Problem 1

Type derivation for:

```
let rec f = fun x -> fun n -> if n <= 0 then [] else x::( f x (n - 1))
      in (f 3 2, f "a" 4) : int list * string list
```

$$\cfrac{\overbrace{\begin{array}{c}\{f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |- fun x -> fun n ->}\\ \text{if n} <= 0 \text{ then [] else x::(f x (n - 1))}\\ :\alpha\to \text{ int }\to\alpha \text{ list}\end{array}}^{e_1} \qquad \overbrace{\begin{array}{c}\{f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |- (f 3 2, f "a" 4)}\\ \text{int list } * \text{ string list}\end{array}}^{e_2}}{\begin{array}{c}\{\} \text{ |- let rec f = fun x } \to \text{ fun n } \to \text{ if n } \le \text{ 0 then [] else x::(f x (n - 1))}\\ \text{in (f 3 2, f "a" 4) : int list * string list}\end{array}}REC$$

(1)

$$\cfrac{\overbrace{\begin{array}{c}\{x:\alpha,f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |- fun n ->}\\ \text{if n } <= 0 \text{ then [] else x::(f x (n - 1)) : int } -> \alpha \text{ list}\end{array}}^{e_3}}{\underbrace{\begin{array}{c}\{f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |- fun x -> fun n ->}\\ \text{if n } <= 0 \text{ then [] else x::(f x (n - 1))}\\ :\alpha\to \text{ int }\to\alpha \text{ list}\end{array}}_{e_1}}FUN$$

(2)

$$\cfrac{\overbrace{\begin{array}{c}\{n: \text{ int },x:\alpha,f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |-}\\ \text{if n } <= 0 \text{ then [] else x::(f x (n - 1)) : }\alpha \text{ list}\end{array}}^{e_4}}{\underbrace{\begin{array}{c}\{x:\alpha,f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |- fun n ->}\\ \text{if n } <= 0 \text{ then [] else x::(f x (n - 1)) : int } -> \alpha \text{ list}\end{array}}_{e_3}}FUN$$

(3)

$$\cfrac{\overbrace{\begin{array}{c}\{n: \text{ int },x:\alpha,\\ f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |-}\\ \text{n} <= 0 \text{ : bool}\end{array}}^{e_5} \quad \overbrace{\begin{array}{c}\{n: \text{ int },x:\alpha,\\ f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |-}\\ \text{[] : }\alpha \text{ list}\end{array}}^{e_6} \quad \overbrace{\begin{array}{c}\{n: \text{ int },x:\alpha,\\ f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |-}\\ \text{x::(f x (n - 1)) : }\alpha \text{ list}\end{array}}^{e_7}}{\underbrace{\begin{array}{c}\{n: \text{ int },x:\alpha,f:\alpha\to \text{ int }\to\alpha \text{ list }\} \text{ |-}\\ \text{if n } <= 0 \text{ then [] else x::(f x (n - 1)) : }\alpha \text{ list}\end{array}}_{e_4}}ITE$$

(4)

$$\dfrac{\dfrac{VAR}{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ n:\ \text{int}}\qquad\dfrac{CON}{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ 0:int}}{\underbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ \text{n}<=0\ :\text{bool}}_{e_5}}REL \qquad (5)$$

$$\dfrac{CON}{\underbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ []\ :\alpha\ \text{list}}_{e_6}} \qquad (6)$$

$$\dfrac{\overbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ (\ (::)\ \text{x}\ )\ :\alpha\ \text{list}\ \to\alpha\ \text{list}}^{e_8}\qquad\overbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ (\text{f x (n - 1)})\ :\alpha\ \text{list}}^{e_9}}{\underbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ \text{x::(f x (n - 1))}\ :\alpha\ \text{list}}_{e_7}}APP \qquad (7)$$

$$\dfrac{\dfrac{CON}{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ (::)\ :\alpha\to\alpha\ \text{list}\ \to\alpha\ \text{list}}\qquad\dfrac{VAR}{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ \text{x}\ :\alpha}}{\underbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ (\ (::)\ \text{x}\ )\ :\alpha\ \text{list}\ \to\alpha\ \text{list}}_{e_8}}APP \qquad (8)$$

$$\dfrac{\overbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ \text{f x}\ :\ \text{int}\ \to\alpha\ \text{list}}^{e_{10}}\qquad\overbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ (\text{n - 1})\ :\ \text{int}}^{e_{11}}}{\underbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ (\text{f x (n - 1)})\ :\alpha\ \text{list}}_{e_9}}APP \qquad (9)$$

$$\dfrac{\dfrac{VAR}{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ \text{f}\ :\alpha\to\ \text{int}\ \to\alpha\ \text{list}}\qquad\dfrac{VAR}{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ \text{x}\ :\alpha}}{\underbrace{\{n:\ \text{int}\ ,x:\alpha,\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ \vdash\ \text{f x}\ :\ \text{int}\ \to\alpha\ \text{list}}_{e_{10}}}APP \qquad (10)$$

