Math 331 — Homework 8
Due: Monday, April 26

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(1) Given a function $f : D \to \mathbb{R}$, suppose that for some $x \in D$, and every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \to x$ as $n \to \infty$, $f(x_n) \to f(x)$ as $n \to \infty$. Prove that f is continuous at x.

Proof. We will proceed by contraposition. We want to show that if f is discontinuous at x, then $f(x_n) \not\to f(x)$ as $n \to \infty$. If f is discontinuous at x, then there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exists a $x_m \in D$ such that $|x_n - x_m| < \delta$ but $|f(x_n) - f(x_m)| \geq \epsilon$.

Fix $\delta > 0$ and let $x_m \in D$ be such that $|x_n - x_m| < \delta$. □

(3) Complete the proof of the extreme value theorem given in class by showing that there exists a point x_{min} such that $f(x_{min}) = y_{min}$.

Proof. We have already shown that given $f : [a, b] \to \mathbb{R}$ where f is continuous everywhere on $[a, b]$, the set of values $\{f(x) : x \in [a, b]\}$ is bounded. By the completeness axiom, its supremum and infimum exist. Denote these values by y_{max} and y_{min}. We have that $y_{min} \leq f(x) \leq y_{max}$ for all $x \in [a, b]$.

To finish the proof we need to find x_{min}, $x_{max} \in [a, b]$ such that $f(x_{min}) = y_{min}$. Construct $\{x_n\}_{n=1}^{\infty}$ as follows. For all $n \in \mathbb{N}$, there exists an $x_n \in [a, b]$ such that $y_{min} < f(x_n) < y_{min} + \frac{1}{n}$, $f(x_n) \geq y_{min}$ since y_{min} is the infimum. If x_n does not exist, then $f(x) \geq y_{min} + \frac{1}{n}$ for all $x \in [a, b]$. This contradicts the fact that y_{min} is the greatest lower bound.

Since $y_{min} + \frac{1}{n} \to y_{min}$ as $n \to \infty$, by the squeeze lemma, $f(x_n) \to y_{min}$ as $n \to \infty$. This gives us a sequence $\{x_n\}_{n=1}^{\infty} \subset [a, b]$. Since this sequence is bounded, it has a convergent subsequence by the Bolzano-Weierstrauss theorem. So there exists $\{x_{n_k}\}_{k=1}^{\infty}$ and x_{min} such that $x_{n_k} \to x_{min}$ as $k \to \infty$.

Claim: $x_{min} \in [a, b]$, since $a \leq x_{n_k} \leq b$ by a homework problem $a \leq x_{min} \leq b$. Therefore $f(x_{min})$ exists and by the lemma from class since f is continuous at x_{min}, $f(x_{n_k}) \to y_{min}$ as $k \to \infty$ by the other in class lemma $f(x_{min}) = y_{min}$. □

(4) Complete the proof of the intermediate value theorem by doing the case where $x_{max} < x_{min}$.

Proof. Case $x_{max} < x_{min}$: Define $E = \{z \in [a, b] : f(z) > 0\}$. Since $f(x_{max}) > 0$, E is nonempty, and since it's contained in $[x_{min}, x_{max}]$, E is bounded. By the completeness axiom, $\inf E$ exists and $\inf E \geq x_{min}$. Let $x = \inf E$. I claim $f(x) = 0$.

Suppose to the contrary that $f(x) \neq 0$. If $f(x) > 0$, by continuity there exists a $\delta > 0$ such that if $|x - y| < \delta$, $|f(x) - f(y)| < \frac{|f(x)|}{2}$. This implies $f(y) - f(x) < \frac{-f(x)}{2}$ and so $f(y) < \frac{-f(x)}{2} < 0$. We can find points y such that $x - \delta < y < x$ and $f(y) < 0$. So $y \in E$, contradicting that $x = \inf E$.

If $f(x) < 0$, since $f(x_{max}) > 0$, $x < x_{max}$. □

(5) Use the definition to prove that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^{1/3}$, is differentiable at every $x \neq 0$.

Proof. Fix $x \in \mathbb{R}\backslash\{0\}$. We need to show $\lim_{y \to x} \frac{f(x)-f(y)}{x-y} = \frac{1}{3x^{2/3}}$. In other words:

$$\lim_{y \to x} \frac{f(x) - f(y)}{x - y} = \lim_{y \to x} \frac{x^{1/3} - y^{1/3}}{x - y}.$$

Using the limit rules, we can write this as:

$$\lim_{y \to x} \frac{x^{1/3} - y^{1/3}}{x - y} = \lim_{y \to x} \frac{1}{x^{2/3} + x^{1/3} \cdot y^{1/3} + y^{2/3}}.$$

By the continuity of $\frac{1}{y^{1/3}}$ and $\frac{1}{y^{2/3}}$, we have:

$$\begin{aligned} \lim_{y \to x} \frac{1}{x^{2/3} + x^{1/3} \cdot y^{1/3} + y^{2/3}} &= \frac{1}{x^{2/3} + x^{1/3} \cdot x^{1/3} + x^{2/3}} \\ &= \frac{1}{x^{2/3} + x^{2/3} + x^{2/3}} \\ &= \frac{1}{3x^{2/3}}. \end{aligned}$$

□

(6) Prove that if $f : D \to \mathbb{R}$ is uniformly continuous, then given any Cauchy sequence $\{x_n\} \subset D$, the sequence $\{f(x_n)\}$ is also a Cauchy sequence.

Hint: Use the definition of Cauchy sequence and the definition of uniform continuity. Choose the value of "ϵ" in the definition carefully!

Proof. Let $f : D \to \mathbb{R}$ be a uniformly continuous function, and $\{x_n\} \subset D$ be a Cauchy sequence. Fix $\epsilon > 0$. By the definition of a Cauchy sequence, there exists an $N > 0$ such that for all $n, m \geq N$, $|x_n - x_m| < \epsilon$. Additionally, by the definition of a uniformly continuous function, there exists a $\delta > 0$ such that for all x_n, $x_m \in D$ such that $|x_n - x_m| < \delta$, $|f(x_n) - f(x_m)| < \epsilon$.

We want to show that there exists an $N > 0$ such that for all $n, m \geq N$, $|f(x_n) - f(x_m)| < \epsilon$. But we already know from the definition of a uniformly continuous function that for all x_n, $x_m \in D$ such that $|x_n - x_m| < \delta$, $|f(x_n) - f(x_m)| < \epsilon$. Let $N = \min(n, m)$ where $|x_n - x_m| < \delta$. Then for all $n, m \geq N$, $|f(x_n) - f(x_m)| < \epsilon$, and thus $\{f(x_n)\}$ is a Cauchy sequence. □

(8) (Extra Credit) Construct a function $S : \mathbb{R} \to \mathbb{R}$, that is continuous only at 0.

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$