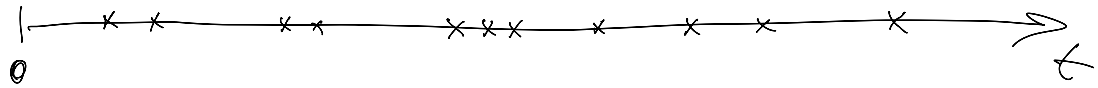
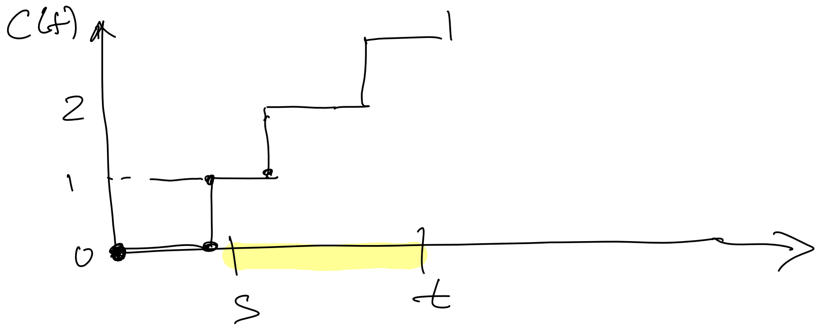
POISSON PROCESS

A COUNTING PROCESS IN CONTINUOUS TIME $t \geq 0$:



$C(t)$ = # OF "EVENTS" ("POINTS") IN $[0, t]$



$C(t) - C(s)$ = # OF POINTS IN $(s, t]$.

RANDOM VARIABLE $X \in \{0, 1, 2, \dots\}$
HAS POISSON DISTRIBUTION
WITH MEAN a, IF

$$\mathbb{P}\{X = i\} = \frac{a^i}{i!} e^{-a} \equiv \pi_i(a), \qquad i = 0, 1, 2, \dots$$

WE WILL USE NOTATION: $X \sim \text{POISSON}(a)$

A COUNTING PROCESS
$N = (N(t),\ t \geq 0)$, $N(t) \in \{0, 1, 2, \dots\}$,
IS CALLED A POISSON PROCESS WITH RATE $\lambda \geq 0$,
IF IT HAS THE FOLLOWING PROPERTIES:

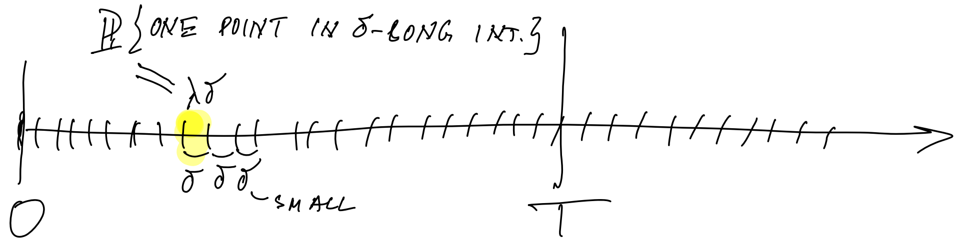
(i) $N(t) - N(s) \sim \text{POISSON}(\lambda(t-s))$, $0 \leq s \leq t$;

(ii) $N(\cdot)$ HAS INDEPENDENT INCREMENTS: $\forall\, 0 \leq s \leq t$,
$N(t) - N(s)$ IS INDEP. OF $(N(u),\ 0 \leq u \leq s)$;

(iii) $N(0) = 0$.

WE WILL USE THIS NOTATION: $\text{POIS-PROC}(\lambda)$

BASIC INTUITION FOR WHERE THE INDEPENDENT POISSON INCREMENTS COME FROM



$p = \lambda\delta$ SMALL

$n = T/\delta$ LARGE

IF $n \to \infty$, THEN $\delta \to 0$ AND $np = \lambda T$ = CONST

AS $n \to \infty$,

$$M = \#\text{ POINTS IN } [0,T] \xrightarrow{\text{IN DISTRIBUTION}} \text{POISSON}(\lambda T)$$

POISSON THEOREM

EXAMPLES. $\quad$ POIS-PROC(λ), $\lambda = 2$.

