Topic 10 - Heterogenous Agents model

PSE Master APE
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1 Bellman Formulation of the Problem

The households solve:

$$\begin{aligned}
V(a_0, y_0) &= \max_{(c_t)_{t=0}^{\infty}, (a_t)_{t=1}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \text{ s.t } a_{t+1} = (1+r)a_t + y_t - c_t \\
&= \max_{(a_{t+i})_{i=1}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{((1+r)a_t + y_t - a_{t+1})^{1-\sigma}}{1-\sigma} \\
&= \max_{(a_{t+i})_{i=1}^{\infty}} \mathbb{E}_0 \left(\frac{((1+r)a_0 + y_0 - a_1)^{1-\sigma}}{1-\sigma} + \sum_{t=1}^{\infty} \beta^t \frac{((1+r)a_t + y_t - a_{t+1})^{1-\sigma}}{1-\sigma} \right) \\
&= \max_{a_1} \frac{((1+r)a_0 + y_0 - a_1)^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_0 V(a_1, y_1)
\end{aligned}$$

In summary:

$$V(a_0, y_0) = \max_{a_1} U(c_0) + \mathbb{E}_0 \beta V(a_1, y_1)$$

Note that the expectation term $\mathbb{E}$ reflects the transition matrix $\mathcal{P}$ between states of good or bad employment and unemployment.

An algorithm to solve this numerically would be to use Value function iteration:

- Choose a grid for consumption choices c_t and for possible wealth today a_t
- Initialize a concave value for $V_0(a_t)$, taking for instance: $V_0(a_t) = U(c_t)$ as a first approximation
- We are going to use the mapping contraction theorem and the Banach's Fixed Point theorem to analyze the convergence of the sequence of $V_n(a)$ defined by:

$$\forall n, V_{n+1}(a_t, y_t) = \max_{a_{t+1}} U((1+r)a_t + y_t - a_{t+1})) + \mathbb{E}_t \beta V_n(a_{t+1}, y_{t+1})$$

  which satisfies the Blackwell conditions of monotonicity and contraction

- Iterate on the sequence of $\{V_n\}$ until, for a given tolerance ϵ:

$$\max_a |V_{n+1}(a) - V_n(a)| < \epsilon$$

– In particular, at each step, for each current possible value of income y_t, loop over the grid for a_t and for each a_t, find $a_{t+1}(a_t, y_t)$ such that $a_{t+1}(a_t, y_t) \geq 0, c_t(a_{t+1}) \geq 0$ and the one maximizing $U(c_t(a_{t+1})) + \mathbb{E}\beta V(a_{t+1})$.
– This defines an optimal policy $a^*_{t+1}(a_t, y_t)$ and then use the budget constraint to find the optimal consumption policy $c^*_t(a_t, y_t)$

- Once a fixed point V^* is found, reapply the iteration to get the final optimal policy for $a^*_{t+1}(a_t, y_t)$ and $c^*_t(a_t, y_t)$

In the code, we are however implementing the Endogenous Grid Method algorithm in order to speed up the computations, in particular for the general equilibrium computations and their adjustment when Γ changes. Regarding the calibrations, we took classical calibrations for the agents problems, and random values for the μs and the incme depending on the state, ensuring simply that the well paid states was indeed better.

Our numerical approximation yields the following policy functions:

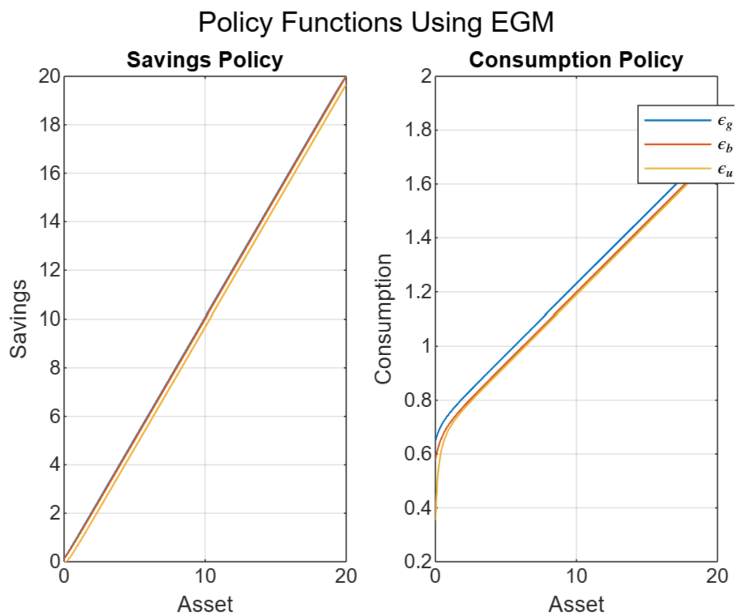

Figure 1: Policy Function

We can observe in Figure 1 that unemployed households, which are also poorer, have logically less assets as well as a lower consumption.

2 General Equilibrium of the Model

We solved the Household Bellman problem using fixed r, w. However, these are actually endogenous to the model. Taking r and w as given, households supply assets through their savings and labor to earn wage. On the other side, the representative firm needs capital and labor in order to run and thus is in demand of capital and labor.

