Math 117 / SyDe 111 Assignment #10 Fall 2006

(Not to be handed in.)

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1. MEM, p. 693 #16 cde

2. Solve the differential equation you derived in MEM p. 495 #7 (Assignment 7). Sketch your solution for $t \in [0, \infty)$. Does it seem physically realistic?

3. Find the length of the curve $y = x^2 - \frac{1}{8}\ln x$, from $(1,1)$ to $(3, 9 - \frac{1}{8}\ln 3)$.

4. MEM, p. 599 #14 ai (You may use the fact that $x^2 \ln x \to 0$ as $x \to 0^+$, without justification.)

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Other Suggested Problems: p. 693 #11, 12, 15, 16ab

p. 599 #14 (there is an error in (b); the integral should be $\int_0^\infty x^2 e^{-x^3}\,dx$.)

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SOLUTIONS

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1. p. 693 #16 c) $\frac{dx}{dt} = (x^2-1)\cos t$ , $x(0)=2$

In differential form this is $\frac{dx}{x^2-1} = \cos t \, dt$

so $\int \frac{dx}{x^2-1} = \int \cos t \, dt$.

Since $\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}$, we have $\frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = \int \cos t \, dt$

$$\Rightarrow \frac{1}{2}\Big[\ln|x-1| - \ln|x+1|\Big] = \sin t + C_1$$

$$\Rightarrow \ln\left|\frac{x-1}{x+1}\right| = 2\sin t + C_2 \qquad (C_2 = 2C_1)$$

$$\Rightarrow \frac{x-1}{x+1} = C_3 e^{2\sin t} \qquad (C_3 = e^{C_2}) \quad *$$

$$\Rightarrow x-1 = (x+1)C_3 e^{2\sin t}$$

$$\Rightarrow x - C_3 x e^{2\sin t} = 1 + C_3 e^{2\sin t}$$

$$\Rightarrow x = \frac{1 + C_3 e^{2\sin t}}{1 - C_3 e^{2\sin t}}$$

What's $C_3$? The easiest place to calculate it is the line marked $*$;

if $x(0)=2$ then $\frac{1}{3} = C_3$, so $x = \frac{1+\frac{1}{3}e^{2\sin t}}{1-\frac{1}{3}e^{2\sin t}} = \frac{3+e^{2\sin t}}{3-e^{2\sin t}}$.

d) $\frac{dx}{dt} = e^{x+t}$, $x(0) = a$

$\frac{dx}{dt} = e^{x}e^{t}$, so $\frac{dx}{e^{x}} = e^{t}dt$, i.e. $\int e^{-x}dx = \int e^{t}dt$.

Therefore $-e^{-x} = e^{t} + C_1$

i.e. $e^{-x} = -e^{t} + C_2$ $\quad *$.

i.e. $-x = \ln(-e^{t} + C_2)$

i.e. $x = -\ln(C_2 - e^{t})$

Finally, $x(0) = a$ means $e^{-a} = -1 + C_2$, from $*$,

so $x = -\ln\left(1 + e^{-a} - e^{t}\right)$.

e) $\frac{dx}{dt} = \frac{4\ln t}{x^2}$, $x(1) = 0$

Separating gives $\int x^2 dx = 4\int \ln t \, dt$

$\Rightarrow \frac{x^3}{3} = 4t\ln t - 4t + C_1$

$\Rightarrow x = \left[12t\ln t - 12t + C_2\right]^{1/3}$

For $\int \ln t \, dt$, let $u = \ln t$, $dv = dt$
$du = \frac{1}{t}dt$, $v = t$

$= t\ln t - \int dt$

$= t\ln t - t + C$

If $x(1) = 0$ then $0 = (-12 + C_2)^{1/3}$, so $C_2 = 12$.

Thus $x = 12^{1/3}\left[t\ln t - t + 1\right]^{1/3}$.

2. Our equation was $\frac{dQ}{dt} = 1 - \frac{Q}{250}$, and we also were told that $Q(0) = 1000\,L \cdot 0.15 \frac{kg}{L} = 150\,kg$.

