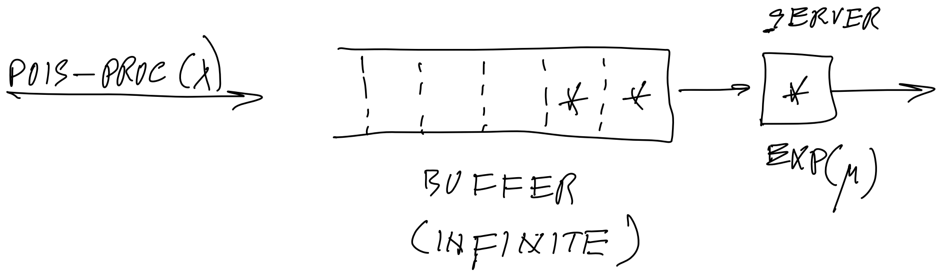
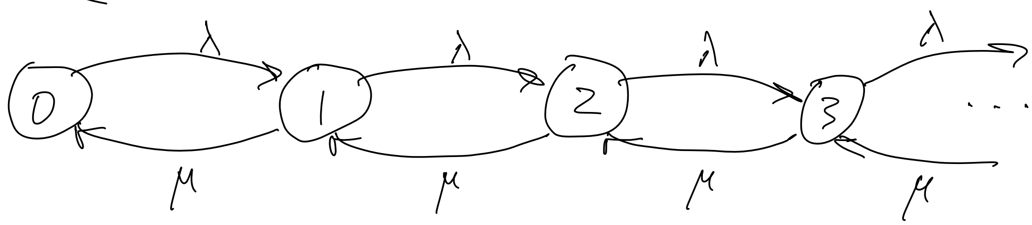
QUEUEING SYSTEMS

M/M/1 QUEUE



$X(t)$ = # OF CUST. AT $t \geq 0$.

$(X(t), t \geq 0)$ CTMC



IRREDUCIBLE.

STAT. DISTR. $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ EXISTS?

IN QUEUEING THEORY

THE EXISTENCE OF STAT. DISTR. IS OFTEN CALLED STOCHASTIC STABILITY

$\rho = \lambda/\mu$ = SYSTEM LOAD = $\frac{\text{RATE OF WORK ARRIVAL}}{\text{SYSTEM CAPACITY}}$

$$\pi_0 \cdot \lambda = \pi_1 \cdot \mu \quad \Rightarrow \quad \pi_1 = \rho \pi_0$$
$$\pi_1 \cdot \lambda = \pi_2 \cdot \mu \quad \Rightarrow \quad \pi_2 = \rho \pi_1 = \rho^2 \pi_0$$
$$\pi_2 \cdot \lambda = \pi_3 \cdot \mu \quad \Rightarrow \quad \pi_3 = \rho \pi_2 = \rho^3 \pi_0$$
$$\cdots \qquad \cdots$$
$$\pi_{i-1} \cdot \lambda = \pi_i \cdot \mu \quad \Rightarrow \quad \pi_i = \rho^i \pi_0$$

CASE $\rho \geq 1 \Leftrightarrow \lambda \geq \mu$: NO STAT. DISTR.

u = SERVER UTILIZATION = PROB. SERVER IS BUSY = 1

CASE $\rho < 1 \Rightarrow \lambda < \mu$: $\sum_i \pi_i = \sum_{i=0}^{\infty} \rho^i \pi_0 = \pi_0 \frac{1}{1-\rho} = 1$

$$\pi_0 = 1 - \rho$$

$\pi_0 = 1 - \rho$ PROB. THAT THE SERVER IS IDLE

LOAD = $\frac{\lambda}{\mu} = \rho = 1 - \pi_0$ = PROB. THAT THE SERVER IS BUSY
= SERVER UTILIZATION = u

$$\boxed{\rho < 1: \qquad \pi_i = (1-\rho)\rho^i, \quad i = 0, 1, 2, \ldots}$$

GEOMETRIC DISTRIBUTION

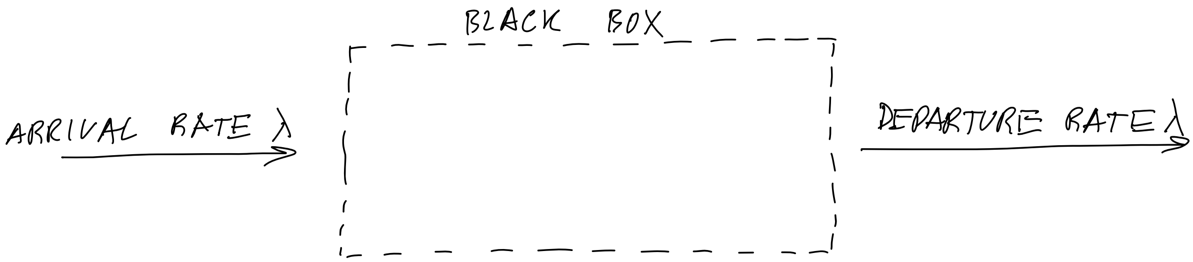
LONG-RUN AVERAGE # OF CUSTOMERS IN THE SYSTEM:

$$\ell = \sum_{i=0}^{\infty} i \pi_i = \sum_{i=0}^{\infty} i (1-\rho) \rho^i = \frac{\rho}{1-\rho}$$

