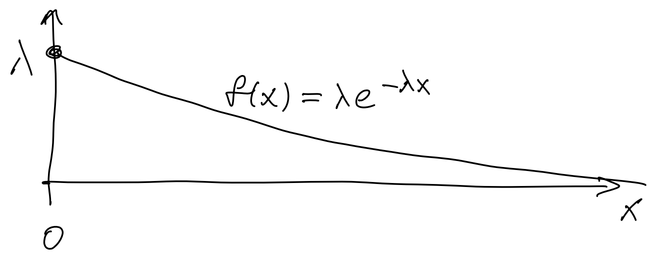
CONTINUOUS TIME MARKOV CHAINS

EXPONENTIAL DISTRIBUTION

R.V. $X \sim EXP(\lambda)$, $\lambda > 0$:

CDF $F(x) = \mathbb{P}\{X \le x\} = 1 - e^{-\lambda x}$, $x \ge 0$

PDF $f(x) = F'(x) = \lambda e^{-\lambda x}$, $x \ge 0$



$$\mathbb{E}X = \int_0^\infty x f(x)\,dx = \int_0^\infty x \lambda e^{-\lambda x} dx = \int_0^\infty x\, d\left[-e^{-\lambda x}\right] =$$

$$= \underbrace{-x e^{-\lambda x}\Big|_0^\infty}_{0} + \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$X \sim EXP(\lambda) \implies \mathbb{E}X = \frac{1}{\lambda}$$

$$\mathbb{E}X^2 \equiv \mathbb{E}\left[X^2\right] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \int_0^\infty x^2 d\left[-e^{-\lambda x}\right] =$$

$$= \int_0^\infty 2x e^{-\lambda x} dx = \frac{2}{\lambda}\int_0^\infty x \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$VAR\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{\lambda^2}$$

$$STD\, X = \sqrt{VAR\, X} = \frac{1}{\lambda}$$

MEMORYLESS PROPERTY OF EXP DIST.

$\| X \sim \mathrm{Exp}(\lambda)$: $\forall s \geq 0, \forall t \geq 0, \ \mathbb{P}\{X > s+t \mid X > s\} = \mathbb{P}\{X > t\}$

$$\mathbb{P}\{X > s+t \mid X > s\} = \frac{\mathbb{P}\{X > s+t, X > s\}}{\mathbb{P}\{X > s\}} = \frac{\mathbb{P}\{X > s+t\}}{\mathbb{P}\{X > s\}} =$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}\{X > t\}$$

REMARK: IN FACT, MEMORYLESS PROPERTY IMPLIES THAT $X \sim EXP(\cdot)$.

DENOTE: $g(t) = \mathbb{P}\{X > t\}$

IF MEMORYLESS PROPERTY HOLDS, THEN

$$\mathbb{P}\{X > s+t\} = \mathbb{P}\{X > s\} \cdot \mathbb{P}\{X > s+t \mid X > s\} =$$

$$= \mathbb{P}\{X > s\} \cdot \mathbb{P}\{X > t\}$$

$$\Downarrow$$

$$g(s+t) = g(s) \cdot g(t)$$

$$\Downarrow \quad \text{CAN BE PROVED}$$

$$g(t) = e^{ct}$$

$$\Downarrow$$

$$c = -\lambda < 0$$

IN OTHER WORDS, EXP($\cdot$) IS THE ONLY DISTRIBUTION CLASS WHICH HAS THE MEMORYLESS PROPERTY (λ MAY BE DIFFERENT, OF COURSE)

STRONGER FORM OF MEMORYLESS PROPERTY:

$X \sim EXP(\lambda)$, $\mathcal{S} \geq 0$ IS A R.V. INDEPENDENT OF X.

THEN $\mathbb{P}\{X > \mathcal{S}+t \mid X > \mathcal{S}\} = \mathbb{P}\{X > t\} = e^{-\lambda t}$.

