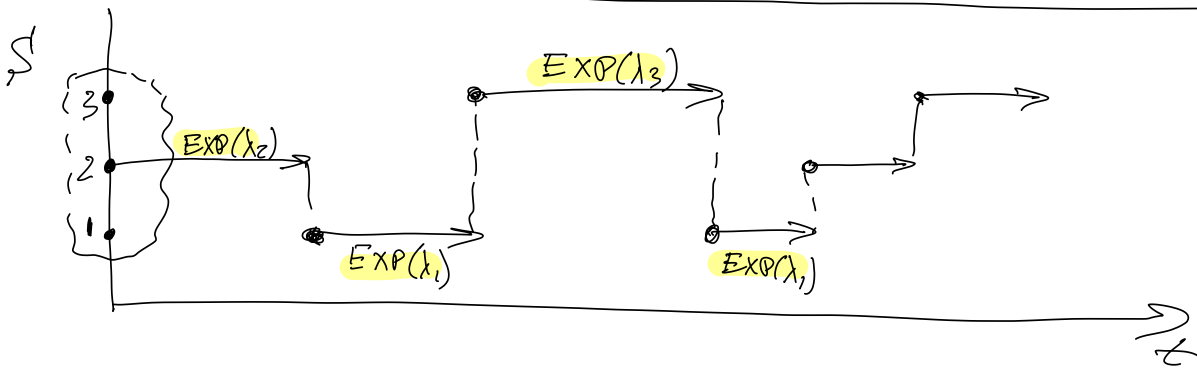
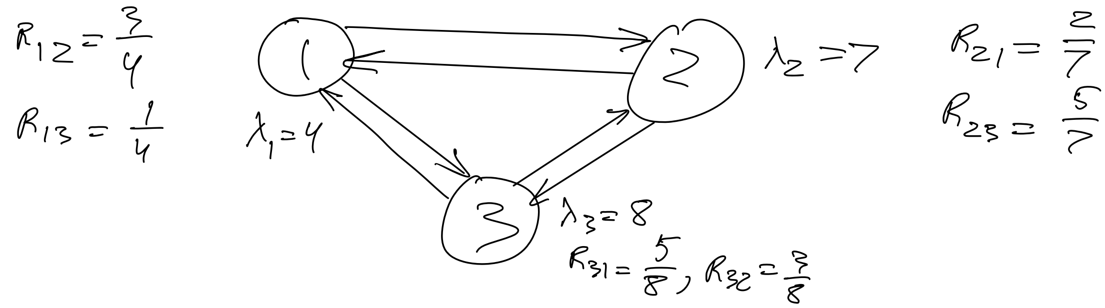
CONTINUOUS TIME MARKOV CHAINS (CTMC)





DEFINITION: CTMC $X(t)$, $t \geq 0$.

$X(t) \in S$ (FINITE OR COUNTABLE).

EXP(λ_i) IS THE DISTR OF TIME THE PROC. $X(\cdot)$ "SITS" IN STATE $i \in S$.

$\forall i$: $R_{ij} = \mathbb{P}\{$ JUMP TO j AFTER LEAVING $i\}$, $j \neq i$.

$$\sum_{j \neq i} R_{ij} = 1.$$

EQUIVALENT DEF. OF CTMC.

INSTEAD OF $\{\lambda_i\}$, $\{R_{ij}\}$, CTMC CAN BE EQUIVALENTLY DEFINED BY THE SET OF PARAMETERS

$$G_{ij} = \lambda_i R_{ij}, \quad j \neq i.$$

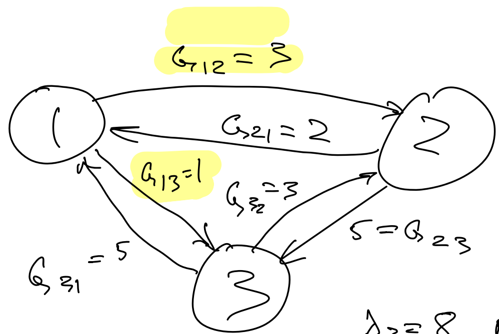
G_{ij} IS THE TRANSITION RATE FROM i TO j.

