Math 331 — Homework 7
Due: Monday, April 19

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(1) Use the definition of continuity to prove that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 1$, is continuous everywhere.

Proof. Fix $\epsilon > 0$ and $x \in \mathbb{R}$. We want to show that there exists a $\delta > 0$ such that for every $y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Let $\delta = \min(1, \frac{\epsilon}{(2|y|+1)^2})$. Fix any $y \in \mathbb{R}$ such that $|x-y| < \delta$. Since $|x-y| < 1$, $|y| - |x| < 1$, and so $|y| < |x| + 1$. Observe that:

$$|f(x) - f(y)| = |x^3 + 1 - y^3 - 1| = |(x-y)(x^2 + xy + y^2)|$$

and

$$|(x-y)(x^2 + xy + y^2)| \le |(x-y)(x^2 + 2xy + y^2)| = |(x-y)(x+y)^2|$$

and finally,

$$|(x-y)(x+y)^2| < |x-y| \cdot (2|y|+1)^2 < \frac{\epsilon}{(2|y|+1)^2} \cdot (2|y|+1)^2 = \epsilon.$$

$\square$

(2) Use the definition of continuity to prove that $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$, is continuous for all $x \ge 0$.

Proof. Fix $\epsilon > 0$ and $x \ge 0$. We want to show that there exists a $\delta > 0$ such that for every $y \in [0, \infty)$, if $|x - y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < \epsilon$.

We will proceed by cases.

Case 1: $x = 0$. Let $\delta = \epsilon^2$. Then if $|x - y| < \delta$,

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{|x-y|}{\sqrt{x} + \sqrt{y}},$$

but since $x = 0$, we have:

$$\frac{|x-y|}{\sqrt{x} + \sqrt{y}} = \frac{y}{\sqrt{y}} = \sqrt{y},$$

and

$$\sqrt{y} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon.$$

Case 2: $x > 0$. Let $\delta = \epsilon \cdot \sqrt{x}$. Then if $|x - y| < \delta$,

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{|x-y|}{\sqrt{x} + \sqrt{y}},$$

and we have that
$$\frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{\sqrt{x}},$$
and
$$\frac{|x-y|}{\sqrt{x}} < \frac{\delta}{\sqrt{x}} = \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon.$$
$\square$

(4) Given a function $f : D \to \mathbb{R}$, suppose $\lim_{x\to a} f(x) = L$. Prove the following lemmas:

(a) There exists $\delta > 0$ such that for all for all $x \in D$, $0 < |x-a| < \delta$, $|f(x)| < |L| + 1$.

Proof. Fix $x \in D$. Let $\epsilon = 1$. By the definition of a limit, there exists a δ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < 1$. Since $|f(x) - L| < 1$, we have that $L - 1 < f(x) < L + 1$, so therefore $|f(x)| \leq \max(|L-1|, |L+1|)$. Hence, if $|f(x) - L| < 1$, then $|f(x)| < |L| + 1$. $\square$

(5) Given functions f, $g : D \to \mathbb{R}$ such that
$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M,$$
Prove the following:

(a) $\lim_{x\to a}(fg)(x) = LM$.

Proof. Fix $\epsilon > 0$. We want to find a $\delta > 0$ such that for all $x \in D$, if $0 < |x-a| < \delta$, then $|f(x)\cdot g(x) - LM| < \epsilon$. By problem (4a), there exists a $\delta_0 > 0$ such that if $0 < |x-a| < \delta_0$, then $f(x) \leq |L| + 1$.
Let $\epsilon_0 = \frac{\epsilon}{2|L|} > 0$. Then there exists a $\delta_1 > 0$ where if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2|M|}$. Similarly there exists a $\delta_2 > 0$ where if $0 < |x-a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. Let $\delta = min(\delta_0, \delta_1, \delta_2)$, then if $0 < |x-a| < \delta$, then $|f(x)| \leq |L| + 1$ and $|f(x) - L| < \frac{\epsilon}{2|L|}$ and $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$.
Observe that
$$|f(x)\cdot g(x) - LM| = |f(x)\cdot g(x) - f(x)M + f(x)M - LM|.$$
By the triangle inequality,
$$|f(x)\cdot g(x) - LM| \leq f(x)\cdot g(x) - f(x)M| + |f(x)M - LM|$$
or
$$|f(x)\cdot g(x) - LM| \leq |f(x)|\cdot|g(x) - M| + |M|\cdot|f(x) - L|.$$
But now we can say:
$$|f(x)|\cdot|g(x) - M| + |M|\cdot|f(x) - L| < (|L|+1)\frac{\epsilon}{2(|L|+1)} + |M|\frac{\epsilon}{2|M|},$$

and

$$(|L|+1)\frac{\epsilon}{2(|L|+1)}+|M|\frac{\epsilon}{2|M|}=\epsilon.$$

$\square$