Math 331 — Take Home Final
Due Wednesday, May 12, 4 pm

I affirm that I have read the instructions given with the exam, have had all questions about them answered, and that I have followed these instructions while working on the exam.

Signature:

The Problems:

(1) Use the definition of the definite integral to show that $f : [0,1] \to \mathbb{R}$, $f(x) = \sqrt{x}$, is integrable and evaluate its integral.

Proof. Let $Q_n = \{(i/n)^2\}_{i=0}^{n}$ be a partition of $[0,1]$. We want to show that $U(f,Q_n) - L(f,Q_n) = 0$.

Since f is an increasing function, we have that:

$$f(x_{i-1}) = m_i(f,Q_n) = \sqrt{\left(\frac{i-1}{n}\right)^2} = \frac{i-1}{n}.$$

Then we have,

$$L(f,Q_n) = \sum_{i=1}^{n} \frac{i-1}{n} \Delta x_i.$$

Since Q_n is not a regular partition, we find that:

$$\Delta x_i = \frac{2i-1}{n^2}.$$

Therefore,

$$\sum_{i=1}^{n} \frac{i-1}{n} \Delta x_i = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{2i-1}{n^2} = \sum_{i=1}^{n} \frac{2i-3i+1}{n^3}.$$

By the summation rules, we can write this as,

$$= \sum_{i=1}^{n} \frac{2}{n^3} i^2 - \sum_{i=1}^{n} \frac{3}{n^3} i + \sum_{i=1}^{n} \frac{1}{n^3},$$

and since n is actually constant in this context, we can factor it out. Then we're left with summations of i^2, i, and 1, all of which we have rules for:

$$= \frac{2}{n^3} \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6} - \frac{3}{n^2} \cdot \frac{n \cdot (n+1)}{2} + \frac{n}{n^3}.$$

After combining terms and expanding, we have the following:

$$= \frac{2n^2 + 3n + 1}{3n^2} - \frac{3n+3}{2n^2} + \frac{1}{n^2}.$$

Since we want to take the limit of this as n gets very large, it would help very much to put it even easier terms. Applying more algebraic rules, we get to:

$$= \frac{4n^2 - 3n - 1}{6n^2} = \frac{2}{3} - \frac{3n+1}{6n^2}.$$

Finally we have an expression we can work with. Observe:

$$\sup\{L(f, Q_n) : n \in \mathbb{N}\} = \frac{2}{3}$$

We're halfway there. Now we must show that:

$$\inf\{U(f, Q_n) : n \in \mathbb{N}\} = \frac{2}{3}.$$

Again, since f is increasing, we have that:

$$f(x_i) = M_i(f, Q_n) = \sqrt{\left(\frac{i}{n}\right)^2} = \frac{i}{n}.$$

So,

$$U(f, Q_n) = \sum_{i=1}^{n} \frac{i}{n} \Delta x_i = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{2i-1}{n^2},$$

which is the same as saying,

$$= \sum_{i=1}^{n} \frac{2i^2 - i}{n^3} = \sum_{i=1}^{n} \frac{2}{n^2} i^2 - \sum_{i=1}^{n} \frac{1}{n^3} i.$$

Again, by pulling out the constant term we're left with the summations of i^2 and i, so we can apply their formulae and get:

$$= \frac{2}{n^3} \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6} - \frac{1}{n^3} \cdot \frac{n \cdot (n+1)}{2}.$$

Now we can combine terms to get:

$$= \frac{2 \cdot (n+1) \cdot (2n+1) - 3 \cdot (n+1)}{6n^2} = \frac{4n^2 + 3n - 1}{6n^2}.$$

Finally, we split up this fraction:

$$= \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}.$$

As n gets very large, the second two terms go to zero. In other words,

$$\inf\{U(f, Q_n) : n \in \mathbb{N}\} = \frac{2}{3},$$

which is exactly what we wanted to show.

Hence,

$$U(f, Q_n) - L(f, Q_n) = \frac{2}{3} - \frac{2}{3} = 0,$$

so f is integrable and its value is $2/3$. $\square$

(2) Define the function $f : [0, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \leq 1 \\ 2 & x > 1. \end{cases}$$

Show that f is integrable.

Proof. We will proceed using the Darboux criterion. Since f is not continuous at $x = 1$, we must construct a partition that has an endpoint there. Suppose we had a partition P_L such that:

$$P_L = \left\{ \frac{2i}{2n+1} \right\}_{i=0}^{n},$$

and a partition P_R such that:

$$P_R = \left\{ \frac{2i}{n} \right\}_{i=n+1}^{2n+1},$$

Define a partition $P = P_L + P_R$. When $i \leq n$, we're working with a partition of $[0, 1]$, and when $i > n$, we're working with a partition of $[1, 2]$. This should work nicely to avoid the discontinuity at $x = 1$.