INDEPENDENCE OF X AND $\mathcal{S}$ IS IMPORTANT

EXAMPLE. $X \sim EXP(\lambda)$; $\mathcal{S} = \frac{X}{2}$ ← NOT INDEP.

$$\mathbb{P}\{X > \mathcal{S}+t \mid X > \mathcal{S}\} = \mathbb{P}\{X > \tfrac{X}{2}+t \mid X > \tfrac{X}{2}\} =$$

$$= \mathbb{P}\{X > \tfrac{X}{2}+t\} = \mathbb{P}\{X > 2t\} = e^{-\lambda \cdot 2t}$$

$$\neq \mathbb{P}\{X > t\} = e^{-\lambda t}$$

$$\begin{cases} X_1 \sim EXP(\lambda_1) \quad , \quad X_2 \sim EXP(\lambda_2); \\ X_1, X_2 \text{ ARE INDEPENDENT} \end{cases}$$

$$X = \min\{X_1, X_2\} \sim \; ?$$

$$\mathbb{P}\{X > t\} = \mathbb{P}\{X_1 > t, X_2 > t\} =$$

$$= \mathbb{P}\{X_1 > t\} \cdot \mathbb{P}\{X_2 > t\} = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} =$$

$$= e^{-(\lambda_1 + \lambda_2) t}$$

$$\Downarrow$$

$$\min\{X_1, X_2\} \sim EXP(\lambda_1 + \lambda_2)$$

MORE GENERAL FORM:

$$\begin{cases} X_i \sim EXP(\lambda_i) \quad , \quad i = 1, \dots, n \\ X_i \text{ ARE INDEP} \end{cases}$$

$$X = \min_i X_i \sim EXP(\lambda_1 + \lambda_2 + \dots + \lambda_n)$$

$$\begin{cases} X_i \sim EXP(\lambda_i) \quad , \quad i = 1, 2 \\ X_1, X_2 \text{ ARE INDEP.} \end{cases}$$

$\min\{a, b\} + \max\{a, b\} = a + b$

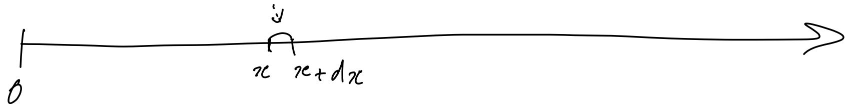
$$\mathbb{E} \max\{X_1, X_2\} = \mathbb{E}(X_1 + X_2) - \mathbb{E} \min\{X_1, X_2\} =$$

$$= \mathbb{E}X_1 + \mathbb{E}X_2 - \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

$$\begin{cases} X_1 \sim EXP(\lambda_1), \; X_2 \sim EXP(\lambda_2) \\ X_1, X_2 \text{ ARE INDEP} \end{cases}$$

$$\mathbb{P}\{X_1 < X_2\} = \int_0^\infty \left[\lambda_1 e^{-\lambda_1 x} dx \cdot e^{-\lambda_2 x}\right] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

X_1 $\qquad \mathbb{P}\{X_1 \in [x, x+dx]\} = \lambda_1 e^{-\lambda_1 x} dx$



$$X_1 \sim EXP(1), \; X_2 \sim EXP(6); \; X_1, X_2 \text{ INDEP.}$$

$$\mathbb{P}\{X_1 < X_2\} = \frac{1}{1+6} = \frac{1}{7}$$

MORE GENERAL FORM:

$$\begin{cases} X_i \sim EXP(\lambda_i), \quad i = 1, 2, \dots, n \\ X_j \text{ ARE INDEP.} \end{cases}$$

$$\forall j: \quad \mathbb{P}\{X_j = \min_i X_i\} =$$

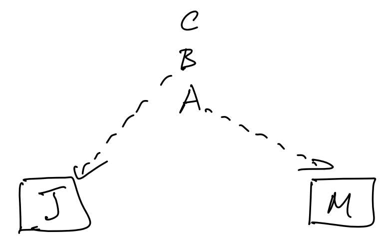
$$= \mathbb{P}\{X_j < X_i, \forall i \neq j\} = \frac{\lambda_j}{\sum_i \lambda_i}$$

