Math 331 — Homework 5
Due: Friday, March 12

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(1) Prove that $\frac{3n+1}{2n+5} \to 3/2$ as $n \to \infty$.

Proof. Fix $\epsilon > 0$. We need to find $N > 0$ such that if $n \geq N$,

$$\left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| < \epsilon.$$

Since $\frac{3n+1}{2n+5} - \frac{3}{2} = \frac{-13}{2(2n+5)}$, we want to show:

$$\epsilon > \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| = \left| \frac{-13}{2(2n+5)} \right| = \frac{13}{2(2n+5)}$$

By the Archimedian Property, there exists $N \in \mathbb{N}$ such that $N > \frac{13}{4\epsilon}$. Therefore, for all $n \geq N$,

$$\frac{1}{n} \leq \frac{1}{N} \leq \frac{13}{4\epsilon}$$

Hence $\frac{13}{2n+5} < \frac{13}{2n} < 2\epsilon$ and so $\frac{13}{2(2n+5)} < \epsilon$. $\square$

(2) Prove that $\{\sqrt{1+1/n}\}_{n=1}^{\infty}$ converges to 1 as $n \to \infty$.
Hint: Use the identity $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = x - y$.

Proof. Fix $\epsilon > 0$. We need to find $N > 0$ such that if $n \geq N$,

$$\left| \sqrt{1+1/n} - 1 \right| < \epsilon.$$

Since $\sqrt{x} \geq 0$ for all $x \in \mathbb{R}$, we can say that for any $\delta > 0$,

$$\delta \cdot (\sqrt{1+1/n} + 1) \geq \delta > 0$$

for all n. Therefore, we can use the property $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = x - y$ and multiply both sides by $(\sqrt{1+1/n} + 1)$.

$$\begin{aligned} (\sqrt{1+1/n} - 1)(\sqrt{1+1/n} + 1) &< \delta \cdot (\sqrt{1+1/n} + 1) \\ (1 + 1/n) - 1) &< \delta \cdot (\sqrt{1+1/n} + 1) \\ 1/n &< \delta \cdot (\sqrt{1+1/n} + 1) \end{aligned}$$

(...stuck) $\square$

(3) Prove that given the sequence $\{a_n\}_{n=1}^{\infty}$, if $a_n \geq 0$ and $a_n \to L$ as $n \to \infty$, then $\sqrt{a_n} \to \sqrt{L}$ as $n \to \infty$.

Proof. Fix $\epsilon > 0$. We need to find $N > 0$ such that if $n \geq N$,

$$\left|\sqrt{a_n} - \sqrt{L}\right| < \epsilon.$$

We also know that for some $\delta > 0$:

$$0 < \delta + 2\sqrt{L} < \delta \cdot (\delta + 2\sqrt{L}).$$

Therefore, for all $n \geq N$,

$$\begin{aligned} a_n - L &< \delta \cdot (\delta + 2\sqrt{L}) \\ a_n &< \delta^2 + 2 \cdot \delta \cdot \sqrt{L} + L \\ &< (\delta + \sqrt{L})^2 \\ \sqrt{a_n} &< \delta + \sqrt{L}. \end{aligned}$$

(…stuck) □

(4) Prove the Squeeze Lemma: given sequences $\{b_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ such that $a_n \leq b_n \leq c_n$, if $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$.

Proof. Fix $\epsilon > 0$. We need to find $N > 0$ such that if $n \geq N$, $|b_n - L| < \epsilon$.

Since $a_n \to L$ as $n \to \infty$, we know that there exists an $N_a > 0$ such that for all $n \geq N_a$, $|a_n - L| < \epsilon$. Similarly, we know that since $c_n \to L$ as $n \to \infty$, there exists an $N_c > 0$ such that for all $n \geq N_c$, $|c_n - L| < \epsilon$.

Let $N = \max(N_a, N_c)$. Then if $n > N, n > N_a$ and $n > N_c$. But if $|a_n - L| < \epsilon$, then $L - \epsilon < a_n < L + \epsilon$ for all $n > N$. Similarly, $L - \epsilon < c_n < L + \epsilon$ for all $n > N$.

Since $a_n \leq b_n \leq c_n$,

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon,$$

so for all $n > N$, $L - \epsilon < b_n < L + \epsilon$. Hence, for all $n > N$, $|b_n - L| < \epsilon$.

□

(5) Given sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $a_n \to L_a$ and $b_n \to L_b$ as $n \to \infty$, then:

(a) if $c \in \mathbb{R}$, then $ca_n \to cL_a$ as $n \to \infty$;

Proof. Fix $\epsilon > 0$. We need to find $N > 0$ such that if $n \geq N$, $|ca_n - cL_a| < \epsilon$. Assume $c \neq 0$, since if $c = 0$ the result is trivial.
Because $|c| > 0$, we have that $\frac{\epsilon}{|c|} > 0$.

Since $a_n \to L_a$ as $n \to \infty$, there exists an $N > 0$ such that for all $n \geq N$, $|a_n - L_a| < \frac{\epsilon}{|c|}$. Equivalently, we can say that:

$$\begin{aligned}\epsilon > |c| \cdot |a_n - L_a| &= |c \cdot (a_n - L_a)| \\ &= |ca_n - cL_a|.\end{aligned}$$

Hence, for all $n > N$, $|ca_n - cL_a| < \epsilon$, and thus $ca_n \to cL_a$ as $n \to \infty$. $\square$

(b) if $L_b \neq 0$, then $\dfrac{a_n}{b_n} \to \dfrac{L_a}{L_b}$ as $n \to \infty$.

Proof. Fix $\epsilon > 0$. We need to find $N > 0$ such that if $n \geq N$, $|\frac{a_n}{b_n} - \frac{L_a}{L_b}| < \epsilon$. Since $L_b \neq 0$, it follows that $|L_b| > 0$. Now consider:

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{L_a}{L_b}\right| &= \left|\frac{a_n \cdot L_b - L_a \cdot b_n}{b_n \cdot L_b}\right| \\ &= \frac{|a_n L_b - a_n b_n + a_n b_n - L_a b_n|}{|b_n||L_b|}\end{aligned}$$

From the lemma provided in the sketch for this proof, since $b_n \to L_b$ as $n \to \infty$, there exists some $n_0 \geq 0$ such that for all $n \geq n_0$, $|b_n - L_b| > \frac{|L_b|}{2}$. (...stuck) $\square$