Math 331 — Homework 1
Due: Wednesday, February 3

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(1) Given sets A and B, prove that $A \cap B = A$ if and only if $A \subset B$.

Proof. Suppose $A \subset B$. Then for every $a \in A, a \in B$. Given $x \in A \cap B, x \in A$ and $x \in B$. Therefore for any $x \in A \cap B, x \in A$. On the other hand, given an $x \in A$, since $A \subset B$, then $x \in B$ as well, so $x \in A \cap B$. Thus, $A \cap B = A$ if $A \subset B$.

Conversely, suppose $A \cap B = A$. We want to show that $A \subset B$. Let $x \in A \cap B$, then $x \in A$ and $x \in B$ by the definition of intersection. Because $A \cap B = A$, then either $A \subset B$ or $B = \emptyset$. If $B = \emptyset$ then $A \cap B = \emptyset$, which is a contradiction because we know $x \in A \cap B$. On the other hand, let $x \in A$. Since $A \cap B = A$, then $x \in A \cap B$ as well, which implies that $x \in A$ and $x \in B$. Hence for every $a \in A, a \in B$. Thus, $A \subset B$ if $A \cap B = A$. □

(2) Prove that if g is a function and A and B are sets contained in the domain of g, then $g(A \cap B) \subset g(A) \cap g(B)$. Give an example to prove that equality need not hold.

Proof. Note that $g(A \cap B)$, the image of $A \cap B$ under g, is a set. By the definition of the image of a function,

$$g(A \cap B) = \{g(x) : x \in (A \cap B)\}$$

Let c be an element in the domain of g such that $g(c) \in g(A \cap B)$. By the definition of the image of g, $c \in (A \cap B)$. This means that $c \in A$ as well as $c \in B$, by the definition of intersection. Hence, $g(c)$ must be an element of both $g(A)$ and $g(B)$, so $g(c) \in g(A) \cap g(B)$. □

(3) Prove that for all natural numbers n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Proof. Instead of using induction, we will prove this using Gauss' method of pairing numbers. Fix $n \in \mathbb{N}$. We will begin by choosing a name for the left side of the equation:

$$x = \sum_{k=1}^{n} k = 1 + 2 + \ldots + (n-1) + n$$

First, we will multiply x by 2:

$$\begin{aligned}
2x &= 2[1 + 2 + \ldots + (n-1) + n] \\
2x &= [1 + 2 + \ldots + (n-1) + n] + [1 + 2 + \ldots + (n-1) + n]
\end{aligned}$$

Notice that we can reverse the second sum, and pair it off with a summand from the first sum:

$$\begin{aligned}
2x &= [1 + 2 + \ldots + (n-1) + n] + [1 + 2 + \ldots + (n-1) + n] \\
&= [1 + 2 + \ldots + (n-1) + n] + [n + (n-1) + \ldots + 2 + 1] \\
&= [1 + 2 + \ldots + (n-1) + n] \\
&\quad [n + (n-1) + \ldots + 2 + 1] \\
&= (1 + n) + [2 + (n-1)] + \ldots + [(n-1) + 2] + (n+1) \\
&= (n+1) + (n+1) + \ldots + (n+1) + (n+1)
\end{aligned}$$

Here we have n copies of $(n+1)$, so:

$$2x = n(n+1)$$

Solving the equation for x and substituting back in $\sum_{k=1}^{n} k$ gets us:

$$\begin{aligned}
x &= \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2}
\end{aligned}$$

$\square$

(4) Prove that for $n \in \mathbb{N}$, $2^{n-1} \leq n!$.

Proof. We will prove this by induction. First we will show it's true when $n = 1$.

$$2^{n-1} = 2^{1-1} = 2^0 = 1 = n! = 1$$

So the inequality holds when $n = 1$. Now suppose the inequality is true for some fixed $n \in \mathbb{N}$. Then we want to show $2^{(n+1)-1} = 2^n \leq (n+1)!$. By our induction hypothesis:

$$2^{n-1} \cdot (n+1) \leq n! \cdot (n+1)$$

Further, for all $n \in \mathbb{N}$, $2 \leq n+1$. Multiplying this by the inequality in our inductive hypothesis gives us:

$$2(2^{n-1}) \leq 2^{n-1} \cdot (n+1)$$

Note that $2(2^{n-1}) = 2^{n-1} + 2^{n-1} = 2^n$. So we have:

$$\begin{aligned} 2^n &\leq 2^{n-1} \cdot (n+1) \leq n! \cdot (n+1) \\ 2^n &\leq n! \cdot (n+1) \\ 2^n &\leq (n+1)! \end{aligned}$$

Therefore, by induction, the inequality holds for all natural numbers n. $\square$

(5) Prove that given any natural number $n \geq 8$, there exist non-negative integers j and k such that $n = 3j + 5k$.

Hint: If you use induction, choose your base case carefully.

Proof. We will prove this by induction. Note that the Principle of Mathematical Induction does not *require* that the base case begins on 1, so we can just as easily start with a different number for this problem.

First we will show the equality holds for $n = 8$. When $n = 8$, j and k can both equal 1, so:

$$3j + 5k = 3(1) + 5(1) = 3 + 5 = 8$$

Now suppose the equality is true for some fixed $n \in \mathbb{N}$, where $n > 8$. We want to show that $n + 1 = 3j' + 5k'$ for some non-negative $j', k' \in \mathbb{Z}$.

Consider j and k from the inductive hypothesis. We will proceed by cases:

Case 1: $k \neq 0$. Since at least one 5 is used in the summation, we can replace this 5 with two 3's to make it equal $n + 1$. In other words, set $k' = k - 1$ and $j' = j + 2$ and you will get:

$$\begin{aligned} 3j' + 5k' &= 3(j+2) + 5(k-1) \\ &= 3j + 6 + 5k - 5 \\ &= 3j + 5k + 1 \\ &= n + 1 \end{aligned}$$

Case 2: $k = 0$. Since there are no 5's in n, then $n = 3j$ for some non-negative $j \in \mathbb{Z}$. Because $n \geq 8$, there must be at least three 3's in n. If you replace these three 3's with two 5's, you will end up with $n + 1$. In other words, set $j' = j - 3$ and $k' = 2$ and you will get:

$$\begin{aligned} 3j' + 5k' &= 3(j-3) + 5(2) \\ &= 3j - 9 + 10 \\ &= 3j + 1 \\ &= n + 1 \end{aligned}$$

Hence, regardless of the value of k, you can find a non-negative $j', k' \in \mathbb{Z}$ such that $n + 1 = 3j' + 5k'$. Therefore, by induction, the equality holds for all $n \geq 8$. $\square$