Quantile-parameterized distributions for expert knowledge elicitation

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ABSTRACT
This paper provides a comprehensive overview of quantile-parameterized distributions (QPDs) as a tool for capturing expert predictions and parametric judgments. We survey a range of methods for constructing distributions that are parameterized by a set of quantile-probability pairs and describe an approach to generalizing them to enhance their tail flexibility. Furthermore, we delve into the extension of QPDs to the multivariate setting, surveying the approaches to construct bivariate distributions, which can be adopted to obtain distributions with quantile-parameterized margins. Through this review and synthesis of the previously proposed methods, we aim to enhance the understanding and utilization of QPDs in various domains.

KEYWORDS
quantile functions; quantile-parameterized distributions; expert knowledge elicitation; statistical distributions

1. Introduction

Judgment plays a crucial role in transforming raw data into meaningful insights. For judgment to be useful, it needs to be translated into the language of mathematical models and assumptions. These models are designed to capture the expert's understanding of the world, including the causal links between relevant entities. The models serve as a representation of this understanding, while also accounting for any limitations in knowledge, which are treated as uncertainties. The process of elicitation involves translating the qualitative understanding of the problem at hand into quantitative models that can provide valuable insights.

Past research

Most of the expert elicitation protocols described in the literature (Hanea et al. 2021; Gosling 2018; O'Hagan et al. 2006; Hemming et al. 2018; Morgan 2014; Welsh and

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Begg 2018; Spetzler and Staël Von Holstein 1975) encode expert judgments about the parameter or quantity of interest as an ordered set of quantiles with corresponding probabilities. This typically includes measures such as the median and the upper and lower quartiles. Assessors are then encouraged to select a probability distribution that reasonably fits the elicited quantile-probability pairs and validate the choice with the expert (Gosling 2018). A distribution is selected from a predefined set of "simple and convenient" distributions (O'Hagan et al. 2006) with boundedness that accounts for the nature of the elicited quantity.

Several specialized distributions have been developed to facilitate smooth interpolation of probabilistic assessments. These distributions, parameterized by quantile-probability pairs, ensure that the elicited QPPs are exactly preserved (Keelin and Powley 2011; Powley 2013; Keelin 2016; Christopher Campbell Hadlock 2017; Kevin J. Wilson et al. 2023). Quantile-parameterized distributions are particularly valuable thanks to the interpretability of their parameters. By leveraging the elicited quantiles, these distributions enable precise capturing of expert knowlegde while maintaining a high level of flexibility in modeling.

We believe that the primary utility of QPDs lies in their ability to simplify the specification of probability distributions for model parameters, known as *prior elicitation* (Mikkola et al. 2021). However, these same distributions can also be employed to describe an expert's predictions for the next observation, referred to as *predictive elicitation* (Winkler 1980; J. B. Kadane 1980; Akbarov 2009; Hartmann et al. 2020), or to capture both uncertainty and variability through a two-dimensional probability distribution in *hybrid elicitation* (Perepolkin, Goodrich, and Sahlin 2021).

This review paper aims to introduce quantile-parameterized distributions (QPDs) to a wide readership. The literature review and the findings are presented through the perspective of quantile functions, building upon the theoretical foundations established by Parzen (Parzen 1979) and Gilchrist (Gilchrist 2000). The derivatives and inverses for each of the quantile functions discussed in the paper are provided in Appendix A, serving as a valuable reference for future research. Through our comprehensive review and identification of research gaps, we aim to contribute to the development of flexible and extensible distributions that can effectively capture expert knowledge. We hope that our overview of quantile-parameterized distributions will be useful for researchers and practitioners, enabling them to make an informed choice of a distribution suitable for the task.

Paper structure

In Section 2, we revisit the approaches to quantile parameterization of probability distributions and explore how QPDs can effectively describe expert beliefs regarding model parameters or predictions. Moving on to Section 3, we conduct a comprehensive review and comparison of various continuous univariate QPDs found in the literature. Specifically, we focus on the Myerson distribution and its generalization accommodating different tail thicknesses. To assess the flexibility and behavior of the QPDs, we compare their robust moments. This comparative analysis can guide the selection of an appropriate distribution to characterize the quantity of interest. In Section 4, we delve into several methods for extending univariate distributions to a multivariate setting. These methods include the utilization of standard multivariate distributions (Drovandi and Pettitt 2011), copulas (Hoff 2007), and bivariate quantiles (Nair and Vineshkumar

2023; Vineshkumar and Nair 2019). We show how these techniques can be applied to develop a bivariate version of the Generalized Myerson distribution and demonstrate its application in parametric and predictive elicitation. Finally, in Section 5, we discuss future research directions and potential applications of QPDs in Bayesian analysis.

2. Quantile parameterization of probability distributions

In Bayesian data analysis, a fundamental principle is that learning from data requires more than just formulating hypotheses and models. It necessitates the articulation of prior beliefs, expressing existing knowledge in a mathematical form and translating it into a probability distribution for the model parameters.

To accurately translate knowledge into the language of statistical models the encoding distribution needs to be flexible, the process should be transparent, and the results must be interpretable. For continuous distributions, elicitation often consists of capturing a series of quantile-probability pairs (QPPs) (J. Kadane and Wolfson 1998; Morgan 2014), and then fitting a distribution to these pairs (O'Hagan 2019). However, in practice, the choice of a parametric distribution to fit the elicited QPPs is often influenced by concerns about conjugacy with the selected statistical model that represents the data-generative process (the likelihood) and/or the availability of required distribution functions and fitting algorithms in the software employed. Frequently, the selected distribution possesses fewer parameters than the number of elicited QPPs, which can result in a less-than-perfect fit (O'Hagan 2019). For instance, it is common to elicit three quantiles (the median along with an upper and lower quartile) and subsequently attempt to fit a normal or lognormal distribution (which features two parameters) to these points.

An alternative approach to characterizing the distribution of predictions or parameters is through quantile-parameterized distributions (QPDs). These distributions are parameterized by the QPPs, allowing the elicited values to directly define the distribution, thereby ensuring a good fit and interpretability of parameters. The QPDs examined in this paper can accommodate a wide range of shapes and boundedness, making them valuable for accurately representing experts' prior beliefs.

Parameterizing distributions using a vector of quantiles is not a novel concept in the scientific community. The earliest mention can be traced back to the *substitution likelihood* proposed by Jeffreys (Jeffreys 1939), which outlines a non-parametric procedure for inferring the median using a set of sample quantiles. Subsequently, similar ideas were further developed in (Boos and Monahan 1986; Lavine 1995; Dunson and Taylor 2005).

All the QPDs found in the literature are constructed using the *quantile function*. These distributions are built either by transforming simpler quantile functions or by simultaneous fitting of parameterizing quantiles, as described below.

Let Y be a random variable with a (cumulative) distribution function (CDF) denoted as $F_Y(y|\theta)$. The quantile function (QF) $Q_Y(u|\theta)$ for Y is defined as

$$Q_Y(u|\theta) = \inf\{y : F_Y(y|\theta) \geq u\},\ u \in [0, 1]$$

Here, θ represents the distribution parameter, and the subscript $_Y$ indicates that the depth u corresponds to the random variable Y.

Both the CDF and the QF are considered equally valid ways of defining a distribution (Tukey 1965). For a quantile function that is right-continuous and strictly increasing over the support of Y, the quantile function $Q_Y(u)$ is simply the inverse of the distribution function, denoted as $Q_Y(u|\theta) = F_Y^{-1}(u|\theta)$. Therefore, the quantile function is often referred to as the *inverse CDF*.

The derivative of the quantile function, known as the *quantile density function* (QDF), is denoted as $q(u) = \frac{dQ(u)}{du}$. It is reciprocally related to the probability density function (PDF) $f(x)$, such that $f(Q(u))q(u) = 1$. The quantity $f_Y(Q_Y(u|\theta)) = [q_Y(u|\theta)]^{-1}$ is referred to as the *density quantile* function (Parzen 1979) or *p-pdf* (Gilchrist 2000). The relationships between these functions are concisely illustrated in the probability function Möbius strip (Figure 1).

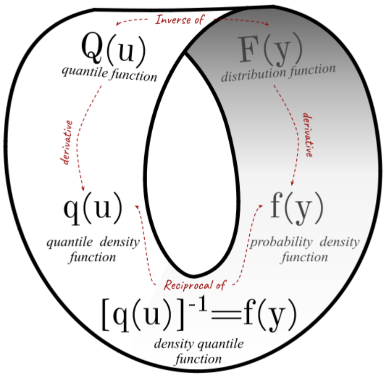

Figure 1. Möbius strip of probability functions (Perepolkin, Goodrich, and Sahlin 2023)

Although many of the distributions discussed in Section 3 have closed-form cumulative distribution functions (CDFs) and probability density functions (PDFs), the functional form of the quantile function (QF) is often simpler and can be reasoned about in terms of other quantile functions, following *Gilchrist's QF transformation rules* summarized in Table 1. This table presents the addition, linear combination, and multiplication rules, which involve two quantile functions Q_1 and Q_2. We will refer to these three rules as *Gilchrist combinations*, as they represent valid ways to combine quantile functions to create new quantile functions.

The quantile-parameterized distributions described in this paper can be categorized into two groups based on their construction method. The first group comprises distributions that are *directly* parameterized by the quantile-probability pairs (QPPs). This group includes the Myerson distribution (Myerson 2005), and the Johnson Quantile-Parameterized Distribution (Christopher C. Hadlock and Bickel 2017, 2019). These distributions are constructed by reparameterizing or transforming existing distributions, following Gilchrist rules (Table 1). The transformations used to construct them are detailed in the next section.

The other group of distributions is *indirectly* parameterized by the QPPs. They require a fitting step where the quantile-probability pairs are translated into distribution parameters, usually through optimization or least-squares methods. This group includes the

Table 1. Gilchrist's quantile function transformation rules (Gilchrist 2000)

[ht]
Table 2.

Original QF	Rule	Resulting QF	Resulting variable
$Q_Y(u)$	Reflection rule	$-Q(1-u)$	QF of -Y
$Q_Y(u)$	Reciprocal rule	$1/Q(1-u)$	QF of $1/Y$
$Q_1(u), Q_2(u)$	Addition rule	$Q_1(u)+Q_2(u)$	valid QF
$Q_1(u), Q_2(u)$	Linear combination rule	$aQ_1(u) + bQ_2(u)$	valid QF for $a, b > 0$
$Q_1(u), Q_2(u) > 0$	Multiplication rule	$Q_1(u)Q_2(u)$	valid QF
$Q_Y(u)$	Q-transformation	$T(Q_Y(u))$	QF of $T(Y)$, $T(Y)$ non-decreasing
$Q_Y(u)$	p-transformation	$Q_Y(H(u))$	p-transformation of $Q_Y(u)$, $H(u)$ non-decreasing

Simple Q-Normal (Keelin and Powley 2011), Metalog (Keelin 2016), quantile mixtures (Peng, Li, and Uryasev 2023), the variant of the Generalized Lambda Distribution (GLD) by Chalabi et al (Chalabi, Scott, and Wuertz 2012), and the quantile-parameterized Triangular (Two-Sided Power) distribution by Kotz and van Dorp (Kotz and Van Dorp 2004). The fitting methods for each of these distributions is described in the respective sub-sections below.

3. Univariate quantile-parameterized distributions

In this section, we review various continuous univariate quantile-parameterized distributions found in the literature. We then discuss the generalized form for these distributions, based on the variations of these QPDs appearing in the literature. For each distribution, we present its quantile function and discuss the parameterization and feasibility conditions. The derivative and inverse of each distribution can be found in Appendix A.

3.1. *Myerson distribution*

One of the earliest examples of a distribution parameterized by quantiles is the *generalized log-normal* distribution defined by the median and the upper and lower quartiles proposed by (Myerson 2005). It relies on a transformation of the normal quantile function.

