Математический анализ

ДЗ 21

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Группа БПМИ248

Задание 19.5

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{k^2+n^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{k}{n}}{\left(\frac{k}{n}\right)^2+1}\cdot\frac{1}{n}=|x_k=\frac{k}{n}|=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{x_k}{x_k^2+1}\cdot\frac{1}{n}=$$

$$=\int_0^1\frac{x}{x^2+1}dx=\frac{1}{2}\int_0^1\frac{1}{x^2+1}dx^2=\frac{1}{2}(\ln 2-\ln 1)=\frac{1}{2}\ln 2$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{k(n-k)}=\frac{1}{n}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\sqrt{k(n-k)}=\frac{1}{n}\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}=\int_0^1\sqrt{x(1-x)}dx=$$

$$=|x=\sin^2 t, dx=2\sin t\cos t dt|=\int_0^{\frac{\pi}{2}}2\sin^2 t\cos^2 t dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2t dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos 4t}{2}=$$

$$=\frac{1}{2}\left(\frac{t}{2}-\frac{\sin 4t}{4}\right)|_0^{\frac{\pi}{2}}=\frac{\pi}{8}$$

Задание 19.6

$$\int_0^{2\pi}\sin^4 x dx=\int_0^{2\pi}(\sin^2 x)^2 dx=\int_0^{2\pi}\left(\frac{1-\cos 2x}{2}\right)^2 dx=\frac{1}{4}\int_0^{2\pi}1-2\cos 2xd+\cos^2 2xdx=\frac{1}{4}(2\pi+0+\pi)=\frac{3\pi}{4}$$

$$\int_0^{2\pi}\frac{dx}{4+\cos^2 x}=2\int_0^{\pi}\frac{dx}{4+\cos^2 x}=2\left(\int_0^{\frac{\pi}{2}}\frac{dx}{4+\cos^2 x}+\int_{\frac{\pi}{2}}^{\pi}\frac{dx}{4+\cos^2 x}\right)=|t=\operatorname{tg} x, dx=\frac{dt}{1+t^2},\cos^2 x=$$

$$=\frac{1}{1+t^2}|=2\left(\int_0^{\infty}\frac{dt}{4t^2+5}+\int_{-\infty}^{0}\frac{dt}{4t^2+5}\right)=2\int_{-\infty}^{\infty}\frac{dt}{4t^2+5}=\frac{\pi}{\sqrt{5}}$$

$$\int_0^{\sqrt{3}}x\arctan x dx=\frac{1}{2}\int_0^{\sqrt{3}}\arctan x dx^2=\frac{1}{2}\left((\arctan x\cdot x^2)|_0^{\sqrt{3}}-\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}\right)=\frac{1}{2}\Big((\arctan x\cdot x^2)|_0^{\sqrt{3}}-$$

$$-(x-\arctan(x))|_0^{\sqrt{3}}\Big)=\frac{1}{2}\left(\pi-\sqrt{3}+\frac{\pi}{3}\right)=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

Задание 20.7

б) $y=a\sin x,\ y=a\cos x,\ 0\le x\le 2\pi.$

$$a\sin x=a\cos x\Rightarrow x=\frac{\pi}{4}+\pi k$$

$$\Rightarrow S=\int_0^{2\pi}a\sin x-a\cos x dx=a\left(\int_0^{\frac{\pi}{4}}\sin x-\cos x dx+\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\sin x-\cos x dx+\right.$$

$$\left.+\int_{\frac{5\pi}{4}}^{2\pi}\sin x-\cos x dx\right)=4a\sqrt{2}$$

Задание 20.8

$$(x,y,z) = (3,3,2) \Rightarrow t = 1$$

$$\Rightarrow L = \int_0^1 \sqrt{x(t)'^2 + y(t)'^2 + z(t)'^2}dt = \int_0^1 \sqrt{9 + 36t^2 + 36t^4}dt = \int_0^1 3\left(1 + 2t^2\right) = 5$$

Задание 20.9

1.

$$y = 0 \Rightarrow x = 0$$

$$\Rightarrow V = \pi \int_0^a xe^{2x}dx$$

$$\int xe^{2x}dx = |u = x, dv = e^{2x}dx| = x\frac{e^{2x}}{2} - \frac{1}{2}\int e^{2x}dx = x\frac{e^{2x}}{2} - \frac{1}{4}e^{2x} + C$$

$$\Rightarrow V = \pi\left(\frac{2a-1}{4}e^{2a} + \frac{1}{4}\right)$$

2.

$$y = 0 \Rightarrow x = 0; \pi^2$$

$$\Rightarrow V = \pi \int_0^{\pi^2} \left(\sin\sqrt{x}\right)^2 dx = \frac{\pi}{2}\int_0^{\pi^2} 1 - \cos\left(2\sqrt{x}\right)dx$$

$$\int \cos\left(2\sqrt{x}\right)dx = |u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx, dx = 2udu| = 2\int u\cos(2u)du = u\sin(2u) + \frac{1}{2}\cos(2u) + C =$$

$$= \sqrt{x}\sin\left(2\sqrt{x}\right) + \frac{1}{2}\cos\left(2\sqrt{x}\right) + C$$

$$\Rightarrow V = \frac{\pi}{2}\int_0^{\pi^2} 1 - \cos\left(2\sqrt{x}\right)dx = \frac{\pi^3}{2} - \frac{\pi}{2}\int_0^{\pi^2} \cos\left(2\sqrt{x}\right)dx = \frac{\pi^3}{2} - 0 = \frac{\pi^3}{2}$$

Задание 20.10

1.

$$S = 2\pi\int_{\frac{5}{4}}^{\frac{21}{4}} \sqrt{x}\cdot\sqrt{1 + \frac{1}{4x}}dx = \pi\int_{\frac{5}{4}}^{\frac{21}{4}} \sqrt{4x+1}dx = |u = 4x + 1, du = 4dx| =$$

$$= \frac{\pi}{4}\int_6^{22} \sqrt{u}du = \frac{\pi}{6}\left(22\sqrt{22} - 6\sqrt{6}\right)$$

2.