LECTURE 5: Regression Discontinuity Design

Duncan Webb (Nova SBE)

October 30, 2025

What is regression discontinuity?

- Regression discontinuity design (RDD) is one of the most powerful and credible research designs for causal inference
- The core idea: take advantage of **institutional features** that generate a **discontinuous change** in treatment at some threshold
- Examples:
  - Scholarship eligibility based on income threshold
  - Election outcomes determined by vote share
  - Class size rules based on enrollment thresholds
  - Program eligibility based on age cutoffs

Example 1: Financial aid and college enrollment

Research question: Does receiving financial aid increase college enrollment?

Setting: Many scholarship programs have sharp income cutoffs

- Example: Students from families earning below \$30,000 receive a \$1,500 scholarship
- Students from families earning \$30,001 receive nothing

Example 1: Financial aid and college enrollment

Research question: Does receiving financial aid increase college enrollment?

Setting: Many scholarship programs have sharp income cutoffs

- Example: Students from families earning below \$30,000 receive a \$1,500 scholarship
- Students from families earning \$30,001 receive nothing

Key insight: Students just below vs. just above the income threshold are likely very similar in all respects *except* scholarship receipt

Example 1: Financial aid and college enrollment

Research question: Does receiving financial aid increase college enrollment?

Setting: Many scholarship programs have sharp income cutoffs

- Example: Students from families earning below $30,000 receive a $1,500 scholarship
- Students from families earning $30,001 receive nothing

Key insight: Students just below vs. just above the income threshold are likely very similar in all respects *except* scholarship receipt

If enrollment rates differ discontinuously at the threshold, this difference can be attributed to the scholarship

Example 2: Electoral advantage and incumbency

Research question: Does barely winning an election give a party an advantage in future elections?

Setting: Close elections (Lee, 2008)

- A Democratic candidate wins if they get $> 50\%$ of the vote
- They lose if they get $< 50\%$ of the vote

Example 2: Electoral advantage and incumbency

Research question: Does barely winning an election give a party an advantage in future elections?

Setting: Close elections (Lee, 2008)

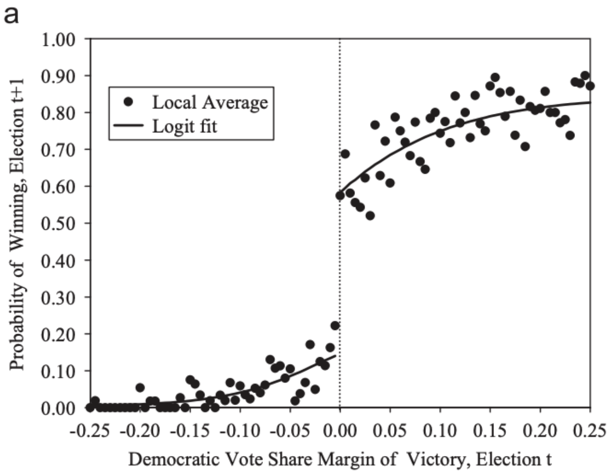
- A Democratic candidate wins if they get $> 50\%$ of the vote
- They lose if they get $< 50\%$ of the vote

Key insight: Elections decided by a very small margin (e.g., 50.1% vs 49.9%) are essentially random

- The parties competing in such close races should be very similar
- Any discontinuous change in future electoral success at the 50% threshold reveals the causal effect of incumbency

Graphical intuition

D.S. Lee / Journal of Econometrics 142 (2008) 675–697



Why is RDD so popular?

RDD has **exploded in popularity** in empirical economics. Key advantages:

- **Credibility**: Exploits institutional rules, not researcher assumptions
  - Often considered nearly as credible as randomized experiments
- **Transparency**: Visual evidence makes results compelling and hard to manipulate
  - The ”eyeball test” is very informative
- **Policy relevance**: Treatment itself is often the policy of direct interest
  - Can also be used as an instrument (fuzzy RDD = IV setup)

Outline

1. Motivation

2. Identification

3. Sharp vs. Fuzzy RDD

4. Estimation

5. Bandwidth selection

6. Threats to validity

7. Conclusion

Setup: Notation

Using the potential outcomes framework from our first lecture:

- $Y_i(0)$, $Y_i(1)$: potential outcomes for individual i
- $D_i \in \{0, 1\}$: treatment status
- Observed outcome: $Y_i = D_i Y_i(1) + (1 - D_i) Y_i(0)$

Setup: Notation

Using the potential outcomes framework from our first lecture:

- $Y_i(0)$, $Y_i(1)$: potential outcomes for individual i
- $D_i \in \{0, 1\}$: treatment status
- Observed outcome: $Y_i = D_i Y_i(1) + (1 - D_i) Y_i(0)$

New element: The **running variable** (or **forcing variable**) Z_i

- Also called the "assignment variable" or "score"
- Examples: test score, age, income, vote share
- Treatment assignment determined by cutoff value z^*

The forcing variable

Key assumption: Treatment status D_i is determined (at least partly) by whether Z_i crosses a threshold z^*

Two cases:

- **Sharp RDD**: Treatment changes *deterministically* at z^*
  - Everyone with $Z_i \geq z^*$ is treated; everyone with $Z_i < z^*$ is untreated
- **Fuzzy RDD**: Treatment *probability* changes at z^*
  - Crossing z^* increases the chance of treatment, but doesn't guarantee it

The forcing variable

Key assumption: Treatment status D_i is determined (at least partly) by whether Z_i crosses a threshold z^*

Two cases:

- **Sharp RDD**: Treatment changes *deterministically* at z^*
  - Everyone with $Z_i \geq z^*$ is treated; everyone with $Z_i < z^*$ is untreated
- **Fuzzy RDD**: Treatment *probability* changes at z^*
  - Crossing z^* increases the chance of treatment, but doesn't guarantee it

We'll first derive the **general identification result** that applies to both cases, then distinguish between them

The identifying assumption

Core identification assumption: The conditional expectation functions $\mathbb{E}[Y_i(0)|Z_i = z]$ and $\mathbb{E}[Y_i(1)|Z_i = z]$ are **continuous** in z at the cutoff $z = z^*$

The identifying assumption

Core identification assumption: The conditional expectation functions $\mathbb{E}[Y_i(0)|Z_i = z]$ and $\mathbb{E}[Y_i(1)|Z_i = z]$ are continuous in z at the cutoff $z = z^*$

What does this mean?

- The average potential outcomes change **smoothly** as Z_i changes
- There are no **other factors** that jump discontinuously at $Z_i = z^*$
- The only thing that changes discontinuously at the threshold is treatment D_i

The identifying assumption

Core identification assumption: The conditional expectation functions $\mathbb{E}[Y_i(0)|Z_i = z]$ and $\mathbb{E}[Y_i(1)|Z_i = z]$ are continuous in z at the cutoff $z = z^*$

What does this mean?