CTMC TRANSITION RATE DIAGRAM (GRAPH):

$\lambda_1 = 4$

$R_{12} = \frac{3}{4}$

$R_{13} = \frac{1}{4}$



$\lambda_2 = 7$

$R_{21} = \frac{2}{7}$

$R_{23} = \frac{5}{7}$

$\lambda_3 = 8$, $R_{31} = \frac{5}{8}$, $R_{32} = \frac{3}{8}$

WHEN PROCESS ENTERS STATE i, FOR EACH $j \neq i$, AN INDEPENDENT "ALARM CLOCK", $EXP(G_{ij})$, IS SET. IF (i,j) ALARM CLOCK RINGS FIRST, THE TRANSITION ("JUMP") FROM i TO j OCCURS.

EQUIVALENT DEFINITION, BECAUSE:

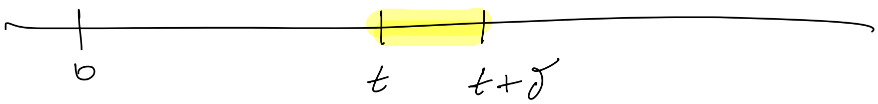
TIME TO "SIT" IN i IS $EXP(\lambda_i)$, $\lambda_i = \sum_j G_{ij}$, $\forall i$;

$$\mathbb{P}\{\text{TRANS. FROM } i \text{ TO } j \neq i\} = R_{ij} = \frac{G_{ij}}{\sum_k G_{ik}}.$$

INTERPRETATION OF TRANSITION RATES G_{ij}

SUPPOSE $X(t) = i$. RECALL $\lambda_i = \sum_{j \neq i}' G_{ij}$, $R_{ij} = \dfrac{G_{ij}}{\sum_{k \neq i}' G_{ik}}$

$\delta > 0$ IS SMALL



$\mathbb{P}\{\text{NO JUMP IN } [t, t+\delta]\} = e^{-\lambda_i \delta} = 1 - \lambda_i \delta + o(\delta)$

$\approx 1 - \left[\sum_{j \neq i}' G_{ij}\right] \delta$

$\mathbb{P}\{\text{JUMP IN } [t, t+\delta]\} \approx \left[\sum_{j \neq i}' G_{ij}\right] \delta$

$\mathbb{P}\{\text{JUMP } i \to j \text{ IN } [t, t+\delta]\} = \mathbb{P}\{\text{JUMP IN } [t, t+\delta]\} \cdot R_{ij} \approx G_{ij}\, \delta$

CTMC GENERATOR (MATRIX)

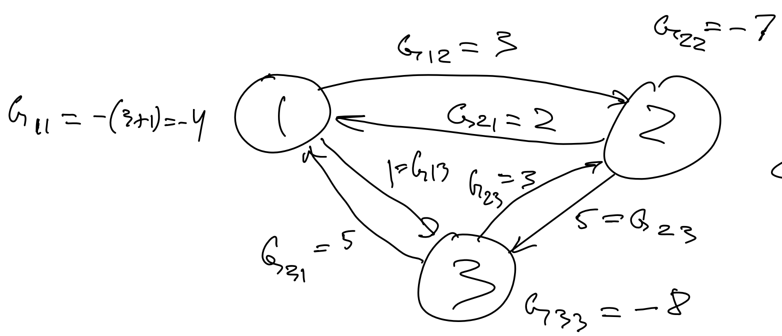
G_{ij} , $j \neq i$

$$G_{ii} = -\sum_{j \neq i} G_{ij} = -\lambda_i$$

$n = |S| =$ TOTAL NUMBER OF STATES (MAYBE $= \infty$)

GENERATOR (MATRIX):

$$G = (G_{ij}) = \underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{n} \Big\} n$$



$\Longleftrightarrow$

GENERATOR:

$$G = \begin{pmatrix} -4 & 3 & 1 \\ 2 & -7 & 5 \\ 5 & 3 & -8 \end{pmatrix}$$

EXAMPLES OF CTMC

EX 0. POISSON PROCESS OF RATE λ.

