# Inertial Measurement Units II

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EE 267 Virtual Reality

Lecture 10

stanford.edu/class/ee267/

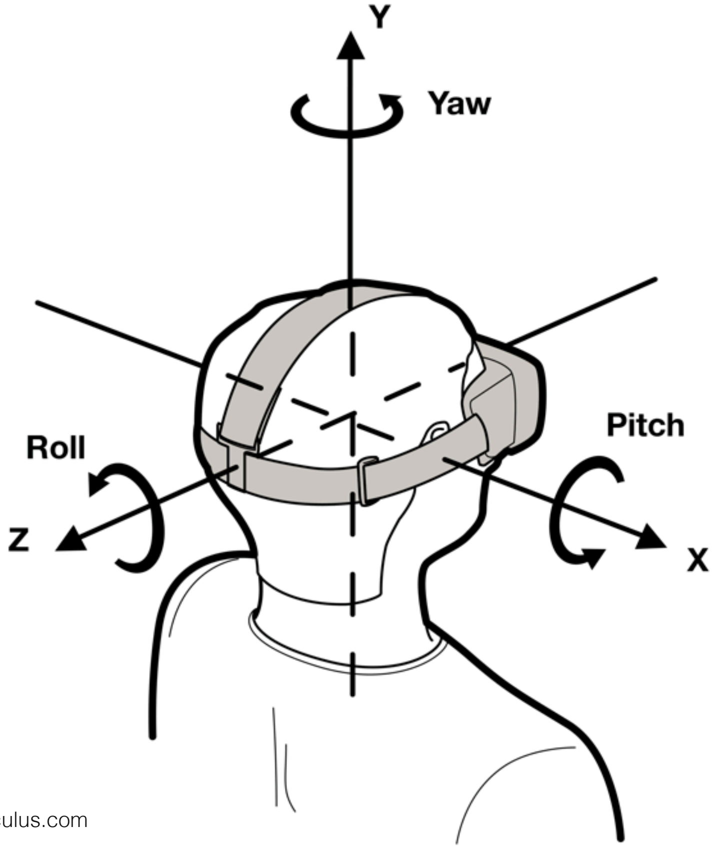
# Polynesian Migration


wikipedia

# Lecture Overview

- short review of coordinate systems, tracking in flatland, and accelerometer-only tracking

- rotations: Euler angles, axis & angle, gimbal lock
- rotations with quaternions
- 6-DOF IMU sensor fusion with quaternions




- primary goal: track orientation of head or device

- inertial sensors required pitch, yaw, and roll to be determined

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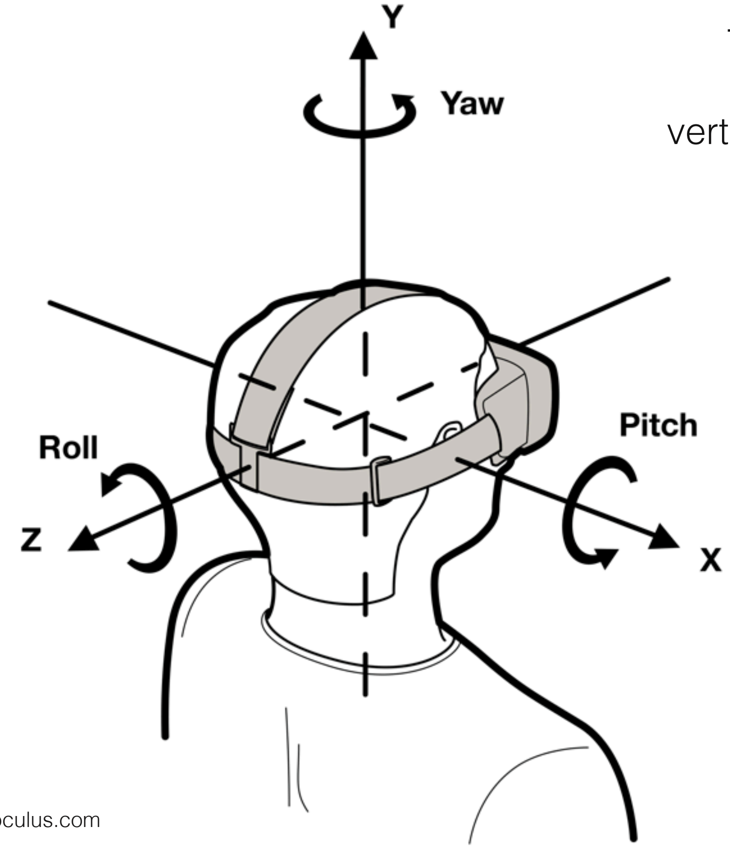

