Math 331 — Take Home Midterm
Due Friday, April 9, in Class

I affirm that I have read the instructions given with the exam, have had all questions about them answered, and that I have followed these instructions while working on the exam.   Signature:

(1) Prove the following result: given a sequence $\{a_n\}_{n=1}^{\infty}$, suppose the subsequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n-1}\}_{n=1}^{\infty}$ both converge to the same limit L. Prove that $a_n \to L$ as $n \to \infty$.

Proof. Fix $\epsilon > 0$. Since $a_{2n} \to L$ as $n \to \infty$, there exists an N_e such that,

$$|a_{2n} - L| < \epsilon$$

where $n > N_e$. Similarly, since $a_{2n-1} \to L$ as $n \to \infty$, there exists an N_o such that,

$$|a_{2n-1} - L| < \epsilon$$

where $n > N_o$.

Now consider the supersequence $\{a_n\}_{n=1}^{\infty}$. We want to show that there exists some N such that if $n > N$,

$$|a_n - L| < \epsilon.$$

Let $N = \max(N_e, N_o)$. Select an arbitrary n such that $n > N$. We will proceed by cases:

First case – where n is even: If n is even, then each a_n belongs to the subsequence $\{a_{2n}\}_{n=1}^{\infty}$ where we have that:

$$|a_n - L| < \epsilon.$$

Second case – where n is odd: Similarly, if n is even, then every a_n belongs to the subsequence $\{a_{2n-1}\}_{n=1}^{\infty}$ where we have that:

$$|a_n - L| < \epsilon.$$

Hence, $\{a_n\}_{n=1}^{\infty}$ must converge to L. $\square$

(2) Prove that for any $r > 1$ and any $k \in \mathbb{N}$, $\frac{n^k}{r^n} \to 0$ as $n \to \infty$.

Proof. Fix $k \in \mathbb{N}$ and let $r = 1 + s$, $s > 0$. By the binomial theorem, we can approximate the value of r^n:

$$\begin{aligned} r^n = (1+s)^n &\geq \binom{n}{k+1} s^k \\ &= \frac{n(n-1)\dots(n-k+1)}{k!} s^k. \end{aligned}$$

If $n > 2k$, then $k < \frac{n}{2}$. Therefore we can observe the properties of the last term in the numerator,

$$(n - k + 1) > \frac{n}{2} + 1 > \frac{n}{2}$$

and it follows that:

$$\frac{n(n-1)\dots(n-k+1)}{k!} > \frac{n^k}{2^k} \cdot \frac{1}{k!}.$$

Hence, we have that:

$$0 \leq \frac{n^k}{r^n} \leq \left(\frac{1}{n^k}\right)\left(\frac{2^k k!}{s^k}\right).$$

Since:

$$\left\{\left(\frac{1}{n^k}\right)\left(\frac{2^k k!}{s^k}\right)\right\}_{n=1}^{\infty} = \left(\frac{2^k k!}{s^k}\right)\left\{\frac{1}{n^k}\right\}_{n=1}^{\infty},$$

and $\frac{1}{n^k} \to 0$ as $n \to \infty$, it follows from the Squeeze Lemma that $\frac{n^k}{r^n} \to 0$ as $n \to \infty$.

□

(3) Prove one of the following convergence problems.

(a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all n, then the series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

Proof. Since the terms of the series are non-negative, by the monotonic convergence theorem it suffices to show that the sequence of partial sums is bounded.
By the Cauchy-Schwarz inequality, we have the following:

$$\left|\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}\right|^2 \leq \sum_{n=1}^{N} |a_n| \cdot \sum_{n=1}^{N} \left|\frac{1}{n^2}\right|.$$

Since both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge, their partial sums are bounded by some values M, V respectively.
Hence, the right side of the Cauchy-Schwarz inequality is bounded by the value MV, and the partial sums of $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ are also bounded. $\square$

(4) Prove one of the following convergence theorems.

(b) (The Limit Comparison Test) Given two series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$, suppose that a_n, $b_n > 0$ for all n. Prove that if the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ converges to a value L, $0 < L < \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges.

Proof. Suppose that $\frac{a_n}{b_n} \to L$ as $n \to \infty$, where $0 < L < \infty$. We can find some K, $M \in \mathbb{R}$ such that $0 < K < L < M < \infty$.
Since $\frac{a_n}{b_n} \to L$ as $n \to \infty$, for a sufficiently large n we know that $\frac{a_n}{b_n}$ gets very close to L. By the definition of convergence, for all $\epsilon > 0$ there exists an $N > 0$ such that if $n > N$, we have:

$$K < \frac{a_n}{b_n} < M.$$

From this we can get:

$$K \cdot b_n < a_n < M \cdot b_n.$$

From this we will proceed by cases.

First case – the sequence $\sum_{n=1}^{\infty} b_n$ diverges: If $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} K \cdot b_n$. Since $K \cdot b_n < a_n$ for all n sufficiently large, by the generalized comparison test, $\sum_{n=1}^{\infty} a_n$ must also diverge.

Second case – the sequence $\sum_{n=1}^{\infty} b_n$ converges: Similarly, if $\sum_{n=1}^{\infty} b_n$ converges, so does the sequence $\sum_{n=1}^{\infty} M \cdot b_n$. Since $a_n < M \cdot b_n$ for all n sufficiently large, by the generalized comparison test $\sum_{n=1}^{\infty} a_n$ must also converge. $\square$

(6) (Extra Credit) Prove that every real number $x \in [0,1]$ has a unique decimal expansion: there exists a unique sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in \{0,1,2,\ldots,9\}$, such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

Proof. Divide the interval $[0,1]$ into 10 equal subintervals (of length $\frac{1}{10}$), $\{I_0, I_1, I_2, \ldots, I_9\}$. We have the following:

$$\begin{aligned} I_0 &= \left[0, \frac{1}{10}\right] \\ &\vdots \\ I_i &= \left[\frac{i}{10}, \frac{i+1}{10}\right] \\ &\vdots \\ I_9 &= \left[\frac{9}{10}, 1\right] \end{aligned}$$

Select an i such that $x \in I_i$ and let $a_1 = i$, then we have that:

$$|x - a_1| < \frac{1}{10}.$$

Next, we will again divide the interval I_{a_1} into 10 equal subintervals (of length $\frac{1}{100}$), $\{I_0^1, I_1^1, I_2^1, \ldots, I_9^1\}$. Then we have the following:

$$I_i^1 = \left[\frac{a_1}{10} + \frac{i}{100}, \frac{a_1}{10} + \frac{i+1}{100}\right],$$

for $i \in \{0,1,2,\ldots,9\}$. Again, select an i such that $x \in I_i^1$ and let $a_2 = i$. Then we have that:

$$\left|x - \left(\frac{a_1}{10} + \frac{a_2}{100}\right)\right| \leq \frac{1}{100}$$

Moving forward inductively, we are constructing a unique sequence $\{a_1, a_2, a_3, \ldots\}$ where $a_n \in \{0,1,2,\ldots,9\}$, and such that we have:

$$\left|x - \sum_{n=1}^{j} \frac{a_n}{10^n}\right| \leq \frac{1}{10^j}$$

From which we get that:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

□