Математический анализ

ДЗ 18

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Задание 1

Найдите точки локальных экстремумов функций

а) $f(x,y) = x^3 + y^3 - 3axy$; б) $f(x,y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$.

Решение:

1.

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\begin{cases} 3x^2 - 3ay = 0 \\ 3y^2 - 3ax = 0 \end{cases} \Longleftrightarrow (x,y) = (0,0); (x,y) = (a,a)$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3a$$

$$G = \begin{pmatrix} 6x & -3a \\ -3a & 6y \end{pmatrix}$$

$$G(0,0) = \begin{pmatrix} 0 & -3a \\ -3a & 0 \end{pmatrix}$$

$$\delta_1 = 0; \delta_2 = -9a^2 < 0$$

$$\text{При } a = 0 : (0,0) - \text{перегиб } \Rightarrow (0,0) - \text{не экстремум}$$

$$G(a,a) = \begin{pmatrix} 6a & -3a \\ -3a & 6a \end{pmatrix}$$

$$\delta_1 = 6a, \delta_2 = 27a^2$$

$$a > 0 \Rightarrow (a,a) \text{ локальный минимум}, a < 0 \Rightarrow (a,a) \text{ локальный максимум}$$

2.

$$\frac{\partial f}{\partial x} = e^{2x+3y}(16x^2 + 16x - 6y - 12xy + 6y^2)$$

$$\frac{\partial f}{\partial y} = e^{2x+3y}(-6x + 6y + 24x^2 - 18xy + 9y^2)$$

$$\begin{cases} e^{2x+3y}(16x^2 + 16x - 6y - 12xy + 6y^2) = 0 \\ e^{2x+3y}(-6x + 6y + 24x^2 - 18xy + 9y^2) = 0 \end{cases} \Longleftrightarrow (x, y) = (0, 0), (x, y) = \left(-\frac{1}{4}, -\frac{1}{2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = e^{2x+3y}(16 - 24y + 32x^2 + 64x - 24xy + 12y^2)$$

$$\frac{\partial^2 f}{\partial y^2} = e^{2x+3y}(6 - 36x + 36y + 72x^2 - 54xy + 27y^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{2x+3y}(-6 + 36x - 6y + 48x^2 - 36xy + 18y^2)$$

$$G = (...)$$

$$G(0, 0) = \begin{pmatrix} 16 & -6 \\ -6 & 6 \end{pmatrix}$$

$$\delta_1 = 16 > 0; \delta_2 = 60 > 0 \Rightarrow (0, 0) \text{ локальный минимум}$$

$$G\left(-\frac{1}{4}, -\frac{1}{2}\right) = e^{-2} \begin{pmatrix} 14 & -9 \\ -9 & 1.5 \end{pmatrix}$$

$$\delta_1 = 14e^{-2} > 0; \delta_2 = -60e^{-4} < 0 \Rightarrow \left(-\frac{1}{4}, -\frac{1}{2}\right) \text{ не экстремум}$$

Задание 2

Найдите $\frac{dy}{dx}$ и $\frac{d^2y}{dx^2}$, если $x^3 + 4y^3 - 3yx^2 = 2$.

Решение:

$$x^3 + 4y^3 - 3yx^2 = 2 \mid \frac{d}{dx}$$

$$3x^2 + 12y^2 \cdot \frac{dy}{dx} - 3\left(\frac{dy}{dx} \cdot x^2 + 2xy\right) = 0$$

$$\frac{dy}{dx} = \frac{x(2y - x)}{4y^2 - x^2} \mid \frac{d}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{2\left(x^3 + x^2y - 4y^3\right)}{\left(4y^2 - x^2\right)^2}$$

Задание 3

Найдите $\frac{dy}{dx}$ и $\frac{d^2y}{dx^2}$, если

$$\begin{cases} x = \ln\left(t + \sqrt{1+t^2}\right) \\ y = \frac{1}{\sqrt{1+t^2}} \end{cases}$$

Решение:

$$\frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}}$$

$$\frac{dy}{dt} = -\frac{t}{(1+t)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{t}{1+t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2} \cdot \frac{dx}{dt} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$\frac{d^2x}{dt^2} = -\frac{t}{(1+t^2)^{\frac{3}{2}}}$$

$$\frac{d^2y}{dt^2} = -\frac{1-2t^2}{(1+t^2)^{\frac{5}{2}}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1-t^2}{(1+t^2)^{\frac{3}{2}}}$$

Задание 4

Найдите $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$, если

$$a)\ x+y+z=\cos(xyz);\ b)\ x^y+y^z=3;\ c)\ x=u^2-v^2, y=uv, z=u^2v.$$

Решение:

1.

$$x+y+z=\cos(xyz) \mid \frac{\partial}{\partial x}$$

$$1+\frac{\partial z}{\partial x}=-\sin(xyz)\cdot\left(yz+xy\cdot\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x}=\frac{-1-yz\sin(xyz)}{1+xy\sin(xyz)}$$

$$x+y+z=\cos(xyz) \mid \frac{\partial}{\partial y}$$

$$1+\frac{\partial z}{\partial y}=-\sin(xyz)\cdot\left(xz+xy\cdot\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial y}=\frac{-1-xz\sin(xyz)}{1+xy\sin(xyz)}$$

$$\frac{\partial z}{\partial x}=\frac{-1-yz\sin(xyz)}{1+xy\sin(xyz)} \mid \frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{z\sin(xyz)(2+xyz\cos(xyz))}{(1+xy\sin(xyz))^2}$$

2.

$$x^y+y^z=3 \mid \frac{\partial}{\partial x}$$

$$yx^{y-1}+y^z\cdot\ln y\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}=-\frac{yx^{y-1}}{y^z\cdot\ln y}$$

$$x^y+y^z=3 \mid \frac{\partial}{\partial y}$$

$$x^y\ln x+zy^{z-1}+y^z\ln y\cdot\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial y}=-\frac{x^y\ln x+zy^{z-1}}{y^z\ln y}$$

$$\frac{\partial z}{\partial x}=-\frac{yx^{y-1}}{y^z\cdot\ln y} \mid \frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{x^{y-1}(y\ln x\ln y-z-1)}{y^z(\ln y)^2}$$

Задание 5

Найдите первый и второй дифференциалы функции $z(x, y)$, заданной неявно

$$a)\ x^2 + zx + z^2 + y = 0;\ b)\ x = e^{u+v}, y = e^{u-v}, z = u^2 + v^2.$$

Решение:

1.

$$J = \begin{pmatrix} 2x + z & 1 & x + 2z \end{pmatrix} \Rightarrow$$

$$dz = -\frac{2x + z}{x + 2z}dx - \frac{1}{x + 2z}dy$$

$$d^2z = -\frac{3zdx^2 - 3x\left(-\frac{2x+z}{x+2z}dx - \frac{1}{x+2z}dy\right)dx}{(x + 2z)^2} + \frac{dxdy + 2\left(-\frac{2x+z}{x+2z}dx - \frac{1}{x+2z}dy\right)dy}{(x + 2z)^2}$$

$$d^2z = \frac{2\left((2x + z)^2 - (x + 2z)^2\right)}{(x + 2z)^3}dx^2 + \frac{4(2x + z)}{(x + 2z)^3}dxdy + \frac{2}{(x + 2z)^3}dy^2$$