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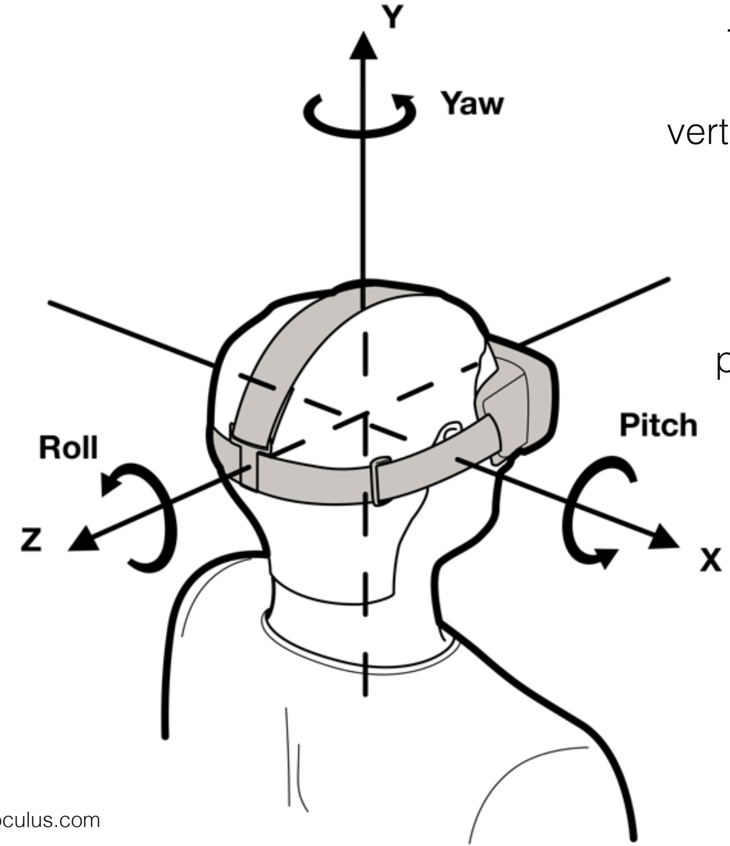
from lecture 2:

