Influence of susceptibility - probability distributions

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If $Z \sim \; N\left(\mu, \sigma^2\right)$ then $E\left(e^z\right) = \int_{-\infty}^{+\infty} e^z \varphi\left(z\right) dz$ with $\varphi\left(z\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

$$\begin{aligned} E\left(e^z\right) &= \int_{-\infty}^{+\infty} e^z \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^z e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \qquad (1) \end{aligned}$$

$$\begin{aligned} z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{1}{2}\left(\frac{-(z-\mu)^2}{\sigma^2} + 2z\right) \\ &= \frac{1}{2}\left(\frac{-(z-\mu)^2}{\sigma^2} + 2(z-\mu) - \sigma^2\right) + \mu + \frac{\sigma^2}{2} \\ &= \frac{-1}{2}\left(\frac{z-\mu}{\sigma} - \sigma\right)^2 \end{aligned}$$

$$\begin{aligned} (1) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}-\sigma\right)^2} e^{\mu+\frac{\sigma^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}-\sigma\right)^2} dz \qquad (2) \end{aligned}$$

Put $w = \frac{z-\mu}{\sigma} - \sigma \Rightarrow dw = \frac{1}{\sigma} dz \Rightarrow dz = \sigma dw$.

$$\begin{aligned} (2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} \sigma dw \\ &= \frac{1}{\sqrt{2\pi}} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} dw \\ &= \frac{1}{\sqrt{2\pi}} e^{\mu+\frac{\sigma^2}{2}} \sqrt{2\pi} \\ &= e^{\mu+\frac{\sigma^2}{2}} \end{aligned}$$

$Var\left(e^{z}\right) = E\left[e^{2z}\right] - \left(E\left[e^{z}\right]\right)^{2}$

$$\begin{aligned} E\left[e^{2z}\right] &= \int_{-\infty}^{+\infty} e^{2z}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(z-\mu)^2}{2\sigma^2}}dz \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{2z-\frac{(z-\mu)^2}{2\sigma^2}}dz \end{aligned} \tag{3}$$

$$\begin{aligned} 2z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{1}{2}\left(\frac{(z-\mu)^2}{\sigma^2} + 4z\right) \\ &= \frac{1}{2}\left(-\frac{(z-\mu)^2}{\sigma^2} + 4(z-\mu) - 4\sigma^2\right) + 2\mu + 2\sigma^2 \\ &= \frac{-1}{2}\left(\frac{z-\mu}{\sigma} - 2\sigma\right)^2 + 2\mu + 2\sigma^2 \end{aligned}$$

$$(3) = \frac{1}{\sqrt{2\pi}\sigma}e^{2\mu+2\sigma^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}-2\sigma\right)^2}dz \tag{4}$$

Put $w = \frac{z-\mu}{\sigma} - 2\sigma \Rightarrow dw = \frac{dz}{\sigma} \Rightarrow dz = \sigma dw$.

$$\begin{aligned} (4) &= \frac{1}{\sqrt{2\pi}\sigma}e^{2\left(\mu+\sigma^2\right)}\int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}}\sigma dw \\ &= \frac{1}{\sqrt{2\pi}}e^{2\left(\mu+\sigma^2\right)}\int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}}dw \\ &= \frac{1}{\sqrt{2\pi}}e^{2\left(\mu+\sigma^2\right)}\sqrt{2\pi} \\ &= e^{2\left(\mu+\sigma^2\right)} \end{aligned}$$

$$\begin{aligned} Var\left(e^{z}\right) &= e^{2\left(\mu+\sigma^2\right)} - e^{2\mu+\sigma^2} \\ &= \left(e^{\sigma^2} - 1\right)e^{2\mu+\sigma^2} \end{aligned}$$

$E\left(e^{z}\right) = 1 \Leftrightarrow \mu = -\frac{\sigma^2}{2}$
$b_0 \sim\ N\left(\mu_0, \sigma_0^2\right)$ and $b_1 \sim\ N\left(\mu_1, \sigma_0^1\right)$
The following conditions have to be fulfilled:
$E\left(e^{b_0+b_1}\right) = e^{\mu_0+\mu_1+\frac{\left(\sigma_0^2+\sigma_1^2\right)}{2}} = 1$
$E\left(e^{b_0+b_2}\right) = e^{\mu_0+\mu_2+\frac{\left(\sigma_0^2+\sigma_2^2\right)}{2}} = 1$
This is a.o. the case if $\mu_0 = -\frac{\sigma_0^2}{2}$, $\mu_1 = -\frac{\sigma_1^2}{2}$ and $\mu_2 = -\frac{\sigma_2^2}{2}$