(i) $\mathbb{P}\{N(7)-N(3)=3\} = \pi_3(2(7-3)) = \pi_3(8)$

(ii) $\mathbb{P}\{N(2)-N(0)=2,\ \overbrace{N(3)-N(1)\geq 3}^{B}\} =$

$$= \mathbb{P}\{N(1)-N(0)=0,\ N(2)-N(1)=2,\ B\} +$$

$$\mathbb{P}\{N(1)-N(0)=1,\ N(2)-N(1)=1,\ B\} +$$

$$\mathbb{P}\{N(1)-N(0)=2,\ N(2)-N(1)=0,\ B\} =$$

$$= \mathbb{P}\{\overbrace{N(1)-N(0)=0}^{\pi_0(2)},\ \overbrace{N(2)-N(1)=2}^{\pi_2(2)},\ \overbrace{N(3)-N(2)\geq 1}^{\sum_{i=1}^{\infty}\pi_i(2) = 1-\pi_0(2)}\} +$$

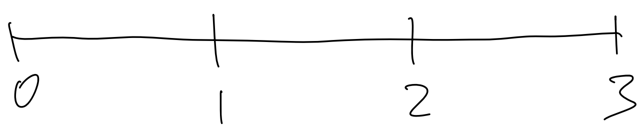
$$\mathbb{P}\{N(1)-N(0)=1,\ N(2)-N(1)=1,\ N(3)-N(2)\geq 2\} +$$

$$\mathbb{P}\{N(1)-N(0)=2,\ N(2)-N(1)=0,\ N(3)-N(2)\geq 3\} =$$

$$= \pi_0(2)\cdot\pi_2(2)\cdot\left(1-\pi_0(2)\right) +$$

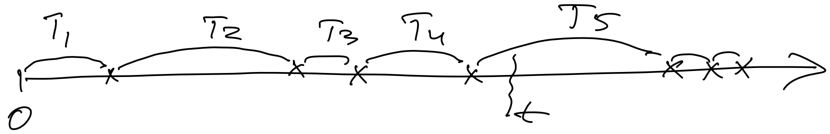
$$+ \pi_1(2)\cdot\pi_1(2)\cdot\left(1-\pi_0(2)-\pi_1(2)\right) +$$

$$+ \pi_2(2)\cdot\pi_0(2)\cdot\left(1-\pi_0(2)-\pi_1(2)-\pi_2(2)\right)$$



EQUIVALENT DEFINITION OF A POISSON PROCESS

$N(\cdot) \sim \text{POIS-PROC}(\lambda)$



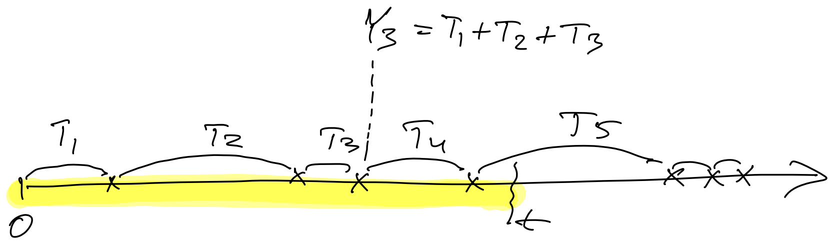
T_i = TIME FROM $(i-1)$-TH POINT TO i-TH POINT

$T_1 \sim \text{EXP}(\lambda)$:

$$\mathbb{P}\{T_1 \le t\} = 1 - \mathbb{P}\{T_1 > t\} = 1 - \mathbb{P}\{N(t) = 0\} =$$
$$= 1 - \pi_0(\lambda t) = 1 - e^{-\lambda t}, \qquad t \geq 0.$$

|| $T_i \sim \text{EXP}(\lambda)$, $i = 1, 2, \ldots$

|| $N(\cdot)$ IS POIS-PROC (λ)
IF AND ONLY IF
$T_i \sim \text{EXP}(\lambda)$, $i = 1, 2, \ldots$ & ALL T_i ARE INDEP.



$N(\cdot)$ IS POIS-PROC (λ).

$Y_3 = T_1 + T_2 + T_3 \sim$ DISTR ?

CDF OF Y_3:

$$\mathbb{P}\{Y_3 \le t\} = \mathbb{P}\{T_1 + T_2 + T_3 \le t\} = \mathbb{P}\{N(t) - N(0) \ge 3\} =$$

$$= 1 - \mathbb{P}\{N(t) - N(0) \le 2\} =$$

$$= 1 - [\pi_0(\lambda t) + \pi_1(\lambda t) + \pi_2(\lambda t)] =$$

$$= 1 - \left[e^{-\lambda t} + \lambda t \cdot e^{-\lambda t} + \frac{(\lambda t)^2}{2} e^{-\lambda t}\right]$$