- The average potential outcomes change **smoothly** as Z_i changes
- There are no **other factors** that jump discontinuously at $Z_i = z^*$
- The only thing that changes discontinuously at the threshold is treatment D_i

Intuition: People just below vs. just above the threshold are essentially identical *except* for their treatment status

Identification: The Wald estimand

Define the limits of the conditional expectation of Y on either side of the cutoff:

$$Y^{+} := \lim_{z \to (z^*)^{+}} \mathbb{E}[Y_i | Z_i = z] \tag{1}$$

$$Y^{-} := \lim_{z \to (z^*)^{-}} \mathbb{E}[Y_i | Z_i = z] \tag{2}$$

Identification: The Wald estimand

Define the limits of the conditional expectation of Y on either side of the cutoff:

$$Y^{+} := \lim_{z \to (z^*)^{+}} \mathbb{E}[Y_i | Z_i = z] \tag{1}$$

$$Y^{-} := \lim_{z \to (z^*)^{-}} \mathbb{E}[Y_i | Z_i = z] \tag{2}$$

Similarly for treatment:

$$D^{+} := \lim_{z \to (z^*)^{+}} \mathbb{E}[D_i | Z_i = z] \tag{3}$$

$$D^{-} := \lim_{z \to (z^*)^{-}} \mathbb{E}[D_i | Z_i = z] \tag{4}$$

Identification: The Wald estimand

Under our model $Y_i = Y_i(0) + D_i(Y_i(1) - Y_i(0))$ and the continuity assumption:

$$
\begin{aligned}
Y^+ &= \lim_{z \to (z^*)^+} \mathbb{E}[Y_i(0) + D_i(Y_i(1) - Y_i(0)) | Z_i = z] && (5) \\
&= \lim_{z \to (z^*)^+} \mathbb{E}[Y_i(0) | Z_i = z] + D^+ \cdot \mathbb{E}[Y_i(1) - Y_i(0) | Z_i = z^*] && (6) \\
&= \mathbb{E}[Y_i(0) | Z_i = z^*] + D^+ \cdot \tau && (7)
\end{aligned}
$$

where $\tau = \mathbb{E}[Y_i(1) - Y_i(0) | Z_i = z^*]$ is the treatment effect at the cutoff

Identification: The Wald estimand

Under our model $Y_i = Y_i(0) + D_i(Y_i(1) - Y_i(0))$ and the continuity assumption:

$$Y^+ = \lim_{z \to (z^*)^+} \mathbb{E}[Y_i(0) + D_i(Y_i(1) - Y_i(0))|Z_i = z] \quad (5)$$

$$= \lim_{z \to (z^*)^+} \mathbb{E}[Y_i(0)|Z_i = z] + D^+ \cdot \mathbb{E}[Y_i(1) - Y_i(0)|Z_i = z^*] \quad (6)$$

$$= \mathbb{E}[Y_i(0)|Z_i = z^*] + D^+ \cdot \tau \quad (7)$$

where $\tau = \mathbb{E}[Y_i(1) - Y_i(0)|Z_i = z^*]$ is the treatment effect at the cutoff

Similarly:

$$Y^- = \mathbb{E}[Y_i(0)|Z_i = z^*] + D^- \cdot \tau \quad (8)$$

Identification: The Wald estimand

Taking the difference:

$$
\begin{aligned}
Y^{+} - Y^{-} &= \mathbb{E}[Y_i(0)|Z_i = z^*] + D^{+} \cdot \tau - \mathbb{E}[Y_i(0)|Z_i = z^*] - D^{-} \cdot \tau && (9) \\
&= (D^{+} - D^{-}) \cdot \tau && (10)
\end{aligned}
$$

Identification: The Wald estimand

Taking the difference:

$$
\begin{aligned}
Y^{+} - Y^{-} &= \mathbb{E}[Y_i(0)|Z_i = z^*] + D^{+} \cdot \tau - \mathbb{E}[Y_i(0)|Z_i = z^*] - D^{-} \cdot \tau && (9) \\
&= (D^{+} - D^{-}) \cdot \tau && (10)
\end{aligned}
$$

Therefore, the treatment effect at the cutoff is:

$$\boxed{\tau = \frac{Y^{+} - Y^{-}}{D^{+} - D^{-}}}$$

This is the **Wald estimand -** the ratio of the jump in outcomes to the jump in treatment

What are we identifying?

The Wald formula identifies the **local average treatment effect** at the cutoff:

$$\tau_{RDD} = \mathbb{E}[Y_i(1) - Y_i(0) | Z_i = z^*] \tag{11}$$

What are we identifying?

The Wald formula identifies the **local average treatment effect** at the cutoff:

$$\tau_{RDD} = \mathbb{E}[Y_i(1) - Y_i(0) | Z_i = z^*] \tag{11}$$

Interpretation depends on sharp vs. fuzzy:

- **Sharp RDD**: This is the ATE for everyone at the cutoff (since everyone's treatment changes)
- **Fuzzy RDD**: This is the LATE for compliers at the cutoff (those induced to take treatment by crossing z^*)

What are we identifying?

The Wald formula identifies the **local average treatment effect** at the cutoff:

$$\tau_{RDD} = \mathbb{E}[Y_i(1) - Y_i(0) | Z_i = z^*] \tag{11}$$

Interpretation depends on sharp vs. fuzzy:

- **Sharp RDD**: This is the ATE for everyone at the cutoff (since everyone's treatment changes)
- **Fuzzy RDD**: This is the LATE for compliers at the cutoff (those induced to take treatment by crossing z^*)

Key limitation: Very local parameter (measure-zero set of population)

RDD for reduced form vs. as an instrument

Important distinction

1. RDD for direct/reduced form effects:

- Interested in effect of treatment D itself on outcome Y
- Example: Effect of scholarship receipt on college enrollment
- Example: Effect of winning election on policy outcomes

RDD for reduced form vs. as an instrument

Important distinction

1. RDD for direct/reduced form effects:

- Interested in effect of treatment D itself on outcome Y
- Example: Effect of scholarship receipt on college enrollment
- Example: Effect of winning election on policy outcomes

2. RDD as an instrument:

- Treatment D is not ultimate interest; it affects some other variable S
- Use RDD assignment as instrument for S to estimate effect of S on Y
- Example: Scholarship $\rightarrow$ years of education $\rightarrow$ labor outcomes
- RDD identifies effect of years of education (not just scholarship)

RDD for reduced form vs. as an instrument

Important distinction

1. RDD for direct/reduced form effects:

- Interested in effect of treatment D itself on outcome Y
- Example: Effect of scholarship receipt on college enrollment
- Example: Effect of winning election on policy outcomes

2. RDD as an instrument:

- Treatment D is not ultimate interest; it affects some other variable S
- Use RDD assignment as instrument for S to estimate effect of S on Y
- Example: Scholarship $\rightarrow$ years of education $\rightarrow$ labor outcomes
- RDD identifies effect of years of education (not just scholarship)

Both uses can occur in sharp or fuzzy RDD!

