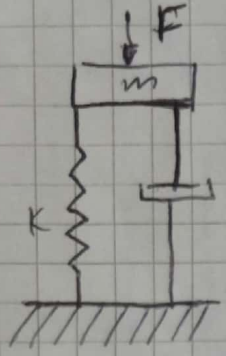
Parcial 3 - Señales y Sistemas

1. Sistema masa-resorte-amortiguador $CI = 0$



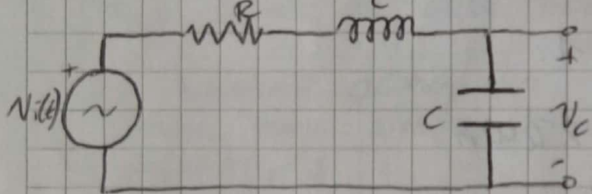
$$F(t) - Kx(t) - c\frac{\partial x(t)}{\partial t} = m\frac{\partial^2 x(t)}{\partial t^2}$$

Aplicando laplace

$$F(s) = (ms^2 + cs + k)X(s)$$

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = \frac{1}{m} \cdot \frac{1}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

Circuito RLC serie $CI = 0$



$$v_i(t) = v_R(t) + v_L(t) + v_c(t)$$

$$v_i(t) = RI_R(t) + L\frac{\partial I_L(t)}{\partial t} + v_c(t)$$

$$I_R(t) = I_L(t) = I_c(t) = C\frac{\partial V_c(t)}{\partial t}$$

Aplicando Laplace

$$V_i(s) = RCsV_c(s) + LsCsV_c(s) + V_c(s)$$

$$V_i(s) = (LCs^2 + RCs + 1)V_c(s)$$

$$\frac{V_c(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

$$\frac{V_c(s)}{V_i(s)} = \frac{1}{LC} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Para sistemas de segundo

Función de transferencia normalizada

$$H(s) \triangleq \frac{K\,W_n^2}{s^2+2\zeta W_n s+W_n^2}$$ Donde

K: Ganancia del sistema

W_n: Frecuencia natural $\Rightarrow 2\pi f_n$

ζ: Factor de amortiguación

Comparando con circuito RLC

$W_n^2 = \frac{1}{LC}$ $\qquad$ $W_n = \frac{1}{\sqrt{LC}}$ $\qquad$ $2\zeta W_n = \frac{R}{L}$ $\qquad$ $\zeta = \frac{R}{2L\frac{1}{\sqrt{LC}}}$

$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$ $\qquad$ $K = 1$

Comparando con sistema masa-resorte-amortiguador

$W_n^2 = \frac{k}{m}$ $\qquad$ $W_n = \sqrt{\frac{k}{m}}$ $\qquad$ $2\zeta W_n = \frac{c}{m}$ $\qquad$ $\zeta = \frac{c}{2m\sqrt{\frac{k}{m}}} = \frac{c}{2\sqrt{\frac{k m^2}{m}}}$

$\zeta = \frac{c}{2\sqrt{km}}$ $\qquad$ $K = \frac{1}{k}$

Sistemas de segundo orden - RLC y MRA

Función de transferencia normalizada

$$H(s) = K \frac{W_n^2}{s^2 + 2\zeta W_n s + W_n^2}$$

Para entrada escalón $v_i(t) = u(t)$

$$V_i(s) = \frac{1}{s}$$

$$V_o(s) = \frac{K W_n^2}{s^2 + 2\zeta W_n s + W_n^2} \cdot \frac{1}{s} = \frac{AK}{s} + \frac{K(Bs + C)}{s^2 + 2\zeta W_n s + W_n^2}$$

$$AKs^2 + AK2\zeta W_n s + AKW_n^2 + KBs^2 + KCs = KW_n^2$$

$$K(A+B)s^2 + K(2\zeta W_n A + C)s + K(W_n^2 A) = KW_n^2$$

Si $K = 1$ $\quad (A+B) = 0 \quad \underline{A = 1} \Rightarrow B = -1$

$$2\zeta W_n A + C = 0 \qquad C = -2\zeta W_n$$

$$V_o(s) = \frac{1}{s} - \frac{s + 2\zeta W_n}{s^2 + 2\zeta W_n s + W_n^2}$$ Polos $s = 0 \quad s_{1,2} = -\zeta W_n \pm \sqrt{(\zeta^2 - 1)}\, W_n$

