Optimal Controller Design for Inverted Pendulum System based on LQR method

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Abstract **- This paper proposes design of optimal controller for inverted pendulum system. The main aim is to control angle of the pendulum and position of the cart simultaneously. Jacobian linearization technique has been used to linearize the system. Linear Quadratic Regulator technique is used here to solve the control problem. The solution of the Riccati is obtained by choosing appropriate weight matrix of continuous algebraic Riccati equation (CARE). The proposed controller able to meet the given performance specification minimizing the cost function efficiently. Finally, the performances of the controller have been evaluated with the presence of disturbances and variable initial conditions.**

Keywords- Inverted pendulum model; Jacobian linearization; LQR

I INTRODUCTION

Inverted pendulum is a common non-linear system used in control theory to analyze the effectiveness of various control algorithms. The LQR method has been appeared several times in the literature to design optimal controller for inverted pendulum. Yong Xin et al.[1] proposed an approach of LQR based optimal controller for pendulum system considering initial angle (θ) very small. This approximation may cause a deviation between the theoretical model and the real system. This paper uses Jacobi linearization technique [2] for this non-linear system which is being linearized around the equilibrium point. Linear Quadratic Regulator (LQR) solves the Riccati equation [2, 3] of a linear system that gives the Riccati solution (P) which can be obtained by choosing weighting matrix (Q) arbitrarily and keeping fixed weighting factor (R). This solution is used to construct a linear state-feedback optimal controller(K) minimizing the performance index (J) of the system. The Inverted pendulum consists of two equilibrium points [4], one of them is stable while the other is unstable. The stable equilibrium corresponds to a state in which the pendulum is pointing downwards.

It requires perhaps no control input to achieve the stability. Whereas the unstable equilibrium point corresponds to a state in which the pendulum points strictly upwards and, thus, requires a control force to maintain this position. The control objective of this paper is to operate the system at unstable equilibrium point with minimum control energy to the system.

On the other hand, Lyapunov stability theorem to design an adaptive controller is proposed for the inverted pendulum in [5]. Other significant work done about the LQR based controller design, modeling and performance analysis of linear inverted pendulum has been proposed in Hongliang Wang et al. [3]. A sliding mode and a variable-gain PID controller is designed to regulate and control of the pendulum and its arm respectively in [6]. W J Grantham et al.[7] have discussed about the Lyapunov optimal feedback control on the nonlinear inverted pendulum.

Rest of the paper is organized as follows: The inverted pendulum model is described in section II. Section III describes about linearized model of inverted pendulum. The LQR technique reports in section IV. Simulation and studies are reported in section V. The paper ends with the conclusion in section VI, followed by the references.

II THE MODEL OF INVERTED PENDULUM

Inverted pendulum on a cart is also known as stick balancer [2]. The free body diagram in Fig.1 represents simplified model of Inverted Pendulum [2]. Horizontal and Vertical reaction forces acting in the pivot are H and V respectively. The mass of the cart is M and the mass of pendulum is m having length of stick equal to$2l$. The angular displacement of the stick from the vertical position is $\theta = \theta(t)$.only frictional force considered in the model is the frictional force of the cart wheel on track denoted byf_c. The moment of inertia of the stick with respect to center of gravity is $I = \frac{ml^2}{3}$

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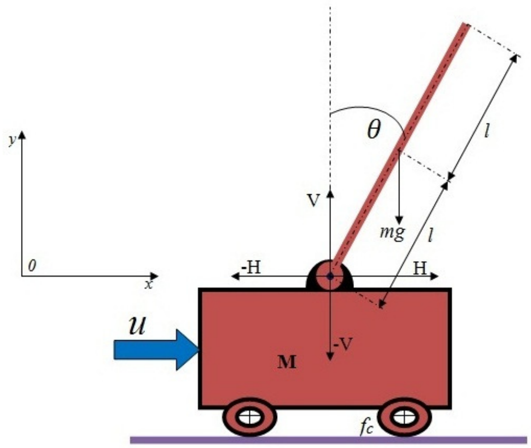

Figure 1 Inverted Pendulum system

Summing the moments about the center of gravity of the stick

$$I\frac{d^2\theta}{dt^2} = Vl\sin\theta - Hl\cos\theta \tag{1}$$

Applying Newton's second law along the *x* axis and y axis yields

$$\left.\begin{aligned} m\left(\ddot{x} + l(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta)\right) &= H \\ ml\left(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta\right) &= V - mg \end{aligned}\right\} \tag{2}$$