RDD as a local randomized experiment

Alternative interpretation: Think of RDD as a randomized experiment in a neighborhood of the cutoff

- For individuals very close to $Z_i = z^*$, whether they end up just above or just below the threshold is essentially random
- Example: In a close election with $Z_i =$ vote margin, whether you get 50.01% or 49.99% is basically random
- This makes treated and untreated individuals near the cutoff comparable

RDD as a local randomized experiment

Alternative interpretation: Think of RDD as a randomized experiment in a neighborhood of the cutoff

- For individuals very close to $Z_i = z^*$, whether they end up just above or just below the threshold is essentially random
- Example: In a close election with $Z_i =$ vote margin, whether you get 50.01% or 49.99% is basically random
- This makes treated and untreated individuals near the cutoff comparable
- This ”local randomization” interpretation is increasingly popular
- Helps with thinking about inference and design

Outline

1. Motivation

2. Identification

3. Sharp vs. Fuzzy RDD

4. Estimation

5. Bandwidth selection

6. Threats to validity

7. Conclusion

Sharp regression discontinuity

Definition: In a **sharp RDD**, treatment status changes *deterministically* at the cutoff

Treatment assignment is a **deterministic function** of the running variable:

$$D_i = \begin{cases} 1 & \text{if } Z_i \geq z^* \\ 0 & \text{if } Z_i < z^* \end{cases} \tag{12}$$

This means: $D^+ = 1$ and $D^- = 0$

Sharp regression discontinuity

Definition: In a **sharp RDD**, treatment status changes *deterministically* at the cutoff

Treatment assignment is a **deterministic function** of the running variable:

$$D_i = \begin{cases} 1 & \text{if } Z_i \geq z^* \\ 0 & \text{if } Z_i < z^* \end{cases} \tag{12}$$

This means: $D^+ = 1$ and $D^- = 0$

The Wald formula simplifies:

$$\tau_{Sharp} = \frac{Y^+ - Y^-}{D^+ - D^-} = \frac{Y^+ - Y^-}{1 - 0} = Y^+ - Y^- \tag{13}$$

The treatment effect is simply the jump in outcomes!

Sharp RDD: Examples

- **Age-based eligibility**: Pension eligibility at age 65
  - $Z_i =$ age, cutoff at 65
  - Everyone 65+ receives pension, nobody under 65 does
- **Test score cutoff**: Admission to selective program
  - $Z_i =$ test score, cutoff at 70
  - Everyone scoring ≥ 70 admitted, everyone below rejected
- **Electoral threshold**: Winning an election
  - $Z_i =$ vote share - 50%
  - Above 50%: you win; below 50%: you lose

Fuzzy regression discontinuity

Definition: In a **fuzzy RDD**, the *probability* of treatment changes discontinuously at the cutoff, but not from 0 to 1

Treatment probability jumps, but not perfectly:

$$\mathbb{P}(D_i = 1|Z_i) = \begin{cases} P_1(Z_i) & \text{if } Z_i \geq z^* \\ P_0(Z_i) & \text{if } Z_i < z^* \end{cases} \tag{14}$$

where $P_1(z^*) > P_0(z^*)$ but the jump is not from 0 to 1

Fuzzy regression discontinuity

Definition: In a **fuzzy RDD**, the *probability* of treatment changes discontinuously at the cutoff, but not from 0 to 1

Treatment probability jumps, but not perfectly:

$$\mathbb{P}(D_i = 1|Z_i) = \begin{cases} P_1(Z_i) & \text{if } Z_i \geq z^* \\ P_0(Z_i) & \text{if } Z_i < z^* \end{cases} \tag{14}$$

where $P_1(z^*) > P_0(z^*)$ but the jump is not from 0 to 1

Equivalently: $0 < D^+ - D^- < 1$

This has the structure of instrumental variables:

- Eligibility: Crossing the cutoff $\mathbb{1}\{Z_i \geq z^*\}$
- Treatment: Actual treatment receipt D_i

Why does fuzzy RDD arise?

Reasons we see fuzzy rather than sharp:

Non-compliance (most common): Eligibility $\neq$ actual treatment

- Not everyone eligible takes up the program
- Some ineligible gain access through exceptions
- Example: Scholarship eligibility vs. actual receipt

Why does fuzzy RDD arise?

Reasons we see fuzzy rather than sharp:

Non-compliance (most common): Eligibility $\neq$ actual treatment

- Not everyone eligible takes up the program
- Some ineligible gain access through exceptions
- Example: Scholarship eligibility vs. actual receipt

When you have fuzzy RDD: Think of it as IV, where crossing z^* is an instrument for treatment D_i

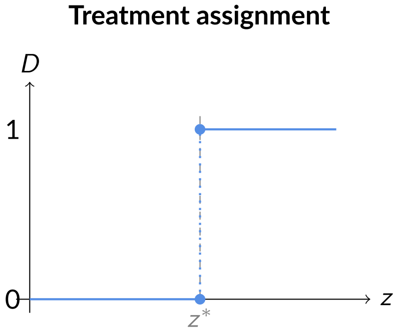
Fuzzy RDD: Examples

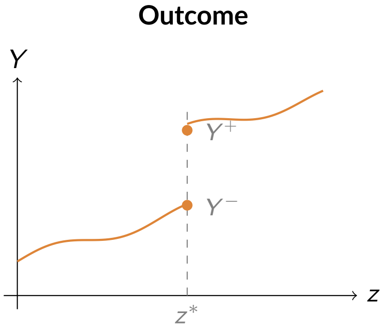
- **Scholarship eligibility**:
  - Z_i = family income, cutoff at $30,000
  - Not everyone eligible applies for or receives the scholarship
  - Some ineligible students receive aid from other sources

- **Class size rules**:
  - Z_i = enrollment, cutoff triggers new class
  - Rules may not be perfectly enforced
  - Some schools may combine classes despite the rule

- **Medicaid eligibility**:
  - Z_i = income relative to poverty line
  - Not all eligible enroll
  - Some ineligible may be covered through other programs

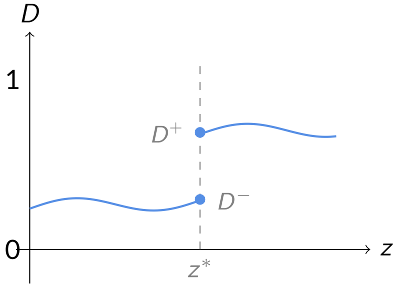
Sharp RDD: Visual representation

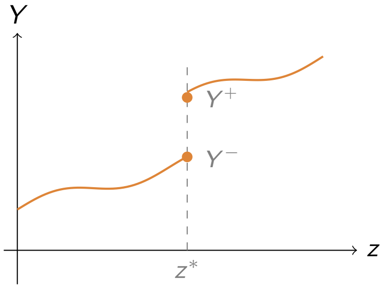




Left: Treatment status jumps from 0 to 1 at cutoff. Right: Outcome shows discontinuous jump.

