Data assimilation for wave propagation problems

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Overview

- Wave propagation (advection) equation
- DA for wave propagation
- Further work: Aims

Advection equation

- The system in 2 dimensions:

$$\begin{aligned}
&\frac{\partial v}{\partial t} = c_1 \frac{\partial v}{\partial x} + c_2 \frac{\partial v}{\partial y}, \quad (x, y, t) \in \mathbb{T}^2 \times [0, T] \\
&v(x, y, 0) = u(x, y), \\
&v(x, 0, t) = v(x, 1, t), \\
&v(0, y, t) = v(1, y, t)
\end{aligned}$$

- Solution is a wave:

$$v(x, y, t) = u(x + c_1 t, y + c_2 t).$$

Visualised advection solution

- Visualisation of solution for:

$$\begin{aligned}
u(x, y) &= \nabla^{\perp}\phi(x, y) \\
\phi(x, y) &= \sin(2\pi x)\cos(2\pi y)
\end{aligned}$$

- Play movie

Set-up

- Idea: use Bayes' Rule to infer on u given v
- Let prior on u be $\mu_0 = \mathcal{N}(m_0, (-\Delta)^{-\alpha})$
- Observations are: $y_{j,k} = g(v(x_j, t_k)) + \eta_{j,k},\ \eta_{j,k} \sim \mathcal{N}(0, \gamma^2)$
- Abstractly, $y = G(u) + \eta,\ \eta \sim \mathcal{N}(0, \Gamma)$
- Aim: to get information from posterior μ where:

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mu_0} \propto \exp\left(-\frac{1}{2}\,|G(u) - y|_\Gamma^2\right)$$

Current progress

- Working on implementing a random walk Metropolis algorithm:
  - Sample ξ from prior:

$$\xi = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{\gamma_k}{(4\pi^2|k|^2)^{\alpha/2}} \exp(2\pi i k \cdot x)$$

  - Let u be current state and make a proposal:

$$w = (1 - \beta^2)^{1/2} u + \beta\xi, \quad \text{some } \beta \in (0, 1)$$

  - Accept w with probability

$$\alpha(u, w) = \min\left\{1, \exp\left(\frac{1}{2}\,|G(u) - y|_B^2 - \frac{1}{2}\,|G(w) - y|_B^2\right)\right\}$$

Further work and aims

- Lagrangian DA
- Eulerian/Lagrangian DA for shallow-water equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= Ju - \nabla h \\ \frac{\partial h}{\partial t} &= -\nabla \cdot u \end{aligned}$$

  J is skew-symmetric
- Possibly implement particle filters for the above