Proof of Idempotence of the Closing-Opening Alternated Filter

1 Proof of Idempotence

The closing-opening alternated filter is defined as:

$$\varphi_B(\psi_B(I))$$

where: - $\psi_B(I)$ is the morphological closing of I with structuring element B. - $\varphi_B(I)$ is the morphological opening of I with structuring element B.

We must prove that applying this operation twice results in the same output:

$$\varphi_B(\psi_B(\varphi_B(\psi_B(I)))) = \varphi_B(\psi_B(I))$$

1.1 First Inequality: $\varphi_B(\psi_B(\varphi_B(\psi_B(I)))) \leq \varphi_B(\psi_B(I))$

Using morphological properties:

1. **Extensivity of closing**:

$$I \leq \psi_B(I)$$

2. **Anti-extensivity of opening**:

$$\varphi_B(I) \leq I$$

3. **Applying opening to a closing**:

$$\varphi_B(\psi_B(I)) \leq \psi_B(I)$$

4. **Applying closing again**:

$$\psi_B(\varphi_B(\psi_B(I))) \geq \varphi_B(\psi_B(I))$$

5. **Applying opening again**:

$$\varphi_B(\psi_B(\varphi_B(\psi_B(I)))) \leq \psi_B(\varphi_B(\psi_B(I)))$$

Thus:

$$\varphi_B(\psi_B(\varphi_B(\psi_B(I)))) \leq \varphi_B(\psi_B(I))$$

1.2 Second Inequality: $\varphi_B(\psi_B(I)) \leq \varphi_B(\psi_B(\varphi_B(\psi_B(I))))$

Using **idempotence**:

1. **Idempotence of closing**:

$$\psi_B(\psi_B(I)) = \psi_B(I)$$

2. **Idempotence of opening**:

$$\varphi_B(\varphi_B(I)) = \varphi_B(I)$$

Applying these:

$$\psi_B(\varphi_B(\psi_B(I))) \geq \varphi_B(\psi_B(I))$$

$$\varphi_B(\psi_B(\varphi_B(\psi_B(I)))) \geq \varphi_B(\psi_B(I))$$

Thus:

$$\varphi_B(\psi_B(I)) \leq \varphi_B(\psi_B(\varphi_B(\psi_B(I))))$$

1.3 Conclusion

Since we have proven both inequalities:

$$\varphi_B(\psi_B(\varphi_B(\psi_B(I)))) \leq \varphi_B(\psi_B(I))$$

$$\varphi_B(\psi_B(I)) \leq \varphi_B(\psi_B(\varphi_B(\psi_B(I))))$$

It follows that:

$$\varphi_B(\psi_B(\varphi_B(\psi_B(I)))) = \varphi_B(\psi_B(I))$$

Thus, the **closing-opening alternated filter is idempotent**. $\square$