Fuzzy RDD: Visual representation





Left: Treatment probability jumps but not from 0 to 1. Right: Outcome shows discontinuous jump.

Fuzzy RDD and LATE

Since fuzzy RDD has IV structure (eligibility vs. take-up), all the IV results apply!

The Wald formula $\tau = \frac{Y^+ - Y^-}{D^+ - D^-}$ identifies a **Local Average Treatment Effect (LATE)**:

- Effect for **compliers**: those induced to take treatment by crossing z^*
- **Not** the effect for always-takers or never-takers

Fuzzy RDD and LATE

Since fuzzy RDD has IV structure (eligibility vs. take-up), all the IV results apply!

The Wald formula $\tau = \frac{Y^+ - Y^-}{D^+ - D^-}$ identifies a **Local Average Treatment Effect (LATE)**:

- Effect for **compliers**: those induced to take treatment by crossing z^*
- **Not** the effect for always-takers or never-takers

Compliers in fuzzy RDD:

- Would take treatment if $Z_i \geq z^*$ (eligible)
- Would not take treatment if $Z_i < z^*$ (ineligible)
- Their treatment status is *changed* by crossing the threshold

Fuzzy RDD and LATE

Since fuzzy RDD has IV structure (eligibility vs. take-up), all the IV results apply!

The Wald formula $\tau = \frac{Y^+ - Y^-}{D^+ - D^-}$ identifies a **Local Average Treatment Effect (LATE)**:

- Effect for **compliers**: those induced to take treatment by crossing z^*
- **Not** the effect for always-takers or never-takers

Compliers in fuzzy RDD:

- Would take treatment if $Z_i \geq z^*$ (eligible)
- Would not take treatment if $Z_i < z^*$ (ineligible)
- Their treatment status is *changed* by crossing the threshold

Standard IV assumptions needed: Relevance, exclusion, monotonicity

Outline

1. Motivation

2. Identification

3. Sharp vs. Fuzzy RDD

4. Estimation

5. Bandwidth selection

6. Threats to validity

7. Conclusion

No exact empirical counterpart to the limit

Important practical issue: We want to estimate limits as $z \to z^*$, but:

- We only have a **finite sample** of observations
- We may not observe anyone *exactly* at $Z_i = z^*$
- We need to use data **away from the cutoff**

No exact empirical counterpart to the limit

Important practical issue: We want to estimate limits as $z \to z^*$, but:

- We only have a **finite sample** of observations
- We may not observe anyone *exactly* at $Z_i = z^*$
- We need to use data **away from the cutoff**

This creates a fundamental estimation challenge:

- How far from the cutoff should we use data?
- How do we approximate the conditional expectations near the cutoff?
- This is the focus of the next section!

From theory to estimation

Recall the identification formula:

$$\tau = \frac{Y^+ - Y^-}{D^+ - D^-} \tag{15}$$

where $Y^+ = \lim_{z \to (z^*)^+} \mathbb{E}[Y_i | Z_i = z]$, etc.

From theory to estimation

Recall the identification formula:

$$\tau = \frac{Y^+ - Y^-}{D^+ - D^-} \tag{15}$$

where $Y^+ = \lim_{z \to (z^*)^+} \mathbb{E}[Y_i | Z_i = z]$, etc.

But: We need to estimate these limits from data

Approach: Assume $\mathbb{E}[Y_i | Z_i = z]$ can be approximated by a smooth function $g(z)$ near the cutoff

Basic regression approach

The simplest specification uses a linear function on each side:

$$Y_i = \alpha + \beta D_i + \gamma_1 Z_i + \gamma_2 D_i \cdot Z_i + \varepsilon_i \tag{16}$$

where $D_i = \mathbb{1}\{Z_i \geq z^*\}$

Basic regression approach

The simplest specification uses a linear function on each side:

$$Y_i = \alpha + \beta D_i + \gamma_1 Z_i + \gamma_2 D_i \cdot Z_i + \varepsilon_i \tag{16}$$

where $D_i = \mathbb{1}\{Z_i \geq z^*\}$

Interpretation:

- α: Intercept for untreated (extrapolated to $Z_i = z^*$)
- γ_1: Slope for untreated
- β: **Treatment effect** at the cutoff
- γ_2: Difference in slopes between treated and untreated

Understanding the specification

The regression can be written separately on each side:

Below the cutoff ($Z_i < z^*$, so $D_i = 0$):

$$\mathbb{E}[Y_i | Z_i = z] = \alpha + \gamma_1 (z - z^*) \tag{17}$$

Understanding the specification

The regression can be written separately on each side:

Below the cutoff ($Z_i < z^*$, so $D_i = 0$):

$$\mathbb{E}[Y_i|Z_i = z] = \alpha + \gamma_1(z - z^*) \tag{17}$$

Above the cutoff ($Z_i \geq z^*$, so $D_i = 1$):

$$\mathbb{E}[Y_i|Z_i = z] = (\alpha + \beta) + (\gamma_1 + \gamma_2)(z - z^*) \tag{18}$$

Understanding the specification

The regression can be written separately on each side:

Below the cutoff ($Z_i < z^*$, so $D_i = 0$):

$$\mathbb{E}[Y_i|Z_i = z] = \alpha + \gamma_1(z - z^*) \tag{17}$$

Above the cutoff ($Z_i \geq z^*$, so $D_i = 1$):

$$\mathbb{E}[Y_i|Z_i = z] = (\alpha + \beta) + (\gamma_1 + \gamma_2)(z - z^*) \tag{18}$$

At the cutoff ($Z_i = z^*$):

- From below: $Y^- = \alpha$
- From above: $Y^+ = \alpha + \beta$
- Difference: $\beta = Y^+ - Y^-$

Allowing for flexible functional forms

Linear may be too restrictive! We can use **polynomial specifications**:

$$Y_i = \alpha + \beta D_i + \sum_{p=1}^{P} \gamma_p Z_i^p + \sum_{p=1}^{P} \delta_p D_i \cdot Z_i^p + \varepsilon_i \tag{19}$$

Allowing for flexible functional forms

Linear may be too restrictive! We can use **polynomial specifications**:

$$Y_i = \alpha + \beta D_i + \sum_{p=1}^{P} \gamma_p Z_i^p + \sum_{p=1}^{P} \delta_p D_i \cdot Z_i^p + \varepsilon_i \tag{19}$$

