Math 331 — Homework 9
Due: Noon, Tuesday, May 4

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(1) Given a function $f : [a, b] \to \mathbb{R}$, suppose f is continuous everywhere on $[a, b]$ and differentiable everywhere on (a, b). Prove that if $f'(x) > 0$ on (a, b), then f is increasing and if $f'(x) < 0$, then f is decreasing.

Proof. Suppose that $f'(x) > 0$ for all $x \in (a, b)$. Let x_1, $x_2 \in [a, b]$. Without loss of generality, we can say $x_1 < x_2$. By the Mean Value Theorem, there exists a value c, $x_1 < c < x_2$, such that:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) > 0$ and $x_1 < x_2$, we know that $f(x_2) - f(x_1) > 0$. Therefore $f(x_1) < f(x_2)$, which implies that f is increasing on $[a, b]$.

Similarly, suppose that $f'(x) < 0$ for all $x \in (a, b)$. Let x_1, $x_2 \in [a, b]$. Without loss of generality, we can say $x_1 < x_2$. By the Mean Value Theorem, there exists a value c, $x_1 < c < x_2$, such that:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_1 < x_2$, we know that $f(x_2) - f(x_1) < 0$. Therefore $f(x_1) > f(x_2)$, which implies that f is decreasing on $[a, b]$. $\square$

(2) Prove the first derivative test: Given a function $f : [a, b] \to \mathbb{R}$, suppose f is continuous everywhere on $[a, b]$ and differentiable everywhere on (a, c) and (c, b) for some $c \in (a, b)$. If f has a critical point at $c \in (a, b)$, (i.e., either $f'(c)$ does not exist or $f'(c) = 0$) and if $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then has a local maximum at c. Formulate and prove the corresponding result for a local minimum.

Proof. Claim: If f has a critical point at $c \in (a, b)$, (i.e., either $f'(c)$ does not exist or $f'(c) = 0$) and if $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then has a local minimum at c.

Suppose f has a critical point at $c \in (a, b)$ and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$. Then there exists a, $b \in [a, b]$ such that $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$. By Problem 1, f is decreasing on $[a, c]$ and increasing on $[c, b]$. Therefore, $f(c)$ is a minimum of f on (a, b). $\square$

(3) Given the function $f : [1, 3] \to \mathbb{R}$, $f(x) = 7 - 2x$, use the definition to prove that f is integrable and determine the value of

$$\int_1^3 f(x)\,dx.$$

Proof. Let $P_n = \{1 + \frac{3-1}{n} i\}_{i=0}^n$ be a regular partition of $[1, 2]$. Fix i and consider f on $[x_{i-1}, x_i]$. Since f is decreasing, we have that $m_i(f, P_n) = f(x_i)$ and $M_i(f, P_n) = f(x_{i-1})$. So:

$$f(x_i) = f\left(1 + \frac{2i}{n}\right) = 7 - 2\left(1 + \frac{2i}{n}\right) = 5 - \frac{4i}{n},$$

and,

$$f(x_{i-1}) = f\left(1 + \frac{2(i-1)}{n}\right) = 7 - 2\left(1 + \frac{2(i-1)}{n}\right) = 5 - \frac{4(i-1)}{n}.$$

Therefore,

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^n m_i(f, P_n)\Delta x = \sum_{i=1}^n \left(5 - \frac{4i}{n}\right)\frac{1}{n} \\ &= \frac{5}{n}\sum_{i=1}^n 1 - \frac{4}{n^2}\sum_{i=1}^n i = \frac{5}{n}n - \frac{4}{n^2}\cdot\frac{n(n+1)}{2} \\ &= 5 - \frac{2(n+1)}{n} = \frac{3n-2}{n}. \end{aligned}$$

Similarly, $U(f, P_n) = \frac{3n+2}{n}$. So we have that,

$$\inf\{U(f, P_n) : n \in \mathbb{N}\} = 3,$$

and,

$$\geq \inf\{U(f, P) : P \text{ partition of } [1, 3]\} = \overline{\int_a^b} f(x)dx.$$

Similarly,

$$\sup\{L(f, P) : P \text{ partition of } [1, 3]\} \geq \sup\{L(f, P_n) : n \in \mathbb{N}\} = 3.$$

In other words,

$$3 \leq \underline{\int_1^3} f(x)dx \leq \overline{\int_1^3} f(x)dx \leq 3,$$

and therefore the upper and lower values are equal. $\square$

(4) Use the definition to prove that the function $f : [a, b] \to \mathbb{R}$, $f(x) = x$, is integrable and that

$$\int_a^b f(x)\,dx = \frac{b^2 - a^2}{2}.$$

Proof. Let $P_n = \{a + \frac{b-a}{n} i\}_{i=0}^{n}$ be a regular partition of $[a, b]$. Fix i and consider f on $[x_{i-1}, x_i]$. Since f is increasing, we have that $m_i(f, P_n) = f(x_{i-1})$ and $M_i(f, P_n) = f(x_i)$. So:

$$f(x_i) = f\left(a + \frac{(b-a)i}{n}\right) = \left(a + \frac{(b-a)i}{n}\right) = \frac{-ai + an + bi}{n},$$

and,

$$\begin{aligned} f(x_{i-1}) &= f\left(a + \frac{(b-a)(i-1)}{n}\right) = \left(a + \frac{(b-a)(i-1)}{n}\right) \\ &= \frac{-ai + an + a + bi - b}{n}. \end{aligned}$$

Therefore,

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} M_i(f, P_n)\Delta x = \sum_{i=1}^{n} \frac{-ai + an + bi}{n} \cdot \frac{1}{n} \\ &= \frac{an - a + bn + b}{2n}. \end{aligned}$$

Similarly,

$$L(f, P_n) = \frac{an + a + bn - b}{2n}.$$

So $\inf\{U(f, P_n) : n \in \mathbb{N}\} = b$ and $\sup\{L(f, P_n) : n \in \mathbb{N}\} = a$. □

(5) Prove that if $f : [a, b] \to \mathbb{R}$ is increasing, then f is integrable.

Proof. Fix P_n, a regular partition. Then each $x_i - x_{i-1} = \frac{b-a}{n} = \Delta x$. Note that if f is increasing on an interval $[c, d]$, then it takes on its infimum at c and its supremum at d. Therefore,

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n} f(x_i)\Delta x - \sum_{i=1}^{n} f(x_{i-1})\Delta x,$$

and after combining the two sums and replacing Δx with $\frac{b-a}{n}$ we get,

$$\begin{aligned} &= \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) \\ &= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_n) - f(x_{n-1}), \end{aligned}$$

which is the same as saying,

$$= \frac{b-a}{n}\left(f(x_n) - f(x_0)\right) = \frac{b-a}{n}\left(f(b) - f(a)\right).$$

Fix $\epsilon > 0$. By the Archimedean principle, we can choose an n for P_n sufficiently large such that:

$$U(f, P_n) - L(f, P_n) = \frac{b-a}{n}\left(f(b) - f(a)\right) < \epsilon.$$

Hence, f is integrable. $\square$