PROOF. WITHOUT LOSS OF GENERALITY, LET $j = 1$.

$$\mathbb{P}\{X_1 < X_i, \forall i \neq 1\} =$$

$$= \int_0^\infty \left[\lambda_1 e^{-\lambda_1 x} dx \cdot e^{-\lambda_2 x} \cdot \ldots \cdot e^{-\lambda_n x}\right] = \frac{\lambda_1}{\sum_i \lambda_i} \qquad \square$$

EXAMPLE: CUSTOMERS A, B, C ARRIVE TOGETHER; J AND M ARE SERVERS



$X_1 \sim \text{EXP}\left(\frac{1}{6}\right)$ $\qquad$ $X_2 \sim \text{EXP}\left(\frac{1}{4}\right)$

X_1, X_2 ARE INDEP.

(i) $\mathbb{E}[\underbrace{\text{TIME WHEN A,B,C LEAVE THE SYSTEM}}_{W}] = ?$

W_1 = THE TIME WHEN THE FIRST OF THE CUSTOMERS LEAVES

$W_1 \sim \min\{X_1, X_2\} \sim \text{EXP}\left(\frac{1}{4} + \frac{1}{6}\right)$

$\mathbb{E}\, W_1 = \dfrac{1}{\frac{1}{4} + \frac{1}{6}}$

$\{W - W_1\} \sim \max\{X_1, X_2\}$

$\mathbb{E}\{W - W_1\} = 4 + 6 - \dfrac{1}{\frac{1}{4} + \frac{1}{6}}$

$\mathbb{E}\, W = \mathbb{E}\, W_1 + \mathbb{E}\{W - W_1\} = 4 + 6 = \mathbb{E}X_1 + \mathbb{E}X_2 =$

$= 10$

(ii) $\mathbb{P}\{$C FINISHES SERVICE BEFORE A$\} = ?$

IMPORTANT: NEED TO KNOW WHERE A GOES.

ASSUME: A GOES TO M

$$\mathbb{P}\{B \text{ FINISHES BEFORE } A\} = \frac{\frac{1}{6}}{\frac{1}{4}+\frac{1}{6}}$$

$$\mathbb{P}\{C \text{ BEFORE } A \mid B \text{ BEFORE } A\} = \frac{\frac{1}{6}}{\frac{1}{4}+\frac{1}{6}}$$

$$\mathbb{P}\{C \text{ BEFORE } A\} = \left[\frac{1/6}{1/4+1/6}\right]^2 = \left[\frac{4}{10}\right]^2$$

(iii) $\mathbb{P}\{C \text{ LEAVES LAST}\} = ?$

IN THIS CASE IT IS NOT IMPORTANT TO KNOW WHERE A GOES, BECAUSE THERE IS A SYMMETRY BETWEEN A AND B.

SO, THE ANSWER WILL BE SAME WHETHER A GOES TO M OR J.

ASSUME A GOES TO M.

$$\mathbb{P}\{C \text{ LEAVES LAST}\} =$$

$$= 1 - \mathbb{P}\{C \text{ NOT LAST}\} =$$

$$= 1 - \left\{\mathbb{P}\{C \text{ BEFORE } A\} + \mathbb{P}\{C \text{ BEFORE } B\}\right\} =$$

$$= 1 - \left(\frac{4}{10}\right)^2 - \left(\frac{6}{10}\right)^2$$

HERE WE USED THE FACT THAT

$$\{C \text{ NOT LAST}\} = \{C \text{ BEFORE } A\} \cup \{C \text{ BEFORE } B\}$$

NON-INTERSECTING