vertex in clip space

vertex

$$v_{clip} = M_{proj} \cdot M_{view} \cdot M_{model} \cdot v$$

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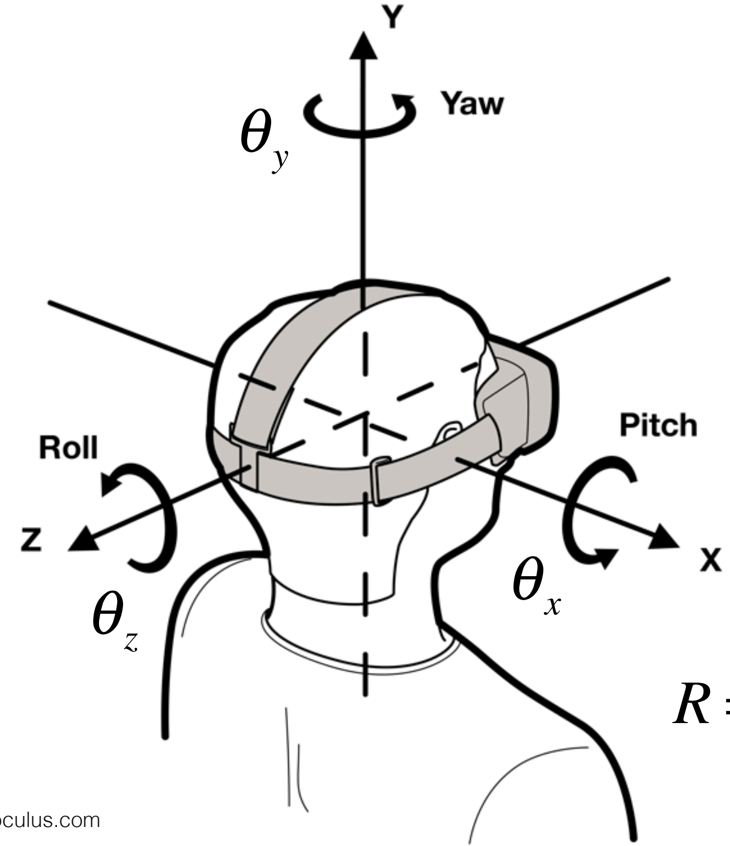
from lecture 2:

$$\underbrace{v_{clip}}_{\text{vertex in clip space}} = \underbrace{M_{proj}}_{\text{projection matrix}} \cdot \underbrace{M_{view}}_{\text{view matrix}} \cdot \underbrace{M_{model}}_{\text{model matrix}} \cdot \underbrace{v}_{\text{vertex}}$$

$$M_{view} = \underbrace{R}_{\text{rotation}} \cdot \underbrace{T(-eye)}_{\text{translation}}$$

# Euler angles

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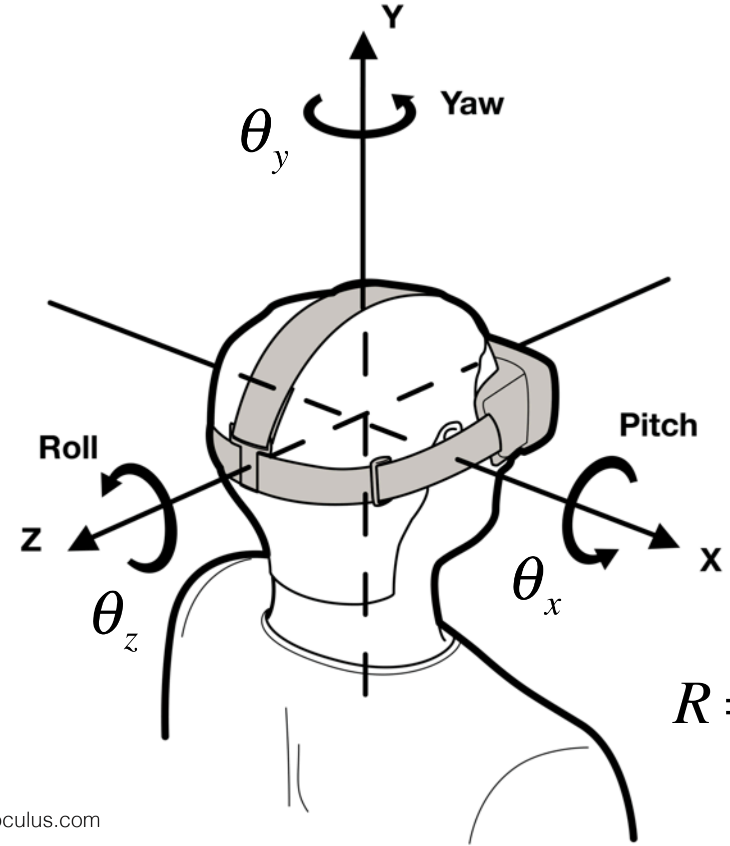
rotation translation