$$0 \xrightarrow{\lambda = G_{01}} 1 \xrightarrow{\lambda = G_{12}} 2 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 4 \xrightarrow{\lambda} \cdots$$

$$S = \{0, 1, 2, 3, \ldots\}.$$

EX 1.

$$\begin{tikzcd} \text{UP} \arrow[r, bend left, "\frac{1}{10}"] & \text{DOWN} \arrow[l, bend left, "1"] \end{tikzcd}$$

$$S = \{\text{UP}, \text{DOWN}\}$$

EX 2.

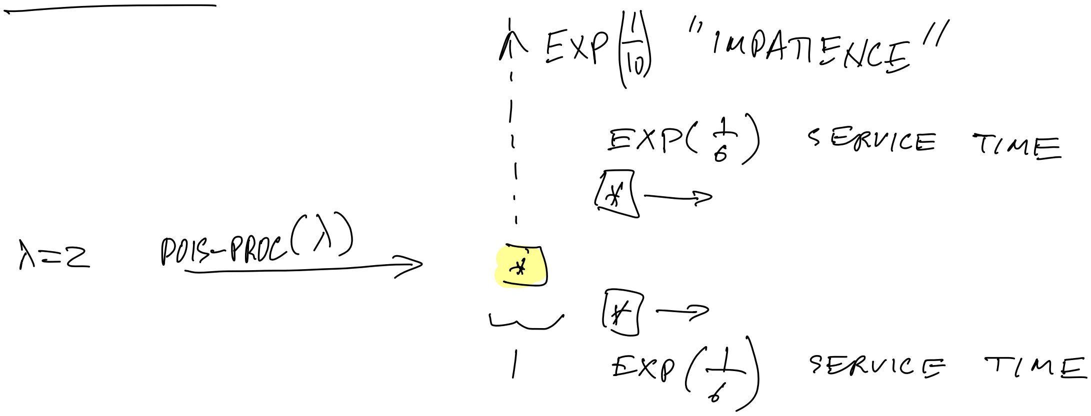
$$\boxed{A} \qquad \boxed{B}$$

$X(t)$ = # OF MACHINES WHICH ARE 'UP' AT TIME t.

$$S = \{0, 1, 2\}.$$

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "\frac{1}{10}"] \arrow[r, bend left, "1"] & 2 \arrow[l, bend left, "\frac{1}{10} + \frac{1}{10}"] \end{tikzcd}$$

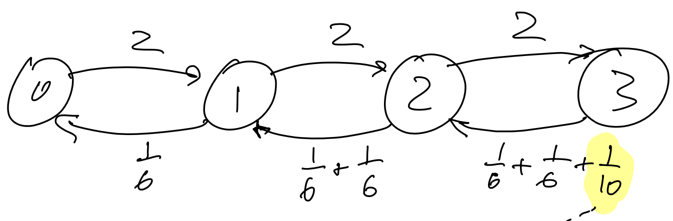
EX 3.



$X(t)$ = # OF CUST. IN THE SYSTEM

IS A CTMC.

$S = \{0, 1, 2, 3\}$



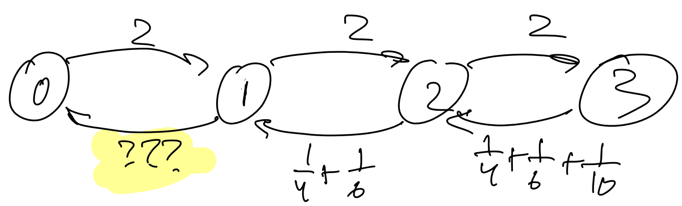
"IMPATIENCE" OF CUSTOMERS WAITING IN THE QUEUE.

EX 4 (EXTENSION OF EX 3).