(SEE GENERATING FUNCTIONS LECT.)

LITTLE'S LAW

(VERY GENERAL LAW, NOT JUST FOR M/M/1)



ASSUME ARR. RATE = DEP. RATE

$Q(t)$ = # OF CUST. IN BLACK BOX AT TIME t.

W_i = TIME i-TH ARRIVING CUST. SPENDS IN BLACK BOX.

$N(t)$ = # OF ARRIVALS IN $[0, t]$.

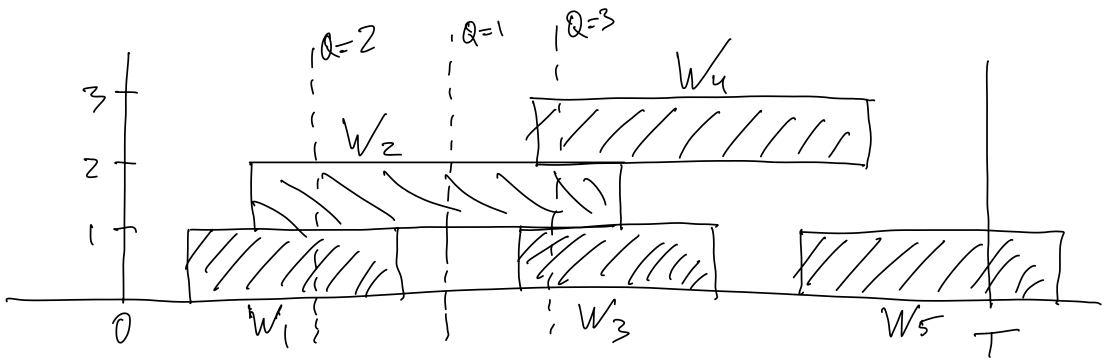
$$\ell = \lim_{T \to \infty} \frac{\int_0^T Q(t)\,dt}{T} = \text{LONG-RUN AVE. QUEUE LENGTH}$$

$$w = \lim_{T \to \infty} \frac{\sum_{i=1}^{N(T)} W_i}{N(T)} = \text{LONG-RUN AVE } \underbrace{\text{TIME IN BLACK BOX}}_{\text{"WAITING TIME"}},$$

$\Downarrow$

LITTLE'S LAW: $\ell = \lambda w$

LITTLE'S LAW PROOF IDEA



$S(T)$ = TOTAL AREA OF RECTANGLES IN $[0,T]$ =

$$= \int_0^T Q(t)\,dt \approx \sum_{i=1}^{N(T)} W_i$$

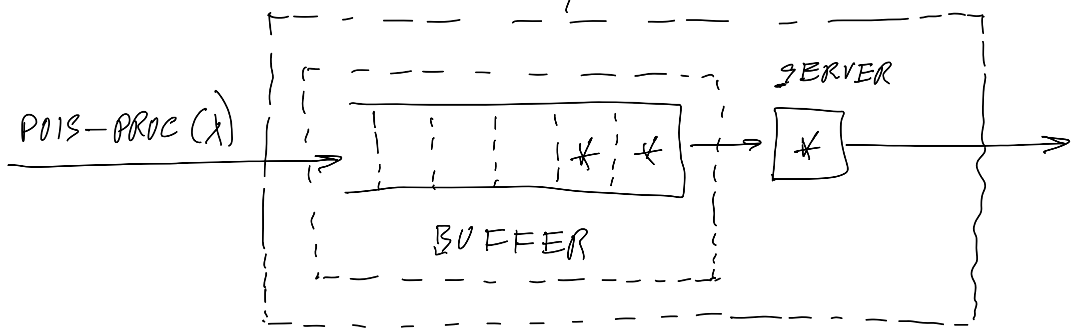
$$\lambda = \lim_{T\to\infty} \frac{N(T)}{T}, \quad w = \lim_{T\to\infty} \frac{S(T)}{N(T)}, \quad \ell = \lim_{T\to\infty} \frac{S(T)}{T}$$

$$\ell = \lim \frac{S(T)}{T} = \lim \left[\frac{N(T)}{T} \cdot \frac{S(T)}{N(T)} \right] = \lambda w \quad \blacksquare$$

LET'S APPLY LITTLE'S LAW TO M/M/1.