$$M_{view} = R \cdot T(-eye)$$

$$R = R_z(-\theta_z) \cdot R_x(-\theta_x) \cdot R_y(-\theta_y)$$

roll pitch yaw

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# Euler angles

2 important coordinate systems:

body/sensor frame

world/inertial frame

$$M_{view} = R \cdot T(-eye)$$

$$R = R_z(-\theta_z) \cdot R_x(-\theta_x) \cdot R_y(-\theta_y)$$

roll pitch yaw

# Gyro Integration aka *Dead Reckoning*

- from gyro measurements to orientation – use Taylor expansion

<u>have</u>: angle at
last time step

<u>have</u>:
time step

$$\theta\left(t+\Delta t\right)\approx\theta\left(t\right)+\underbrace{\frac{\partial}{\partial t}\theta\left(t\right)}_{=\omega}\Delta t+\varepsilon,\quad \varepsilon\sim O\left(\Delta t^{2}\right)$$

<u>want</u>: angle at
current time step

approximation error!

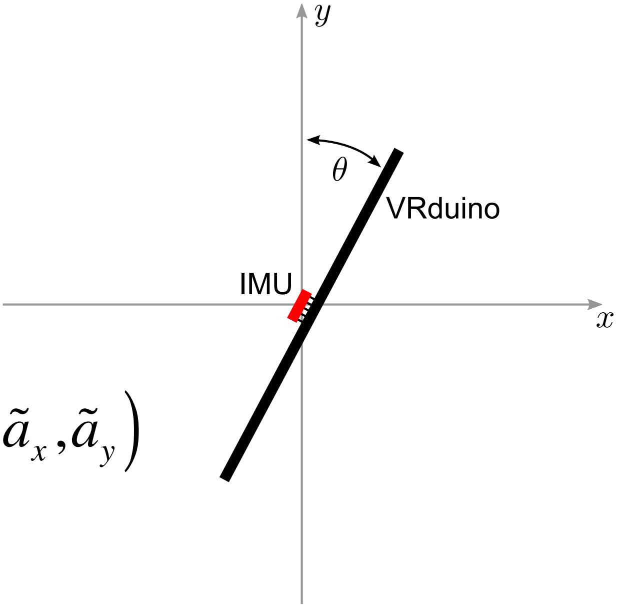
<u>have</u>: gyro measurement
(angular velocity)

# Orientation Tracking in *Flatland*

- problem: track 1 angle in 2D space
- sensors: 1 gyro, 2-axis accelerometer
- sensor fusion with complementary filter, i.e. linear interpolation:

$$\theta^{(t)} = \alpha\left(\theta^{(t-1)} + \tilde{\omega}\Delta t\right) + (1-\alpha)\,\mathrm{atan2}\left(\tilde{a}_x, \tilde{a}_y\right)$$

- no drift, no noise!



# Tilt from Accelerometer

- assuming acceleration points up (i.e. no external forces), we can compute the tilt (i.e. pitch and roll) from a 3-axis accelerometer

$$\hat{a} = \frac{\tilde{a}}{\|\tilde{a}\|} = R \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = R_z(-\theta_z) \cdot R_x(-\theta_x) \cdot R_y(-\theta_y) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\cos(-\theta_x)\sin(-\theta_z) \\ \cos(-\theta_x)\cos(-\theta_z) \\ \sin(-\theta_x) \end{pmatrix}$$

$$\theta_x = -\mathrm{atan2}\left(\hat{a}_z, \mathrm{sign}\left(\hat{a}_y\right) \cdot \sqrt{\hat{a}_x^2 + \hat{a}_y^2}\right)$$