SAME AS EX 3, EXCEPT TWO AGENTS ARE DIFFERENT

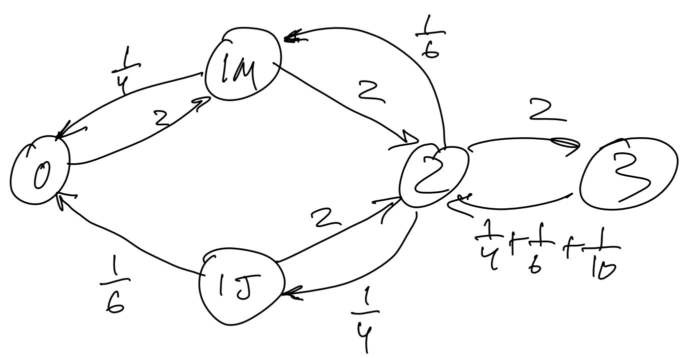
AGENT JOHN SERVICE TIME $\sim EXP(\frac{1}{6})$

— " — MARY — " — $\sim EXP(\frac{1}{4})$



DOES NOT WORK.

NOT A CTMC



CTMC

MARKOV PROPERTY

PROCESS $(X(t),\ t \geq 0)$, $X(t) \in S$,

IS A CTMC IF AND ONLY IF

$\forall\ t_0 < t_1 < \dots < t_{n-1} < t$, $\forall\ s \geq 0$,

$\forall\ i_0, i_1, \dots, i_{n-1}, i, j \in S$

$$\mathbb{P}\{X(t+s)=j \mid X(t_0)=i_0, \dots, X(t_{n-1})=i_{n-1}, X(t)=i\} =$$

$$= \mathbb{P}\{X(t+s)=j \mid X(t)=i\} \equiv P_{ij}(s)$$

$P(t) = (P_{ij}(t))$, $t \geq 0$ TRANSITION PROBABILITIES' MATRIX

$$P(t+s) = P(t)\,P(s)$$

$$\Updownarrow$$

$$P_{ij}(t+s) = \sum_k{}' P_{ik}(t)\,P_{kj}(s)$$

HOW TO COMPUTE $P(t)$?

$$P(0) = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$P'(t+s) = P(t) \cdot P'(s) \Leftarrow \frac{d}{ds} P(t+s) = \frac{d}{ds}\left[P(t)P(s)\right]$$

$$s \downarrow 0$$

$$P'(t) = P(t) \cdot P'(0)$$

$$P'(0) = \lim_{\delta \downarrow 0} \frac{P(\delta) - \overbrace{P(0)}^{I}}{\delta} = G$$

PROOF:

$$P(\delta) \approx \begin{array}{c} \begin{array}{ccccc} & & & j & \end{array} \\ \begin{pmatrix} 1+G_{11}\cdot\delta & & & & \\ & \ddots & & & \\ & & 1+G_{ii}\delta & G_{ij}\cdot\delta & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \end{array} \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}$$

$$P(\delta) - I \approx G \cdot \delta$$

$$P'(t) = P(t)\, G$$

SOLUTION FOR $P(t)$:

$$\begin{cases} P'(t) = P(t)G \\ P(0) = I \end{cases} \Longrightarrow P(t) = e^{Gt}$$

WHAT IS e^A? A IS A MATRIX

FOR A REAL NUMBER x:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots$$

FOR A SQUARE MATRIX A:

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

VERIFICATION THAT e^{Gt} IS INDEED THE SOLUTION OF $P'(t) = P(t)G$

$$P(t) = e^{Gt} = I + Gt + \frac{1}{2}G^2t^2 + \frac{1}{3!}G^3t^3 + \dots$$

$$P'(t) = G + G^2t + \frac{1}{2}G^3t^2 + \dots$$

$$= \underbrace{\left[I + Gt + \frac{1}{2}G^2t^2 + \dots\right]}_{P(t)} G =$$

$$= P(t)G$$

$\mu(t) = (\mu_i(t))$, ROW-VECTOR

$\mu_i(t) = \mathbb{P}\{X(t) = i\}$, $i \in S$

$\mu(t)$: DISTRIBUTION AT TIME $t \geq 0$.