$\rho < 1$: $\quad \ell = \dfrac{\rho}{1-\rho} \Rightarrow w = \dfrac{\ell}{\lambda} = \dfrac{1/\mu}{1-\rho} = \dfrac{m}{1-\rho}$

$m = \dfrac{1}{\mu}$ = MEAN SERVICE TIME



w_q = MEAN TIME SPENT IN THE BUFFER

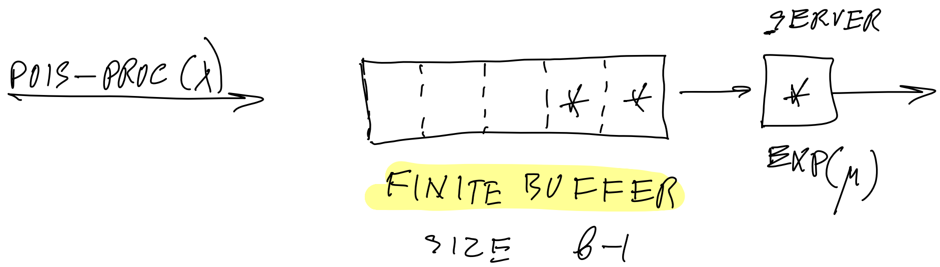
$$w_q = w - m = \frac{m}{1-\rho} - m = \frac{m\rho}{1-\rho} = \frac{\lambda/\mu^2}{1-\rho}$$

ℓ_q = MEAN # OF CUST IN THE BUFFER

$$\ell_q = \lambda w_q = \frac{\rho^2}{1-\rho}$$

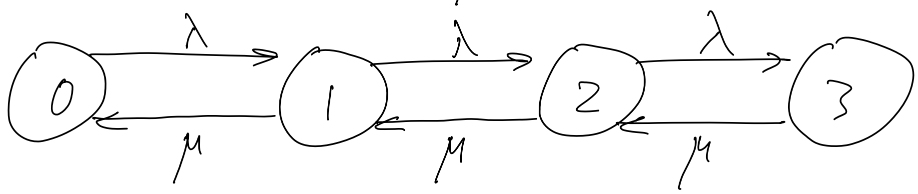
$$\ell_q = \ell - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho}$$

M/M/1/b SYSTEM



b = MAX TOTAL # OF CUST.

FOR EXAMPLE: $b=3$, $\lambda = \frac{1}{2}$, $\mu = 1$. $\rho = \frac{1}{2}$



$$\pi_i = \pi_0 \rho^i, \qquad i = 0, 1, 2, \ldots, b$$

$$\pi_0 = \frac{1}{\sum_{i=0}^{b} \rho^i} = \frac{1-\rho}{1-\rho^{b+1}} = \frac{8}{15}$$

$$\pi_1 = \frac{4}{15}, \quad \pi_2 = \frac{2}{15}, \quad \pi_3 = \frac{1}{15}$$

AVE. # IN THE SYSTEM $= \ell = \sum_{i=1}^{b} i\pi_i = \frac{11}{15}$

AVE # IN THE BUFFER $= \ell_q = \sum_{i=1}^{b} (i-1)\pi_i = \ell - (1-\pi_0) = \frac{4}{15}$