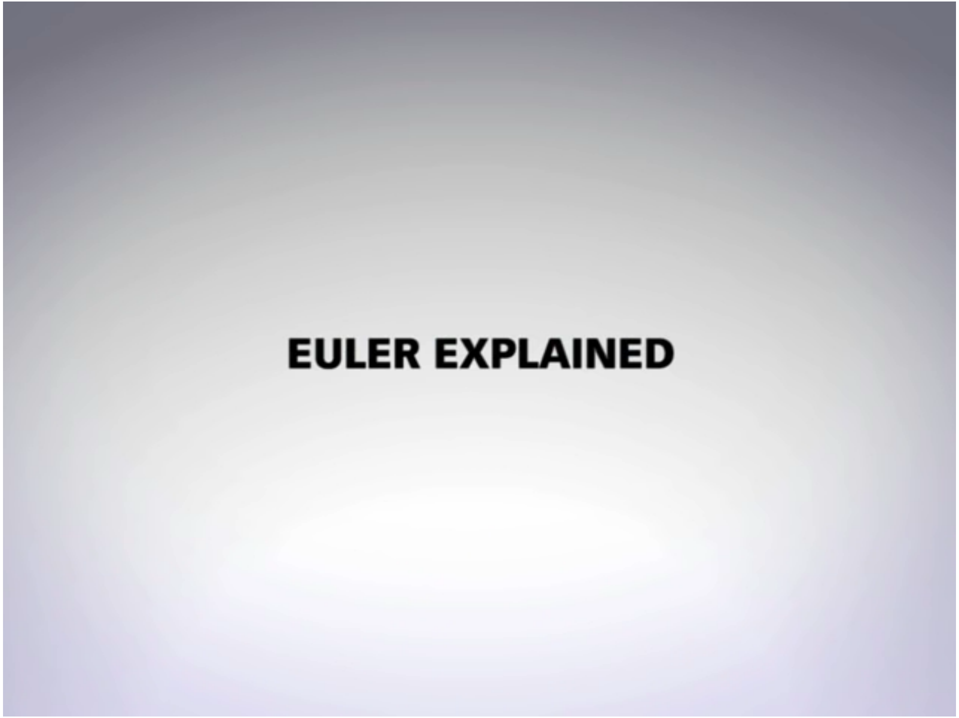
$$\theta_z = -\mathrm{atan2}\left(-\hat{a}_x, \hat{a}_y\right) \text{ both in rad}$$

# Euler Angles and Gimbal Lock

- so far we have represented head rotations with Euler angles: 3 rotation angles around the axis applied in a specific sequence

- problematic when interpolating between rotations in keyframes (in computer animation) or integration → singularities

# Gimbal Lock

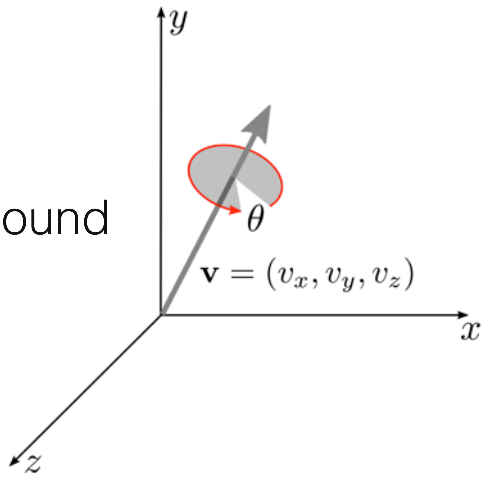



The Guerrilla CG Project, The Euler (gimbal lock) Explained – see: https://www.youtube.com/watch?v=zc8b2Jo7mno

# Rotations with Axis-Angle Representation and Quaternions

# Rotations with Axis and Angle Representation

- solution to gimbal lock: use axis and angle representation for rotation!
- simultaneous rotation around a *normalized* vector *v* by angle $\theta$
- no "order" of rotation, all at once around that vector