Common choices:

- $P = 1$: Linear (most common)
- $P = 2$: Quadratic
- $P = 3$ or $P = 4$: Higher-order polynomials

Allowing for flexible functional forms

Linear may be too restrictive! We can use **polynomial specifications**:

$$Y_i = \alpha + \beta D_i + \sum_{p=1}^{P} \gamma_p Z_i^p + \sum_{p=1}^{P} \delta_p D_i \cdot Z_i^p + \varepsilon_i \tag{19}$$

Common choices:

- $P = 1$: Linear (most common)
- $P = 2$: Quadratic
- $P = 3$ or $P = 4$: Higher-order polynomials

Trade-off:

- Higher $P \rightarrow$ more flexible, fits data better
- But: risk of overfitting, especially near endpoints
- **Warning**: Gelman and Imbens (2019) show high-order polynomials can be problematic

Local linear regression

Better approach: Use data only *close to the cutoff* with linear specification

Estimate:

$$Y_i = \alpha + \beta D_i + \gamma_1 Z_i + \gamma_2 D_i \cdot Z_i + \varepsilon_i \quad (20)$$

but only using observations where $|Z_i| \leq h$ for some bandwidth h

Local linear regression

Better approach: Use data only *close to the cutoff* with linear specification

Estimate:

$$Y_i = \alpha + \beta D_i + \gamma_1 Z_i + \gamma_2 D_i \cdot Z_i + \varepsilon_i \tag{20}$$

but only using observations where $|Z_i| \leq h$ for some bandwidth h

Why is this better?

- Linear approximation is better when we're close to the cutoff
- Avoids relying on functional form assumptions far from cutoff
- More robust to misspecification

Local linear regression

Better approach: Use data only *close to the cutoff* with linear specification

Estimate:

$$Y_i = \alpha + \beta D_i + \gamma_1 Z_i + \gamma_2 D_i \cdot Z_i + \varepsilon_i \tag{20}$$

but only using observations where $|Z_i| \leq h$ for some bandwidth h

Why is this better?

- Linear approximation is better when we're close to the cutoff
- Avoids relying on functional form assumptions far from cutoff
- More robust to misspecification

But: How do we choose the bandwidth h? (More on this soon!)

Weighted local linear regression

Possible other specification: weight observations based on distance from cutoff

Use a **kernel function** $K(u)$ that gives more weight to observations closer to the cutoff

Weighted local linear regression

Possible other specification: weight observations based on distance from cutoff

Use a **kernel function** $K(u)$ that gives more weight to observations closer to the cutoff

Weight for observation i: $w_i = K\left(\frac{Z_i - z^*}{h}\right)$

Weighted local linear regression

Possible other specification: weight observations based on distance from cutoff

Use a **kernel function** $K(u)$ that gives more weight to observations closer to the cutoff

Weight for observation i: $w_i = K\left(\frac{Z_i - z^*}{h}\right)$

Common kernels:

- **Uniform**: $K(u) = 0.5$ for $|u| \leq 1$, zero otherwise
- **Triangular**: $K(u) = 1 - |u|$ for $|u| \leq 1$
- **Epanechnikov**: $K(u) = 0.75(1 - u^2)$ for $|u| \leq 1$

Weighted local linear regression

Possible other specification: weight observations based on distance from cutoff

Use a **kernel function** $K(u)$ that gives more weight to observations closer to the cutoff

Weight for observation i: $w_i = K\left(\frac{Z_i - z^*}{h}\right)$

Common kernels:

- **Uniform**: $K(u) = 0.5$ for $|u| \leq 1$, zero otherwise
- **Triangular**: $K(u) = 1 - |u|$ for $|u| \leq 1$
- **Epanechnikov**: $K(u) = 0.75(1 - u^2)$ for $|u| \leq 1$

In practice: Choice of kernel matters less than choice of bandwidth

Estimation with fuzzy RDD

For **fuzzy RDD**, we use instrumental variables (2SLS):

First stage (treatment on eligibility):

$$D_i = \pi_0 + \pi_1 \mathbb{1}\{Z_i \geq z^*\} + \pi_2 (Z_i - z^*) + \pi_3 \mathbb{1}\{Z_i \geq z^*\} \cdot (Z_i - z^*) + \nu_i \tag{21}$$

Second stage (outcome on predicted treatment):

$$Y_i = \alpha + \beta \hat{D}_i + \gamma_1 (Z_i - z^*) + \gamma_2 \mathbb{1}\{Z_i \geq z^*\} \cdot (Z_i - z^*) + \varepsilon_i \tag{22}$$

Estimation with fuzzy RDD

For **fuzzy RDD**, we use instrumental variables (2SLS):

First stage (treatment on eligibility):

$$D_i = \pi_0 + \pi_1 \mathbb{1}\{Z_i \geq z^*\} + \pi_2(Z_i - z^*) + \pi_3 \mathbb{1}\{Z_i \geq z^*\} \cdot (Z_i - z^*) + \nu_i \tag{21}$$

Second stage (outcome on predicted treatment):

$$Y_i = \alpha + \beta \hat{D}_i + \gamma_1(Z_i - z^*) + \gamma_2 \mathbb{1}\{Z_i \geq z^*\} \cdot (Z_i - z^*) + \varepsilon_i \tag{22}$$

Or equivalently: Reduced form divided by first stage

$$\beta = \frac{Y^+ - Y^-}{D^+ - D^-} = \frac{\text{Reduced form effect}}{\text{First stage effect}} \tag{23}$$

Outline

1. Motivation
2. Identification
3. Sharp vs. Fuzzy RDD
4. Estimation
5. Bandwidth selection
6. Threats to validity
7. Conclusion

The fundamental trade-off

Central challenge in RDD: choosing the bandwidth h

Narrow bandwidth (small h):

- Good: Better approximation (less bias from functional form)
- Bad: Fewer observations (more variance/noise)

The fundamental trade-off

Central challenge in RDD: choosing the bandwidth h

Narrow bandwidth (small h):

- Good: Better approximation (less bias from functional form)
- Bad: Fewer observations (more variance/noise)

Wide bandwidth (large h):

- Good: More observations (less variance)
- Bad: Worse approximation (more bias from functional form)

The fundamental trade-off

Central challenge in RDD: choosing the bandwidth h

Narrow bandwidth (small h):

