Advanced Topics in Applied Math
4/27/2017

Topic: Stratification

$$\Omega \ni X \sim \mu$$

$$\mathbf{E}f(x) = \int\limits_{\Omega} f(x)d\mu(x)$$

$$\bigcup_{i\in S} B_i = \Omega$$

$$\varphi_i : \Omega \to \mathbb{R}$$

$$i \in S \tag{1}$$

$$\sum_{i\in S} \varphi_i(x) = 1 \tag{2}$$

$$\varphi_i(x) = C^{-1}\mathbf{1}_{\mathbf{B_i}}(x)$$

$$\pi_i = \frac{\int_\Omega f(x)d\mu(x)}{\int_\Omega \varphi_i(x)d\mu(x)} = \int_\Omega f(x)d\mu_i(x)$$

$$p_{ij} = \frac{\int_\Omega \varphi_j(x)\varphi_i d\mu(x)}{\int_\Omega \varphi_i(x)d\mu(x)} = \int_\Omega \varphi_j(x)d\mu_i(x)$$

$$\pi_i p_{ij} = \int_\Omega \varphi_j(x)p_i d\mu(x) \geq 0 \overset{(1)}{\Rightarrow} \sum_{j\in S} p_{ij} = 1 \forall i \in S$$

$$\sum_{i\in S} \pi_i p_{ij} = \sum_{i\in S} \int_\Omega \varphi_j(x)\varphi_i d\mu(x) = \int_\Omega \varphi_j(x)d\mu(x) = \pi_j \to \pi_i$$

$$\sum_{i\in S} \pi_i \int_\Omega f(x)d\mu_i(x) = \mathbf{E}f(x)$$

$p_{ij} \to$ ergodic
Let $dZ(t) \to \mu$ measure on path space

$$dZ(t) = b(Z(t))dt + \sigma dWt$$

And let path be

$$X = Z[0, T]$$

$$\mathbf{P} = p(x, y)$$

$$\mathbf{P}(X_{t+1} = j | X_t = i) = p(x, y) \geq 0 \forall x, y \in \Omega$$

$$\sum_{j \in \Omega} p(x, y) = 1 \forall i \in \Omega$$

$$\mathbf{P}(X_0 = i_0 ... X_1 = i_1 ... X_T = i_T) = \pi_0(i_0) p(i_0 i_1) p(i_1 i_2) ... p(i_{T-1} i_T)$$

$$\mathbf{E} f(X_{[0,T]}) = \mathbf{E} f(\underline{X}) = \sum_{\underline{x} \in \Omega_T} f(\underline{x}) \mu(\underline{x})$$

Example:

$p(x, y) = min(e^{-\beta(\mathbf{E}(y) - \mathbf{E}(x))}, 1)$ Metropolis $\beta > 0$

$$a(x, y) = a(y, x) \geq 0$$

$$a(x, x) = 0$$

$$\sum_y a(x, y) = 1$$

$$x \neq y$$

$$p(x, x) = \sum_{y \neq x} p(x, y)$$

$$\mathbf{E} : \Omega \to \mathbb{R}$$

$$e^{-\beta \mathbf{E}(x)} p(x, y) = e^{-\beta \mathbf{E}(y)} p(y, x)$$

Take

$$a(x, y) = \begin{cases} \frac{1}{2} \quad if \quad y = x \pm 1 \\ 0 \qquad\qquad else \end{cases}$$

$$\mathbf{E}(x) = \frac{x^2}{2}$$

when $\beta >>>$ large

$$\mathbf{P}(X_T = x | X_0 = 0)$$

example, $\begin{cases} T = 20, & 4 \\ x = 10 & 2 \end{cases}$

Then

$$\mathbf{P}(X_4 = 2 | X_0 = 0)$$

$$\tilde{\varphi}_i(\underline{x}) = \tilde{\varphi}_i(x_1 ... x_4) = \begin{cases} \frac{1}{2} & if \quad x_4 = i, i+1 \\ 0 & \quad else \end{cases}$$

$$i \in Z$$

$$p(0, x)p(x_1, x_2)p(x_2, x_3)p(x_3, i)$$

$$\sum_{x_1 ... x_4 \in Z} p(0, x)p(x_1, x_2)p(x_2, x_3)p(x_3, x_4)$$

where $x_4 = i, i+1$

What's the way to make this work ?

(i) Need to generate path where you put certain conditions

(ii) Start at 0

(iii) End at the end state where the endpoint remains at i or $i+1$