MATH 119 – Calculus 2 for Electrical and Computer Engineering.

LECTURE NOTES

by

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Chapter 4 – Approximation Methods.

§ 4.1 – Introduction.

A fundamental problem in science is the extraction of a continuous function from a set of experimental points. But measurements always occur in discrete sets of points — as shown, for instance in Fig. 4.1 — while the functions we postulate as the basis of our measurements

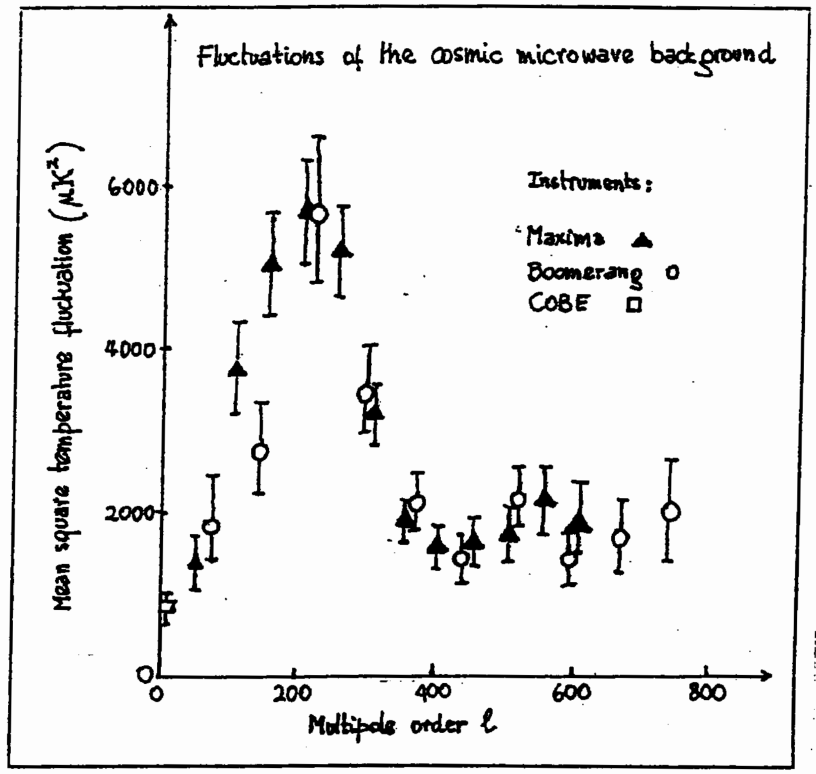

Fig. 4.1

exist for a continuous range of the variable. Essentially, what we get from measurement is a tabulated function; hence, the problem arises of defining the unknown function between the points of tabulation. This is the problem of interpolation, which fascinated mathematical researchers from the early pioneering work of James Gregory (1638–1675) to the contributions of Newton, Stirling (1692–1770), and Bessel (1784–1846). The rigorous treatment of interpolation theory started with D. Hahn and L. Fejér (1918), while the dangers of equidistant polynomial interpolation were discovered independently by C. Runge (1901) and E. Borel (1903). The subject is still alive and well today, and is usually covered in depth in courses on Numerical Analysis. For our purposes only the simplest notions will suffice.

§ 4.2 - Approximation of functions : Polynomial interpolation.

▲ Example (Newton, 1676). Suppose we are given $n+1$ points

$$(x_i, y_i) \quad ; \quad i = 0,1,2,\dots,n ,$$

and would like to find a polynomial of degree n passing through all these points. To make things easy, consider the case in which $n=3$ and the x_i's are equidistant and given by

$$x_0 = 0 \quad ; \quad x_1 = 1 \quad ; \quad x_2 = 2 \quad ; \quad x_3 = 3 .$$

Let the polynomial we seek be represented by

$$(4.1) \qquad y = a + bx + cx^2 + dx^3,$$

where the coefficients $a, b, c,$ and d are as yet unknown. The question is, How can we determine them?