We want to find the sum of the area under the left half of the graph and the right half of the graph, so we have:

$$\sum_{i=0}^{n} f\left(\frac{2i}{2n+1}\right) \Delta x_i = \sum_{i=0}^{n} f\left(\frac{2i}{2n+1}\right) \cdot \frac{2}{2n+1},$$

for the left half and,

$$\sum_{i=n+1}^{2n+1} f\left(\frac{2i}{2n+1}\right)\Delta x_i = \sum_{i=n+1}^{2n+1} 2 \cdot \frac{2}{n+1},$$

for the right half. $\square$

(3) Prove that for all $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$, $f(x) = x^n$, is integrable and compute its integral.

Proof. First we must show that x^n is differentiable for all $n \in \mathbb{N}$. I claim that $f'(x) = nx^{n-1}$.

First, we will show that $f(x) = x^1 = x$ is differentiable. I claim that its derivative is 1 for all $x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$. We can fairly trivial to show show that:

$$\lim_{y\to x} \frac{f(x) - f(y)}{x - y} = \lim_{y\to x} \frac{x - y}{x - y} = 1.$$

Next we will show that $f(x) = x^n$ is differentiable for all $n \in \mathbb{N}$ and that its derivative is nx^{n-1}. We will proceed by induction.

Base case: n = 2 – Fix $x \in \mathbb{R}$. We need to show:

$$\lim_{y\to x} \frac{f(x) - f(y)}{x - y} = 2x.$$

We'll use the limit rules:

$$\lim_{y\to x} \frac{f(x) - f(y)}{x - y} = \lim_{y\to x} \frac{x^2 - x^2}{x - y} = \lim_{y\to x} \frac{(x - y)(x + y)}{x - y}.$$

Since in the definition of the limit as $y \to x$ $y \neq x$, we know $x - y \neq 0$ and so $\frac{x-y}{x-y} = 1$. Define $h : \mathbb{R} \to \mathbb{R}$, $h(y) = y + x$, then h is continuous, so therefore,

$$\lim_{y\to x} h(y) = h(x),$$

but then,

$$\lim_{y\to x} y + x = x + x = 2x.$$

Inductive case: Assume that x^k is differentiable for all $k \in \mathbb{N}$ and that $\frac{d}{dx}x^k = kx^{k-1}$. We want to show that $f(x) = x^{k+1}$ is differentiable, and that $f'(x) = (k+1)x^k$.

Fix $x \in \mathbb{R}$. We need to show that,

$$\lim_{y\to x} \frac{f(x) - f(y)}{x - y} = (k+1)x^k.$$

Again, we'll use the limit rules:

$$\lim_{y\to x}\frac{f(x)-f(y)}{x-y}=\lim_{y\to x}\frac{x^{k+1}-y^{k+1}}{x-y}=\lim_{y\to x}\frac{x\cdot x^k-y\cdot y^k}{x-y},$$

which is the same as saying,

$$\lim_{y\to x}(k+1)y^k=(k+1)\lim_{y\to x}y^k=(k+1)\left(\lim_{y\to x}y\right)^k.$$

Since the limit of y as $y\to x$ is x, we have that,

$$(k+1)\left(\lim_{y\to x}y\right)^k=(k+1)x^k.$$

Hence, our assertion holds. Next, we must show that f is integrable.

Since f is differentiable, it is continuous. Since f is continuous, it is integrable. □

(4) Prove the rule for integration by parts: if f, $g : [a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), and if f' and g' are integrable on $[a,b]$, then

$$\int_a^b f(x)g'(x)\,dx=\big(f(b)g(b)-f(a)g(a)\big)-\int_a^b f'(x)g(x)\,dx.$$

Proof. Define a function $h:[a,b]\to\mathbb{R}$ as $h(x)=f(x)g(x)$. We want to show that:

$$\int_a^b f(x)g'(x)dx=(h(b)-h(a))-\int_a^b f'(x)g(x)dx$$

Since f, g are differentiable at $x\in(a,b)$ we can use the product rule to get:

$$h'(x)=(fg)'(x)=f'(x)g(x)+f(x)g'(x)$$

Assuming that h is integrable, we can say that h is an antiderivative of the function $f'(x)g(x)+f(x)g'(x)$. By the Fundamental Theorem of Calculus,

$$\begin{aligned}h(b)-h(a)&=\int_a^b f'(x)g(x)+f(x)g'(x)dx\\&=\int_a^b f'(x)g(x)dx+\int_a^b f(x)g'(x)dx\end{aligned}$$

If we rearrange the equation, we get:

$$\int_a^b f(x)g'(x)dx = (h(b) - h(a)) - \int_a^b f'(x)g(x)dx,$$

which is precisely what we wanted to prove. $\square$