(i) $\zeta = 0 \Rightarrow$ Oscilador

$$V_o(s) = \frac{1}{s} - \frac{s}{s^2 + W_n^2} \Rightarrow v_o(t) = u(t) - \cos(W_n t)u(t)$$

(ii) $\zeta = 1 \Rightarrow$ Críticamente amortiguado

$$V_o(s) = \frac{1}{s} - \frac{s + 2W_n}{(s + W_n)^2} = \frac{1}{s} - \frac{\cancel{(s + W_n)}}{(s + W_n)^{\cancel{2}}} - \frac{W_n}{(s + W_n)^2}$$

$$V_o(s) = \frac{1}{s} - \frac{1}{(s + W_n)} - \frac{W_n}{(s + W_n)^2}$$

$$v_o(t) = u(t) - e^{-W_n t}u(t) - W_n t e^{-W_n t}u(t) = 1 - (1 + W_n t)e^{-W_n t}u(t)$$

(iii) $\zeta < 1 \Rightarrow$ Sub-amortiguado

$$V_o(s) = \frac{1}{s} - \frac{s + 2\zeta W_n}{s^2 + 2\zeta W_n s + W_n^2}$$

$$s^2 + 2\zeta W_n s + W_n^2 = (s + \zeta W_n)^2 + W_n^2 - \zeta^2 W_n^2$$

$$W_d^2 = (1 - \zeta^2) W_n^2 \qquad W_d = \sqrt{1 - \zeta^2}\, W_n$$

$$V_o(s) = \frac{1}{s} - \frac{s + \zeta W_n}{(s + \zeta W_n)^2 + W_d^2} - \frac{\zeta W_n}{(s + \zeta W_n)^2 + W_d^2} \rightarrow W_d$$

$$= \frac{1}{s} - \frac{s + \zeta W_n}{(s + \zeta W_n)^2 + W_d^2} - \frac{\zeta}{\sqrt{1 - \zeta^2}} \frac{\sqrt{1 - \zeta^2}\, W_n}{(s + \zeta W_n)^2 + W_d^2}$$

$$v_o(t) = u(t) - e^{-\zeta W_n t} \cos(W_d t) u(t) - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta W_n t} \operatorname{sen}(W_d t)\, u(t)$$

(iv) $\zeta > 1 \Rightarrow$ Sobreamortiguado

$$V_o(s) = \frac{1}{s} - \frac{s + 2\zeta W_n}{(s + \zeta W_n)^2 + W_d^2} \qquad W_d^2 = (1 - \zeta^2) W_n^2 \quad \text{como } \zeta > 1$$

$$= -(\zeta^2 - 1) W_n^2 = -W_K^2$$

$$\rightarrow W_K$$

$$V_o(s) = \frac{1}{s} - \frac{s + \zeta W_n}{(s + \zeta W_n)^2 - W_K^2} - \frac{\zeta}{\sqrt{\zeta^2 - 1}} \frac{\sqrt{\zeta^2 - 1}\, W_n}{(s + \zeta W_n)^2 - W_K^2}$$

$$v_o(t) = u(t) - e^{-\zeta W_n t} \cosh(W_K t) - \frac{\zeta}{\sqrt{\zeta^2 - 1}} e^{-\zeta W_n t} \operatorname{senh}(W_K t)$$

$$= u(t) - \frac{e^{-\zeta W_n t}}{\sqrt{\zeta^2 - 1}} u(t) \left(\sqrt{\zeta^2 - 1} \cosh(W_K t) + \zeta \operatorname{senh}(W_K t) \right)$$

$$W_K = \sqrt{\zeta^2 - 1}\, W_n$$

Para entrada impulso $v_i(t) = \delta(t)$

$V_i(s) = 1$

$$V_o(s) = \frac{k\, W_n^2}{s^2 + 2\zeta W_n s + W_n^2}$$

Si $k = 1$

Oscilador

(i) $\zeta = 0$ $\quad V_o(s) = \frac{W_n^2}{s^2 + W_n^2} = W_n \cdot \frac{W_n}{s^2 + W_n^2}$