PROB. OF A CUSTOMER BEING BLOCKED $= \pi_b = \frac{1}{15}$

PROB. OF A CUSTOMER BEING ADMITTED $= 1-\pi_b = \frac{14}{15}$

$\lambda_{eff} = \lambda(1-\pi_b) = \frac{7}{15}$ = RATE OF ADMISSIONS

w_q = AVE. WAITING TIME AMONG THOSE CUST. THAT ARE ADMITTED

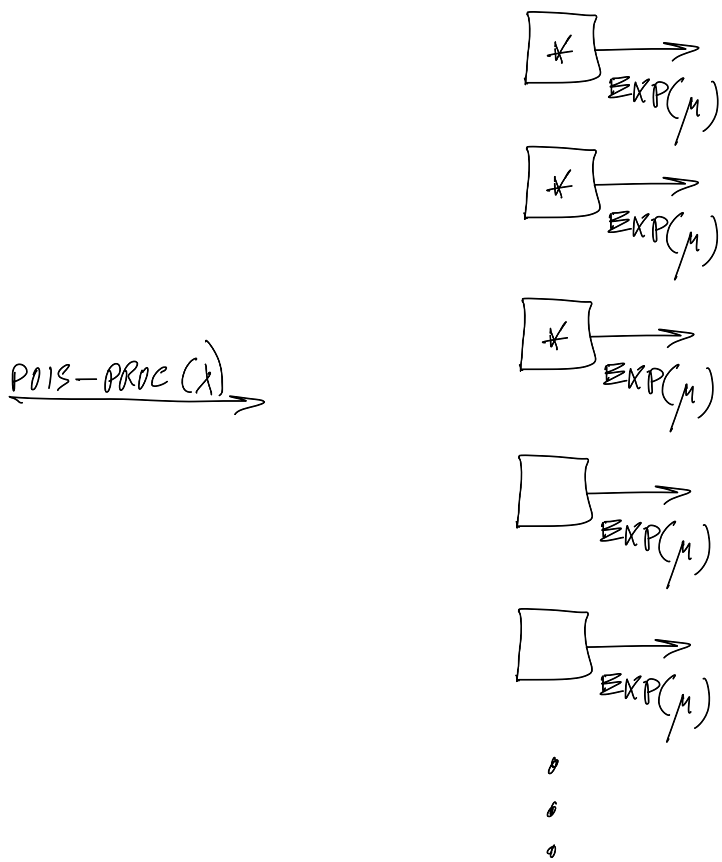
$\lambda_{eff}\, w_q = \ell_q \implies w_q = \frac{4}{7}$

M/M/∞ SYSTEM

INFINITE # OF IDENTICAL SERVERS,
EACH WITH INDEP. SERVICE TIME $\sim EXP(\mu)$.

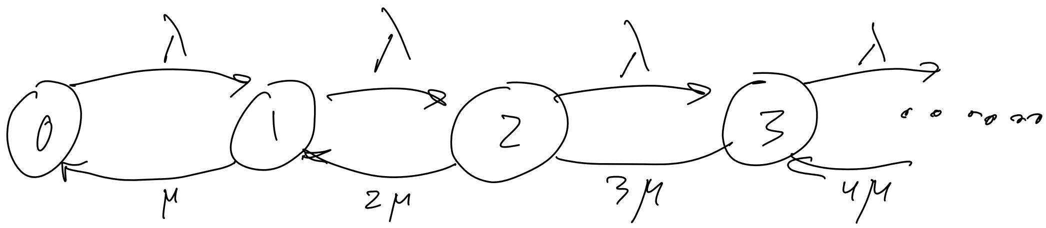
ARRIVALS : POIS-PROC (λ).

NO BUFFER — NOBODY WAITS



$X(t) = $ # OF CUST. AT $t \geq 0$, $(X(t), t \geq 0)$ — CTMC

STATE SPACE $S = \{0, 1, 2, \dots\}$



STAT. DISTR. π EXISTS ? (STABLE ?)

$$\frac{\lambda}{\mu} = \nu$$

$$\pi_0 \cdot \lambda = \pi_1 \cdot \mu \implies \pi_1 = \nu \pi_0 = \frac{1}{1!} \nu \pi_0$$

$$\pi_1 \cdot \lambda = \pi_2 \cdot 2\mu \implies \pi_2 = \frac{1}{2} \nu \pi_1 = \frac{1}{2!} \nu^2 \pi_0$$

$$\pi_2 \cdot \lambda = \pi_3 \cdot 3\mu \implies \pi_3 = \frac{1}{3} \nu \pi_2 = \frac{1}{3!} \nu^3 \pi_0$$

$$\pi_n = \frac{\nu^n}{n!} \pi_0$$

$$\sum_n \pi_n = 1 \implies \pi_0 \sum_{n=0}^{\infty} \frac{\nu^n}{n!} = \pi_0 e^{\nu} \implies \pi_0 = e^{-\nu}$$