The Myerson distribution can be viewed as parameterized by three quantile values $\{q_1, q_2, q_3\}$, which correspond to the cumulative probabilities $\{\alpha, 0.5, 1-\alpha\}$. These quantiles are symmetrical around the median and are defined by the tail parameter $0 < \alpha < 0.5$. This type of parameterization is known as the Symmetric Percentile Triplet

(SPT, α-level SPT or α-SPT) and is also used in several other quantile-parameterized distributions that we will describe below. The Myerson quantile function is

$$
\begin{gathered}
\rho = q_3 - q_2;\ \beta = \frac{\rho}{q_2 - q_1};\ \kappa(u) = \frac{S(u)}{S(1-\alpha)} \\
Q_Y(u|q_1, q_2, q_3, \alpha) = \begin{cases} q_2 + \rho\frac{\beta^{\kappa(u)}-1}{\beta-1}, & \beta \neq 1 \\ q_2 + \rho\kappa(u), & \beta = 1 \end{cases}
\end{gathered}
$$

Here, u represents the depth of the observations of the random variable Y given the parameterizing α-SPT $\{q_1, q_2, q_3, \alpha\}$, with $0 < \alpha < 0.5$. The parameter ρ is the *upper p-difference*, and β is the ratio of the inter-percentile ranges, known as the *skewness ratio* (Gilchrist 2000, 72). The *kernel* quantile function $S(u)$ is equal to the quantile function of the standard normal distribution, also referred to as the probit, defined as $S(u) = \Phi^{-1}(u)$. The formulas for the derivative and the inverse quantile function of the Myerson QPD can be found in Appendix A.

It is important to note that while the Myerson distribution includes the normal distribution as a special case when the skewness parameter $\beta = 1$, it can exhibit right-skewness or left-skewness for other values of β. In the symmetrical case, the range of the quantile function is $(-\infty, \infty)$. For the right-skewed distribution ($\beta > 1$), the range is $(q_2 - \frac{\rho}{\beta-1}, \infty)$, and for the left-skewed distribution ($0 < \beta < 1$), the range is $(-\infty, q_2 - \frac{\rho}{\beta-1})$. The limiting case of the skewed Myerson distribution $\lim_{u\to 0} Q_Y(u|\theta)$ for $\beta > 1$ (and the other limit for $0 < \beta < 1$) possesses some important properties that we discuss in Section 3.3 below.

The basic quantile function (Gilchrist 2000; Lampasi 2008) underlying the Myerson distribution is a simple probit, $S(u) = \Phi^{-1}(u)$, transformed using the exponentiation function $T(x) = \beta^x$, where $\beta > 0$ represents the skewness ratio (Gilchrist 2000). The quantile parameterization is facilitated by $\kappa(u)$, which takes values $\{-1, 0, 1\}$ for the three quantiles $\{q_1, q_2, q_3\}$, such that $Q(\alpha) = q_1$, $Q(0.5) = q_2$, and $Q(1-\alpha) = q_3$.

3.2. *Johnson Quantile-Parameterized Distribution*

Hadlock and Bickel (Christopher Campbell Hadlock 2017) reviewed the existing quantile-parameterized distributions and proposed the quantile parameterization of the Johnson SU family of distributions (N. L. Johnson, Kotz, and Balakrishnan 1994). In their paper, Hadlock and Bickel (Christopher C. Hadlock and Bickel 2017) presented two versions of the distribution: the bounded (J-QPD-B) and the semi-bounded (J-QPD-S), both parameterized by an SPT $\{q_1, q_2, q_3, \alpha\}$ and the bound(s).

The J-QPD-B distribution is obtained by applying the inverse-probit transformation to the Johnson SU quantile function $Q_{SU}(u) = \xi + \lambda \sinh(\delta(S(u) + \gamma))$, where δ and γ are two shape parameters. This function is then rescaled to the compact interval $[l_b, u_b]$. The J-QPD-B quantile function is

$$
Q_B(u|q_1, q_2, q_3, \alpha) = \begin{cases} l + (u_b - l_b)S^{-1}(\xi + \lambda \sinh(\delta(S(u) + nc))), & n \neq 0 \\ l + (u_b - l_b)S^{-1}\left(B + \left(\frac{H-L}{2c}\right)S(u)\right), & n = 0 \end{cases}
$$

where

$$S(u) = \Phi^{-1}(u); \quad c = S(1-\alpha);$$
$$L = S\left(\frac{q_1 - l_b}{u_b - l_b}\right); \quad B = S\left(\frac{q_2 - l_b}{u_b - l_b}\right);$$
$$H = S\left(\frac{q_3 - l_b}{u_b - l_b}\right); \quad n = \operatorname{sgn}(L + H - 2B)$$
$$\xi = \begin{cases} L, & n = 1, \\ B, & n = 0, \\ H, & n = -1, \end{cases}$$
$$\delta = \frac{1}{c}\cosh^{-1}\left(\frac{H-L}{2\min(B-L, H-B)}\right)$$
$$\lambda = \frac{H-L}{\sinh(2\delta c)}$$

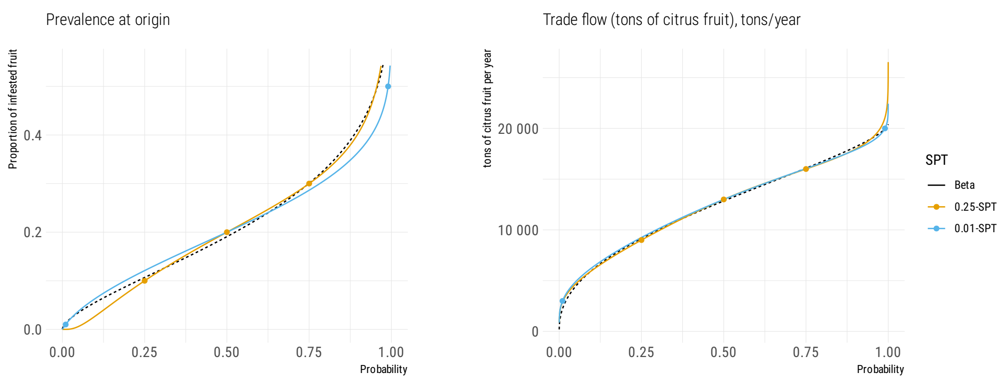

Figure 2. Fitted J-QPD-B (left) and J-QPD-S (right) distribution for prevalence at origin and total trade flow, respectively

The left panel in Figure 2 showcases the J-QPD-B quantile function, which is parameterized using 0.25-SPT and 0.01-SPT assessments of the proportion of fruit infested with *Citripestis sagittiferella*, as elicited by (EFSA et al. 2023). The dashed line represents the Beta distribution fitted by the authors. The J-QPD-B, being parameterized by an SPT, effectively captures three of the five parameterizing quantiles, while the Beta distribution only provides an approximation. Finding parameters of Beta distribution requires an optimization step.

The J-QPD-S distribution is a semi-bounded variant of the distribution that employs exponentiated hyperbolic arcsine transformations of the Johnson's SU quantile function (Christopher C. Hadlock and Bickel 2017)

$$
Q_S(u|q_1,q_2,q_3,\alpha) = \begin{cases} l_b + \theta \exp\left(\lambda \sinh\left(\sinh^{-1}(\delta S(u)) + \sinh^{-1}(nc\delta)\right)\right), & n \neq 0 \\ l_b + \theta \exp\left(\lambda \delta S(u)\right), & n = 0 \end{cases}
$$

where

$$
\begin{gathered}
S(u) = \Phi^{-1}(u); \quad c = S(1-\alpha); \\
L = \ln(q_1 - l_b); \quad B = \ln(q_2 - l_b); \\
H = \ln(q_3 - l_b); \quad n = \mathrm{sgn}(L + H - 2B) \\
\theta = \begin{cases} q_1 - l_b, & n = 1, \\ q_2 - l_b, & n = 0, \\ q_3 - l_b, & n = -1, \end{cases} \\
\delta = \frac{1}{c} \sinh\left(\cosh^{-1}\left(\frac{H-L}{2\min(B-L, H-B)}\right)\right) \\
\lambda = \frac{1}{\delta c} \min(H-B, B-L)
\end{gathered}
$$

When $n = \mathrm{sgn}(L + H - 2B)$ evaluates to zero, he resulting distribution is a lognormal distribution with parameters $\mu = \ln(\theta) = \ln(q_2 - l_b)$ and $\sigma = \lambda\delta = (H - B)/c$. This distribution has support on the interval $[l_b, \infty]$.

The right panel in Figure 2 depicts the J-QPD-S quantile function, which is parameterized using 0.25-SPT and 0.01-SPT assessments of the total trade flow for citrus fruit imported by the EU from Indonesia, Malaysia, Thailand, and Vietnam in tons/year (EFSA et al. 2023).

3.3. *Generalisations of QPDs*

3.3.1. Generalized Johnson Quantile-Parameterized Distribution

Hadlock and Bickel (Christopher C. Hadlock and Bickel 2019) introduced the *generalized* version of the Johnson Quantile-Parameterized distribution system, denoted as G-QPD, by replacing the Normal distribution in the core of the Johnson SU quantile function with the quantile functions of the logistic and Cauchy distributions.

The generalized quantile function (QF) shares similarities with the probit-based distribution described earlier, with $S(u)$ defined as the quantile function of either the logistic or Cauchy distribution.

The standard quantile function and distribution function of the logistic distribution are given by:

$$
S(u) = \ln\left(\frac{u}{1-u}\right); \quad S^{-1}(y) = [\exp(-y) + 1]^{-1}
$$

The standard quantile function and distribution function of the Cauchy distribution are given by:

$$S(u) = \tan\left[\pi\left(u - \frac{1}{2}\right)\right]; \quad S^{-1}(y) = \frac{1}{\pi}\arctan(y) + \frac{1}{2}$$

Hadlock and Bickel (Christopher C. Hadlock and Bickel 2019) show that the *kernel* quantile function $S(u)$ can be any standardized ($S(0.5) = 0$), symmetrical ($s(u) = s(1-u)$), and unbounded ($S(u) \in (-\infty; \infty)$) quantile function with a smooth quantile density $dS(u)/du = s(u)$. The authors further showed that if $S(u)$ and $S^{-1}(y)$ are expressible in closed-form, the quantile function and distribution function of G-QPD will also be closed-form.

For the *logistic* kernel, the G-QPD-S represents the generalized log-logistic distribution, characterized by two shape parameters, λ and δ. For the Cauchy kernel, the G-QPD-S corresponds to the shifted log-Cauchy distribution (Christopher C. Hadlock and Bickel 2019).

3.3.2. Generalized Myerson distributions

Following the approach in Hadlock and Bickel (Christopher C. Hadlock and Bickel 2019), Myerson distribution can be generalized by substituting the Normal kernel quantile function $S(u) = \Phi^{-1}(u)$ with an alternative symmetrical quantile function based on the depth u. Below, we discuss possible kernels and the resulting distributions:

Logit-Myerson distribution. Recently Wilson et al (Kevin J. Wilson et al. 2023) reparameterized *log-logistic distribution* in terms of a Symmetric Percentile Triplet. Even though the authors do not recognize it as such, the resulting quantile-parameterized distribution is a Myerson distribution with logit kernel QF $S(u) = \ln\left(\frac{u}{1-u}\right)$).

There could be several reasons why one might prefer the logit function over the probit function (Berkson 1951). For example, distribution based on logit may exhibit greater numerical stability due to its simple closed-form quantile function, which does not rely on numerical approximation during sampling. Logit-Myerson distribution displays slightly heavier tails compared to the standard (probit-based) Myerson distribution (Figure 3).

Sech-Myerson distribution. Following the same principle adopted by Wilson et al. (Kevin J. Wilson et al. 2023) a variant of Myerson distribution may be created using the hyperbolic secant quantile function:

$$S(u) = \ln\left[\tan\left(\frac{\pi}{2}u\right)\right]$$

The Sech-Myerson distribution possesses thicker tails than the Logit-Myerson distribution for the same parameterizing SPT $\{-5, 4, 16, 0.25\}$ (Figure 3). In Section 3.8, we conduct a comparative analysis of different variations of the Generalized Myerson distribution alongside their parametric counterparts and other quantile distributions.