Separate variables: $\frac{dQ}{1 - Q/250} = dt$

i.e. $\frac{250\, dQ}{250 - Q} = dt$

i.e. $\int \frac{dQ}{250-Q} = \int \frac{1}{250}\, dt$

Integrating gives $-\ln|250-Q| = \frac{1}{250}t + C_1$

$\Rightarrow \quad |250-Q| = e^{-\frac{t}{250}-C_1} = C_2 e^{-t/250} \qquad (C_2 = e^{-C_1})$

$\Rightarrow \quad 250 - Q = \pm C_2 e^{-t/250}$
$= C_3 e^{-t/250} \qquad (C_3 = \pm C_2)$

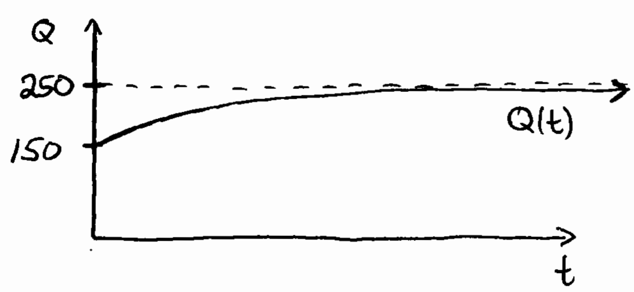
$\Rightarrow \quad Q(t) = 250 - C_3 e^{-t/250}$

What's $C_3$? If $Q(0) = 150$, then $C_3 = 100$, so

$$Q(t) = 250 - 100\, e^{-t/250}.$$

To sketch this, note that $Q'(t) = \frac{10}{25} e^{-t/250}$, and $Q''(t) = \frac{-1}{625} e^{-t/250}$, so $Q(t)$ is always increasing and concave down. Also, as $t \to \infty$, $Q(t) \to 250$:

$Q(0) = 150$, $Q'(0) = \frac{10}{25}$ →




- This shows that the concentration of salt in the tank gradually adjusts to the concentration of the inflow.

3. From MEM p.564 the length of the curve is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If $y = x^2 - \frac{1}{8}\ln x$

then $\frac{dy}{dx} = 2x - \frac{1}{8x}$.

Therefore

$$\begin{aligned} L &= \int_1^3 \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}\, dx \\ &= \int_1^3 \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}}\, dx \\ &= \int_1^3 \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}}\, dx \\ &= \int_1^3 \sqrt{\left(2x + \frac{1}{8x}\right)^2}\, dx \\ &= \int_1^3 \left(2x + \frac{1}{8x}\right) dx \\ &= \left(x^2 + \frac{1}{8}\ln x\right)\Big|_1^3 \\ &= \left(9 + \frac{1}{8}\ln 3\right) - (1 + 0) \\ &= 8 + \frac{1}{8}\ln 3. \end{aligned}$$

(Yes, this was cleverly constructed so that the integral could be evaluated exactly. Most arc length integrals don't work out so nicely; we'll need to approximate their values instead, with techniques you'll learn later.)

4. p.599 *14 a) $\int_0^1 -x \ln x \, dx$ (Note that the integrand is undefined at 0)

$$= \lim_{t \to 0^+} \int_t^1 -x \ln x \, dx$$

$u = \ln x, \quad dv = -x\,dx$

$du = \frac{1}{x}\,dx, \quad v = -\frac{x^2}{2}$

$$= \lim_{t \to 0^+} \left( -\frac{x^2}{2} \ln x \Big|_t^1 + \int_t^1 \frac{x}{2}\,dx \right)$$

$$= \lim_{t \to 0^+} \left[ 0 + \frac{t^2 \ln t}{2} + \frac{x^2}{4} \Big|_t^1 \right]$$

$$= \lim_{t \to 0^+} \left[ \frac{t^2 \ln t}{2} + \frac{1}{4} - \frac{t^2}{4} \right] = \frac{1}{4}.$$

i) $\int_0^\infty \frac{x}{1+x^4}\,dx = \lim_{t \to \infty} \int_0^t \frac{x}{1+x^4}\,dx.$ Try letting $u = x^2$. Then $du = 2x\,dx$

$\downarrow$

$$= \lim_{t \to \infty} \int_0^{t^2} \frac{1}{1+u^2} \cdot \frac{du}{2}$$

$$= \lim_{t \to \infty} \frac{1}{2} \tan^{-1} u \Big|_0^{t^2}$$

$$= \lim_{t \to \infty} \frac{1}{2} \tan^{-1}(t^2) = \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}.$$