$$\boxed{\pi_n = \frac{\nu^n}{n!} e^{-\nu}, \quad n = 0, 1, 2, \dots}$$

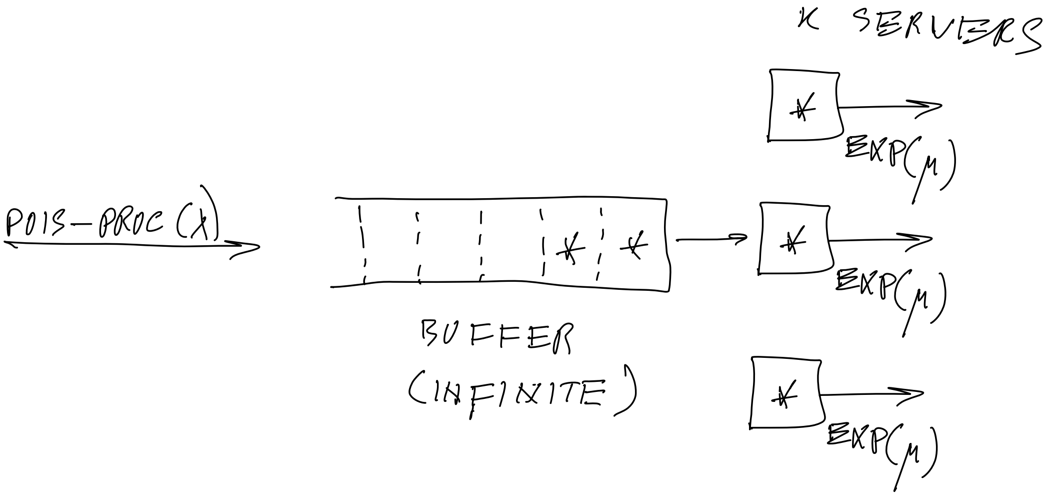
POISSON (λ/μ)

|| M/M/∞ SYSTEM IS STABLE FOR ANY $\frac{\lambda}{\mu}$.

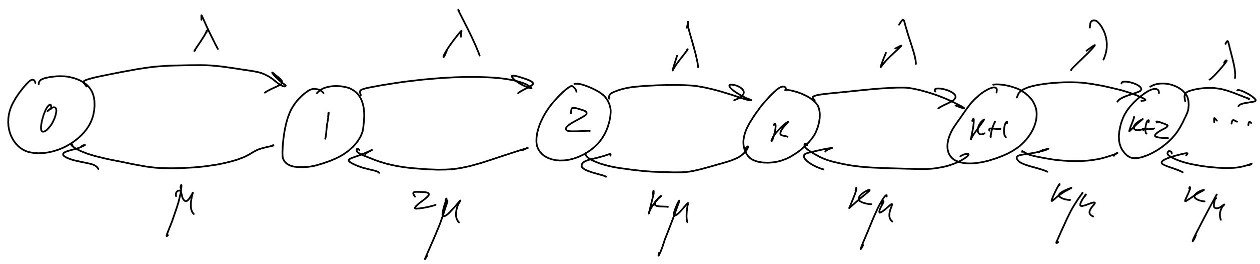
ℓ = AVE. # IN THE SYSTEM $= \frac{\lambda}{\mu}$

SAME FOLLOWS FROM LITTLE'S LAW: $\lambda \cdot \frac{1}{\mu} = \ell$

M/M/K SYSTEM



TRANSITION RATE DIAGRAM (FOR K=3):



$$\rho = \frac{\lambda}{K\mu} \qquad \text{LOAD}$$

$$\pi_1 = \frac{\lambda}{\mu}\pi_0$$

$$\pi_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 \pi_0$$

$$\cdots\cdots$$

$$\pi_K = \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \pi_0$$

$$\pi_{K+1} = \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \frac{\lambda}{K\mu}\pi_0$$

$$\pi_{K+2} = \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \left(\frac{\lambda}{K\mu}\right)^2 \pi_0$$

$$\cdots\cdots\cdots$$

$$\sum_n \pi_n = 1 \implies \pi_0\left[\sum_{n=0}^{K-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=K}^{\infty}\frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \rho^{n-K}\right] = 1$$

STABILITY CONDITION: $\rho = \frac{\lambda}{K\mu} < 1$

THEN

$$\pi_0 = \left[\sum_{n=0}^{K-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \frac{1}{1-\rho}\right]^{-1}$$

$$\boxed{\mathbb{P}\{\text{WAITING}\} = \mathbb{P}\{\text{ALL SERVERS ARE BUSY}\} = \sum_{n=K}^{\infty}\pi_n = \pi_K\frac{1}{1-\rho} = \frac{1}{1-\rho}\frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \pi_0}$$

ERLANG-C FORMULA

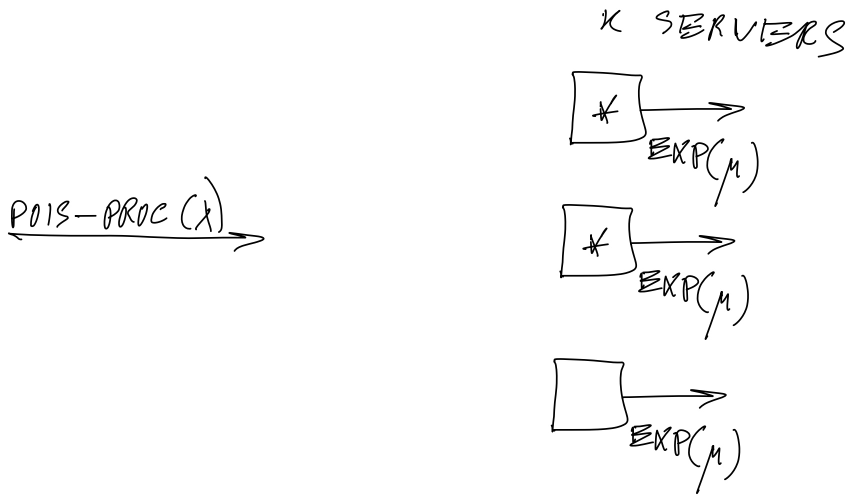
AVE # IN THE BUFFER $= \ell_q = \sum_{i=1}^{\infty} i \cdot \pi_{K+i} =$

