Econometrics III: Applied Methods — NovaSBE, 2026

Problem Set 1 Solutions: Exercise A — Rubin Model and Roy Model

Exercise 1: Rubin Model and Roy Model

Setup: Training increases wages from w_0 to $w_1 = w_0 + \delta$ at cost c. Both $\delta > 0$ and w_0 are heterogeneous; initially $\delta \perp w_0$.

> **Q1.** *(0.5 pts)* What is the treatment impact of a given individual i? What is the average treatment impact in the population?

Solution:

Individual treatment effect: $\tau_i = w_{1i} - w_{0i} = \delta_i$.
Average Treatment Effect (ATE):

$$\text{ATE} = \mathbb{E}[\delta_i] = \mathbb{E}[w_{1i} - w_{0i}]$$

> **Q2.** *(0.5 pts)* Write the decision model of attending the training or not (called a Roy model).

Solution:

Individual i attends training if the benefit exceeds the cost:

$$D_i = \mathbb{1}\{w_{1i} - w_{0i} \geq c\} = \mathbb{1}\{\delta_i \geq c\}$$

Since agents know their counterfactual wages with certainty, they select into treatment based on comparing their individual gain δ_i to the common cost c.

> **Q3.** *(1.5 pts)* Based on that decision rule, people sort themselves into the training program. We observe average wages of treated and untreated. What does each of those averages measure?

Solution:

Observed wage: $w_i = D_i \cdot w_{1i} + (1 - D_i) \cdot w_{0i}$.
Average wage of treated (those with $D_i = 1$, i.e., $\delta_i \geq c$):

$$\begin{aligned}
\mathbb{E}[w_i|D_i = 1] &= \mathbb{E}[w_{1i}|\delta_i \geq c] \\
&= \mathbb{E}[w_{0i} + \delta_i|\delta_i \geq c] \\
&= \mathbb{E}[w_{0i}|\delta_i \geq c] + \mathbb{E}[\delta_i|\delta_i \geq c] \\
&= \mathbb{E}[w_{0i}] + \mathbb{E}[\delta_i|\delta_i \geq c] \quad (\text{using } \delta \perp w_0)
\end{aligned}$$

So this observed average measures the average baseline wage, plus the average treatment effect on the treated (the δ for those who select into treatment).

Average wage of untreated (those with $D_i = 0$, i.e., $\delta_i < c$):

$$\begin{aligned} \mathbb{E}[w_i|D_i = 0] &= \mathbb{E}[w_{0i}|\delta_i < c] \\ &= \mathbb{E}[w_{0i}] \quad \text{(using } \delta \perp w_0\text{)} \end{aligned}$$

The independence assumption $\delta \perp w_0$ is crucial: conditioning on δ (which determines selection) does not affect the distribution of baseline wages w_0. Both groups have the same expected baseline wage $\mathbb{E}[w_{0i}]$.

Q4. *(1.5 pts)* If you compare average wages of the treated and untreated, what parameter do you estimate? Interpret that parameter.

Solution:

The naive comparison estimates:

$$\mathbb{E}[w_i|D_i = 1] - \mathbb{E}[w_i|D_i = 0] = \mathbb{E}[w_{1i}|\delta_i \geq c] - \mathbb{E}[w_{0i}|\delta_i < c]$$

Add and subtract $\mathbb{E}[w_{0i}|\delta_i \geq c]$:

$$= \underbrace{\mathbb{E}[w_{1i} - w_{0i}|\delta_i \geq c]}_{\text{ATT}=\mathbb{E}[\delta_i|\delta_i \geq c]} + \underbrace{\mathbb{E}[w_{0i}|\delta_i \geq c] - \mathbb{E}[w_{0i}|\delta_i < c]}_{\text{Selection bias}}$$

Since $\delta \perp w_0$, the selection bias term equals zero:

$$\mathbb{E}[w_{0i}|\delta_i \geq c] = \mathbb{E}[w_{0i}|\delta_i < c] = \mathbb{E}[w_{0i}]$$

Therefore, the naive comparison identifies the **Average Treatment Effect on the Treated (ATT)**:

$$\boxed{\mathbb{E}[\delta_i|\delta_i \geq c]}$$

This is the average gain for those who *chose* to participate—people with above-average treatment effects (since they self-selected on $\delta_i \geq c$).