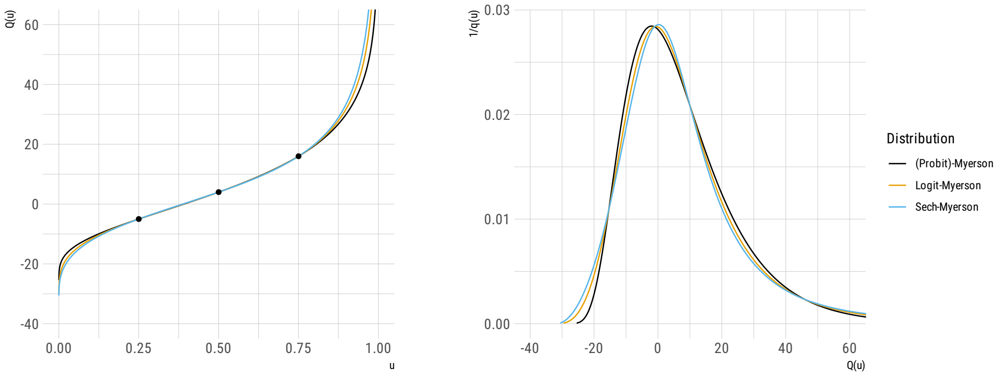

Figure 3. Quantile function and quantile density of Generalized Myerson Distributions

Theoretically, there is an infinite range of quantile function (QF) kernels that can be utilized to generate new variations of the Generalized Myerson distribution. These candidate kernel distributions can even include shape parameters, as long as the resulting $S(u)$ remains standardized, symmetrical, and unbounded, as specified above. For instance, it is possible to incorporate the basic QF of the Tukey Lambda distribution $S(u|\lambda) = u^\lambda - (1-u)^\lambda$ for a fixed $\lambda \neq 0$, or the Cauchy distribution $S(u) = \tan[\pi(u-0.5)]$, as employed by (Christopher C. Hadlock and Bickel 2019). However, it is important to note that not all standard quantile functions are created equal. To illustrate the issue of unreliable kernels, let us consider Myerson distributions based on the Cauchy and Tukey Lambda quantile functions (for $\lambda = -0.5$). As can be observed in Figure 4, the density of Generalized Myerson distribution with these kernels exhibits unexpected spike near the lower bound.

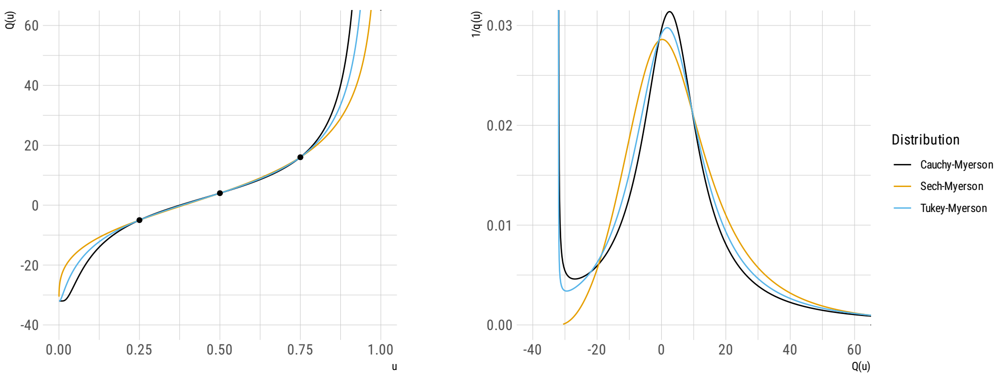

Figure 4. Quantile function and quantile density of Generalized Myerson Distributions with unreliable kernels

While all right-skewed Generalized Myerson distributions are bounded on the left at $\lim_{u\to 0} Q(u|\theta) = q_2 - \rho\frac{1}{\beta-1}$ regardless of the kernel used, the quantile density at the left limit $\lim_{u\to 0}[q(u|\theta)]^{-1}$ is not independent of the kernel. Although we can assume that $q(0) = \infty$, the lower tail of the density quantile function $[q(u)]^{-1}$ may exhibit a curling effect for certain kernels, resulting in an increase in density for lower values of u. This effect is caused by the non-monotonic behavior of the quantile convexity function $c(u) = dq(u)/du$. This can be easily verified by taking the second derivative of $\beta^{S(u)}$ for $\beta > 0$. While such kernels are mathematically valid and yield a non-decreasing Generalized Myerson QF, we believe that they may be less useful due to the counter-intuitive concentration of density in the bounded tail. Consequently, we do not recommend using Cauchy or Tukey Lambda kernels in practical applications.

3.4. *Simple Q-Normal, Metalog distributions*

An alternative system of quantile-parameterized distributions was proposed by Keelin and Powley (Keelin and Powley 2011; Powley 2013). This approach relies on the finite Taylor expansion of parameters in the standardized quantile functions. Within this framework, two distributions were introduced: the Simple Q-Normal distribution and the Metalog distribution.

The Simple Q-Normal (SQN) distribution was developed by expanding the parameters in the normal quantile function. Keelin et al. (2011) used this method to express the parameters of the normal quantile function $Q(u|\mu, \sigma) = \mu + \sigma z(u)$ as linear functions of the depth u. Specifically, $\mu(u) = a_1 + a_4 u$ and $\sigma(u) = a_2 + a_3 u$, where $z(u) = \Phi^{-1}(u)$ denotes the standard normal quantile function. Therefore, the quantile function of the SQN distribution can be expressed as follows:

$$Q(u) = a_1 + a_2 z(u) + a_3 u z(u) + a_4 u \tag{1}$$

where $z(u) = \Phi^{-1}(u)$, and $a = \{a_1, a_2, a_3, a_4\}$ represents a vector of parameters.

Consider a quantile-probability tuple of size 4, denoted as $\{\mathbf{p}, \mathbf{q}\}_4$, which consists of an ordered vector of cumulative probabilities $\mathbf{p} = \{p_1, p_2, p_3, p_4\}$ and an ordered vector of corresponding quantiles $\mathbf{q} = \{q_1, q_2, q_3, q_4\}$. Substituting these vectors into the SQN quantile function for u and $Q(u)$, respectively, we obtain the following matrix equation:

$$\mathbf{q} = \mathbb{P} a \tag{2}$$

where

$$\mathbb{P} = \begin{bmatrix} 1 & z(p_1) & p_1 z(p_1) & p_1 \\ 1 & z(p_2) & p_2 z(p_2) & p_2 \\ 1 & z(p_3) & p_3 z(p_3) & p_3 \\ 1 & z(p_4) & p_4 z(p_4) & p_4 \end{bmatrix}$$

and $a = \{a_1, a_2, a_3, a_4\}$ represents the parameter vector of the SQN distribution.

The parameter vector a can be obtained by solving the matrix Equation 2, given the 4-element quantile-probability tuple $\{\mathbf{p}, \mathbf{q}\}_4$ (Keelin and Powley 2011; Perepolkin, Goodrich, and Sahlin 2021).

The same approach was later employed by (Keelin 2016) in creating the metalog (metalogistic) distribution. Starting with the quantile function of the logistic distribution $Q(u|\mu, s) = \mu + s\text{logit}(u)$, where μ corresponds to the mean and s is proportional to the standard deviation $\sigma = s\pi/\sqrt{3}$, (Keelin 2016) expanded the parameters μ and s using a finite Taylor series centered at 0.5. Specifically, $\mu(u) = a_1 + a_4(u - 0.5) + a_5(u - 0.5)^2 + ...$ and $s(u) = a_2 + a_3(u - 0.5) + a_6(u - 0.5)^2 + ...$, where $a_i,\ i = \{1, 2, \dots, n\}$ are real constants.

Therefore, the metalog quantile function is:

$$Q(u) = a_1 + a_2\text{logit}(u) + a_3(u - 0.5)\text{logit}(u) + a_4(u - 0.5) + a_5(u - 0.5)^2 \cdots,$$

Given a QPT of size m denoted by $\{\mathbf{p}, \mathbf{q}\}_m$, where $\mathbf{p}$ and $\mathbf{q}$ are ordered vectors of cumulative probabilities and corresponding quantiles, respectively, the vector of coefficients $\mathbf{a} = a_1, \dots, a_m$ can be determined by solving the matrix equation $\mathbf{q} = \mathbb{P}\mathbf{a}$, where $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{a}$ are column vectors, and $\mathbb{P}$ is an $m \times n$ matrix:

$$\mathbb{P} = \begin{bmatrix} 1 & \text{logit}(p_1) & (p_1 - 0.5)\text{logit}(p_1) & (p_1 - 0.5) & \cdots \\ 1 & \text{logit}(p_2) & (p_2 - 0.5)\text{logit}(p_2) & (p_2 - 0.5) & \cdots \\ & & \vdots & & \\ 1 & \text{logit}(p_m) & (p_m - 0.5)\text{logit}(p_m) & (p_m - 0.5) & \cdots \end{bmatrix} \tag{3}$$

The vector of coefficients $\mathbf{a}$ can be determined as $\mathbf{a} = [\mathbb{P}^T\mathbb{P}]^{-1}\mathbb{P}^T\mathbf{q}$. If $\mathbb{P}$ is a square matrix, meaning the number of terms n is equal to the size of the parameterizing QPT m, the equation can be further simplified to $\mathbf{a} = \mathbb{P}^{-1}\mathbf{q}$. Metalog is said to be *approximated* when the number of quantile-probability pairs used for parameterization exceeds the number of terms in the metalog QF (Keelin 2016; Perepolkin, Goodrich, and Sahlin 2021).

The SQN and Metalog distributions are families of extended distributions that, in theory, can have an arbitrary number of terms. Keelin (Keelin 2016) demonstrated the flexibility of the metalog distribution and its ability to approximate arbitrarily complex probability density functions with high precision, given enough terms in the metalog specification. In practice, 10-15 terms are sufficient to approximate the distributional shapes of virtually any complexity (Keelin and Howard 2021). Keelin (Keelin 2016) introduced the bounded logit-metalog, the semi-bounded log-metalog, and a special case of a 3-term metalog parameterized by α-SPT (SPT-metalog).

However, not all combinations of parameters $\mathbf{a}$ in metalog and SQN distributions result in a feasible (non-decreasing) quantile function. For an arbitrary $\mathbf{a}$-vector, feasibility must be checked (Keelin and Powley 2011). In the case of 3-term metalogs, the feasibility conditions are straightforward (Keelin 2016). But as the number of terms increases, such conditions become increasingly complex (Keelin 2017). Having to deal with such feasibility requirements stands in contrast with QF's that are constructed using Gilchrist rules Table 1, which guarantee feasibility.