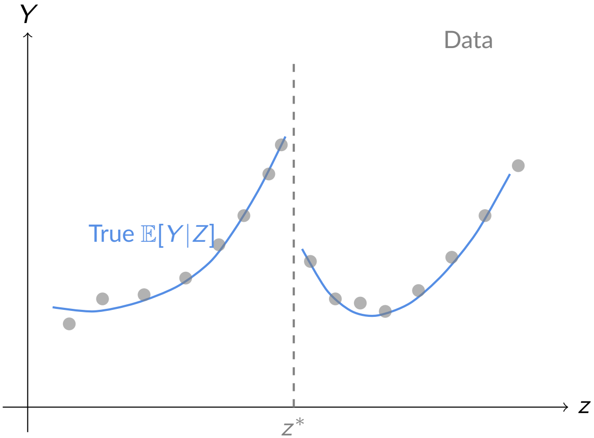
- Good: Better approximation (less bias from functional form)
- Bad: Fewer observations (more variance/noise)

Wide bandwidth (large h):

- Good: More observations (less variance)
- Bad: Worse approximation (more bias from functional form)

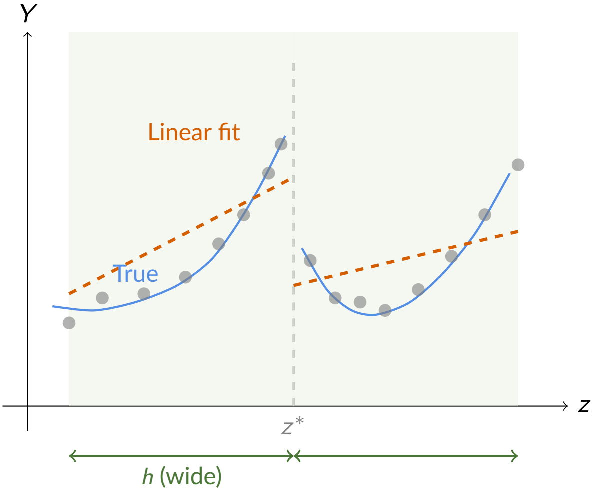
This is the classic **bias-variance trade-off**!

Visualizing the trade-off: True CEF



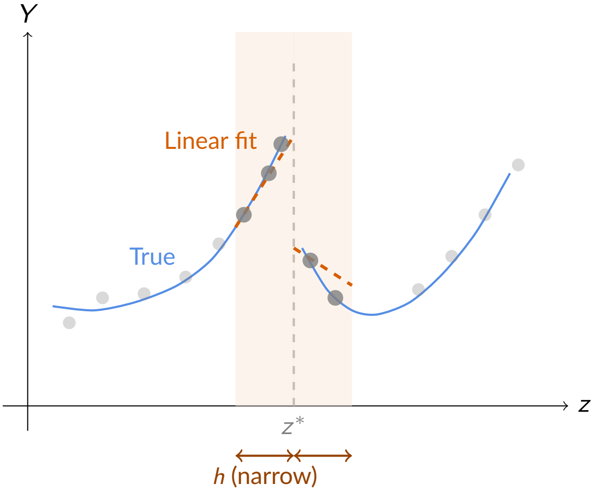
True conditional expectation function with curvature and observed data points

Wide bandwidth: Low variance, high bias



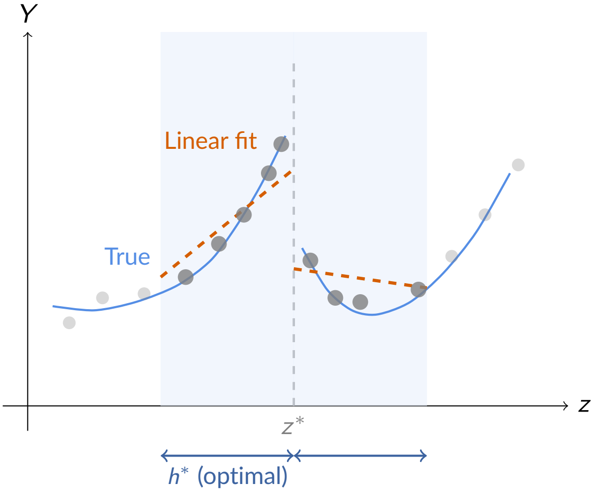
Many observations used, but poor approximation due to curvature

Narrow bandwidth: High variance, low bias



Few observations used, but good local approximation

Optimal bandwidth: Balance bias and variance



Optimal h^* balances moderate sample size with reasonable approximation

Optimal bandwidth selection

Question: Can we choose bandwidth optimally?

Optimal bandwidth selection

Question: Can we choose bandwidth optimally?

Yes! Modern approaches minimize mean squared error (MSE):

$$MSE(\hat{\tau}) = \text{Bias}^2(\hat{\tau}) + \text{Var}(\hat{\tau}) \tag{24}$$

Optimal bandwidth selection

Question: Can we choose bandwidth optimally?

Yes! Modern approaches minimize mean squared error (MSE):

$$MSE(\hat{\tau}) = \text{Bias}^2(\hat{\tau}) + \text{Var}(\hat{\tau}) \tag{24}$$

Key insight:

- Bias typically grows with h^{p+1} (where p is polynomial order)
- Variance typically shrinks with $1/(nh)$ (where n is sample size)
- Optimal h balances these two

Practical bandwidth selection

In practice: Use data-driven bandwidth selection

Popular methods:

- **rdrobust** (Calonico, Cattaneo, Titiunik): Most common, includes bias correction
- **RDHonest** (Kolesar, Rothe): For discrete running variables
- Cross-validation approaches

Practical bandwidth selection

In practice: Use data-driven bandwidth selection

Popular methods:

- **rdrobust** (Calonico, Cattaneo, Titiunik): Most common, includes bias correction
- **RDHonest** (Kolesar, Rothe): For discrete running variables
- Cross-validation approaches

Important: Always show **robustness** to bandwidth choice!

- Plot estimates for a range of bandwidths
- Show that results are not driven by one specific choice

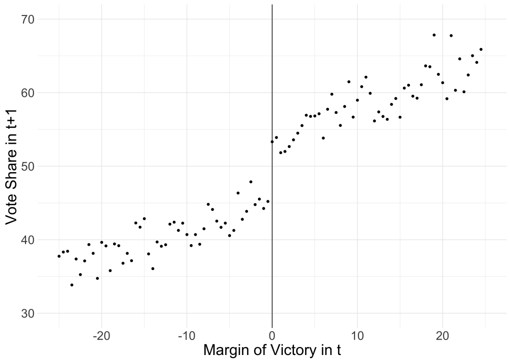
Graphical presentation: Choosing bin size

Similar trade-off for graphical presentation:

- Too many bins → noisy (hard to interpret)
- Too few bins → masks true pattern
- Need to find the right balance

Ideally: Plot binned means

- Similar intuition to binscatter
- But how do we choose bins?