$$= \sum_{i=1}^{\infty} i \, \pi_K \rho^i = \frac{1}{1-\rho} \pi_K \sum_{i=1}^{\infty} i(1-\rho)\rho^i = \frac{1}{1-\rho} \pi_K \frac{\rho}{1-\rho} =$$

$$= \pi_K \frac{\rho}{(1-\rho)^2}$$

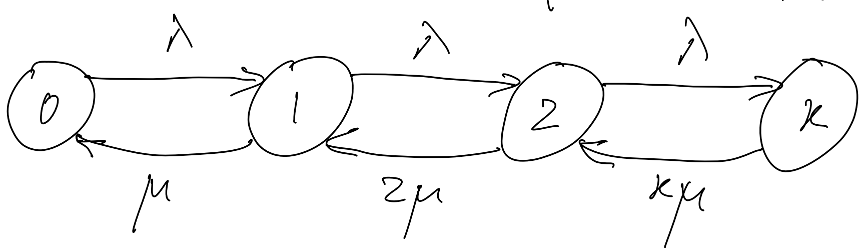
AVE TIME IN THE BUFFER: $w_q = \frac{\ell_q}{\lambda} = \pi_K \frac{\rho}{\lambda(1-\rho)^2}$

M/M/K/K SYSTEM



NO BUFFER: ARRIVING CUST. EITHER GOES TO AN AVAILABLE SERVER OR IS BLOCKED (LOST).

TRANSITION RATE DIAGRAM (FOR $K=3$):



$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 \quad , \qquad n = 0, 1, \dots, K$$

$$\pi_0 = \left[\sum_{n=0}^{K} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$$

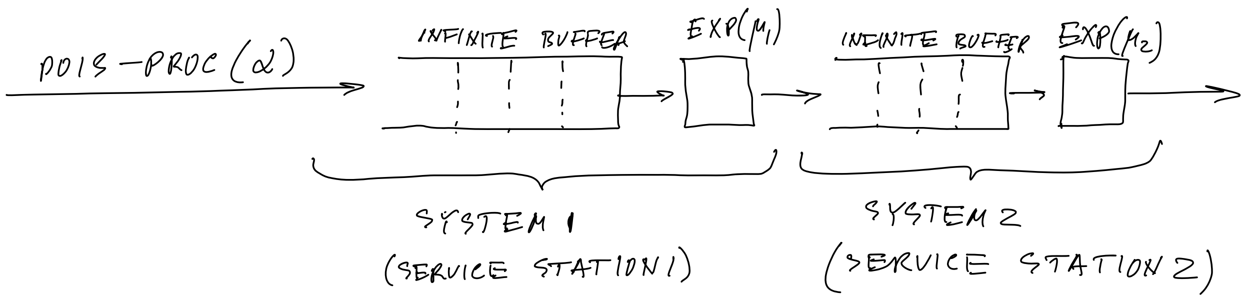
$$\boxed{\mathbb{P}\{\text{BLOCKING}\} = \pi_K = \frac{\frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K}{\sum_{n=0}^{K} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n}}$$

ERLANG-B FORMULA

QUEUEING NETWORKS.

SPECIFICALLY, JACKSON NETWORKS.