DENSITY OF Y_3:

$$f_{Y_3}(t) = -\left[-\lambda e^{-\lambda t} + \lambda e^{-\lambda t} + \lambda t(-\lambda e^{-\lambda t}) + \lambda^2 t e^{-\lambda t} + \frac{(\lambda t)^2}{2}(-\lambda e^{-\lambda t})\right] =$$

$$= \frac{\lambda^3 t^2}{2} e^{-\lambda t}$$

$Y_i = T_1 + \dots + T_i$; DENSITY OF (DISTR. OF) Y_i:

$$f_{Y_i}(t) = \frac{\lambda^i t^{i-1}}{(i-1)!} e^{-\lambda t}, \quad t \ge 0.$$

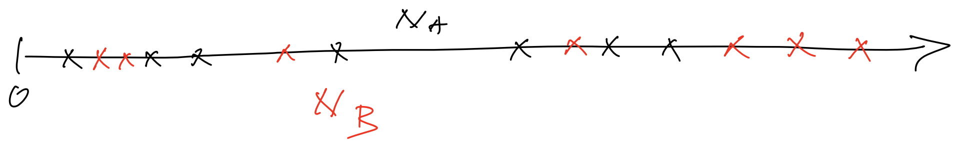
ERLANG DISTRIBUTION

MERGING POISSON PROCESSES

$N_A(\cdot) \sim \text{POIS-PROC}(\lambda_A)$, $N_B(\cdot) \sim \text{POIS-PROC}(\lambda_B)$;

$N_A(\cdot)$ AND $N_B(\cdot)$ ARE INDEP.

$N(\cdot) = N_A(\cdot) + N_B(\cdot) \sim \text{POIS-PROC}(\lambda_A + \lambda_B)$.



(iii) $N(0) = 0$ OBVIOUS

(i) $N(t) - N(s) = N_A(t) + N_B(t) - N_A(s) - N_B(s) =$

$$= \left[N_A(t) - N_A(s)\right] + \left[N_B(t) - N_B(s)\right]$$

$$\sim \text{POISSON}\left(\lambda_A(t-s) + \lambda_B(t-s)\right) =$$

$$= \text{POISSON}\left((\lambda_A + \lambda_B)(t-s)\right)$$

(ii) $N(t) - N(s)$ IS INDEP. OF HISTORY IN $[0, s]$.

$$\parallel$$

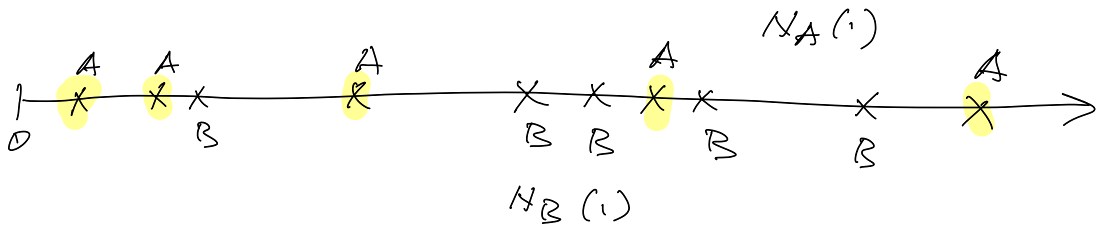
$$\left[N_A(t) - N_A(s)\right] + \left[N_B(t) - N_B(s)\right]$$

"SPLITTING" POISSON PROCESS

(IN PARTICULAR, "THINNING")

$N(\cdot) \sim$ POIS-PROC(λ). $0 \le p \le 1$ IS FIXED.

SPLITTING: EACH POINT, INDEPENDENTLY, WITH PROB. p LABELED A, OTHERWISE — B.



THEN:

$N_A(\cdot) \sim$ POIS-PROC$(p\lambda)$

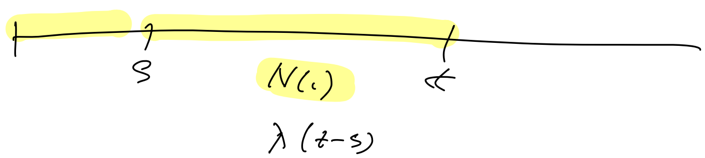
$N_B(\cdot) \sim$ POIS-PROC$((1-p)\lambda)$

$N_A(\cdot)$ AND $N_B(\cdot)$ ARE INDEP.

(iii) $N_A(0) = 0$

(i) $N_A(t) - N_A(s) \sim$ POISSON$(p\lambda(t-s))$

SEE EXAMPLE IN GENER. FUNC. LECTURE



(ii) DIRECTLY FOLLOWS FROM INDEPENDENCE OF INCREMENTS OF $N(\cdot)$