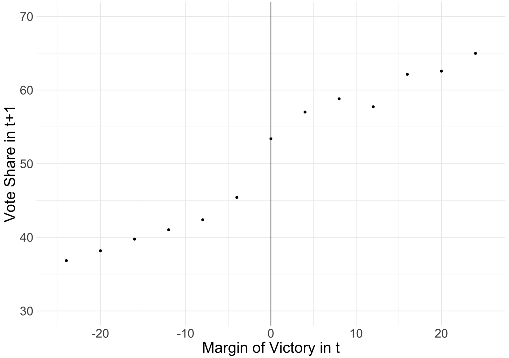
Graphical presentation: Choosing bin size

Similar trade-off for graphical presentation:

- Too many bins $\rightarrow$ noisy (hard to interpret)
- Too few bins $\rightarrow$ masks true pattern
- Need to find the right balance

Ideally: Plot binned means

- Similar intuition to binscatter
- But how do we choose bins?



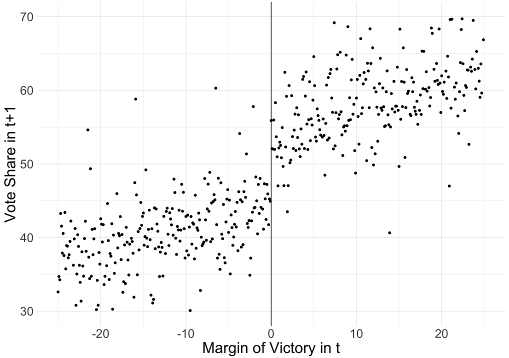
Graphical presentation: Choosing bin size

Similar trade-off for graphical presentation:

- Too many bins → noisy (hard to interpret)
- Too few bins → masks true pattern
- Need to find the right balance

Ideally: Plot binned means

- Similar intuition to binscatter
- But how do we choose bins?



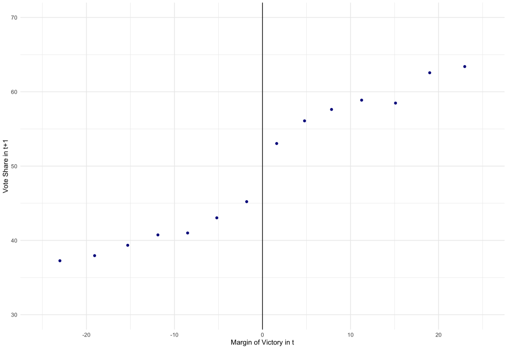
Graphical presentation: Optimal bin selection

Modern approach: Choose bins optimally

Cattaneo et al. (2020) propose two approaches:

1. **IMSE-minimizing** ($\propto n^{1/3}$ bins)
   - Trades off bias vs. variance over whole range
2. **Mimicking-variance** ($\propto n/\log(n)^2$ bins)
   - Matches variance of raw data in binned plots
   - Typically more bins

Software: `rdplot` in `rdrobust` (Stata/R)



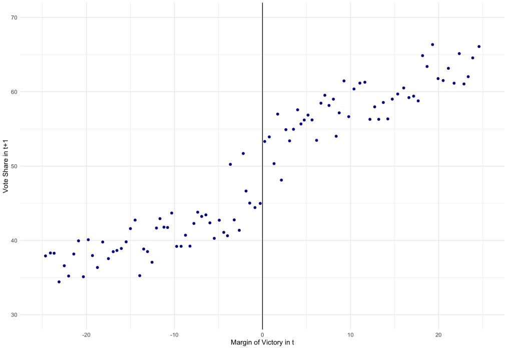
Graphical presentation: Optimal bin selection

Modern approach: Choose bins optimally

Cattaneo et al. (2020) propose two approaches:

1. **IMSE-minimizing** ($\propto n^{1/3}$ bins)
    - Trades off bias vs. variance over whole range
2. **Mimicking-variance** ($\propto n/\log(n)^2$ bins)
    - Matches variance of raw data in binned plots
    - Typically more bins

Software: `rdplot` in `rdrobust` (Stata/R)



Graphical presentation: Bin placement

Other decision to make: How to place bins: equal-spaced vs. quantile

- Quantile binning more transparent
- Equal-spaced can mask underlying density

Outline

1. Motivation
2. Identification
3. Sharp vs. Fuzzy RDD
4. Estimation
5. Bandwidth selection
6. Threats to validity
7. Conclusion

Validating the RDD

Recall: RDD relies on the assumption that potential outcomes are continuous at the cutoff

This assumption is not directly testable!

- We never observe both $Y_i(0)$ and $Y_i(1)$ for same individual
- Cannot verify continuity of counterfactuals

Validating the RDD

Recall: RDD relies on the assumption that potential outcomes are continuous at the cutoff

This assumption is not directly testable!

- We never observe both $Y_i(0)$ and $Y_i(1)$ for same individual
- Cannot verify continuity of counterfactuals

But: We can test **implications** of the assumption

Main threats to validity:

1. **Manipulation/sorting** of the running variable
2. **Imbalance** in covariates at cutoff
3. **Other discontinuities** at the same threshold

Threat 1: Manipulation and bunching

Problem: If individuals can manipulate their value of Z_i to cross the threshold, then treated and untreated are no longer comparable

Example (McCrary, 2008):

- Government announces income support for those earning $<$\$14,000
- Some people might underreport income to qualify
- Would see ”bunching” just below the cutoff

Threat 1: Manipulation and bunching

Problem: If individuals can manipulate their value of Z_i to cross the threshold, then treated and untreated are no longer comparable

Example (McCrary, 2008):

- Government announces income support for those earning $<$\$14,000
- Some people might underreport income to qualify
- Would see "bunching" just below the cutoff

Why is this a problem?

- Those who manipulate may be systematically different
- Example: More sophisticated, more desperate, etc.
- Violates continuity assumption

Testing for manipulation: McCrary density test

McCrary (2008) test: Test whether the *density* of the running variable is continuous at the cutoff

Intuition:

- If no manipulation, density should be smooth through cutoff
- Manipulation creates a ”jump” or ”hole” in the density

Testing for manipulation: McCrary density test

McCrary (2008) test: Test whether the *density* of the running variable is continuous at the cutoff

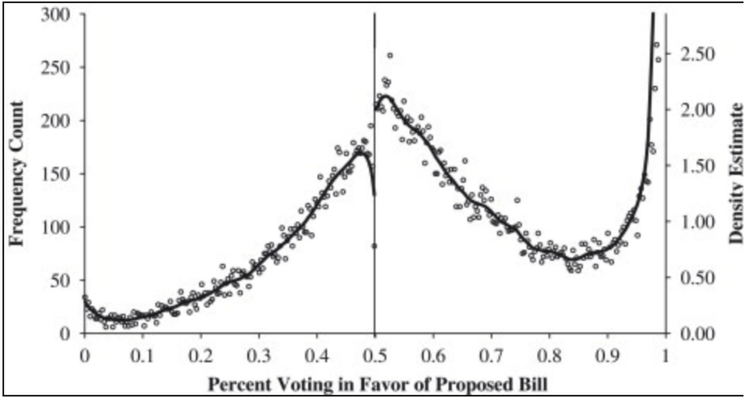
Intuition:

- If no manipulation, density should be smooth through cutoff
- Manipulation creates a ”jump” or ”hole” in the density

Implementation:

- Estimate density on each side of cutoff (using kernel methods)
- Test for discontinuity at $Z_i = z^*$
- Software: `rddensity` (Cattaneo, Jansson, Ma, 2020)

McCrary test: Example



Lee (2008): No evidence of manipulation in close elections

When manipulation happens anyway...