The firm maximizes its profit $\Pi = Y - wN - rK$, with $Y = AK^\alpha N^{1-\alpha}$, so that:

$$\frac{\partial \Pi}{\partial K} = 0 \iff \frac{\partial Y}{\partial K} = r \iff \alpha \frac{Y}{K} = r$$
$$\frac{\partial \Pi}{\partial N} = 0 \iff \frac{\partial Y}{\partial N} = w \iff (1-\alpha) \frac{Y}{N} = w$$

To solve for the equilibrium r, w: for r_n, w_n given, we have to compute the aggregates for A, L on the household side and the demand for K, N on firm's side and adjust r_{n+1}, w_{n+1} in order to arrive at $A = K$ and $L = N$.

When solving, adjusting r will have an impact on capital supply and demand, thus also on labor demand and wages. This in turn will also impact the income levels. This means that our grid for y will be evolving as we are solving for the equilibrium values.

When solving, first note that $N = \mu_g N_g + \mu_b N_b$, with N_g, N_b given by the steady-state distribution of employment, depending only on exogenous values of the transition matrix $\mathcal{P}$. As such, when solving for general equilibrium values, we can consider that N is fixed, depending on external parameters of the economy.

This yields the following equilibrium values:

	Equilibrium Values
A	4.2354
K	4.2354
N	0.76538
r	0.051641
w	1.1808

Figure 2: Equilibrium Values with our initial problem

3 Stationary Distribution of Wealth $\lambda(a, \epsilon)$

In order to compute the stationary distribution, we resort to the Young method, where we build the transition matrix by allocating the policy function of assets next period onto our grid for assets. This allows also to have the distribution by states.

Doing so yields the following distributions:

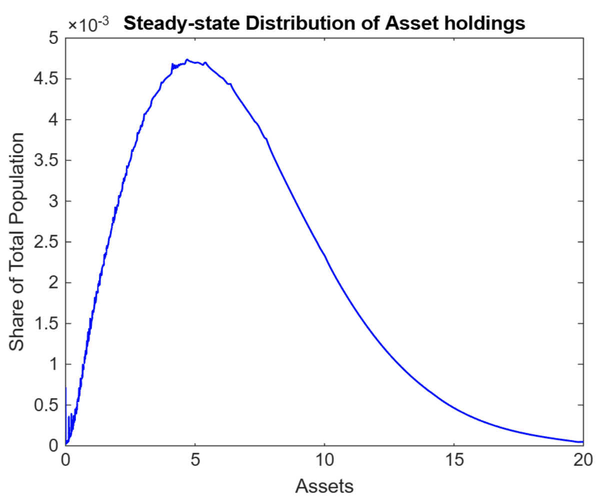

Figure 3: Distribution of Agents, aggregate level

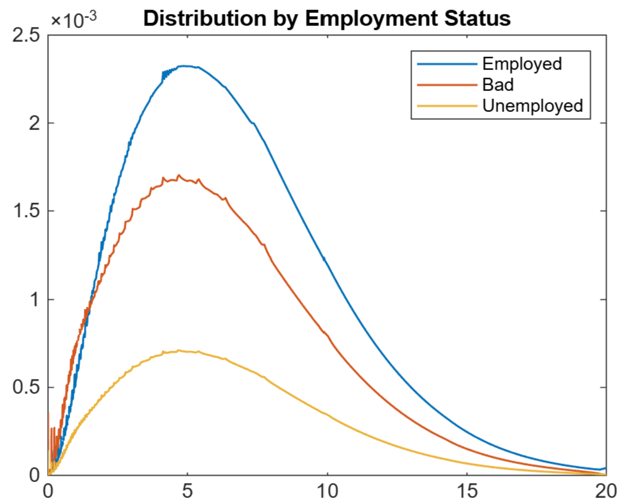

Figure 4: Distribution of Agents, by employment status

We see clearly the different in asset distribution for people with good employment. We also see that unemployed agents tend to be asset constraint, as their borrowing constraint is binding, much more than for the two other types of agents, reflecting the incompletude of markets.

4 Measures of Wealth Inequalities

To compute new measures by changing the spread Γ between well-paid workers and less-paid workers, we need each time to recompute the new general equilibrium, as Γ distorts the transition matrix and thus the problem as a whole.

We compute the metrics for the Gini coefficient, the ratio of asset between the top 10% and the bottom 10% as well as the consumption ratio. As expected, all the ratio increase for larger values of Γ, from a situation where $\Gamma = 0$ where there are only employed vs non-employed agents to a situation where well-paid employed agents make two times as much as less paid-employed agents, when $\Gamma = 1$. We see that the ratio for consumption are in line with the ratio from the lecture for a large Γ.

	Gamma = 0.0	Gamma = 0.2	Gamma = 0.4	Gamma = 0.6	Gamma = 0.8	Gamma = 1.0
Gini	0.36478	0.39741	0.4183	0.43142	0.44137	0.4496
Asset top10/btm10	13.405	19.634	27.133	35.53	41.981	49.762
Consumption top10/btm10	1.79	1.9217	2.0809	2.2516	2.4197	2.6013

Figure 5: Measures of Inequalities

5 Some Potential Next Steps

To conclude, we just wanted to leave some remarks as to how to improve the model:

- calibrate the transition matrix such that the invariant distribution of unemployed agents match the data
- increase the width of the transition matrix to include more potential states as well as more Γ spreads in order to calibrate to match real data distributions

- allow for movements between quality of employment, for instance from bad employment to good employment and vice-versa, to include promotion and demotion
- include illiquid assets with higher return to increase the richness of the households' choices and allow for more inequality in the model