3.5. ***Quantile mixtures***

Recently (Peng, Li, and Uryasev 2023) proposed a novel framework for extended quantile-parameterized distributions based on quantile mixtures (not to be confused with CDF/PDF mixtures, (Gilchrist 2000, 107)). They introduced a formulation in which a QPD quantile function is expressed as a linear combination of I standardized quantile functions, following Gilchrist's *linear combination rule* (Table 1):

$$G(u|\theta) = \sum_{i=0}^{I} \theta_i Q_i(u)$$

Here, $Q_i(u)$ represent basis quantile functions for the random variable Y with $Q_0(u) = 1$, and $\boldsymbol{\theta} = \{\theta_0, \theta_1, \dots, \theta_I\}$ is a non-negative parameter vector that determines the contribution of each QF component in the quantile mixture. To compute the coefficients $\boldsymbol{\theta}$, the system of equations is solved

$$\mathbf{q} = \mathbb{Q}\boldsymbol{\theta} + \boldsymbol{\epsilon}$$

where $\mathbf{q} = \{q_1, q_2, \dots, q_j\}$ is an ordered vector of J parameterizing quantiles, corresponding to an ordered vector of cumulative probabilities $\mathbf{p} = \{p_1, p_2, \dots, p_j\}$, $\boldsymbol{\theta}$ is a non-negative vector of $I+1$ parameters, $\boldsymbol{\epsilon}$ is a J-size vector of errors to be minimized, and $\mathbb{Q}$ is a $J \times (I+1)$ matrix of regression factors

$$\mathbb{Q} = \begin{bmatrix} 1 & Q_1(p_1) & Q_2(p_1) & \cdots & Q_I(p_1) \\ 1 & Q_1(p_2) & Q_2(p_2) & \cdots & Q_I(p_2) \\ \vdots & & \vdots & \ddots & \\ 1 & Q_1(p_J) & Q_2(p_J) & \cdots & Q_I(p_J) \end{bmatrix}$$

By ensuring non-negativity of weights ($\theta_i \geq 0$), the solution guarantees a proper non-decreasing quantile function. To estimate the values of the vector $\boldsymbol{\theta} \in \Theta$, the authors suggest using constrained weighted least squares regression with optional regularization. The authors demonstrated that the estimator $\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\Theta}{\operatorname{argmin}} \left(\frac{1}{J} \sum_{j=1}^{J} w_j \mathcal{E}_q(y_j - Q_j\boldsymbol{\theta}) \right)^{\frac{1}{q}}$, $\mathcal{E}_q(x) = |x|^q$, $w_j > 0$, is asymptotically a q-Wasserstein distance estimator, which converges in distribution to a Normal distribution. The paper (Peng, Li, and Uryasev 2023) includes the application of the quantile mixture model using a large number of asymmetric t-distributions, and a quantile mixture of Generalized Beta II distributions.

The quantile mixtures method of creating new QPDs guarantees feasibility by construction, while affording nearly infinite flexibility, provided that the component quantile functions are selected from a wide set of distributions of varying shapes. Besides, a QPD constructed as a linear combination of QFs is guaranteed to be unimodal, unless one of the component in the mixture is multimodal (see Gilchrist 2000 for examples). In addition, the method proposed by (Peng, Li, and Uryasev 2023) offers an advantage of resulting in closed-form quantile function and quantile density function, provided that each of the components can be expressed analytically. Unfortunately, neither asymmetric t-distribution nor Generalized Beta II distribution, used by the authors, has a closed-form QF. However, one can construct a highly flexible quantile function using Gilchrist rules (Table 1) or use one of the existing well-studied QFs discussed in the literature. In Section 3.7, we provide an example of using a quantile mixture of diversely-shaped quantile functions to construct a bespoke highly flexible QPD.

3.6. *Other distributions*

3.6.1. Triangular and Two-Sided Power distributions

Several other distributions with at least some parameters mapped to quantiles were proposed, including the reparameterization of the Generalized Lambda Distribution by (Chalabi, Scott, and Wuertz 2012) and the quantile-parameterized triangular (two-sided power) distribution by (Kotz and Van Dorp 2004).

Kotz and van Dorp (Kotz and Van Dorp 2004) describe the quantile-parameterized version of the triangular distribution (D. Johnson 1997). This bounded distribution is widely used in the finance and insurance industry and is popularized by the @Risk software package, developed by Palisade (Palisade Corporation 2009). The triangular distribution is parameterized by the two quantiles q_a and q_b, and the mode m, subject to the constraint that $a \leq q_a \leq m \leq q_b \leq b$, where a and b represent the lower and upper bounds, respectively. The standard quantile function for the triangular distribution is expressed in terms of the bounds a, b, and the mode m.

$$Q(u|a,m,b) = \begin{cases} a + \sqrt{u(m-a)(b-a)}, & \text{for } 0 \leq u \leq \frac{m-a}{b-a} \\ b - \sqrt{(1-u)(b-m)(b-a)}, & \text{for } \frac{m-a}{b-a} \leq u \leq 1 \end{cases}$$

In (Kotz and Van Dorp 2004) the authors show that given the two parameterizing quantile-probability pairs q_a, p_a and q_b, p_b and the mode value m, there exists a unique value of depth $p_a < p < p_b$ corresponding to the root of the function

$$g(p) = \frac{(m-q_a)(1-\sqrt{\frac{1-p_b}{1-p}})}{(q_b-m)(1-\sqrt{\frac{p_a}{p}}) + (m-q_a)(1-\sqrt{\frac{1-p_b}{1-p}})} - p$$

The root value $p \in (p_a, p_b)$ of the function $g(p)$ can be found using any of the bracketing root-finding algorithms (Perepolkin, Goodrich, and Sahlin 2023). It can then be substituted into the following expressions to find the lower a and upper b limit parameters of the triangular distribution:

$$
a(p) \equiv \frac{q_a - m\sqrt{\frac{p_a}{p}}}{1-\sqrt{\frac{p_a}{p}}}, \quad a(p) < q_a
$$

$$
b(p) \equiv \frac{q_b - m\sqrt{\frac{1-p_b}{1-p}}}{1-\sqrt{\frac{1-p_b}{1-p}}}, \quad b(p) > q_b
$$

The book (Kotz and Van Dorp 2004) provides an algorithm for fitting a four-parameter generalization of the triangular distribution called the Two-Sided Power Distribution (TSP), using three quantile-probability pairs and a mode value. For more information on fitting the Quantile-Parameterized TSP Distribution by quantiles, refer to Section 4.3.3 of (Kotz and Van Dorp 2004).

3.6.2. Generalized Lambda Distribution

Chalabi, Scott and Würtz (CSW) (Chalabi, Scott, and Wuertz 2012) proposed an asymmetry-steepness reparameterization of the Generalized Lambda Distribution (GLD) (Freimer et al. 1988) with four parameters. This reparameterization involves mapping the location to the median and the scale to the interquartile range (IQR), which corresponds to the first and second robust moments (Kim and White 2004; Moors 1988).

The reparameterized Generalized Lambda Distribution (CSW GLD) has a quantile function given by

$$
Q(u|\tilde{\mu}, \tilde{\sigma}, \chi, \xi) = \tilde{\mu} + \tilde{\sigma}\frac{S\left(u|\chi,\xi\right) - S\left(\frac{1}{2}|\chi,\xi\right)}{S\left(\frac{3}{4}|\chi,\xi\right) - S\left(\frac{1}{4}|\chi,\xi\right)}
$$

where $\tilde{\mu}, \tilde{\sigma}, \chi, \xi$ represent the location, scale, asymmetry, and steepness parameters, respectively. The specific form of the basic function $S(u)$ depends on the values of the parameters χ and ξ

$$
S(u|\chi,\xi) = \begin{cases}
\ln(u) - \ln(1-u), & \text{if } \chi = 0, \xi = 0.5 \\
\ln(u) - \frac{1}{2\alpha}\left[(1-u)^{2\alpha} - 1\right], & \text{if } \chi \neq 0, \xi = \frac{1}{2}(1+\chi) \\
\frac{1}{2\beta}\left[u^{2\beta} - 1\right] - \ln(1-u), & \text{if } \chi \neq 0, \xi = \frac{1}{2}(1-\chi) \\
\frac{1}{\alpha+\beta}\left[u^{\alpha+\beta} - 1\right] - \frac{1}{\alpha-\beta}\left[(1-u)^{\alpha-\beta} - 1\right], & \text{otherwise}
\end{cases}
$$

where $\alpha = 0.5\frac{0.5-\xi}{\sqrt{\xi(1-\xi)}}$ and $\beta = 0.5\frac{\chi}{\sqrt{1-\chi^2}}$. The bounds of the distribution are given by

$$S(0|\chi,\xi) = \begin{cases} -\dfrac{1}{\alpha+\beta}, & \text{if } \xi < \dfrac{1}{2}(1+\chi) \\ -\infty, & \text{otherwise} \end{cases}$$

$$S(1|\chi,\xi) = \begin{cases} \dfrac{1}{\alpha-\beta}, & \text{if } \xi < \dfrac{1}{2}(1-\chi) \\ \infty, & \text{otherwise} \end{cases}$$

The CSW GLD can have unbounded, bounded, and semi-bounded support, accommodating a wide range of shapes, including unimodal, monotone, U-shaped, and S-shaped densities (Chalabi, Scott, and Wuertz 2012). Although the CSW GLD is not strictly parameterized by quantiles, the mapping of the location and scale parameters to the median and IQR makes it a suitable candidate for expert-informed distribution specification.

Several specialized methods have been developed for fitting the GLD to samples (Karian and Dudewicz 2003). The parameterization of the CSW GLD simplifies the fitting process because two of the four parameters can be directly calculated from the sample: the location parameter is equal to the sample median, and the scale parameter is equal to the interquartile range. The remaining parameters can be estimated using various methods, including robust moment matching, quantile matching, trimmed L-moments, distributional least squares/absolutes, as well as maximum likelihood estimation (Chalabi, Scott, and Wuertz 2012; Gilchrist 2000). The range of feasible values for the steepness and asymmetry parameters can be further reduced with the shape conditions specified in Section 3.5 of (Chalabi, Scott, and Wuertz 2012).

Recently, (Dedduwakumara, Prendergast, and Staudte 2021) proposed a new method of matching the shape of the GLD distribution to data using the probability density quantile (pdQ) function (Staudte 2017). For the quantile function $Q(v),\ v \in [0,1]$ and the corresponding density quantile function $f(Q(v)) = [q(v)]^{-1}$, the pdQ is defined as

$$f^*(v) = \frac{f(Q(v))}{E\left[f(Q(v))\right]}$$

The probability density quantile function is defined on the unit square and is independent of the location and scale parameters.

Since integrating the GLD density quantile function is difficult, (Staudte 2017, sec. 2.2), proposed using the kernel density method to estimate the empirical QDF and, thus, an empirical pdQ for samples from continuous distributions. Fitting the CSW GLD to a sample can be reduced to finding the asymmetry and steepness parameters that minimize

$$\underset{\chi,\xi}{\operatorname{argmin}} \int_0^1 \left[f^*(v,\chi,\xi) - f^*_e(v)\right]^2 du$$

where $f^*(v, \chi, \xi)$ is the pdQ of the CSW GLD, and $f_e^*(v)$ is the empirical pdQ of the sample. The authors (Dedduwakumara, Prendergast, and Staudte 2021) suggest approximating the integral by a discrete set of depths v, replacing the integral with a sum.

3.7. *Example*

As an illustration of a faithful approximation of a large number of quantile-probability pairs by QPD, we take 4000 posterior samples (4 chains of 1000 samples each) of one of the random intercepts in the Eight Schools example model included in the `cmdstanr` package (Gabry and Češnovar 2022) in R. The Eight Schools problem (Rubin 1981) measuring the effectiveness of SAT coaching program in 8 US schools is often used as an example model in introductory classes on Bayesian Statistics. In `cmdstanr` it is modeled using a hierarchical Bayesian model with normal priors for each of the 8 random intercepts `theta`. However, due to the low number of posterior samples and the heterogeneity in the data, the marginal posterior distributions of the intercept parameters `theta` deviate from the Gaussian shape in various ways (Figure 5).

An empirical distribution of posterior samples from a Bayesian model can be viewed as a large number of quantile-probability pairs. Although it is unlikely that such number of quantile-probability pairs could ever be elicitable from an expert (in our case 4000), it could still be of interest to approximate such marginal posterior distribution with a highly flexible quantile function, e.g. for the purpose of posterior passing (Brand et al. 2019; Pritsker 2021). Closed-form QF expression for the posterior margins would allow reusing it as a prior in a similar model at a later stage. We discuss multivariate extension of this idea in Section 4.