What if there IS bunching?

Option 1: Bounds on treatment effects

- Gerard, Rokkanen, and Rothe (2020): `rdbounds`
- Even with sorting, can bound the magnitude of effects
- Asks: how much could sorting explain the results?

When manipulation happens anyway...

What if there IS bunching?

Option 1: Bounds on treatment effects

- Gerard, Rokkanen, and Rothe (2020): `rdbounds`
- Even with sorting, can bound the magnitude of effects
- Asks: how much could sorting explain the results?

Option 2: Donut-hole RDD

- Exclude observations very close to cutoff
- If manipulation only affects those right at threshold
- But: loses precision, changes estimand

Threat 2: Covariate imbalance

Covariate balance test: Check whether predetermined covariates change discontinuously at cutoff

Idea: If RDD is valid, things that were determined *before* treatment assignment shouldn't jump at the cutoff

Threat 2: Covariate imbalance

Covariate balance test: Check whether predetermined covariates change discontinuously at cutoff

Idea: If RDD is valid, things that were determined *before* treatment assignment shouldn't jump at the cutoff

What to test:

- Demographic characteristics (age, gender, race, etc.)
- Baseline/lagged outcomes (outcomes before treatment)
- Other predetermined variables

Threat 2: Covariate imbalance

Covariate balance test: Check whether predetermined covariates change discontinuously at cutoff

Idea: If RDD is valid, things that were determined *before* treatment assignment shouldn't jump at the cutoff

What to test:

- Demographic characteristics (age, gender, race, etc.)
- Baseline/lagged outcomes (outcomes before treatment)
- Other predetermined variables

Implementation: Run the same RDD regression, but use covariate as outcome

$$X_i = \alpha + \beta \mathbb{1}\{Z_i \geq z^*\} + \gamma_1 (Z_i - z^*) + \gamma_2 \mathbb{1}\{Z_i \geq z^*\} \cdot (Z_i - z^*) + \varepsilon_i \tag{25}$$

Test $H_0 : \beta = 0$ for each covariate

Threat 3: Other discontinuities at cutoff

Problem: If *other things* besides treatment change at the cutoff, we can't isolate the treatment effect

Example: Retirement age

- At age 65: eligibility for pension, Medicare, mandatory retirement, etc.
- Hard to say which one drives any outcome changes

Threat 3: Other discontinuities at cutoff

Problem: If *other things* besides treatment change at the cutoff, we can't isolate the treatment effect

Example: Retirement age

- At age 65: eligibility for pension, Medicare, mandatory retirement, etc.
- Hard to say which one drives any outcome changes

Solutions:

- Find a setting where only one thing changes
- Try to measure all the things that change
- Focus on outcomes that should only be affected by one channel

Additional robustness tests

Placebo cutoffs:

- Test for discontinuities at points *other than* the true cutoff
- If you find effects at fake cutoffs, something's wrong!
- Permutation tests: randomly assign cutoff location

Additional robustness tests

Placebo cutoffs:

- Test for discontinuities at points *other than* the true cutoff
- If you find effects at fake cutoffs, something's wrong!
- Permutation tests: randomly assign cutoff location

Donut-hole specification:

- Exclude observations very close to cutoff
- Tests whether results driven by a few observations at boundary
- Also useful if worried about measurement error in Z_i

Additional robustness tests

Placebo cutoffs:

- Test for discontinuities at points *other than* the true cutoff
- If you find effects at fake cutoffs, something's wrong!
- Permutation tests: randomly assign cutoff location

Donut-hole specification:

- Exclude observations very close to cutoff
- Tests whether results driven by a few observations at boundary
- Also useful if worried about measurement error in Z_i

Alternative specifications:

- Different polynomial orders
- Different bandwidths
- With/without covariates

External validity: The local nature of RDD

Key limitation: RDD identifies $\tau_{RDD} = \mathbb{E}[Y_i(1) - Y_i(0)|Z_i = z^*]$ **-** a **very local** parameter

When is this a problem?

- Treatment effects are heterogeneous (marginal vs. inframarginal individuals)
- Policy question is about broader population, not just at cutoff
- Example: Class size effect at 41 vs. 40 may not apply to 30 vs. 20

External validity: The local nature of RDD

Key limitation: RDD identifies $\tau_{RDD} = \mathbb{E}[Y_i(1) - Y_i(0)|Z_i = z^*]$ **- a very local** parameter

When is this a problem?

- Treatment effects are heterogeneous (marginal vs. inframarginal individuals)
- Policy question is about broader population, not just at cutoff
- Example: Class size effect at 41 vs. 40 may not apply to 30 vs. 20

When is it less of a concern?

- Marginal policy changes (moving the cutoff slightly)
- Effect at cutoff IS the policy-relevant parameter
- Using RDD to understand mechanisms, not for extrapolation

RDD estimation checklist

A credible RDD paper should present:

1. **Graphical evidence**: Plot outcome vs. running variable with discontinuity clearly visible (`rdplot`)
2. **Density test**: McCrary/`rddensity` test to rule out manipulation at cutoff
3. **Covariate balance**: Test for discontinuities in predetermined variables
4. **Main estimates**: Use local linear with MSE-optimal bandwidth (`rdrobust`)
   - For fuzzy RDD: report first stage (check F-stat > 10)
5. **Robustness**: Show sensitivity to bandwidth, polynomial order, covariates
6. **Placebo tests**: Check for effects at fake cutoffs, lagged outcomes, donut-hole

Outline

1. Motivation
2. Identification
3. Sharp vs. Fuzzy RDD
4. Estimation
5. Bandwidth selection
6. Threats to validity
7. Conclusion

Summary

Key takeaways:

- RDD exploits discontinuous changes in treatment at a threshold
- **Identification**: Relies on continuity of potential outcomes at cutoff
- **Sharp vs. Fuzzy**: Deterministic vs. probabilistic treatment assignment
- **Estimation**: Local linear regression with optimal bandwidth
- **Validity**: Test for manipulation, covariate balance, other discontinuities
- **External validity**: Effects are local to the cutoff