Math 331 — Homework 6
Due: Friday, March 19
Pushed back to Monday, March 29

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(1) Prove that if $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then it is bounded.
Hint: follow the proof that convergent sequences are bounded.

Proof. If $\{a_n\}_{n=1}^{\infty}$ is Cauchy, then for any $\epsilon > 0$, there exists an $N > 0$ such that for all n, $m \geq N$, $|a_n - a_m| < \epsilon$. If $\epsilon = 1$, there exists an N_1 such that for all m, $n > N_1$, $|a_n - a_m| < 1$.

Using the triangle inequality, we know that for all a, b, $c \in \mathbb{Z}$, the following holds:

$$\big||a-c| - |b-c|\big| \leq |a-b|.$$

Let $m = N_1 + 1$. By the triangle inequality, for any $n > N_1$, we have:

$$\begin{aligned}\big||a_n - a_{N_1}| - |a_n - a_m|\big| &\leq |a_{N_1} - a_m| \\ \big||a_n - a_{N_1}| - |a_n - a_{(N_1+1)}|\big| &\leq |a_{N_1} - a_{(N_1+1)}| \\ &< 1\end{aligned}$$

Therefore, $|a_n - a_{N_1}| < |a_n - a_{(N_1+1)}| + 1$.
Let $M = \max\big(|a_{N_1} - a_1|, |a_{N_1} - a_2|, \ldots, |a_{N_1} - a_{N_1}|, |a_{N_1} - a_{(N_1+1)}|\big) + 1$.
Then for all $n \in \mathbb{N}$, $|a_n - a_{N_1}| \leq M$. Hence, $\{a_n\}_{n=1}^{\infty}$ is bounded by M. $\square$

(2) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and find its sum.

Hint: Use induction and the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ to find the partial sums.

Proof. Let $\{S_N\}_{n=1}^{\infty}$ be the series of partial sums where $S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}$. We want to show that $S_N \to S$ as $n \to \infty$.

I claim that:

$$S_N = 1 - \frac{1}{N+1}.$$

We will proceed by induction. When $N = 1$, then we have:

$$\frac{1}{1 \cdot (1+1)} = 1 - \frac{1}{1+1} = \frac{1}{2}.$$

Suppose that $S_k = 1 - \frac{1}{k+1}$, we want to show that $S_{k+1} = 1 - \frac{1}{k+2}$. We have that $S_k = 1 - \frac{1}{k+1}$, so S_{k+1} must equal $1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$. But since we can write $\frac{1}{n(n+1)}$ as $\frac{1}{n} - \frac{1}{n+1}$, we have the following:

$$\begin{aligned} S_{k+1} &= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= 1 - \frac{1}{k+1} + \frac{1}{k+1} - \frac{1}{k+2} \\ &= 1 - \frac{1}{k+2} \\ &= 1 - \frac{1}{(k+1)+1}. \end{aligned}$$

Hence, our assertion that $S_N = 1 - \frac{1}{N+1}$ holds. □

(3) Given two convergent series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$, and any constant $c \in \mathbb{R}$, prove the following:

(a) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

Proof. Since $\sum_{n=1}^{\infty} a_n$ converges, we have a series of partial sums $\{S_N\}_{n=1}^{\infty}$ where:

$$S_N = \sum_{n=1}^{N} a_n.$$

Further, we know that $S_N \to S$ as $n \to \infty$.
Since we know that $c\{S_N\}_{n=1}^{\infty} = \{cS_N\}_{n=1}^{\infty}$, it follows that:

$$c\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ca_n.$$

□

(b) $\displaystyle\sum_{n=1}^{\infty} \big(a_n + b_n\big) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

Proof. Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, we have two series of partial sums $\{S_N^a\}_{n=1}^{\infty}$ and $\{S_N^b\}_{n=1}^{\infty}$ where:

$$S_N^a = \sum_{n=1}^{N} a_n \text{ and } S_N^b = \sum_{n=1}^{N} b_n$$

Further, we know that $S_N^a \to S^a$ and $S_N^b \to S^b$ as $n \to \infty$.
We know from the arithmetic of series that:

$$\{S_N^a\}_{n=1}^{\infty} + \{S_N^b\}_{n=1}^{\infty} = \{S_N^a + S_N^b\}_{n=1}^{\infty},$$

and that $\left(S_N^a + S_N^b\right) \to \left(S^a + S^b\right)$ as $n \to \infty$. It follows that:

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(a_n + b_n\right).$$

□

(4) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Hint: Show by induction that the partial sums S_{2^k} are bounded below by $k/2$. Use this to deduce that the sequence of partial sums diverges.

Proof. Notice we have the following:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= (1) + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots \\ &= S_1 + S_2 + S_3 + \cdots\end{aligned}$$

Note that there are 2^k summands in each S_k. Since each of the summands in a given S_k is greater than $\frac{1}{2^{k+1}}$, we have that $S_k > \frac{2^k}{2^{k+1}} = \frac{1}{2}$.

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{k=1}^{\infty} S_k$, we have that:

$$\sum_{k=1}^{\infty} S_k > \sum_{k=1}^{\infty} \frac{1}{2},$$

which diverges. Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ must diverge. □