TWO QUEUES IN TANDEM

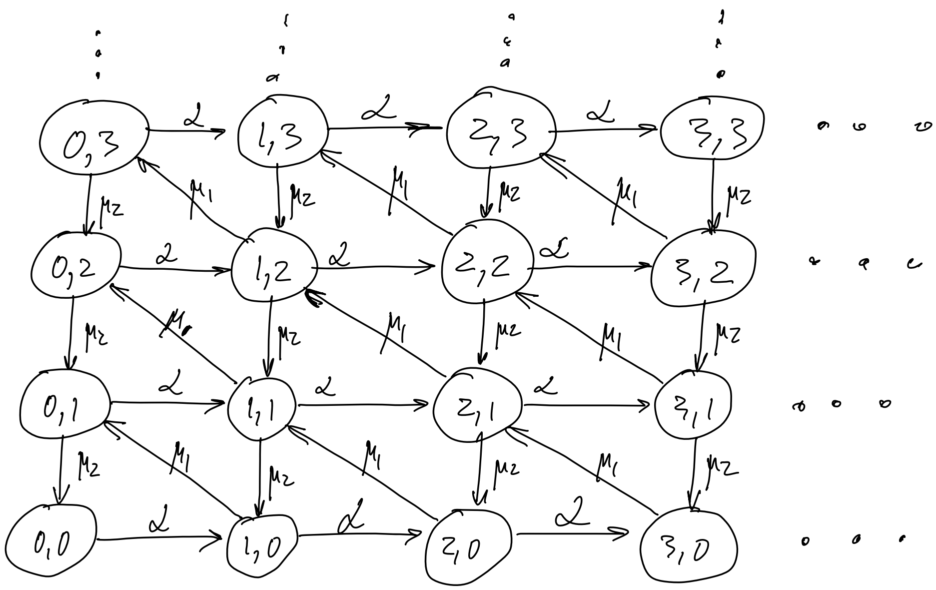


PICK SOME PARAMETER VALUES: $\alpha = 2$, $\mu_1 = \frac{1}{0.45} > 2$, $\mu_2 = \frac{1}{0.40} > 2$.

$X_i(t)$ = # OF CUST. IN STATION i

$X(t) = (X_1(t), X_2(t))$ IS CTMC.

STATE SPACE $S = \{(n_1, n_2) \mid n_i \in \{0, 1, 2, \ldots\}\}$



IS THIS SYSTEM STABLE?
(DOES THE STAT. DISTR. EXIST?)

CONSIDER CASE: $\rho_1 = \frac{\lambda}{\mu_1} < 1$, $\rho_2 = \frac{\lambda}{\mu_2} < 1$.

DENOTE: $\pi_n^{(1)} = (1-\rho_1)\rho_1^n$, $\pi_n^{(2)} = (1-\rho_2)\rho_2^n$, $n = 0, 1, 2, \ldots$

STATION 1 IS M/M/1 $\Rightarrow$ $\{\pi_n^{(1)}\}$ IS THE STAT. DISTR. OF X_1

STATION 2 IS NOT M/M/1; BUT IF WE "PRETEND" THAT IT IS, THEN $\{\pi_n^{(2)}\}$ WOULD BE THE STAT. DISTR. OF X_2

DENOTE: $\{\pi_{n_1, n_2}\}$ THE STAT. DISTR. OF (X_1, X_2) (IF IT EXISTS)

THEOREM. ASSUME $\rho_1 < 1$, $\rho_2 < 1$.
THEN $\pi_{n_1, n_2} = \pi_{n_1}^{(1)} \cdot \pi_{n_2}^{(2)} = (1-\rho_1)\rho_1^{n_1} (1-\rho_2)\rho_2^{n_2}$

THIS IMPLIES THAT:

- STAT. DISTR. OF X_2 IS INDEED $\{\pi_n^{(2)}\}$
- STAT. DISTR. OF (X_1, X_2) IS SUCH THAT X_1 AND X_2 ARE INDEPENDENT

THIS FACT IS TOTALLY AMAZING!

TO PROVE THE THEOREM, NEED TO CHECK THAT $\{\pi_{n_1,n_2}\}$ IS INDEED THE STATIONARY DISTRIBUTION.

OBVIOUSLY, $$\sum' \pi_{n_1,n_2} = \sum_{n_1}' \sum_{n_2}' \pi^{(1)}_{n_1} \cdot \pi^{(2)}_{n_2} = \sum_{n_1}' \pi^{(1)}_{n_1} \cdot \sum_{n_2}' \pi^{(2)}_{n_2} = 1$$

LET'S CHECK BALANCE EQ. FOR ONE OF THE STATES, SAY (2,1).

$$\pi_{2,1} \cdot (\alpha + \mu_1 + \mu_2) \overset{?}{=} \pi_{1,1} \cdot \alpha + \pi_{3,0} \cdot \mu_1 + \pi_{2,2} \cdot \mu_2$$

$$\rho_1^2 \cdot \rho_2 (\alpha + \mu_1 + \mu_2) = \rho_1 \rho_2 \alpha + \rho_1^3 \mu_1 + \rho_1^2 \rho_2^2 \mu_2$$

$$\rho_1^2 \rho_2^2 \cdot \mu_2 + \rho_1 \rho_2 \alpha + \rho_1^3 \mu_1$$

TRUE

AVE. # OF CUST. IN STATION i:

$$\ell^{(i)} = \sum_{n_i}' n_i \cdot \pi^{(i)}_{n_i} = \sum_n n (1-\rho_i)\rho_i^n = \frac{\rho_i}{1-\rho_i}$$