Finally, applying Newton's second law to the cart, the dynamic equations are obtained as follows:

$$M\frac{d^2x}{dt^2} = u - H - f_c \tag{3}$$

$$m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + f_c = u - M\ddot{x} \tag{4}$$

Substituting equation (2) and (3) in equation (1) gives following relation

$$I\ddot{\theta} = \left(mg - ml\dot{\theta}^2\cos\theta - ml\ddot{\theta}\sin\theta\right)l\sin\theta + f_c - (u - M\ddot{x})l\cos\theta \tag{5}$$

Putting value of $(u - M\ddot{x})$ in equation (5) and representing $a = \frac{1}{M+m}$ the equation (6) can be represented as

$$\ddot{x} = ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta - af_c + au \tag{6}$$

Putting equation (6) in equation (5) gives relation

$$\ddot{\theta} = \frac{mgl\sin\theta - m^2l^2a\dot{\theta}^2\sin(2\theta)/2 + mal\cos\theta(f_c - u)}{I - m^2l^2a\cos^2\theta + ml^2} \tag{7}$$

Inverted pendulum system is then represented in state space form by using the system equation which is derived as above by considering $x_1 = \theta, x_2 = \dot{\theta},\ x_3 = x$ and $x_4 = \dot{x}$ as,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{g\sin x_1 - mlax_2{}^2\sin(2x_1)/2}{4l/3 - mla\cos^2 x_1} \\ x_4 \\ \dfrac{-mag\sin(2x_1)/2 + alx_2{}^2\sin x_1\, 4m/3}{4/3 - ma\cos^2 x_1}\end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{-a\cos x_1}{4l/3 - mla\cos^2 x_1} \\ 0 \\ \dfrac{4a/3}{4/3 - ma\cos^2 x_1}\end{bmatrix}(u - f_c) \tag{8}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}x \tag{9}$$

III Linearized model of inverted pendulum

Jacobian linearization technique [2] linearize a set of nonlinear differential equation.
Set of Nonlinear differential equation is given by

$$\frac{dx_1}{dt} = f_1(x_1, \dots\dots\dots, x_n, u_1, \dots\dots\dots, u_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, \dots\dots\dots, x_n, u_1, \dots\dots\dots, u_n)$$

If $f_i,\ i = 1,2,3, \dots\dots, n,$ are continuous and differentialble. It can be represented in vector form as

$$\dot{x} = f(x, u)$$

Equilibrium state of above system is given by

$$f\left(x_{eq}, u_{eq}\right) = 0$$

where, x_{eq} *and* u_{eq} are corresponding equilibrium state and equilibrium input respectively.

Now,

$$x = x_{eq} + \Delta x, \qquad u = u_{eq} + \Delta u,$$

where Δx *and* Δu are perturbations around the equilibrium point.

From Taylor series expansion,

$$\frac{dx}{dt} = f\left(x_{eq}+\Delta x, u_{eq} + \Delta u\right)$$

$$= f\left(x_{eq},u_{eq}\right) + \frac{\partial f}{\partial x}\left(x_{eq},u_{eq}\right) + \frac{\partial f}{\partial u}\left(x_{eq},u_{eq}\right)\Delta u + \ higher\ order\ terms$$

Where,

$$\frac{\partial f}{\partial x}\left(x_{eq},u_{eq}\right) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\substack{x=x_{eq}\\u=u_{eq}}} = A$$

and,

$$\frac{\partial f}{\partial u}\left(x_{eq},u_{eq}\right) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}\right|_{\substack{x=x_{eq}\\u=u_{eq}}} = B$$

are Jacobian matrices of f with respect to x and u, evaluated at the equilibrium point (x_{eq}, u_{eq}).

Again, $f\left(x_{eq},u_{eq}\right) = 0$ and neglecting the higher order terms,

The linear approximation is given by

$$\frac{d}{dt}\Delta x = A\Delta x + B\Delta x$$

Now the system can be linearized as follows:

Equation (9) when linearized by using above technique at the equilibrium points

$$x_1 = 0, \quad x_2 = 0, \quad x_4 = 0, \quad u = 0$$

results in linear matrix defined in the form

$$\dot{x} = Ax + Bu$$

Where, matrix A and B defined as follows:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{10}$$