$$\frac{\begin{array}{c}\overline{\{n:\ \text{int}\ ,x:\alpha,\quad}^{VAR}\\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ |\text{-}\\ n\ :\ \text{int}\end{array}\quad\begin{array}{c}\overline{\{n:\ \text{int}\ ,x:\alpha,\quad}^{CON}\\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ |\text{-}\\ 1\ :\ \text{int}\end{array}\quad (-):\ \text{int}\ \to\ \text{int}\ \to\ \text{int}}{\underbrace{\begin{array}{c}\{n:\ \text{int}\ ,x:\alpha,\\ f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ |\text{-}\\ (n\text{ - }1)\ :\ \text{int}\end{array}}_{e_{11}}}PO$$

$$(11)$$

$$\frac{\overbrace{\begin{array}{c}\{f:\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\ \}\ |\text{-}\\ f\ 3\ 2\ :\ \text{int}\ \text{list}\end{array}}^{e_{12}}\quad\overbrace{\begin{array}{c}\{f:\ \text{string}\ \to\ \text{int}\ \to\ \text{string}\ \text{list}\ \}\ |\text{-}\\ f\ "a"\ 4\ :\ \text{string}\ \text{list}\end{array}}^{e_{13}}\quad\begin{array}{c}(,):\ \text{int}\ \text{list}\\ \to\ \text{string}\ \text{list}\\ \to\ \text{int}\ \text{list}\ *\\ \text{string}\ \text{list}\end{array}}{\underbrace{\begin{array}{c}\{f:\alpha\to\ \text{int}\ \to\alpha\ \text{list}\ \}\ |\text{-}\ (f\ 3\ 2,\ f\ "a"\ 4\ )\\ \text{int}\ \text{list}\ *\ \text{string}\ \text{list}\end{array}}_{e_2}}PO$$

$$(12)$$

$$\frac{\overbrace{\begin{array}{c}\{f:\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\ \}\ |\text{-}\\ f\ 3\ :\ \text{int}\ \to\ \text{int}\ \text{list}\end{array}}^{e_{14}}\quad\overline{\begin{array}{c}{}^{CON}\\ \{f:\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\ \}\ |\text{-}\\ 2\ :\ \text{int}\end{array}}}{\underbrace{\begin{array}{c}\{f:\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\ \}\ |\text{-}\\ f\ 3\ 2\ :\ \text{int}\ \text{list}\end{array}}_{e_{12}}}APP\quad(13)$$

$$\frac{\overline{\begin{array}{c}{}^{VAR}\\ \{f:\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\ \}\ |\text{-}\\ f\ :\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\end{array}}\quad\overline{\begin{array}{c}{}^{CON}\\ \{f:\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\ \}\ |\text{-}\\ 3\ :\ \text{int}\end{array}}}{\underbrace{\begin{array}{c}\{f:\ \text{int}\ \to\ \text{int}\ \to\ \text{int}\ \text{list}\ \}\ |\text{-}\\ f\ 3\ :\ \text{int}\ \to\ \text{int}\ \text{list}\end{array}}_{e_{14}}}APP\quad(14)$$

$$\frac{\overbrace{\begin{array}{c}\{f:\ \text{string}\ \to\ \text{int}\ \to\ \text{string}\ \text{list}\ \}\ |\text{-}\\ f\ "a"\ :\ \text{int}\ \to\ \text{string}\ \text{list}\end{array}}^{e_{15}}\quad\overline{\begin{array}{c}{}^{CON}\\ \{f:\ \text{string}\ \to\ \text{int}\ \to\ \text{string}\ \text{list}\ \}\ |\text{-}\\ 4\ :\ \text{int}\end{array}}}{\underbrace{\begin{array}{c}\{f:\ \text{string}\ \to\ \text{int}\ \to\ \text{string}\ \text{list}\ \}\ |\text{-}\\ f\ "a"\ 4\ :\ \text{string}\ \text{list}\end{array}}_{e_{13}}}APP$$

$$(15)$$

$$\cfrac{\cfrac{}{\{f: \text{ string } \rightarrow \text{ int } \rightarrow \text{ string list }\} \;|\text{-} \atop \text{f} : \text{ string } \rightarrow \text{ int } \rightarrow \text{ string list}} \; VAR \qquad \cfrac{}{\{f: \text{ string } \rightarrow \text{ int } \rightarrow \text{ string list }\} \;|\text{-} \atop \text{"a"} : \text{ int}} \; CON}{\underbrace{\{f: \text{ string } \rightarrow \text{ int } \rightarrow \text{ string list }\} \;|\text{-} \atop \text{f "a"} : \text{ int } \rightarrow \text{ string list}}_{e_{15}}} \; APP$$

$$(16)$$