Well, we know the four points $(0, y_0)$, $(1, y_1)$, $(2, y_2)$, $(3, y_3)$; hence, writing down (4.1) for each of these we get the system

$$(4.2) \quad \begin{cases} y_0 = a , & \text{(and so } a \text{ is immediately found)} \\ y_1 = a+b+c+d , & \\ y_2 = a+2b+4c+8d , & \\ y_3 = a+3b+9c+27d , & \end{cases} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \begin{matrix} a, y_1, y_2, y_3 \text{ known,} \\ b, c, d \text{ unknown.} \end{matrix}$$

Next Newton noticed that the first coefficient (namely, a) disappears if we subtract the equations term by term: The first of (4.2) from the second, the second from the third, and the third from the fourth. The result is

$$(4.3) \quad \begin{cases} b+c+d = y_1 - y_0 := \Delta y_0 , \\ b+3c+7d = y_2 - y_1 := \Delta y_1 , \\ b+5c+19d = y_3 - y_2 := \Delta y_2 ; \end{cases}$$

The last equalities define the first finite differences

furthermore, b disappears if we subtract the eqs. of (4.3) term by term once again:

$$(4.4)\quad \begin{cases} 2c + 6d = \Delta y_1 - \Delta y_0 := \Delta^2 y_0 , \\ 2c + 12d = \Delta y_2 - \Delta y_1 := \Delta^2 y_1 . \end{cases}$$

Def^n of the second finite differences

Finally, one last subtraction term by term leaves us with

$$(4.5)\qquad 6d = \Delta^2 y_1 - \Delta^2 y_0 := \Delta^3 y_0 ,$$ (third finite difference)

which can be solved for d (since all the finite differences are known). Then we can go backwards, and get c from (4.4), b from (4.3), and a from (4.2):

$$d = \frac{1}{6}\Delta^3 y_0 \;;\quad c = \frac{1}{2}\left(\Delta^2 y_0 - \Delta^3 y_0\right) \;;\quad b = \Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 \;;\; a = y_0$$

All that is left now is to substitute these values into the assumed polynomial (4.1) and order the terms in accordance with the order of the finite differences; the result is (check it!)

$$(4.6)\qquad y = y_0 + x\,\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 .$$

And the problem is solved. ▲

■ Remark 4.1 - The Newton method illustrated above can be clearly generalized to an arbitrary number of (equidistant) points. In fact, it is not too difficult to prove the following

Theorem 4.1. The interpolating polynomial of degree n taking the values

$$y_0 \text{ (for } x=0) \;,\quad y_1 \text{ (for } x=1) \;,\; \cdots \;,\; y_n \text{ (for } x=n)$$

is given by the formula

$$(4.7)\quad \boxed{y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \cdots + \frac{x(x-1)\cdots(x-n+1)}{n!}\Delta^n y_0 .}$$

Since Newton's time, it is customary to arrange the finite differences in a table as follows (e.g., for $n=4$)

$$
\begin{array}{ccccccc}
x_0 & : & \underline{y_0} & & & & \\
 & & & \underline{\Delta y_0} & & & \\
x_1 & : & y_1 & & \underline{\Delta^2 y_0} & & \\
 & & & \Delta y_1 & & \underline{\Delta^3 y_0} & \\
x_2 & : & y_2 & & \Delta^2 y_1 & & \underline{\Delta^4 y_0} \\
 & & & \Delta y_2 & & \Delta^3 y_1 & \\
x_3 & : & y_3 & & \Delta^2 y_2 & & \\
 & & & \Delta y_3 & & & \\
x_4 & : & y_4 & & & &
\end{array}
$$

where the underlined differences (in the top diagonal) are the coefficients used in the interpolating polynomials.

• A slight generalization of Thm 4.1 is possible with little work. The interpolating abscissae in (4.7) are given by $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, ..., $x_n = n$. Obviously, these can be written as a single formula, namely $x_n = x_0 + n$. More generally, we may have

$$x_n = x_0 + nh \quad , \quad \text{with } x_0 \neq 0 \text{ and } h \neq 1 ;$$

in that case (4.7) becomes

$$
(4.7') \quad \boxed{\begin{aligned} y = y_0 + (x-x_0)\frac{\Delta y_0}{1!\,h} + (x-x_0)(x-x_1)\frac{\Delta^2 y_0}{2!\,h^2} + \cdots \\ + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\frac{\Delta^n y_0}{n!\,h^n} \ . \end{aligned}}
$$

This is usually referred to as <u>the Newton forward difference formula</u>.