# Quaternions

- think about quaternions as an extension of complex numbers to having 3 (different) imaginary numbers or fundamental quaternion units *i,j,k*

$$q = q_w + iq_x + jq_y + kq_z$$

$$i \neq j \neq k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = k$$

$$ki = -ik = j$$

$$jk = -kj = i$$

# Quaternions

- think about quaternions as an extension of complex numbers to having 3 (different) imaginary numbers or fundamental quaternion units *i,j,k*

$$q = q_w + iq_x + jq_y + kq_z$$

- quaternion algebra is well-defined and will give us a powerful tool to work with rotations in axis-angle representation in practice

# Quaternions

- axis-angle to quaternion (need normalized axis ***v***)

$$q(\theta, v) = \underbrace{\cos\left(\frac{\theta}{2}\right)}_{q_w} + i\,\underbrace{v_x \sin\left(\frac{\theta}{2}\right)}_{q_x} + j\,\underbrace{v_y \sin\left(\frac{\theta}{2}\right)}_{q_y} + k\,\underbrace{v_z \sin\left(\frac{\theta}{2}\right)}_{q_z}$$

# Quaternions

- axis-angle to quaternion (need normalized axis *v*)

$$q(\theta,v)=\underbrace{\cos\left(\frac{\theta}{2}\right)}_{q_w}+i\underbrace{v_x\sin\left(\frac{\theta}{2}\right)}_{q_x}+j\underbrace{v_y\sin\left(\frac{\theta}{2}\right)}_{q_y}+k\underbrace{v_z\sin\left(\frac{\theta}{2}\right)}_{q_z}$$

- valid rotation quaternions have unit length

$$||q||=\sqrt{q_w^2+q_x^2+q_y^2+q_z^2}=1$$

# Two Types of Quaternions

- vector quaternions represent 3D points or vectors ***u***$=(u_x,u_y,u_z)$ can have arbitrary length

$$q_u = 0 + i\,u_x + j\,u_y + k\,u_z$$

- valid rotation quaternions have unit length

$$\|q\| = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} = 1$$

# Quaternion Algebra

- quaternion addition:

$$q + p = (q_w + p_w) + i(q_x + p_x) + j(q_y + p_y) + k(q_z + p_z)$$

- quaternion multiplication:

$$\begin{aligned} qp &= (q_w + iq_x + jq_y + kq_z)(p_w + ip_x + jp_y + kp_z) \\ &= \quad (q_w p_w - q_x p_x - q_y p_y - q_z p_z) + \\ &\quad i(q_w p_x + q_x p_w + q_y p_z - q_z p_y) + \\ &\quad j(q_w p_y - q_x p_z + q_y p_w + q_z p_x) + \\ &\quad k(q_w p_z + q_x p_y - q_y p_x + q_z p_w) + \end{aligned}$$

# Quaternion Algebra

- quaternion conjugate: $q^* = q_w - iq_x - jq_y - kq_z$
- quaternion inverse: $q^{-1} = \dfrac{q^*}{\|q\|^2}$
- rotation of vector quaternion $q_u$ by $q$ : $q'_u = q q_u q^{-1}$
- inverse rotation: $q_u = q^{-1} q'_u q$
- successive rotations by $q_1$ then $q_2$ : $q'_u = q_2\, q_1\, q_u\, q_1^{-1}\, q_2^{-1}$

# Quaternion Algebra

- detailed derivations and reference of general quaternion algebra and rotations with quaternions in course notes

- please read *course notes* for more details!

# Quaternion-based

# 6-DOF Orientation Tracking

# Quaternion-based Orientation Tracking

1. 3-axis gyro integration
2. computing the tilt correction quaternion
3. applying a complementary filter

# Gyro Integration with Quaternions

- start with initial quaternion: $q^{(0)} = 1 + i0 + j0 + k0$

- convert 3-axis gyro measurements $\tilde{\omega} = \left(\tilde{\omega}_x, \tilde{\omega}_y, \tilde{\omega}_z\right)$ to instantaneous rotation quaternion as

