Math 331 — Homework 2
Due: Wednesday, February 10

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(1) Given x, $y > 0$, prove the following:
(a) $x < y$ if and only if $x^2 < y^2$.

Proof. Let $x < y$. It follows from the ordering axioms that $xx < yx$ and $xy < yy$. Therefore, by the Transitive Property, $xx < yy$, or, $x^2 < y^2$.
On the other hand, suppose on the contrary that $x \geq y$. We want to show that this implies $x^2 \geq y^2$. By an argument similar to the implication, by the ordering axioms we have that $xx \geq yx$ and $xy \geq yy$. We can once again invoke the Transitive Property to get $xx \geq yy$ or $x^2 \geq y^2$. □

(b) $x < y$ if and only if $x^{1/2} < y^{1/2}$.

Proof. Let $x^{1/2} < y^{1/2}$. Then by (1a) we have:

$$\left(x^{1/2}\right)^2 < \left(y^{1/2}\right)^2$$
$$x < y$$

On the other hand, let $x < y$. Suppose on the contrary that $y^{1/2} \leq x^{1/2}$. Then by (1a) we have:

$$\left(y^{1/2}\right)^2 \leq \left(x^{1/2}\right)^2$$
$$y \leq x$$

This contradicts the first hypothesis. □

(2) Prove Bernoulli's inequality: for $n \in \mathbb{N}$ and $x > -1$, $(1 + x)^n \geq 1 + nx$.

Proof. Fix $x > -1$. We will proceed by induction on n. As a base case, when $n = 1$, we have:

$$(1 + x)^0 \geq 1 + 0x$$
$$1 \geq 1$$

Which is a true statement. Suppose the statement $(1 + x)^k \geq 1 + kx$ is true. Then we want to show that $(1+x)^{(k+1)} \geq 1+(k+1)x$ is also true. Because we've fixed $x > -1$, we can say that the statement $(1+x) \geq 0$ is true. By the ordering axioms, we now have:

$$\begin{aligned}
(1+x)^k &\geq 1+kx \\
(1+x)^k(1+x) &\geq (1+kx)(1+x) \\
(1+x)^{(k+1)} &\geq 1+x+kx+kx^2 \\
&\geq 1+(k+1)x+kx^2
\end{aligned}$$

This isn't exactly what we were looking for, because of the extraneous kx^2 term. Consider, however, that since we've fixed $x > -1$,

$$1+(k+1)x+kx^2 \geq 1+(k+1)x$$

Therefore we can say that the statement $(1+x)^{(k+1)} \geq 1+(k+1)x$ is true, and thus $(1+x)^k \geq 1+kx$ is true for all $k \geq 1$. □

(3) Given non-negative numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, $n \geq 1$, prove the Cauchy-Schwartz inequality

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

Hint: Denote the terms on the right-hand side by A and B, let $c_k = Ba_k - Ab_k$, and evaluate

$$0 \leq \sum_{k=1}^{n} c_k^2.$$

(Why is this inequality true?) You will have to re-arrange terms to get the desired answer—you cannot work "left to right."

Proof. Let $A = (\sum_{k=1}^{n} a_k^2)^{1/2}$ and $B = (\sum_{k=1}^{n} b_k^2)^{1/2}$. Rewrite the expression as the following:

$$\begin{aligned}
\sum_{k=1}^{n} a_k b_k &\leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}. \\
&\leq AB
\end{aligned}$$

Let $c_k = Ba_k - Ab_k$. Consider the expression:

$$\begin{aligned}
0 &\leq \sum_{k=1}^{n} c_k^2 \\
0 &\leq \sum_{k=1}^{n} (Ba_k - Ab_k)^2 \\
0 &\leq \sum_{k=1}^{n} (Ba_k)^2 - 2(Ba_k Ab_k) - (Ab_k)^2 \\
0 &\leq \sum_{k=1}^{n} (Ba_k)^2 - \sum_{k=1}^{n} 2(Ba_k Ab_k) - \sum_{k=1}^{n} (Ab_k)^2 \\
2\sum_{k=1}^{n} (Ba_k Ab_k) &\leq \sum_{k=1}^{n} (Ba_k)^2 - \sum_{k=1}^{n} (Ab_k)^2 \\
&\leq \vdots \quad (?)
\end{aligned}$$

$\square$

(4) Given $a,\, b \in \mathbb{R}$, prove the following:

(a) $|a| \geq a$;

Proof. Let $a \in \mathbb{R}$. We will proceed by cases:

(i) If $a \geq 0$, then $|a| = a$, so $|a| \geq a$.
(ii) If $a \leq 0$, then $|a| \geq 0 \geq a$ and thus $|a| \geq a$.