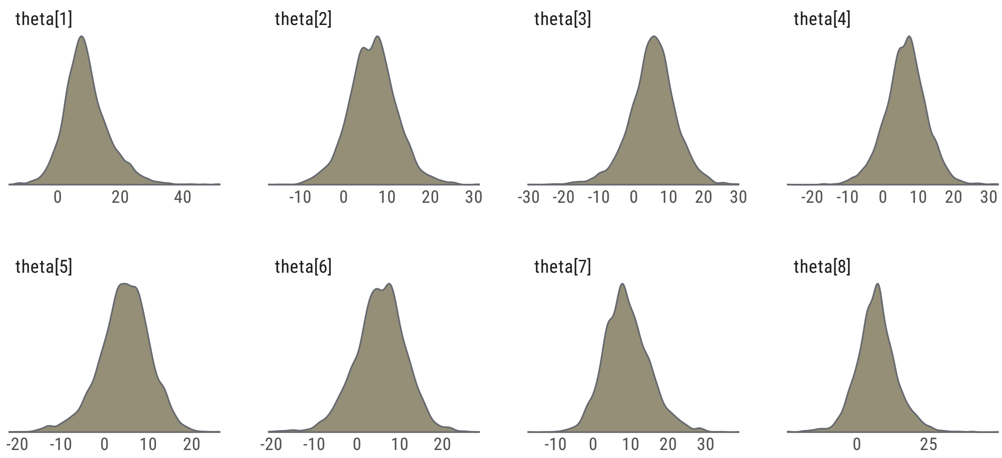

Figure 5. Posterior distributions of random interecept parameters `theta` in the Eight Schools example model (Gabry and Češnovar 2022)

Figure 7 shows a QPD approximation of the marginal distribution of `theta[5]` using a quantile mixture of standardized (centered at zero and with the scale parameter set to one) Chalabi, Scott, and Wuertz (2012) Generalized Lambda Distributions (CSW GLD). In order to ensure the diversity of mixture components we generated 400 independent uniformly distributed pairs of the two shape parameters for GLD components using Hubbard (2019) pseudo random number generator.

We constructed the matrix $\mathbb{Q}$ above following the method outlined by Peng, Li, and Uryasev (2023) and used Lawson-Hanson non-negative least squares algorithm (implemented in `nnls` package (Mullen and van Stokkum 2023) in R) to find the weights for each of the mixture components. The non-zero elements are shows in Figure 6 along with the weights (which become the scale parameters of the quantile mixture components).

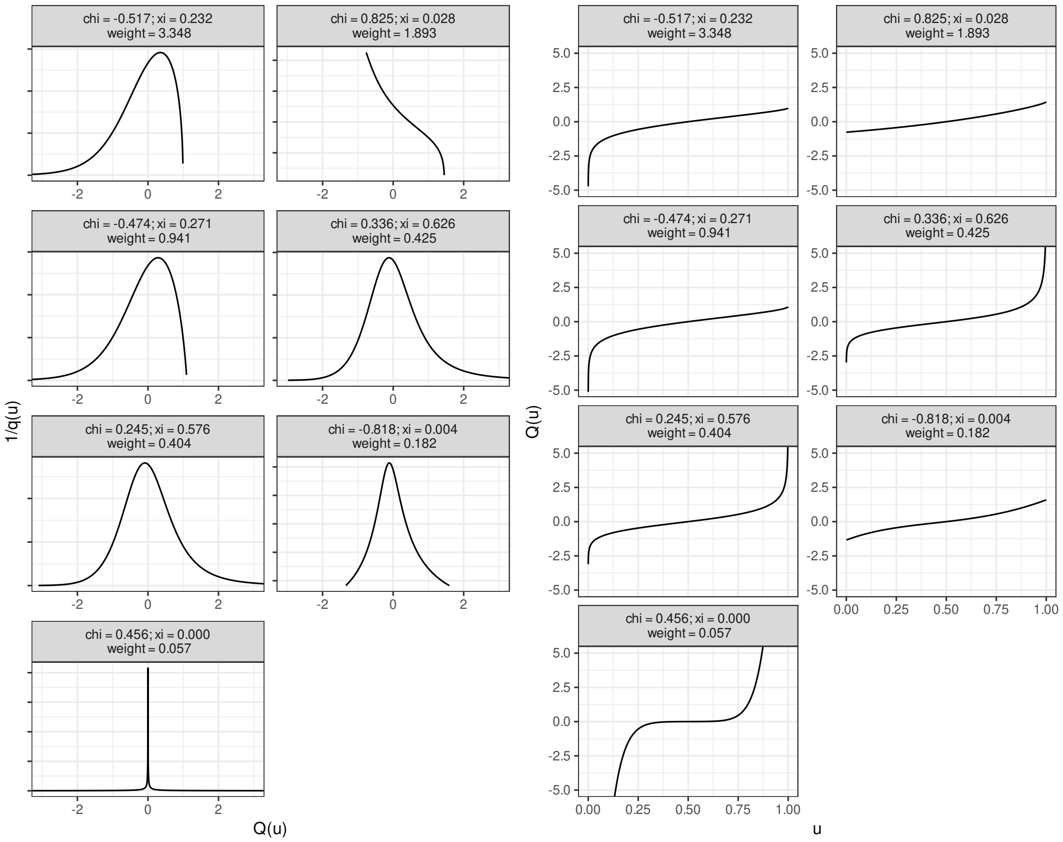

Figure 6. Density functions and quantile functions of GLD components in approximating quantile mixture

Figure 7 shows the histogram of 4000 parameter values for `theta[5]` along with the approximation using the quantile mixture with the components shown in Figure 6. The resulting mixture is a linear combination of GLD quantile functions with a closed form QF and DQF, which makes it possible to reuse this distribution as a quantile-based prior in a Bayesian model (Perepolkin, Goodrich, and Sahlin 2023).

3.8. *Choosing quantile-parameterized distribution*

A common approach to assess the properties of probability distributions is through central moments, denoted by $\mu_k = \mathbb{E}[(Y-\mu)^k]$, where μ represents the expected value of Y. Karl Pearson introduced a classification system for distributions using moment ratios associated with skewness and kurtosis (Fiori and Zenga 2009):

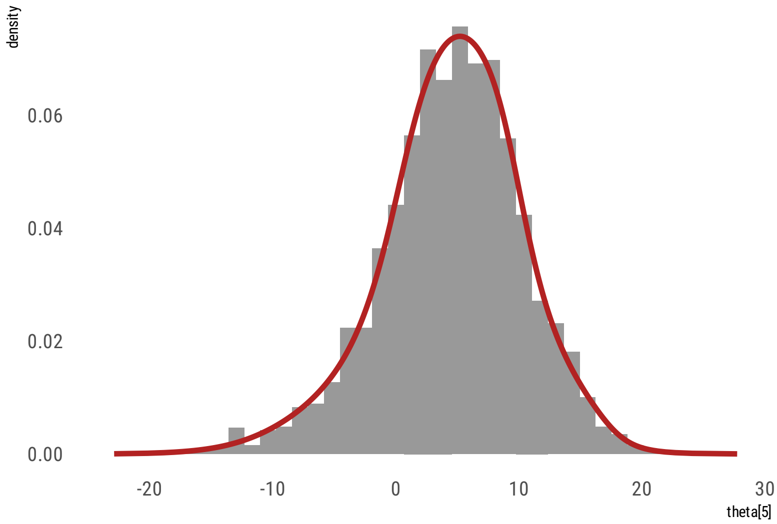

Figure 7. Distribution of posterior samples approximated by the quantile mixture

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \quad \beta_2 = \frac{\mu_4}{\mu_2^2}$$

While computing moments using the quantile function is straightforward (the n-th raw moment is $\mu_k = \int_0^1 Q(u)^k du$), it may not be possible to calculate higher-order moments for certain distributions.

Alternatively, robust alternatives to moments can be utilized, such as the sample median μ_r, the interquartile range σ_r, the quartile-based robust coefficient of skewness s_r (Kim and White 2004), also known as Bowley's skewness (Bowley 1920) or Galton's skewness (Gilchrist 2000), and the octile-based robust coefficient of kurtosis κ_r, also known as Moors' kurtosis (Moors 1988).

$$\begin{aligned}
\mu_r &= Q(1/2) \\
\sigma_r &= Q(3/4) - Q(1/4) \\
s_r &= \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{\sigma_r} \\
\kappa_r &= \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{\sigma_r}
\end{aligned}$$

(Kim and White 2004; Arachchige, Prendergast, and Staudte 2022) have proposed to standardize robust moments to facilitate their comparison with the corresponding robust moments of the standard normal distribution. (Groeneveld 1998; Jones, Rosco, and Pewsey 2011) have introduced generalizations of robust moments to other quantiles.

Unlike moments, quantiles are always well defined, and since QPDs are parameterized by quantile-probability pairs, quantile-based robust moments can sometimes be directly computed from the parameters. For instance, if the basic quantile function $S(u)$ in $Q(u) = \mu + \sigma S(u)$ is standardized (such that $S(0.5) = 0$), where μ and σ are the location and scale parameters of $Q(u)$ respectively, then $\mu_r = \mu$. Moreover, σ_r is always independent of location, and s_r and κ_r are independent of both location and scale.

Figure 8, Figure 9, and Figure 10 resemble the Cullen and Frey (Cullen, Frey, and Frey 1999) plots (Pearson plots), but instead of using central moments, they employ quartile/octile-based robust metrics of skewness s_r and kurtosis κ_r to compare the quantile-parameterized distributions to some of their parametric counterparts.

In these plots, Metalog3 and Metalog4 refer to 3- and 4-term metalog distributions, respectively, and GLDcsw refers to Chalabi et al (Chalabi, Scott, and Wuertz 2012) parameterization of GLD. As can be seen in Figure 8, all generalizations of Myerson distributions have higher robust kurtosis for the same robust skewness. Additionally, GLD CSW is more flexible than the unbounded 4-term metalog. The *log*-transformed metalog distribution appears to be the best among the semi-bounded distributions (Figure 9). Furthermore, the flexibility of the bounded J-QPD-B is at least as good as that of the Beta and Kumaraswamy distributions (Figure 10).

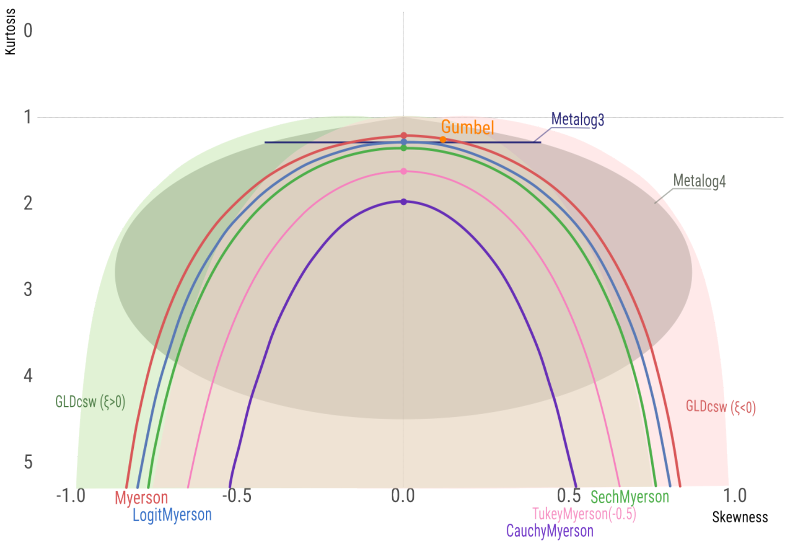

Figure 8. Robust skewness vs robust kurtosis for some unbounded distributions

4. Multivariate quantile-parameterized distributions

Quantile-parameterized distributions can serve as marginal distributions in multivariate models, where the dependency structure is captured by a standard (parametric) multivariate distribution, a copula, or described by bivariate quantiles. However, the marginal distributions alone are insufficient to determine the corresponding bivariate distribution, resulting in an infinite number of bivariate distributions with the same margins (Gumbel 1960, 1961). In this section, we describe several methods for extending