Where,

$$a_{11} = 0\,, a_{12} = 1,\ a_{13} = 0,\ a_{14} = 0$$

$$a_{21} = \frac{(f_c-u)sinx_1}{\frac{cos^2 x_1}{2} - \frac{22}{3}} - \frac{20cosx_1 - (x_2{}^2 \cos 2x_1/11)}{\frac{cos^2 x_1}{11} - \frac{4}{3}} - \frac{(f_c-u)\cos^2 x_1\, sinx_1}{\left(\frac{cos^2 x_1}{2} - \frac{22}{3}\right)^2} - \frac{(2cosx_1 sinx_1 20 sinx_1 - x_2{}^2 sin2x_1/22)}{11\left(\frac{cos^2 x_1}{11} - \frac{4}{3}\right)^2}$$

$$a_{22} = \frac{x_2 sin2x_1}{\frac{cos^2 x_1}{11} - \frac{4}{3}}, \quad a_{23} = 0, \quad a_{24} = 0$$

$$a_{31} = 0, \quad a_{32} = 1, \quad a_{33} = 0, \quad a_{34} = 1$$

$$a_{41} = \frac{\frac{10\cos 2x_1}{11} - \frac{2x_2{}^2 cosx_1}{33}}{\frac{cos^2 x_1}{11} - \frac{4}{3}} + \frac{24(f_c-u)cosx_1 sinx_1}{(3\cos^2 x_1 - 44)^2} + \frac{2cosx_1 sinx_1\left(\frac{5sin2x_1}{11} - \frac{2x_2{}^2 sinx_1}{33}\right)}{11\left(\frac{cos^2 x_1}{11} - \frac{4}{3}\right)^2}$$

$$a_{42} = -\frac{(8x_2 \sin x_1)}{\frac{33cos^2 x_1}{11} - \frac{133}{100}}, \quad a_{43} = 0, \quad a_{44} = 0$$

$$\tilde{B} = \begin{bmatrix} 0 \\ \frac{cosx_1}{\left(\frac{cos^2 x_1}{2} - \frac{22}{3}\right)} \\ 0 \\ \frac{-4}{3\left(\frac{cos^2 x_1}{2} - 44\right)} \end{bmatrix} \tag{11}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

IV Design of LQR

LQR is a state feedback optimal controller which minimize the performance index J of a controllable, linear system

$$\dot{x} = Ax + Bu \tag{13}$$

It gives the Riccati solution P which is symmetric positive definite matrix of the Algebraic Riccati Equation (ARE) given by (13)

$$PA + A^T P + Q - PBR^{-1}B^T = 0 \tag{14}$$

It minimize the Performance Index

$$J = \int_0^\infty (X^T Q X + u^T R u)dt \tag{15}$$

By control input $u = -Kx(t)$,

Where,

$$K = R^{-1} B^T P, \tag{16}$$

Q and R are positive semi definite matrices which are user defined. It is often that $Q = C^T C$, $R = I$.

V SIMULATION RESULTS

System parameters are taken for the inverted pendulum system as given in table 1.

TABLE 1 Systems Parameter

Symbol	Parameter name	Value
M	Mass of cart	*10 kg*
M	Mass of pendulum	*1 kg*
2l	Length of pendulum	*1 m*
G	Universal Gravitational Constant	$10\ m/s^2$

Putting the value of system parameter gives following results

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 16.0976 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.73177 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -0.1463 \\ 0 \\ 0.0976 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = 0$$

$$Q = C^T C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R = 1$$

By using the MATLAB command

$$[P, L, K] = care(A, B, Q, R)$$

The state feedback gain K is obtained as

$$K = [-244.6776 \quad -61.2078 \quad -1.0 \quad -5.1899].$$

So the corresponding closed loop system can be represented as

$$A_c = (A - BK)$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -19.7089 & -8.9572 & -0.1463 & -0.7595 \\ 0 & 0 & 0 & 1 \\ 23.1393 & 5.9715 & 0.0976 & 0.5063 \end{bmatrix}$$

Figure 2 represents the step response of the pendulum angle and cart position. With the proper optimal gain K it has been seen that the system states are stabilized within 20 seconds. The same responses are studied with different initial conditions (0.1 rad for angle and 1 m for cart position) and are shown in Figure 3. Further study have been made with a step disturbance which is shown in Figure 4 and proposed optimal controller able to stabilize the system successfully. It gives the Riccati solution P which is symmetric positive definite matrix of the Algebraic Riccati Equation (ARE) given by (13)