$\mu(0)$: INITIAL DISTRIBUTION

$$\mu(t) = \mu(0)\, P(t)$$

$$\Updownarrow$$

$$\mu_i(t) = \sum_k{}' \mu_k(0) \cdot P_{ki}(t), \quad \forall i \in S$$

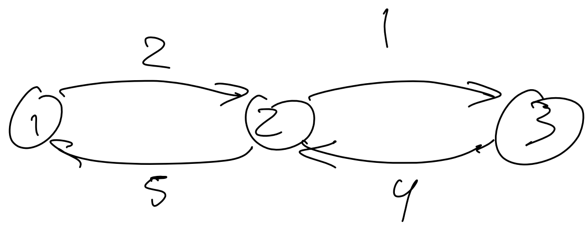
STATIONARY DISTRIBUTION

π IS STAT. DISTR. IF $\pi = \pi P(t)$, $\forall t \geq 0$

$$[\pi]' = \pi P'(0) \Rightarrow 0 = \pi G$$

$$\begin{cases} 0 = \pi G \\ \sum_i{}' \pi_i = 1 \end{cases}$$

EX.



$$G = \begin{pmatrix} -2 & 2 & 0 \\ 5 & -6 & 1 \\ 0 & 4 & -4 \end{pmatrix}$$

$$(0,0,0) = (\pi_1, \pi_2, \pi_3) \cdot G$$

$$\begin{cases} 0 = \pi_1(-2) + \pi_2 \cdot 5 + \pi_3 \cdot 0 \\ 0 = \pi_1 \cdot 2 + \pi_2(-6) + \pi_3 \cdot 4 \\ 0 = \pi_1 \cdot 0 + \pi_2 \cdot 1 + \pi_3 \cdot (-4) \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \pi$$

EQUATIONS FOR A STATIONARY DISTRIBUTION USUALLY EASIER TO WRITE AS BALANCE EQ'S.

(SEE ALSO NEXT PAGE)

$$\begin{cases} \pi_1 \cdot 2 = \pi_2 \cdot 5 \\ \pi_2(5+1) = \pi_1 \cdot 2 + \pi_3 \cdot 4 \\ \pi_3 \cdot 4 = \pi_2 \cdot 1 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

STATIONARY DISTRIBUTION EQUATIONS IN SCALAR FORM (BALANCE EQUATIONS)

$$0 = \pi G$$

$$0 = \sum_j \pi_j G_{ji}, \qquad \forall i$$

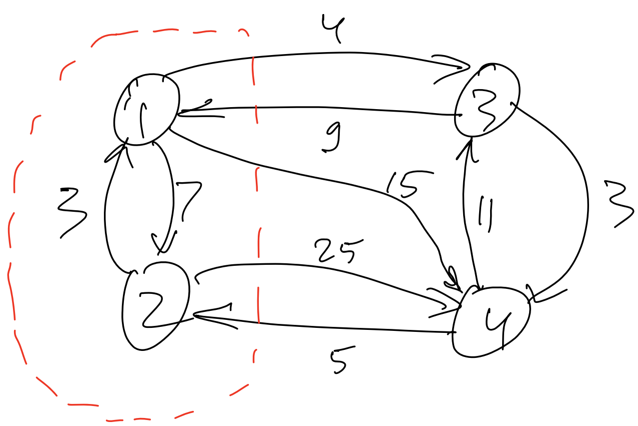
$$-\pi_i G_{ii} = \sum_{j \neq i} \pi_j G_{ji}$$

$$\pi_i \left[\sum_{j \neq i} G_{ij} \right] = \sum_{j \neq i} \pi_j G_{ji}$$

$$\sum_{j \neq i} \pi_i G_{ij} = \sum_{j \neq i} \pi_j G_{ji}, \quad \forall i \in S$$

PROB. FLOW BALANCE EQUATION FOR STATE i

BALANCE EQ. FOR A SUBSET (CUT):



$$\pi_1 \cdot 4 + \pi_1 \cdot 15 + \pi_2 \cdot 25 = \\ = \pi_3 \cdot 9 + \pi_4 \cdot 5$$