*avoid division by 0!*

$$q_\Delta = q\left( \Delta t \left\| \tilde{\omega} \right\|, \frac{\tilde{\omega}}{\left\| \tilde{\omega} \right\|} \right)$$

angle rotation axis

- integrate as $q_\omega^{(t+\Delta t)} = q^{(t)} q_\Delta$

# Gyro Integration with Quaternions

- integrated gyro rotation quaternion $q_\omega^{(t+\Delta t)}$ represents rotation from body to world frame, i.e.

$$q_u^{(world)} = q_\omega^{(t+\Delta t)} q_u^{(body)} q_\omega^{(t+\Delta t)^{-1}}$$

- last estimate $q^{(t)}$ is either from gyro-only (for dead reckoning) or from last complementary filter

- integrate as

$$q_\omega^{(t+\Delta t)} = q^{(t)} q_\Delta$$

# Tilt Correction with Quaternions

- assume accelerometer measures gravity vector in body (sensor) coordinates $\tilde{a} = \left(\tilde{a}_x, \tilde{a}_y, \tilde{a}_z\right)$

- transform vector quaternion of $\tilde{a}$ into current estimation of world space as

$$q_a^{(\text{world})} = q_\omega^{(t+\Delta t)} q_a^{(\text{body})} q_\omega^{(t+\Delta t)^{-1}}$$

$$q_a^{(body)} = 0 + i\tilde{a}_x + j\tilde{a}_y + k\tilde{a}_z$$

# Tilt Correction with Quaternions

- assume accelerometer measures gravity vector in body (sensor) coordinates $\tilde{a} = \left(\tilde{a}_x, \tilde{a}_y, \tilde{a}_z\right)$

- transform vector quaternion of $\tilde{a}$ into current estimation of world space as

$$q_a^{(\text{world})} = q_\omega^{(t+\Delta t)} q_a^{(\text{body})} q_\omega^{(t+\Delta t)^{-1}}$$

- if gyro quaternion is correct, then accelerometer world vector points up, i.e. $q_a^{(\text{world})} = 0 + i0 + j9.81 + k0$

# Tilt Correction with Quaternions

- gyro quaternion likely includes drift
- accelerometer measurements are noisy and also include forces other than gravity, so it's unlikely that accelerometer world vector actually points up
- if gyro quaternion is correct, then accelerometer world vector points up, i.e.

$$q_a^{(\text{world})} = 0 + i0 + j9.81 + k0$$

# Tilt Correction with Quaternions

solution: compute tilt correction quaternion that would rotate $q_a^{(\text{world})}$ into up direction

how? get normalized vector part of vector quaternion $q_a^{(\text{world})}$

$$v = \left( \frac{q_{a_x}^{(\text{world})}}{\left\| q_a^{(\text{world})} \right\|}, \frac{q_{a_y}^{(\text{world})}}{\left\| q_a^{(\text{world})} \right\|}, \frac{q_{a_z}^{(\text{world})}}{\left\| q_a^{(\text{world})} \right\|} \right)$$

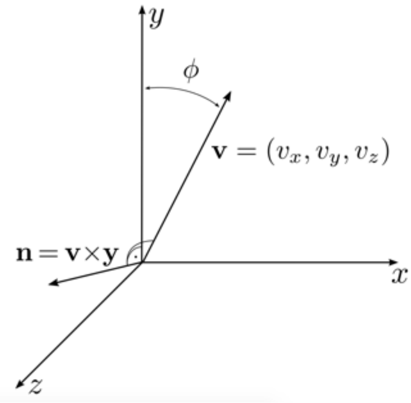
# Tilt Correction with Quaternions

solution: compute tilt correction quaternion that would rotate $q_a^{(\text{world})}$ into up direction

$$q_t = q\left(\phi, \frac{n}{\|n\|}\right)$$

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \cos(\phi) \quad \Rightarrow \quad \phi = \cos^{-1}\left(v_y\right)$$

$$n = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -v_z \\ 0 \\ v_x \end{pmatrix}$$



