark</u>. Recall the definition of <u>first difference</u>:

$$(1.1) \qquad \Delta y_n := y_{n+1} - y_n , \qquad (n \text{ integer})$$

where $\{y_n\}$ is a given sequence of numbers. The symbol Δ can be seen as <u>an operator</u>,

$$y_n \longrightarrow \boxed{\Delta} \longrightarrow y_{n+1} - y_n ,$$

analogous to the derivative operator $\frac{d}{dx}$. Note that <u>Δ operates on discrete inputs</u>, while of course $\frac{d}{dx}$ operates only on smooth inputs. ■

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6. Let $\{y_n\}$ and $\{z_n\}$ be two sequences of numbers, and c an arbitrary constant. Show that

$$(1.2) \qquad \Delta(y_n + z_n) = \Delta y_n + \Delta z_n ,$$

and that

$$(1.3) \qquad \Delta(c\, y_n) = c\, \Delta y_n$$

by using the definition of Δ in (1.1). These two properties guarantee that <u>Δ is a linear operator</u> (just like $\frac{d}{dx}$).

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7. Show that

$$\Delta^2 y_n = \Delta(\Delta y_n) = y_{n+1} - 2y_{n+1} + y_n, \tag{1.4}$$

$$\Delta^3 y_n = \Delta(\Delta^2 y_n) = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n \tag{1.5}$$

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(NOTE. In fact, we also have $\Delta(\Delta^2 y_n) = \Delta^2(\Delta y_n)$, and in general $\Delta^i(\Delta^j y_n) = \Delta^{i+j} y_n = \Delta^j(\Delta^i y_n)$.)

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8. Use the definition (1.1) to show that each member of the sequence $\{y_n\}$ can be expressed solely in terms of y_0 and $\Delta^k y_0$; that is,

$$y_0 = y_0,$$
$$y_1 = y_0 + \Delta y_0,$$
$$y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0,$$

(You don't need an inductive proof; just go as far as y_3.)

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The coefficients on the r.h.s. can be arranged in the scheme

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ \cdots & & \cdots & & \cdots & & \cdots \end{array}$$

see MEM, p.15

in which each number is the sum of the two numbers directly above it. This is called the Pascal triangle (Pascal, 1654).

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9. Show that the entries of Pascal's triangle can be expressed in terms of the combinatorial coefficients $\binom{n}{k}$ by proving that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} . \tag{1.6}$$

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THE END

Answer to warm-up: $y = 2 + x$. (This can be thought of as a 4th-order polynomial; it simply has some zero coefficients because in this case the data happen to be linear!)