Hence, regardless of whether a is positive or negative, the assertion holds. $\square$

(b) $|ab| = |a||b|$;

Proof. Let $a,\, b \in \mathbb{R}$. We will proceed by cases:

(i) If $a,\, b \geq 0$, then $|a| = a$ and $|b| = b$, so $ab \geq 0$ and $|ab| = ab = |a||b|$.
(ii) If $a,\, b \leq 0$, then $|a| = -a$ and $|b| = -b$, so $ab \geq 0$ and $|ab| = ab = (-a)(-b) = |a||b|$.
(iii) If $a \geq 0$ and $b \leq 0$, then $|a| = a$ and $|b| = -b$, so $ab \leq 0$ and $|ab| = -ab = a(-b) = |a||b|$.
(iv) If $a \leq 0$ and $b \geq 0$, then $|a| = -a$ and $|b| = b$, so $ab \leq 0$ and $|ab| = -ab = (-a)b = |a||b|$.

Hence, regardless of whether a is positive or negative, the assertion holds.

$\square$

(c) $||a| - |b|| \leq |a - b|$.

Proof. From the Triangle Inequality we have:

$$|x + y| \leq |x| + |y|$$

For all x, $y \in \mathbb{R}$. Let $x = (a - b)$ and $y = b$. Then we get:

$$|(a - b) + b| = |a| \leq |(a - b)| + |b|$$
$$|a| - |b| \leq |a - b|$$

$\square$

(5) Use the Cantor diagonalization argument to prove that the set of sequences

$$S = \big\{\{a_n\}_{n=1}^{\infty} : a_n \in \mathbb{Z}, 0 \leq a_n \leq 9\big\}$$

is uncountable.

Proof. We will prove this by contradiction. Suppose the set S is not uncountable. Then it is either countable or finite. However, S is not finite. Consider the family of sequences:

$$V = \{\{1, 0, 0, \ldots\}, \{0, 1, 0, \ldots\}, \{0, 0, 1, \ldots\}, \ldots\}$$

This set is clearly a subset of S. Since V is countable, S cannot be finite. Therefore S is countable. In other words, we can enumerate $S = \{a_k\}_{k=1}^{\infty}$ where $V = \{a_n^k\}_{n=1}^{\infty}, a_n^k = 0, 1$. Define a new sequence $\{b_n\}_{n=1}^{\infty}$ where $b_n = 9 - a_n^n, n \in \mathbb{N}, 0 \leq b_n \leq 9$. For example:

$$\begin{aligned} A_1 &= \{a_1^1, a_2^1, a_3^1, \ldots\} \\ A_2 &= \{a_1^2, a_2^2, a_3^2, \ldots\} \\ A_3 &= \{a_1^3, a_2^3, a_3^3, \ldots\} \\ B_1 &= \{b_1, b_2, b_3, \ldots\} \\ &= \{9 - a_1^1, 9 - a_2^1, 9 - a_3^1, \ldots\} \end{aligned}$$

Further, $b_n \neq a_n^n$ for any n. I claim that $\{b_n\}_{n=1}^{\infty}$ is different from every sequence $\{a_n^k\}_{n=1}^{\infty}$, because $\{b_n\}_{n=1}^{\infty}$ differs from A_k in the kth element. But this contradicts the fact that A_k's are an enumeration of S, i.e. every element of S is given by one of the A_k's. Hence, our original assumption is false and S is uncountable. $\square$

(7) (Extra Credit) Define the binomial coefficients by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad 0 \leq k \leq n.$$

(Recall that $0! = 1$.) Prove Pascal's identity: for each $n \geq 0$ and for each k, $1 \leq k \leq n$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof. By the definition of the binomial coefficient:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(n-k)!k!} \\
&= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n+1-k)!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

$\square$

(8) (Extra Credit) Prove the binomial theorem: given real numbers a and b, and $n \geq 0$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. We will proceed by induction on n. As a base case, when $n = 1$, the assertion holds, because:

$$(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k}.$$

Suppose that the assertion is true for $n \geq 1$. Then:

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)\left[\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\right] \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k} \\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k} \\
&= \vdots \quad (?) \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}.
\end{aligned}$$

$\square$