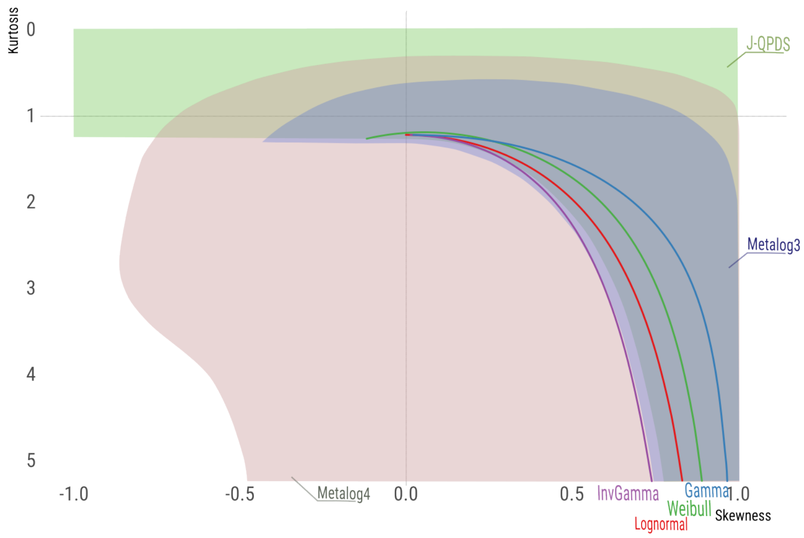

Figure 9. Robust skewness vs robust kurtosis for some left-bounded distributions

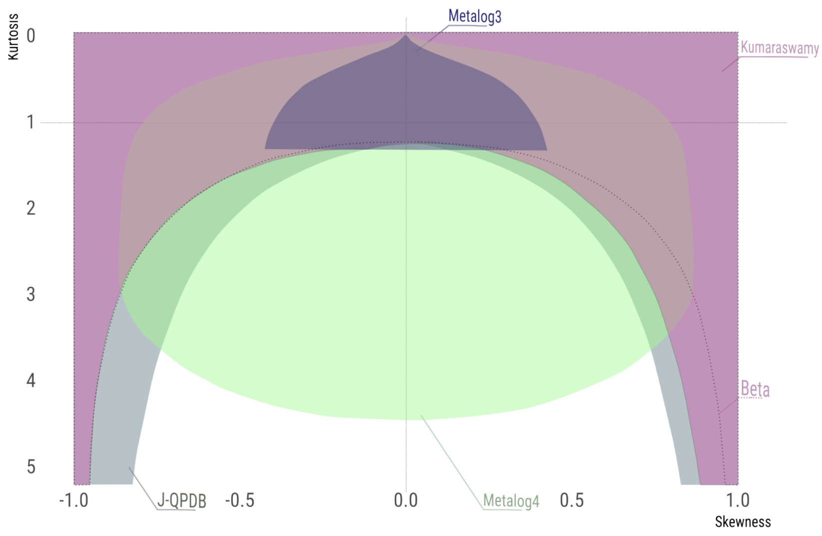

Figure 10. Robust skewness vs robust kurtosis for some bounded distributions

the distributions parameterized by the quantile-probability pairs to become Multivariate Quantile-Parameterized Distributions (MQPDs).

4.1. *MQPDs based on standard multivariate distributions*

4.1.1. Normal distribution

In the simplest case, multivariate Quantile-Parameterized Distributions (MQPDs) can be created by using the multivariate normal distribution, following the approach of (Hoff 2007). The Myerson, J-QPD, and SQN quantile functions are Q-transformations of the probit $Q(z(u)|\theta)$, where $z(u) = \Phi^{-1}(u)$ represents the standard normal quantile function. The multivariate versions of these distributions can be viewed as the Q-transformations of the multivariate normal distribution. To extend these QPDs to J dimensions using the multivariate normal distribution, we employ the method outlined in (Drovandi and Pettitt 2011).

The i-th component of a single observation y_i can be described by the quantile function:

$$y_i = Q(z(u_i)|\theta_i), \text{ for } i = 1, \dots, J$$

where θ_i represents the set of parameters for component i (e.g., $\{q_1, q_2, q_3, \alpha\}_i$) for Myerson or J-QPD distributions). The vector $(z(u_1), \dots, z(u_j))^T \sim N(0, \Sigma)$, where Σ denotes the covariance matrix.

For invertible distributions, the inverse quantile function is the cumulative distribution function (CDF) $Q^{-1}(y_i|\theta) = F(y_i|\theta)$, otherwise, the inverse can be computed numerically as $\widehat{F}(y_i|\theta) = \widehat{Q^{-1}}(y_i|\theta)$ (Perepolkin, Goodrich, and Sahlin 2023).

Drovandi and Pettitt (Drovandi and Pettitt 2011) show that the joint density of a single (multivariate) observation $(y_i, \dots, y_J)$ can be expressed as:

$$f(y_1, \dots, y_J|\theta) = \varphi(z(Q^{-1}(y_1|\theta_1)), \dots, z(Q^{-1}(y_J|\theta_J)); \Sigma) \prod_{i=1}^{J} \frac{dQ^{-1}(y_i|\theta_i)}{dy_i}$$

where $z(Q^{-1}(y_i|\theta_i)) = z_i$, $\varphi(z_1, \dots, z_J; \Sigma)$ represents the multivariate normal density with a mean of zero and a covariance matrix of Σ, and $\frac{dQ^{-1}(y_i)}{dy_i} = f(y_i)$ is the probability density function (PDF) of the QPD (refer to Appendix A).

For distributions without a PDF, the same joint density can be expressed as a joint density quantile function

$$[q(u_1, \dots, u_j)]^{-1} = \varphi(z(u_1), \dots, z(u_J); \Sigma) \prod_{i=1}^{J} [q(u_i|\theta_i)]^{-1}$$

since $Q^{-1}(y_i|\theta_i) = u_i$ and $f(y_i|\theta_i) = [q(u_i|\theta_i)]^{-1}$ (Gilchrist 2000).

It's worth noting that this method of creating multivariate distributions does not require every component to follow the same distributional form. As illustrated earlier, it is entirely possible to combine several different QPDs using the multivariate Gaussian distribution (Drovandi and Pettitt 2011).

To use the MQPD for the prior, both the density of the multivariate normal and the marginal densities need to be explicitly added to the log-likelihood. This is possible when the marginal QPDs used to define the multivariate prior are invertible, such as Myerson and J-QPD, as both the CDF ($Q^{-1}(y_i|\theta_i)$) and PDF ($dQ^{-1}(y_i|\theta_i)/dy_i$) are required.

When a quantile-based prior specification is used, only the multivariate normal log-density needs to be added because the Jacobian for the marginal QF transformation is reciprocal to the DQF of the prior (Perepolkin, Goodrich, and Sahlin 2023).

4.1.2. Logistic distribution

The same approach of joining the marginal QPDs can be applied by using the base quantile functions of other distributions. For instance, the Logit-Myerson distribution (Kevin J. Wilson et al. 2023) is based on the logistic quantile function. Two Logit-Myerson distributions can be connected using the bivariate logistic distribution. (Gumbel 1961) proposed three different formulations for the bivariate logistic distribution. The Type II distribution from the Morgenstern Family (Sajeevkumar and Irshad 2014; Basikhasteh, Lak, and Tahmasebi 2021) has the following joint distribution and density functions:

$$
\begin{aligned}
F(y_1, y_2|\beta) =& F_1(y_1)F_2(y_2)[1 + \beta(1 - F_1(y_1))(1 - F_2(y_2))] \\
f(y_1, y_2|\beta) =& f_1(y_1)f_2(y_2)[1 + \beta(1 - 2F_1(y_1))(1 - 2F_2(y_2))]
\end{aligned}
$$

where $F_i(y_i)$ and $f_i(y_i)$ for $i \in \{1, 2\}$ refer to the univariate logistic distribution and density funcitons, respectively and $-1 \leq \beta \leq 1$. Since $y_i = Q_i(u_i)$ we can express the bivariate density in the quantile form

$$
\begin{aligned}
f(Q(u_1), Q(u_2)|\beta) =& f_1(Q(u_1))f_2(Q(u_2))[1 + \beta(1 - 2F_1(Q_1(u_1)))(1 - 2F_2(Q_2(u_2)))] \\
[q(u_1, u_2|\beta)]^{-1} =& [q_1(u_1)]^{-1}[q_2(u_2)]^{-1} [1 + \beta(1 - 2u_1)(1 - 2u_2)]
\end{aligned}
$$

For logistic distribution $Q(u) = \ln(u) - \ln(1 - u)$ and $[q(u)]^{-1} = u(1 - u)$. Therefore, the bivariate logistic density quantile function can be expressed as

$$
[q_L(u_1, u_2|\beta)]^{-1} = u_1(1 - u_1)u_2(1 - u_2) [1 + \beta(1 - 2u_1)(1 - 2u_2)]
$$

If we combine the QPD marginals, the result is the joint quantile-based density for the bivariate logistic-based QPD, where the dependency is captured by the bivariate logistic distribution with the coupling parameter β, and the margins are QPDs. The joint density quantile function is given by:

$$\left[q_{MQPD}(u_1, u_2|\theta_1, \theta_2, \beta)\right]^{-1} = u_1(1-u_1)u_2(1-u_2)\left[1+\beta(1-2u_1)(1-2u_2)\right] \times [q_1(u_1|\theta_1)]^{-1}[q_2(u_2|\theta_2)]^{-1}$$

Here, $[q_i(u_i|\theta_i)]^{-1}$, for $i = 1, 2$, represents the marginal QPD density quantile functions, such as the density quantile function (DQF) of the Logit-Myerson distribution (see Appendix A).

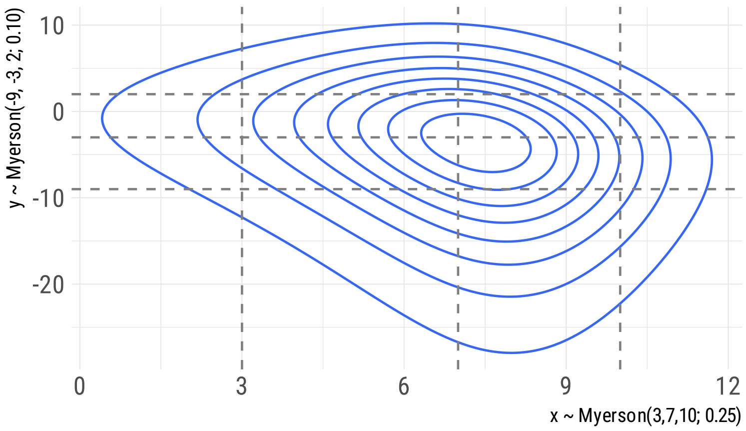

Figure 11. Density of Generalized Myerson distributions joined by Type II bivariate logistic distribution

Figure 11 presents the Bivariate Logit-Myerson Distribution, parameterized by $\Theta = \{\theta_1, \theta_2, \rho\}$, where the marginal Myerson distributions are given by $y_{ij} = Q_j(z(u_{ij}), \theta_j)$ for $j = 1, 2$, with parameter vectors $\theta_1 = \{3, 7, 10; 0.25\}$, $\theta_2 = \{1, 10, 20; 0.1\}$, and the dependence parameter $\beta = 0.6$.

4.2. *Copula-based MQPDs*

The approach we have used so far is similar to constructing the joint distribution using the Gaussian copula (Hoff 2007). Copulas provide a general approach to modeling joint distributions, separating the bivariate dependence from the effects of marginal distributions (Kurowicka and Cooke 2006). The literature describes a wide range of copulas (Genest and Favre 2007; Smith 2013; Kurowicka and Joe 2011), and new copulas can be created using generator functions (Durrleman, Nikeghbali, and Roncalli 2000). When a copula is used to connect QPDs, the joint density is calculated as follows:

$$f_{MQPD}(y_1, y_2|\theta_1, \theta_2, \Xi) = c(F(y_1|\theta_1), F(y_2|\theta_2)|\Xi) f_1(y_1|\theta_1)\, f_2(y_2|\theta_2)$$

where c represents the copula density function with parameter Ξ, and $F(y_i|\theta_i)$ and

$f_i(y_i|\theta_i)$ are the CDF and PDF of the marginal quantile-parameterized distributions, respectively.

The same density can be expressed in quantile-based form (Perepolkin, Goodrich, and Sahlin 2023):

$$[q_{MQPD}(u_1, u_2|\theta, \Xi)]^{-1} = c\,(u_1, u_2|\Xi)\,[q_1(u_1|\theta_1)]^{-1}[q_2(u_2|\theta)]^{-1}$$

where c is the copula density function with parameter Ξ, and $[q_i(u_i|\theta_i)]^{-1}$, for $i = 1, 2$, are the marginal DQFs of QPDs. Figure 12 presents 10,000 samples from the bivariate Myerson distribution joined by the Joe copula with $\theta = 3$.