RCT Setup: Three groups: $Z = 0$ (no access), $Z = 1$ (normal cost c), $Z = 2$ (subsidized cost $c - s$).

Q5. *(1.5 pts)* Compute the value of the average wage in each of these populations.

Solution:

Let $p = \Pr(\delta_i \geq c)$ denote the share of trainees under normal cost, and $q = \Pr(\delta_i \geq c - s)$ under the subsidy.

Group $Z = 0$ (no access): Everyone is untreated.

$$\mathbb{E}[w_i|Z = 0] = \mathbb{E}[w_{0i}]$$

Group $Z = 1$ (cost c): Fraction p trains.

$$\begin{aligned} \mathbb{E}[w_i|Z = 1] &= p \cdot \mathbb{E}[w_{1i}|\delta_i \geq c] + (1 - p) \cdot \mathbb{E}[w_{0i}|\delta_i < c] \\ &= \mathbb{E}[w_{0i}] + p \cdot \mathbb{E}[\delta_i|\delta_i \geq c] \end{aligned}$$

(using $\delta \perp w_0$ to simplify)

Group $Z = 2$ (cost $c - s$): Fraction $q > p$ trains.

$$\mathbb{E}[w_i|Z = 2] = \mathbb{E}[w_{0i}] + q \cdot \mathbb{E}[\delta_i|\delta_i \geq c - s]$$

Q6. *(1 pt)* Note that you can estimate the proportion of trainees in each random group. Using this, show how you can recover the same parameter as in Q4 from the wage difference between groups $Z = 1$ and $Z = 0$.

Solution:

Let $p_1 = \Pr(D = 1|Z = 1)$ be the observed treatment rate in group 1 (and $p_0 = 0$ for group 0). This measures $p = \Pr(\delta_i \geq c)$.

The wage difference is:

$$\mathbb{E}[w_i|Z = 1] - \mathbb{E}[w_i|Z = 0] = p_1 \cdot \mathbb{E}[\delta_i|\delta_i \geq c]$$

Since we can estimate p_1 directly from the data, we recover the ATT via:

$$\boxed{\frac{\mathbb{E}[w_i|Z = 1] - \mathbb{E}[w_i|Z = 0]}{p_1} = \mathbb{E}[\delta_i|\delta_i \geq c] = \text{ATT}}$$

This is a **Wald estimator** (IV with Z as instrument for D).

Q7. *(1 pt)* Explain why the independence between δ and w_0 ensures that the naive wage comparison can estimate a treatment parameter without the experiment. Why can't we obtain ATE, though?

Solution:

Independence $\delta \perp w_0$ eliminates **selection on levels**: treated and untreated have the same baseline wage distribution. This removes the selection bias term in Q4, so naive comparison identifies ATT.

However, we **cannot obtain ATE** because there is still **selection on gains**: people with $\delta_i \geq c$ self-select into treatment. Since trainees have above-average treatment effects by construction, $\mathbb{E}[\delta_i|\delta_i \geq c] > \mathbb{E}[\delta_i]$ (unless δ is constant).

The ATE would require observing treatment effects for non-participants, which is impossible under self-selection.

Q8. *(1.5 pts)* Show that it is possible to identify the impact of the training on the population that is induced to participate by the subsidy s (and would not participate otherwise), using group $Z = 2$ and group $Z = 1$. Define that parameter formally and explain how you can compute it.

Solution:

Define the **compliers**: individuals with $c - s \leq \delta_i < c$. These would train with subsidy but not without.