$$v_o(t) = W_n \operatorname{sen}(W_n t)\, u(t)$$

(ii) $\zeta = 1$ $\quad V_o(s) = \frac{W_n^2}{s^2 + 2W_n s + W_n^2} = \frac{W_n^2}{(s + W_n)^2}$

$$v_o(t) = W_n^2 t e^{-W_n t} u(t)$$

(iii) $\zeta < 1$ $\quad s^2 + 2\zeta W_n s + W_n^2 = (s + \zeta W_n)^2 + W_n^2 - \zeta^2 W_n^2$

$$W_d^2 = (1 - \zeta^2) W_n^2$$

$$V_o(s) = \frac{W_n^2}{(s + \zeta W_n)^2 + W_d^2} = W_n^2 \cdot \frac{1}{(s + \zeta W_n)^2 + W_d^2} = \frac{W_n^2}{W_d} \frac{W_d}{(s + \zeta W_n)^2 + W_d^2}$$

$$\frac{W_n^2}{W_d} = \frac{W_n^2}{\sqrt{1 - \zeta^2}\, W_n} = \frac{W_n}{\sqrt{1 - \zeta^2}}$$

$$v_o(t) = \frac{W_n}{\sqrt{1 - \zeta^2}} e^{-\zeta W_n t} \operatorname{sen}(W_d t)\, u(t)$$

(iv) $\zeta > 1$ $\quad (1 - \zeta^2) W_n^2 = W_d^2 \Rightarrow -(\zeta^2 - 1) W_n^2 = -W_k^2$

$$V_o(s) = \frac{W_n^2}{(s + \zeta W_n)^2 - W_k^2} = W_n^2 \cdot \frac{1}{(s + \zeta W_n)^2 + W_d^2} = \frac{W_n^2}{W_k} \frac{W_k}{(s + \zeta W_n)^2 - W_k^2}$$

$$\frac{W_n^2}{W_k} = \frac{W_n^2}{\sqrt{\zeta^2 - 1}\, W_n} = \frac{W_n}{\sqrt{\zeta^2 - 1}}$$

$$v_o(t) = \frac{W_n}{\sqrt{\zeta^2 - 1}} e^{-\zeta W_n t} \operatorname{senh}(W_k t)\, u(t)$$

2. $c(t) = A_c \operatorname{sen}(2\pi F_c t)$ $\quad A_c, F_c \in \mathbb{R}$; $m(t) \in \mathbb{R}$

$y(t) = \left(1 + \frac{m(t)}{A_c}\right) c(t) \quad \Rightarrow Y(\omega) = F\{c(t)\} + \frac{F}{A_c}\{m(t)c(t)\}$

$F\{c(t)\} = A_c F\{\operatorname{sen}(2\pi F_c t)\} = \frac{A_c}{2j} F\{e^{j2\pi F_c t} - e^{-j2\pi F_c t}\}$

$C(\omega) = \frac{A_c}{2j}\left[2\pi\delta(\omega - 2\pi F_c) - 2\pi\delta(\omega + 2\pi F_c)\right]$

$F\{m(t)c(t)\} = \frac{A_c}{2j}\left[M(\omega - 2\pi F_c) - M(\omega + 2\pi F_c)\right]$

$Y(\omega) = \frac{A\pi}{j}\left[\delta(\omega - 2\pi F_c) - \delta(\omega + 2\pi F_c)\right] + \frac{1}{2j}\left[M(\omega - 2\pi F_c) - M(\omega + 2\pi F_c)\right]$

La señal modulada es la seña original, pero transportada por una señal de alta frecuencia.

Es necesario demodular esta señal con el fin de obtener una señal muy similar a la original. Para ello se usa un filtro pasabajos

Filtro pasabajos Butterworth

Función de transferencia $H(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^n + 1}$. Donde n = orden

$\omega_c = 2\pi F_c$; para nuestro caso $F_c = 3500$ $\quad n = 6$

Para pasar a $z \Rightarrow s = \frac{2}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}}$; Donde T_s: Periodo de muestreo

$$H(z) = \frac{1}{\left(\frac{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}{\omega_c}\right)^6 + 1}$$