Elicitation of multivariate distributions may require a specialized approach (Elfadaly and Garthwaite 2017; Kevin J. Wilson et al. 2021). For examples of expert-specified multivariate distributions encoded with copulas, we refer to (Kevin James Wilson 2018; Holzhauer et al. 2022; Sharma and Das 2018; Aas et al. 2009). When fitting copulas to empirical observations, the "blanket" goodness of fit measure (Wang and Wells 2000) based on Kendall's transform (Genest, Quessy, and Rémillard 2006; Genest, Rémillard, and Beaudoin 2009) can be used.

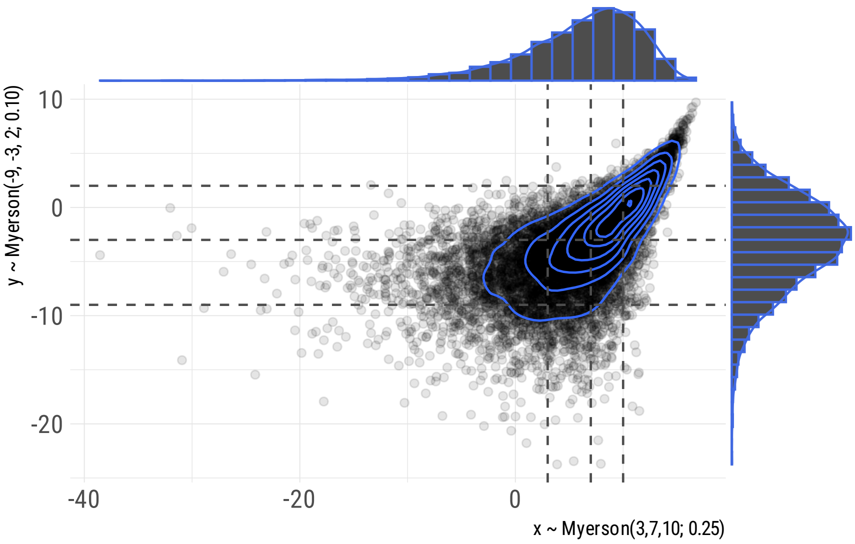

Figure 12. Samples from the bivariate Myerson distribution joined by the Joe copula ($\theta = 3$)

4.3. *Bivariate quantiles*

The formal definition of bivariate quantile functions and the method for constructing bivariate quantile distributions using marginal and conditional quantile functions are provided by (Nair and Vineshkumar 2023; Vineshkumar and Nair 2019). They define the bivarate quantile function (bQF) of (X_1, X_2) as the pair $Q(u_1, u_2) = (Q_1(u_1), Q_{21}(u_2|u_1))$, where $Q_1(u_1) = \inf\{x_1 : F_1(x_1) \geq u_1\}$, $u_1 \in [0, 1]$ and $Q_{21}(u_2|u_1) = \inf\{x_2 : F_{21}(Q_1, x_2) \geq u_2\}$.

The conditional quantile function $Q_{21}(u_2|u_1)$ can be obtained by inverting the conditional distribution function $F_{21}(x_1, x_2)$, which is computed from the factorization of the joint survival function. The joint survival function is defined as $\bar{F}(x_1, x_2) = P(X_1 > x_1)P(X_2 > x_2|X_1 > x_1) = \bar{F}(x_1)\bar{F}_{21}(x_1, x_2)$. Note that the joint survival function $\bar{F}(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2)$, and the conditional survival function $\bar{F}_{21}(x_1, x_2) = 1 - F_{21}(x_1, x_2)$.

Another approach for creating bivariate quantile functions is through Gilchrist's QF transformation rules (Gilchrist 2000), which can be generalized to bivariate quantile functions. According to (Nair and Vineshkumar 2023) (Property 6), the conditional QF can be constructed as a sum of two univariate QFs: $Q_{21}(u_2|u_1) = Q_1(u_1)+Q_2(u_2)$. This means that the pair $(Q_1(u_1), ; Q_1(u_1) + Q_2(u_2))$ is a valid bivariate quantile function, which generalizes Gilchrist's *addition rule* (Table 1). The addition rule also works for quantile density functions (Property 7). If Q_1 is left-bounded at zero, i.e., $Q_1(0) = 0$, then the margins of such a bQF are $X_1 = Q_1(u_1)$ and $X_2 = Q_2(u_2)$. Otherwise, the marginal distribution of X_2 will be $\lim_{u_1 \to 0} Q_{21}(u_2|u_1)$, which in many cases is not tractable.

If $Q_1(u_1)$ and $Q_2(u_2)$ are positive on $u_i \in [0, 1]$, then their product is also a valid conditional QF (Property 8), generalizing Gilchrist's "product rule". Finally, Property 9 generalizes the "Q-transformation rule," stating that for every increasing transformation functions T_1 and T_2, $(T_1(Q_1(u_1)), T_1(Q_1(u_1)) + T_2(Q_2(u_2)))$ is also a valid bQF.

Therefore, valid bivariate quantile-parameterized QFs can be created by constructing the conditional quantile functions as Gilchrist combinations of univariate quantile-parameterized QFs. Figure 13 shows 1000 samples from the bivariate distribution created by adding together two Myerson distributions. Note that in this case, only the marginal distribution of $x_1 = Q_1(u_1)$ is available in closed form.

$$
\begin{aligned}
(u_1, u_2) &\overset{X_1,X_2}{\backsim} (Q_1(u_1), Q_1(u_1) + Q_2(u_2)) \\
Q_1(u_1) &\sim \text{Myerson}(3, 7, 10; 0.1) \\
Q_2(u_2) &\sim \text{Myerson}(-9, -3, 2; 0.25)
\end{aligned}
$$

This bQF is easy to elicit and interpret, since $Q_2(u_2)$ can be thought of as a random adjustment to the value of $Q_1(u_1)$. In fact, the conditional quantile function $Q_{21}(u_2|u_1)$ can be thought of as having the classical form $Q_{21}(u_2|u_1) = \mu(u_1) + \sigma Q_2(u_2)$ (Gilchrist 2000), where the location is randomly varying with $\mu(u_1) = Q_1(u_1)$ and the scale parameter $\sigma = 1$. First, the marginal distribution $Q_1(u_1)$ is elicited, and then the difference between the values x_1 and x_2 can be elicited as a QPT and encoded as $Q_2(u_2)$.

5. Discussion

Quantile-based distributions have garnered significant attention in the research community. Several distributions, such as the Generalized Lambda Distribution (GLD) (Freimer et al. 1988; Ramberg and Schmeiser 1974), the g-and-k distribution (M. A. Haynes, MacGillivray, and Mengersen 1997; M. Haynes and Mengersen 2005; Jacob 2017; Prangle 2017), the g-and-h distribution (Field and Genton 2006; Mac Gillivray

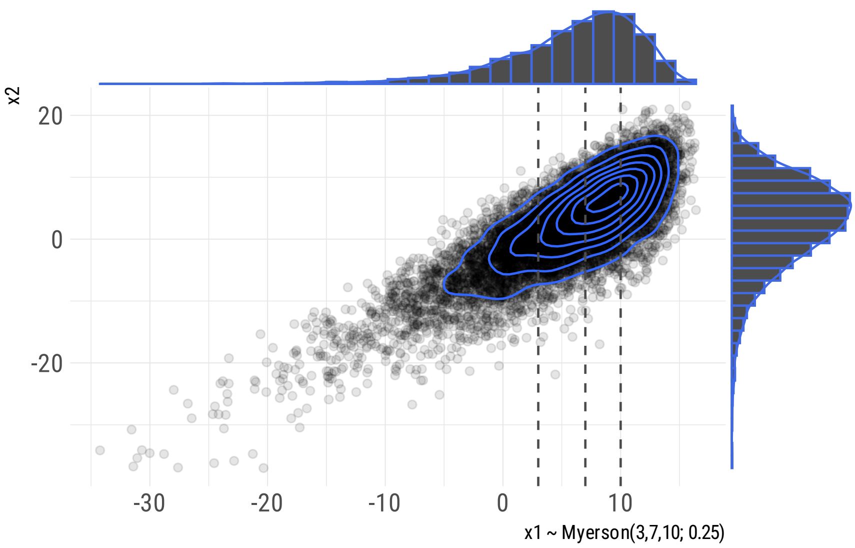

Figure 13. Samples from the Bivariate Myerson quantile function

1992; Rayner and MacGillivray 2002), and the Wakeby distribution (Jeong-Soo 2005; Rahman et al. 2015; Tarsitano 2005a), have been extensively studied and documented in the literature. These distributions are defined by non-invertible quantile functions (Perepolkin, Goodrich, and Sahlin 2023). However, the research on quantile-parameterized distributions remains relatively unexplored. These distributions offer interpretable parameters that are defined on the same scale as the quantities of interest, simplifying the elicitation process for experts. Many popular elicitation protocols for both predictive and parametric elicitation rely on the assessment of quantile-probability pairs (QPPs). Instead of fitting a parametric distribution to the elicited QPPs (Best, Dallow, and Montague 2020; O'Hagan 2019), assessors could directly use the elicited QPPs as inputs into one of the QPD quantile functions, which can be easily employed in both quantile-parameterized and parametric models.

Provided that the expert and the elicitor agree on the scientific model to be used for representing the expert's understanding of the world (Burgman, Layman, and French 2021), several types of inputs may be required to inform the model. Among those are the expert's judgement about the model *parameters* (Mikkola et al. 2021; O'Hagan 2019) and their *predictions* of the next observation (Akbarov 2009; J. Kadane and Wolfson 1998; Winkler 1980). Both parametric and predictive judgments should be captured together with corresponding uncertainties to reflect the expert's state of knowledge. Quantile-parameterized distributions offer distinct advantages as high-fidelity priors that precisely capture expert assessments. These distributions are particularly beneficial for domain experts who may not be well-versed in statistics, as they provide high flexibility while retaining parameter interpretability. As a result, QPDs can faithfully represent an expert's beliefs without compromising convenience or precision.

Different quantile-parameterized distributions fitted to the same set of quantile-probability pairs may exhibit slight variations in shape. However, given the diverse range of QPDs proposed in the literature a knowledgeable assessor should be able to select an appropriate distribution and validate the choice with the expert, taking into account the thickness of the distribution tails.

Most QPDs we reviewed are parameterized by a symmetric percentile triplet (SPT). These distributions rely on the symmetric property of underlying *kernel* distributions and can be generalized by swapping the distribution with another one that exhibits different tail shapes. Hadlock and Bickel (Christopher C. Hadlock and Bickel 2019) utilized this method to generalize Johnson Quantile Parameterized distributions (J-QPDs). We show that the variants of Myerson distribution appearing in the literature (Myerson 2005; Kevin J. Wilson et al. 2023) represent similar generalization. This principle can be extended to include other kernels which result in varying thickness of the tails.

Quantile function perspective

The distributions discussed in this paper are defined using the quantile function and, therefore, they can be considered *quantile-based* quantile-parameterized distributions. Myerson, J-QPD, and several other quantile-parameterized distributions reparameterize conventional distributions, utilizing Gilchrist's Quantile Function (QF) transformations (Gilchrist 2000).

Perepolkin et al. (Perepolkin, Goodrich, and Sahlin 2023) demonstrated that the distributions defined by the quantile function can be used both as prior and as likelihood in Bayesian models. Priors defined by the quantile function eliminate the need to compute prior density. The quantile function acts as a non-linear transformation of a uniform degenerate random variate with the resulting Jacobian adjustment reciprocal to the density quantile function. Therefore, both the Jacobian and the density quantile function are omitted from the Bayesian updating equation (Perepolkin, Goodrich, and Sahlin 2023). When using quantile-based QPDs as likelihood, special care needs to be taken with regards to the suitable prior for the QPP parameters. (Perepolkin, Goodrich, and Sahlin 2021) used the Dirichet-based prior for the metalog likelihood model and descibed the *hybrid* elicitation process for encoding the expert judgments into the two-dimensional prior distribution implied by the model.