EXISTENCE AND UNIQUENESS OF STAT. DISTRIBUTIONS

CTMC $(X(t), t \geq 0)$, $X(t) \in S$ (STATE SPACE)

$X(\cdot)$ IS IRREDUCIBLE IF IT CAN GET FROM ANY STATE i TO ANY OTHER STATE j (ALL STATES COMMUNICATE)

STATE i IS POSITIVE RECURRENT IF THE AVERAGE TIME TO RETURN TO IT (AFTER LEAVING) IS FINITE

STATE i IS NULL RECURRENT IF THE PROBABILITY OF RETURNING TO IT (AFTER LEAVING) IS 1, BUT THE AVERAGE TIME TO RETURN IS INFINITE

STATE i IS TRANSIENT IF THE PROBABILITY OF RETURNING TO IT (AFTER LEAVING) IS LESS THAN 1.

IF $X(\cdot)$ IS IRREDUCIBLE, ALL STATES HAVE THE SAME RECURRENCE STATUS. IN PARTICULAR, IF ONE STATE IS POSITIVE RECURRENT, THEN ALL STATES ARE POSITIVE RECURRENT

$X(\cdot)$ IS POSITIVE RECURRENT IF ALL STATES ARE POSITIVE RECURRENT

THERE IS NO SUCH THING AS PERIODICITY OR APERIODICITY FOR CONTINUOUS TIME MC

TH. CONSIDER CTMC $(X(t), t \geq 0)$.

(i) IF X IS IRREDUCIBLE, THEN AT MOST ONE STAT. DISTR. EXISTS.

(ii) IF X IS FINITE (I.E. $\mathcal{S}$ IS FINITE), THEN AT LEAST ONE STAT. DISTR. EXISTS.

(iii) SUPPOSE X IS IRREDUCIBLE. THEN

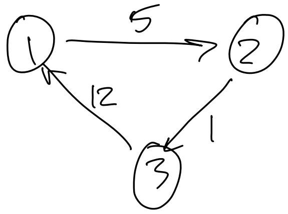
$$\{\text{STAT. DISTR. } \pi \text{ EXISTS}\} \iff$$
$$\iff \{\forall \mu(0),\ \mu(t) = \mu(0) P(t) \to \pi\} \iff$$
$$\iff X(\cdot) \text{ IS POSITIVE RECURRENT}$$

(iv) SUPPOSE X IS IRREDUCIBLE AND STAT. DISTR. π EXISTS. THEN, $\forall \mu(0)$,

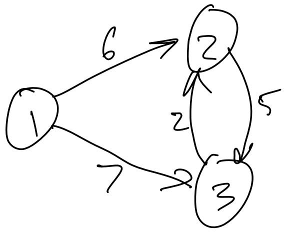
π_i = AVERAGE LONG-TERM FRACTION OF TIME $X(t)$ SPENDS IN STATE i,

I.E. $$\lim_{t\to\infty} \frac{1}{t} \int_0^t I\{X(u)=i\}\, du = \pi_i, \quad \forall i.$$

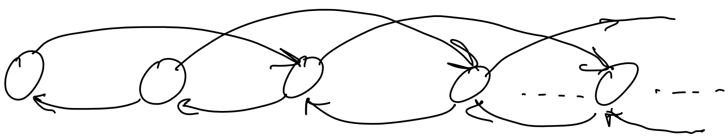
COROLLARY. IF CTMC IS IRREDUCIBLE AND FINITE, THEN IT HAS UNIQUE STAT. DISTR.
(AND THEN ALL THE NICE PROPERTIES IN (iii), (iv) IN ABOVE TH HOLD).



IRREDUCIBLE, FINITE
⇓
UNIQUE STAT. DISTR.