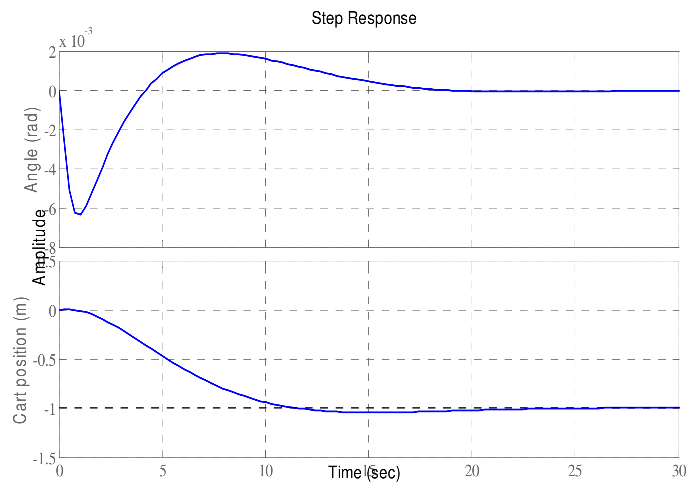

Figure 2 Step response of inverted pendulum

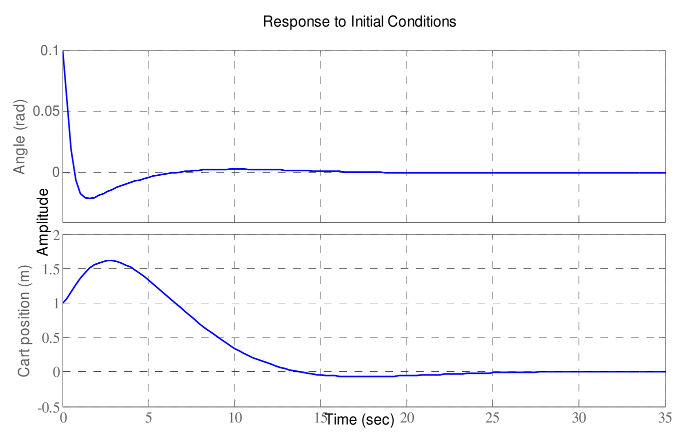

Figure 3 Step response of inverted pendulum having nonzero initial condition

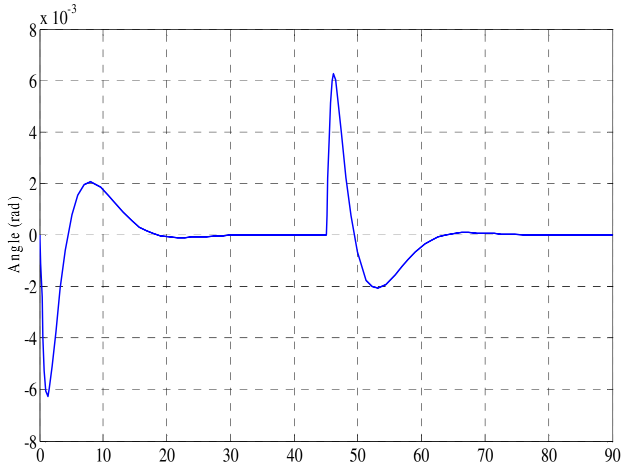

Figure 4 Response of inverted pendulum angle with the presence of step disturbance.

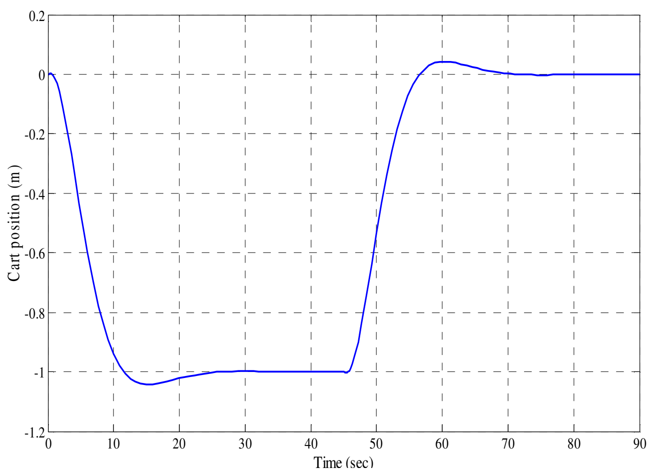

Figure 5 Response of cart position with the presence of step disturbance.

VI CONCLUSION

The non-linear model is linearized and then linear optimal control theory has been applied successfully on the system such that the wide variations of the angle and cart position do not lead the system get unstable. The proposed control approach shows good stability performance also with the presence of disturbance.

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