Math 331 — Homework 4
Due: Friday, February 26

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(1) Prove that for any natural number N, if $\{I_1, I_2, \ldots, I_N\}$ is a finite collection of of closed intervals, then the set

$$E = \bigcup_{n=1}^{N} I_n$$

is compact: i.e., every open cover of E has a finite subcover.

Hint: use induction, the Heine-Borel theorem, and the proof of the Heine-Borel theorem.

Proof. First we will show that every I_n is both closed and bounded. Each I_n is closed by their definition, and bounded by the fact that they are closed.

We will proceed by induction on N. As a base case, if $N = 2$, then we're working with the set $I_1 \cup I_2$. Let O be an open cover of this set. Then O must cover both I_1 and I_2. Since every I_n is closed and bounded, by the Heine-Borel theorem both I_1 and I_2 are compact, there must be a finite subcover of each. Let S_1 be the finite subcover of I_1, and S_2 be the finite subcover of I_2. Then $S_1 \cup S_2$ is finite and covers $I_1 \cup I_2$.

Assume that the set

$$A = \bigcup_{n=1}^{k} I_n$$

is compact for some integer k. In other words, every open cover of A has a finite subcover. We want to show that every open cover of the set

$$B = \bigcup_{n=1}^{k+1} I_n = A \cup I_{k+1}$$

also has a finite subcover.

Let O_B be an open cover of B. Then O_B must cover both A and I_{k+1}. By our inductive hypothesis, every open cover of the set A contains a finite subcover. Since I_{k+1} is both closed and bounded, by the Heine-Borel theorem every open cover of it contains a finite subcover. Let S be the union of the finite subcover of the cover O_B for the sets A and I_{k+1}. Since both subcovers are finite and cover $A \cup I_{k+1}$, then the set B must be compact.

$\square$

(2) Construct a counter-example to show that previous result is not true if E is the union of a countable set of closed intervals $\{I_n : n \in \mathbb{N}\}$.

Proof. Define I_n as $[-n, n]$. Then each I_n is closed but

$$E = \bigcup_{n \in \mathbb{N}} I_n$$

is unbounded. Hence, by the Heine-Borel theorem, E is not compact. $\square$