The **Local Average Treatment Effect (LATE)** for compliers:

$$\text{LATE} = \mathbb{E}[\delta_i|c - s \leq \delta_i < c]$$

From Q5:

$$\mathbb{E}[w_i|Z=2] - \mathbb{E}[w_i|Z=1] = q \cdot \mathbb{E}[\delta_i|\delta_i \geq c-s] - p \cdot \mathbb{E}[\delta_i|\delta_i \geq c]$$

The additional trainees in group 2 are exactly the compliers. Their contribution:

$$\mathbb{E}[w_i|Z=2] - \mathbb{E}[w_i|Z=1] = (q-p) \cdot \mathbb{E}[\delta_i|c-s \leq \delta_i < c]$$

Therefore:

$$\boxed{\text{LATE} = \frac{\mathbb{E}[w_i|Z=2] - \mathbb{E}[w_i|Z=1]}{q-p}}$$

where $q - p = \Pr(D=1|Z=2) - \Pr(D=1|Z=1)$ is the compliance rate.

Q9. *(0.5 pts)* What can you estimate if $s = c$?

Solution:

If $s = c$, the training is **free** for group $Z = 2$. Since $\delta > 0$ for everyone, *all* individuals in group 2 will take the training: $q = 1$.

Comparing $Z = 2$ (all trained) with $Z = 0$ (none trained):

$$\mathbb{E}[w_i|Z=2] - \mathbb{E}[w_i|Z=0] = \mathbb{E}[w_{1i}] - \mathbb{E}[w_{0i}] = \boxed{\text{ATE} = \mathbb{E}[\delta_i]}$$

With full subsidy, we eliminate selection entirely and identify the population average treatment effect.

Now assume δ and w_0 are correlated: $w_0 = a + \rho\delta + \varepsilon$ where $\varepsilon \perp \delta$.

Q10. *(1.5 pts)* Show that the naive comparison of the wages of treated and untreated absent an experiment would no longer identify a treatment parameter. Discuss the sign of the bias depending on ρ.

Solution:

The naive comparison now includes selection bias (from Q4):

$$\begin{aligned} &\mathbb{E}[w_i|D=1] - \mathbb{E}[w_i|D=0] \\ &= \underbrace{\mathbb{E}[\delta_i|\delta_i \geq c]}_{\text{ATT}} + \underbrace{\mathbb{E}[w_{0i}|\delta_i \geq c] - \mathbb{E}[w_{0i}|\delta_i < c]}_{\text{Selection bias}} \end{aligned}$$

Since $w_{0i} = a + \rho\delta_i + \varepsilon_i$ and $\varepsilon \perp \delta$:

$$\mathbb{E}[w_{0i}|\delta_i \geq c] - \mathbb{E}[w_{0i}|\delta_i < c] = \rho \left(\mathbb{E}[\delta_i|\delta_i \geq c] - \mathbb{E}[\delta_i|\delta_i < c] \right)$$

The term in parentheses is **positive** (treated have higher δ).
Sign of bias:

- $\rho > 0$: Upward bias. High-gain individuals also have high baseline wages. The naive comparison overstates ATT.
- $\rho < 0$: Downward bias. High-gain individuals have low baseline wages (e.g., training helps the disadvantaged most). The naive comparison understates ATT.
- $\rho = 0$: No bias (back to the independent case).

Q11. *(1 pt)* Show that comparing group $Z = 1$ with group $Z = 0$ identifies the same treatment parameter as before.

Solution:

The key insight: **randomization breaks the selection bias**.

In the RCT:

$$\mathbb{E}[w_i|Z = 1] - \mathbb{E}[w_i|Z = 0] = \mathbb{E}[w_i|Z = 1] - \mathbb{E}[w_{0i}]$$

Since Z is randomly assigned, $Z \perp (w_0, \delta)$. The treatment probability in group 1 is still $p = \Pr(\delta_i \geq c)$, and:

$$\mathbb{E}[w_i|Z = 1] = \mathbb{E}[w_{0i}] + p \cdot \mathbb{E}[\delta_i|\delta_i \geq c]$$

Therefore:

$$\frac{\mathbb{E}[w_i|Z = 1] - \mathbb{E}[w_i|Z = 0]}{p} = \mathbb{E}[\delta_i|\delta_i \geq c] = \text{ATT}$$

The Wald/IV estimator recovers ATT regardless of the correlation between δ and w_0, because randomization ensures the control group provides a valid counterfactual for baseline outcomes.