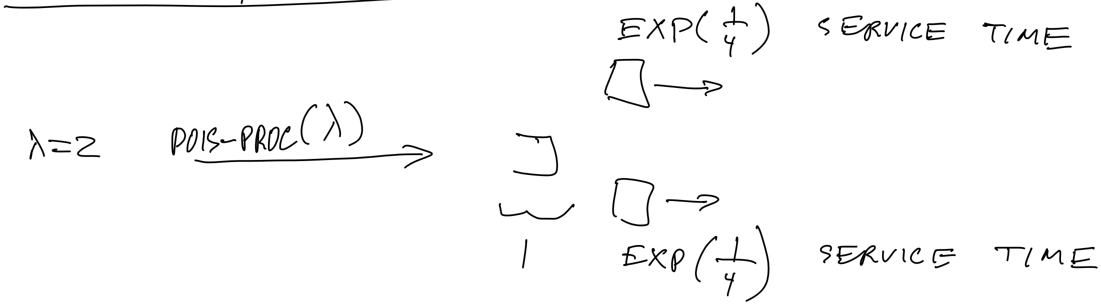


NOT IRREDUCIBLE, FINITE.
STILL HAS UNIQUE STAT. DISTR.
AND ALL THE NICE PROPERTIES
(iii), (iv) HOLD.
SO, IRREDUCIBILITY IS NOT
A NECESSARY CONDITION FOR
THE EXISTENCE/UNIQUENESS OF
STAT. DISTR.



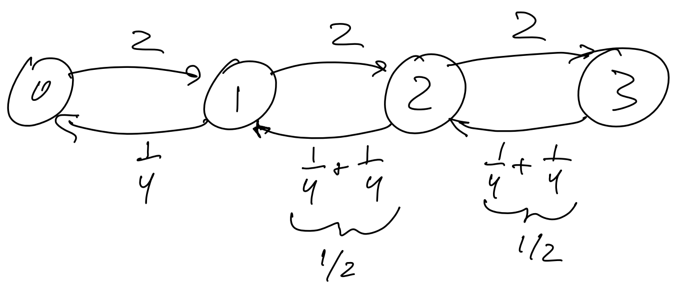
IRREDUCIBLE,
BUT NOT FINITE

EX. M/M/2/3 QUEUEING SYSTEM



$X(t)$ = # OF CUST. IN THE SYSTEM IS A CTMC.

$S = \{0, 1, 2, 3\}$



$(\pi_0, \pi_1, \pi_2, \pi_3)$ UNIQUE STAT. DISTR.

$$\begin{cases} \pi_0 \cdot 2 = \pi_1 \cdot \frac{1}{4} & \Rightarrow \pi_1 = 8\pi_0 \\ \pi_1 \cdot 2 = \pi_2 \cdot \frac{1}{2} & \Rightarrow \pi_2 = 4\pi_1 = 32\pi_0 \\ \pi_2 \cdot 2 = \pi_3 \cdot \frac{1}{2} & \Rightarrow \pi_3 = 4\pi_2 = 128\pi_0 \\ \sum \pi_i = 1 & \Rightarrow \pi_0 + 8\pi_0 + 32\pi_0 + 128\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{169} \end{cases}$$

$$\pi = \left(\frac{1}{169}, \frac{8}{169}, \frac{32}{169}, \frac{128}{169}\right)$$

WHAT KIND OF QUANTITIES CAN WE COMPUTE?
CONSIDER THE ABOVE M/M/2/3 SYSTEM AS AN EXAMPLE.

π_0 = LONG-RUN FRAC. OF TIME WHEN THERE ARE NO CUST.

$\pi_2 + \pi_3$ = ——— 〃 ——— 〃 ——— 〃 ——— BOTH SERVERS ARE BUSY

π_3 = ——— 〃 ——— 〃 ——— 〃 ——— NO PLACE TO TAKE ADDL. CUST.

$\lambda \pi_3$ = RATE (CUST/UNIT TIME) AT WHICH CUSTOMERS ARE BLOCKED (LOST)

$\lambda(1-\pi_3) = \pi_1 \cdot \frac{1}{4} + \pi_2 \cdot \frac{1}{2} + \pi_3 \cdot \frac{2}{2}$ = RATE AT WHICH CUSTOMERS ARE ACCEPTED

= RATE AT WHICH CUSTOMERS LEAVE SYSTEM

= SYSTEM THROUGHPUT