■ <u>Remark 4.2</u>. The point of view taken so far has been that of regarding the ordinates $y_0, y_1, \dots, y_n$

as given (for instance, by an experiment). Graphically, we have solved approximately the problem of fitting a polynomial to a set of data, as illustrated in Fig. 4.2:

x	0	1	2	3	4	5
y	4	5	2	5	2	2

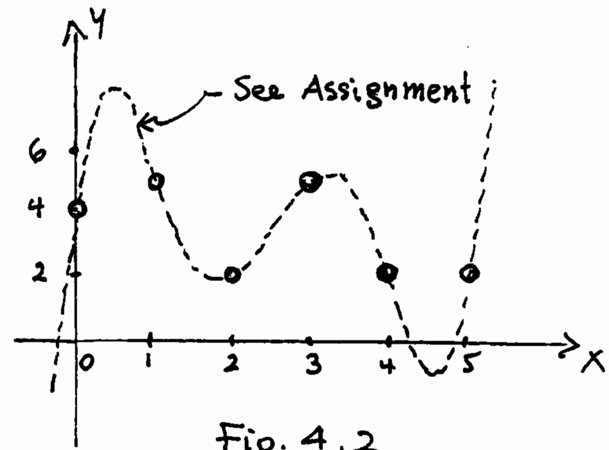

Fig. 4.2

Note that the dashed line in this figure is the graph of the 5^{th} degree polynomial that can be obtained from Eq. (4.7) using the above table (and that you are required to find). Here, the "actual" function is not known — only the polynomial that coincides with it at the given nodes (that's the name of the given x_n's) is produced by this method.

• On the other hand, sometimes the problem may be the following: We are given two values of a function f, say $f_0 = f(x_0)$ and $f_1 = f(x_1)$, and want to find by interpolation the value of f at a point $x \in [x_0, x_1]$. The simplest approximate answer to this question is obtained by assuming that the graph of f between x_0 and x_1 is a straight segment (see Fig. 4.3). This is the so-called linear interpolation method, and it is easily obtained from (4.7′) with $y = f(x)$, $y_0 = f(x_0)$, $\Delta y_0 = f(x_1) - f(x_0)$, and $h = x_1 - x_0$. Thus

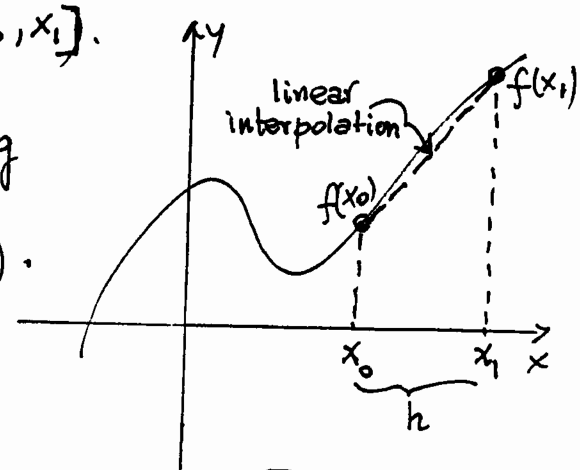

Fig. 4.3

$$f(x) \approx f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)}\left(f(x_1) - f(x_0)\right) = f_0 + \frac{(x-x_0)}{(x_1-x_0)}(f_1 - f_0)$$

which may also be rewritten as

$$f(x) \approx \frac{(x-x_1)}{(x_0-x_1)} f_0 + \frac{(x-x_0)}{(x_1-x_0)} f_1 ,$$

MEM, p. 77

which is usually called the Lagrange form of linear interpolation. Furthermore, since this argument can be repeated for any other interval $[x_i, x_{i+1}]$, the Lagrange formula may be stated more generally as

$$f(x) \approx \frac{(x-x_{i+1})}{(x_i-x_{i+1})} f_i + \frac{(x-x_i)}{(x_{i+1}-x_i)} f_{i+1} . \tag{4.8}$$

MEM, p. 149

• Exercise. Do on your own Ex. 2.12 (p. 77 of MEM), and Ex. 2.58 (p. 149).

More examples will be found in the Assignments. ■

NOTE. This, of course, is just the beginning of Interpolation Theory — for instance, it would be a simple matter to go from what we have studied to more sophisticated methods, such as interpolation by splines. Unfortunately, we don't have the time, and so we must stop here.