# Complementary Filter with Quaternions

- complementary filter: rotate into gyro world space first, then rotate “a bit” into the direction of the tilt correction quaternion

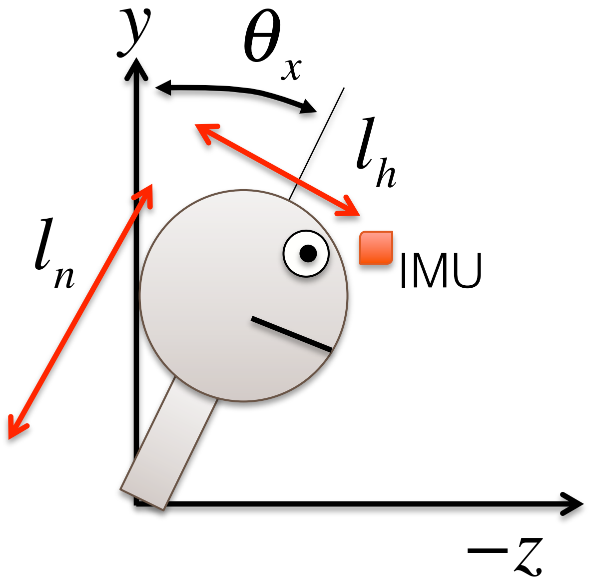
$$q_c^{(t+\Delta t)} = q\left( (1-\alpha)\phi, \frac{n}{\|n\|} \right) q_\omega^{(t+\Delta t)} \qquad 0 \le \alpha \le 1$$

- rotation of any vector quaternion is then $q_u^{(\text{world})} = q_c^{(t+\Delta t)} q_u^{(\text{body})} {q_c^{(t+\Delta t)}}^{-1}$

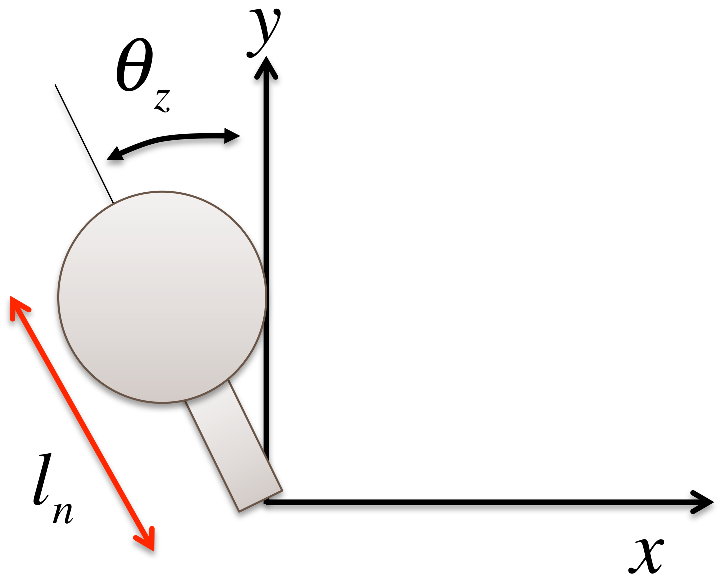
# Integration into Graphics Pipeline

- compute $q_c^{(t+\Delta t)}$ via quaternion complementary filter first
- stream from microcontroller to PC
- convert to 4x4 rotation matrix (see course notes) $\quad q_c^{(t+\Delta t)} \Rightarrow R_c$
- set view matrix to $M_{view} = R_c^{-1}$ to rotate the world in front of the virtual camera

# Head and Neck Model




pitch around base of neck!




roll around base of neck!

# Head and Neck Model

- why? there is not always positional tracking! this gives some motion parallax
- can extend to torso, and using other kinematic constraints

- integrate into pipeline as

$$M_{view} = T\left(0,-l_n,-l_h\right)\cdot R\cdot T\left(0,l_n,l_h\right)\cdot T\left(-eye\right)$$

*Must read: course notes on IMUs!*