AVE WAITING TIME IN STATION i:

$$w^{(i)} = \frac{1}{\alpha} \ell^{(i)} = \frac{1/\mu_i}{1-\rho_i}$$

AVE WAITING TIME IN SYSTEM:

$$w = w^{(1)} + w^{(2)} = \frac{\ell^{(1)} + \ell^{(2)}}{\alpha}$$

GENERAL JACKSON NETWORK

I STATIONS;

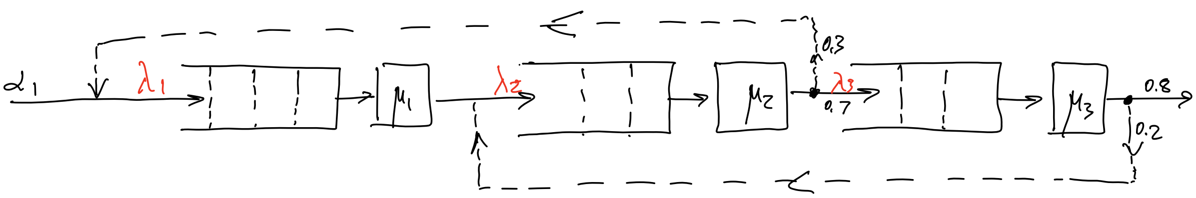
EACH STATION i HAS A SINGLE SERVER WITH INDEPENDENT, EXP(μ_i) SERVICE TIMES;

EACH STATION HAS INFINITE BUFFER;

STATION i RECEIVES EXOGENOUS ARRIVALS AS POIS-PROC(α_i);

AFTER LEAVING STATION i, A CUST. IS SENT TO STATION j WITH PROB. r_{ij} ($i=j$ IS OK), AND LEAVES NETWORK WITH PROB. $1-\sum_j r_{ij}$.

EXAMPLE: $I=3$, $r_{12}=1$, $r_{21}=0.3$, $r_{23}=0.7$, $r_{32}=0.2$,
$\alpha_1>0$, $\alpha_2=\alpha_3=0$.



λ_i = AVE. RATE OF ALL ARRIVALS INTO STATION i

$$\boxed{\lambda_i = \alpha_i + \sum_j \lambda_j r_{ji}, \quad \forall i}$$

TRAFFIC BALANCE EQUATIONS

NEED ALL OF THEM, FOR ALL i

$$\begin{cases} \lambda_1 = \alpha_1 + \lambda_2 \cdot 0.3 \\ \lambda_2 = \lambda_1 + \lambda_3 \cdot 0.2 \\ \lambda_3 = \lambda_2 \cdot 0.7 \end{cases} \Rightarrow \lambda_1, \lambda_2, \lambda_3$$

DENOTE: $\rho_i = \dfrac{\lambda_i}{\mu_i}$ (NOMINAL LOAD OF STATION i)

$$\pi_n^{(i)} = (1-\rho_i)\rho_i^n, \quad n = 0, 1, 2, \ldots$$

$$X_i(t) = \# \text{ OF CUST. IN STATION } i$$

THEOREM: SUPPOSE $\rho_i < 1$, $\forall i \in I$.
THEN $(X_i(t), i \in I)$ IS A CTMC,
AND ITS STATIONARY DISTR. IS:

$$\{\pi_{n_1, \ldots, n_I}\},$$

$$\pi_{n_1, \ldots, n_I} = \prod_{i=1}^{I} \pi_{n_i}^{(i)}$$ (PRODUCT FORM)

THIS IMPLIES THAT:

- STAT. DISTR. OF EACH X_i IS $\{\pi_n^{(i)}\}$
- STAT. DISTR. OF $(X_1, X_2, \ldots, X_I)$ IS SUCH THAT ALL X_i ARE MUTUALLY INDEPENDENT

EVEN MORE AMAZING!!

AVE. # OF CUST. IN STATION $i = \ell^{(i)} = \dfrac{\rho_i}{1-\rho_i}$

AVE. WAITING TIME IN STATION $i = w^{(i)} = \dfrac{\ell^{(i)}}{\lambda_i} = \dfrac{1/\mu_i}{1-\rho_i}$

QUESTION: WHAT IS THE AVE TIME A CUST. SPENDS IN THE NETWORK ?

IS IT $w = \sum_i w^{(i)}$? NO

HAVE TO USE LITTLE'S LAW:

$$w = \frac{\sum_i \ell^{(i)}}{\sum_i \alpha_i}$$

IN OUR 3-STATION EXAMPLE:

$$w = \frac{\ell^{(1)} + \ell^{(2)} + \ell^{(3)}}{\alpha_1}$$