HW 6: Solutions

1 Theoretical Questions (30 points)

Problem 1: (10 points)

For the following two problems give a regular expression that represents the strings in the described language. In both cases, assume that our alphabet is the set $\{a, b, c\}$.

1. The language where a occurs in every third position.
2. The language where each string contains exactly 3 c's.

Solution:

1. $(\Sigma\Sigma a)^*\Sigma?\Sigma?$
2. $(a|b)^*c(a|b)^*c(a|b)^*c(a|b)^*$

Problem 2: (20 points)

Decide whether the following (informally described) languages are regular. For each regular language, write a **regular expression** and a **regular grammar** of the language. For each language that is not regular, make an argument of why the language is not regular.

1. $L_1 = \{w|w$ starts with symbol 0 and contains the symbol 1 at least once$\}$ where $\Sigma = \{0, 1\}$
2. $L_2 = \{w|w$ contains an equal number of 0s and 1s$\}$ where $\Sigma = \{0, 1\}$
3. $L_3 = \{w|$the length of w is odd$\}$ where $\Sigma = \{a, b\}$
4. $L_4 = \{w|w$ does *not* contain symbol a immediately followed by symbol $b\}$ where $\Sigma = \{a, b, c\}$
5. $L_5 = \{w|$the length of w is a perfect cube$\}$ where $\Sigma = \{a, b, c\}$

Solution:

1. $L_1 = \{w|w$ starts with symbol 0 and contains the symbol 1 at least once$\}$ where $\Sigma = \{0, 1\}$
   Solution:
   a. $0(0^*)1((0 \vee 1)^*)$

b.
$$\begin{array}{lcl} S & \rightarrow & 0T \\ T & \rightarrow & 0T \\ T & \rightarrow & 1V \\ T & \rightarrow & 1 \\ V & \rightarrow & 0V \\ V & \rightarrow & 1V \\ V & \rightarrow & 0 \\ V & \rightarrow & 1 \end{array}$$

The language L_2 is not regular. Informally, for any bound n, there is a string in L_2 that has an initial substring with n more 0's than 1's, so it would take at least n distinct states to count down having the same number of each. I.e., it would take an infinite number of states to track the unbounded extent of possible imbalance between the number of 0s that have occurred and the number of 1s.

More formally, let us assume L_2 is regular. Then, by the pumping lemma, there exists a $p \geq 1$ such that for all strings $w \in L_2$ there exist substrings x, y, and z such that $w = xyz$ and $|y| \geq 1$ and $|xy| \leq p$ and for all $i \geq 0$, we have $x(y^i)z \in L_2$. Let $w = 0^p 1^p$, and let x, y, and z be such that $w = xyz$ and $|y| \geq 1$ and $|xy| \leq p$. Let $n = |y|$. Then $y = 0^n$, and $xz = 0^{(p-n)}1^P$, and since $n \geq 1$, $xz \notin L_2$, by the definition of L_2. This contradiction proves that L_2 can not be regular.

3. $L_3 = \{w|\text{the length of } w \text{ is odd}\}$ where $\Sigma = \{a, b\}$
Solution:

a. $(0 \vee 1)(((0 \vee 1)(0 \vee 1))^*)$

b.
$$\begin{array}{lcl} S & \rightarrow & 0T \\ S & \rightarrow & 1T \\ S & \rightarrow & 0 \\ S & \rightarrow & 1 \\ T & \rightarrow & 0S \\ T & \rightarrow & 1S \end{array}$$

4. $L_4 = \{w|w \text{ does } \textit{not} \text{ contain symbol } a \text{ immediately followed by symbol } b\}$ where $\Sigma = \{a, b, c\}$
Solution:

a. $(b \vee (a^*c))^*(a^*)$

b.
$$\begin{array}{lcl} S & \rightarrow & \varepsilon \\ S & \rightarrow & bS \\ S & \rightarrow & cS \\ S & \rightarrow & aT \\ T & \rightarrow & aT \\ T & \rightarrow & cS \\ T & \rightarrow & \varepsilon \end{array}$$

5. $L_5 = \{w|\text{the length of } w \text{ is a perfect cube}\}$ where $\Sigma = \{a, b, c\}$ The language L_5 is not regular. Very informally, the finite state automaton accepting L_5 would have to be able to count

exactly $3n^2 + 3n + 1 = (n+1)^3 - n^3$ for every natural number $n \geq 0$, an unbounded amount of information.

More formally, let us assume L_5 is regular. Then, by the pumping lemma, there exists a $p \geq 1$ such that for all strings $w \in L_5$ there exist substrings x, y, and z such that $w = xyz$ and $|y| \geq 1$ and $|xy| \leq p$ and for all $i \geq 0$, we have $x(y^i)z \in L_5$. Choose w such that $|w| = p^3$, and let x, y, and z be such that $w = xyz$ and $|y| \geq 1$ and $|xy| \leq p$. Let $n = |y|$. Note that $1 \leq n \leq p$. Then $|xyyz| = p^3 + n < p^3 + 3p^2 + 3p + 1 = (p+1)^3$. Therefore, $xyyz \notin L_5$, by the definition of L_5. This contradiction proves that L_5 can not be regular.