Feasibility of parameters

Not all QPDs are equally reliable in approximating the underlying distributions. Violating the QF transformation rules imposes additional constraints on the feasibility of parameters, as certain combinations of parameters may result in locally decreasing quantile functions (Keelin 2016; Christopher Campbell Hadlock 2017). We discussed this limitation in relation to SQN and metalog distributions, but the same challenges affect other distributions with QF violating Gilchrist QF transformation rules. In this regard, the quantile-parameterized model, which relies on Gilchrist combination of basic quantile functions, proposed by (Peng, Li, and Uryasev 2023), represents a highly promising advancement. Weighted constrained optimization algorithm ensuring that the quantile mixture weights remain non-negative opens new possibilities for other QPDs using monotonic transformations of quantile functions. The estimator proposed by (Peng, Li, and Uryasev 2023) is asymptotically a q-Wasserstein distance, which has also been used for parameter estimation in Approximate Bayesian Computation (Bernton et al. 2019).

The feasibility conditions for the Generalized Lambda Distribution (GLD) have been a focal point of numerous research endeavors in the past (Dean 2013; Fournier et al. 2007; Karian and Dudewicz 2019; King and MacGillivray 2007; Tarsitano 2005b, etc). Various reparameterizations have been explored to enhance parameter identifiability

(Ramberg and Schmeiser 1974). Recently, (Chalabi, Scott, and Wuertz 2012) proposed a novel asymmetry-skewness reparameterization for the previously popular Freimer, Kollia, Mudholkar & Lin version of GLD (FKML GLD) (Freimer et al. 1988), wherein two of the four parameters are mapped to robust quantile-based moments, namely the median and Interquartile Range (IQR). This reduction in the number of parameters required for data fitting simplifies the previously computationally intensive fitting algorithms. As demonstrated in the plot of robust moments (Figure 8) GLD remains one of the most flexible unbounded distributions, capable of accommodating a wide range of shapes. (Dedduwakumara, Prendergast, and Staudte 2021) described a two-step method for fitting FKML GLD using the probability density quantile function (Staudte 2017). However, when applying their method to fitting the CSW GLD, the second step becomes unnecessary as the location and scale can be directly mapped to the empirical first and second robust moments.

CSW GLD represents a prime example of clever reparameterization aiming at alleviating the deficiencies of QF construction through setting consistent parameter boundaries and defining fall-back cases for an impossible combination of parameters. This degree of reparameterization is difficult for QPDs because the objective is to retain the mapping of parameters to the valid set of quantile-probability pairs. Therefore, for improperly constructed QPDs the feasibility conditions will have to be expressed as ratios of quantiles.

Multivariate extensions

Quantile-parameterized distributions can be readily extended to the multivariate setting by leveraging traditional multivariate distributions. The combination of quantile-based marginal distributions joined by the multivariate normal has been previously discussed in the literature (Drovandi and Pettitt 2011; Hoff 2007). Building on this approach, we proposed the use of Gumbel's bivariate logistic distribution (Gumbel 1961) to combine quantile-parameterized Logit-Myerson distributions (Kevin J. Wilson et al. 2023).

Copulas offer a natural extension of univariate QPDs into the multivariate domain. Bivariate copulas can be assembled into more complex structures using vine copulas (Czado 2019; Kurowicka and Joe 2011; Kevin James Wilson 2018). Flexible QPDs serve as a viable alternative to empirical copulas, where the margins are represented by kernel density estimation (KDE) or other non-parametric approaches. Poorly fitted marginal distributions mean *less-than-ideal* starting point for copula modeling, because of deviations from uniformality of the copula margins.

Quantile-parameterized distributions defined by the quantile function are particularly well-suited for constructing new distributions using bivariate quantiles (Nair and Vineshkumar 2023; Vineshkumar and Nair 2019). The ability to construct a conditional quantile function as a Gilchrist combination of univariate quantile functions offers a convenient and interpretable approach to defining bivariate distributions, especially when the univariate quantile functions are parameterized by quantiles. These distributions are easy to sample from and construct. However, fitting these distributions to data or posterior samples can be challenging. As shown by (Castillo, Sarabia, and Hadi 1997) the fitting process requires all marginal and conditional quantile functions to be available in closed form, which is often unattainable.

Further research

There appears to be a limited availability of unbounded quantile-parameterized distributions in the current literature. Among the distributions we examined, only the metalog distribution and quantile mixtures can extend across the entire real line. The G-QPD system provides clear distributional bounds explicitly defined by the expert during elicitation. In contrast, the (Generalized) Myerson distribution system relies on implicit bounds that need to be communicated to the expert. Most of the distributions we reviewed are characterized by a symmetrical percentile triplet (SPT), as they rely on the symmetrical property of their kernels. However, there may be situations where an arbitrary (non-symmetrical) quantile parameterization could prove valuable (as shown by Perepolkin, Goodrich, and Sahlin 2021). The development of flexible quantile-parameterized distributions defined by an arbitrary set of quantile-probability pairs using quantile mixtures (Peng, Li, and Uryasev 2023) can enhance versatility of QPDs and facilitate their broader adoption.

In conclusion, quantile-parameterized distributions offer a valuable framework for capturing expert assessments and incorporating them into statistical models. They provide high flexibility and parameter interpretability, making them particularly beneficial for domain experts. The diverse range of quantile-parameterized distributions explored in the literature allows for customized modeling approaches that align with the expert's beliefs and uncertainties. By embracing these innovative distributions, researchers and practitioners can enhance the accuracy and reliability of their statistical models while leveraging expert knowledge effectively.

Miscellaneous

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Appendix A. Distribution functions

Myerson Distribution

The derivative of the quantile function with respect to the depth u is the Quantile Density Function, which for Myerson distribution has the following form

$$q(u|q_1, q_2, q_3, \alpha) = \begin{cases} \rho \frac{\beta^\kappa \ln(\beta)}{(\beta-1)} \frac{q_{norm}(u)}{\Phi^{-1}(1-\alpha)}, & \beta \neq 1 \\ \rho \frac{q_{norm}(u)}{\Phi^{-1}(1-\alpha)}, & \beta = 1 \end{cases}$$

where $q_{norm} = \frac{d\Phi^{-1}(u)}{du}$ is the quantile density function for the standard normal distribution.

The Myerson distribution is invertible. The distribution function of random variable X has the form

$$\psi = \Phi^{-1}(1-\alpha)\left(\frac{\ln\left(1+\frac{(x-q_2)(\beta-1)}{\rho}\right)}{\ln(\beta)}\right)$$

$$F(x|q_1,q_2,q_3,\alpha) = \begin{cases} \Phi(\psi), & \beta \neq 1 \\ F_{normal}(x|q_2, \rho/\Phi^{-1}(1-\alpha)), & \beta = 1 \end{cases}$$

where $\Phi()$ is the CDF of the standard normal distribution and $\Phi^{-1}()$ is its inverse. $F_{normal}(x|q_2, \rho/\Phi^{-1}(1-\alpha))$ is the CDF of the normal distribution with mean $\mu = q_2$ and standard deviation $\sigma = \rho/\Phi^{-1}(1-\alpha)$.

The derivative of the distribution function with respect to the random variable X is the probability density function, which for the Myerson distribution takes the following form

$$f(x|q_1,q_2,q_3,\alpha) = \begin{cases} \frac{\Phi^{-1}(1-\alpha)(\beta-1)}{(\rho+(x-q_2)(\beta-1))\ln(\beta)}\varphi(\psi), & \beta \neq 1 \\ f_{normal}(x|q_2, \rho/\Phi^{-1}(1-\alpha)), & \beta = 1 \end{cases}$$

where $\varphi()$ is the probability density function of the standard normal distribution, $f_{normal}(x|q_2, \frac{\rho}{\Phi^{-1}(1-\alpha)})$ is the PDF of the normal distribution with the mean $\mu = q_2$ and standard deviation $\sigma = \rho/\Phi^{-1}(1-\alpha))$.

Generalized Myerson Distributions

The Quantile Density Function of Generalized Myerson Distribution for $u \neq 0, u \neq 1$ is

$$q_M(u|q_1,q_2,q_3,\alpha) = \begin{cases} \rho\frac{\beta^\kappa \ln(\beta)}{(\beta-1)}\frac{s(u)}{S(1-\alpha)}, & \beta \neq 1 \\ \rho\frac{s(u)}{S(1-\alpha)}, & \beta = 1 \end{cases}$$

where $S(u)$ is the quantile function and $s(u) = \frac{dS(u)}{du}$ is the quantile density function for the kernel distribution. When $u = 0$ or $u = 1$ the $q_M(u) = \infty$.

The Generalized Myerson distribution is invertible. The distribution function of random variable X has the form

$$\psi = S(1-\alpha)\left(\frac{\ln\left(1+\frac{(x-q_2)(\beta-1)}{\rho}\right)}{\ln(\beta)}\right)$$

$$F_M(x|q_1,q_2,q_3,\alpha) = \begin{cases} F(\psi), & \beta \neq 1 \\ q_2 + \frac{\rho}{S(1-\alpha)}F(x), & \beta = 1 \end{cases}$$

where $F()$ is the standard CDF of the kenel distribution and $S()$ is its inverse.

The derivative of the distribution function with respect to the random variable X is the probability density function, which for the Myerson distribution takes the following form

$$f_M(x|q_1,q_2,q_3,\alpha) = \begin{cases} \frac{S(1-\alpha)(\beta-1)}{(\rho+(x-q_2)(\beta-1))\ln(\beta)}f(\psi), & \beta \neq 1 \\ f\left(\frac{x-q_2}{\rho/S(1-\alpha)}\right), & \beta = 1 \end{cases}$$

where $f()$ is the probability density function of the standard kernel distribution. Compare it to the simplicity of the Quantile Density Function above.

Johnson Quantile-Parameterized Distribution

The JQPD-B quantile density function can be computed as

$$q_B(p) = \begin{cases} (u_b - l_b)\varphi(\xi + \lambda\sinh(\delta(z(p)+nc)))\times \\ \quad \lambda\cosh(\sigma(z(p)+nc))\sigma q_{norm}(p), & n \neq 0 \\ (u_b - l_b)\varphi\left(B + \left(\frac{H-L}{2c}\right)z(p)\right) \times \left(\frac{H-L}{2c}\right)q_{norm}(p), & n = 0 \end{cases}$$

The JQPD-B distribution function

$$F_B(x) = \begin{cases} \Phi\left((2c/(H-L))(-B + z\left(\frac{x-l}{u-l}\right))\right), & n = 0 \\ \Phi\left(\frac{1}{\delta}\sinh^{-1}\left(\frac{1}{\lambda}\left(z\left(\frac{x-l}{u-l}\right)-\xi\right)\right)-nc\right), & n \neq 0 \end{cases}$$

The JQPD-B probability density function (PDF) is

$$f(x) = \begin{cases} \frac{2c}{(H-L)(u_b-l_b)}\frac{1}{\varphi\left(z\left(\frac{x-l_b}{u_b-l_b}\right)\right)}\varphi\left(\frac{2c}{H-L}\left(-B+z\left(\frac{x-l_b}{u-l_b}\right)\right)\right), & n = 0 \\ \frac{1}{\delta}\frac{1}{u_b-l_b}\varphi\left(-nc+\frac{1}{\delta}\sinh^{-1}\left(\frac{1}{\lambda}\left(-\xi+z\left(\frac{x-l_b}{u_b-l_b}\right)\right)\right)\right)\times \\ \frac{1}{\varphi\left(z\left(\frac{x-l_b}{u_b-l_b}\right)\right)}\frac{1}{\sqrt{\lambda^2+\left(-\xi+z\left(\frac{x-l_b}{u_b-l_b}\right)\right)^2}}, & n \neq 0 \end{cases}$$

J-QPD-S quantile density function

$$q_S(p) = \begin{cases} \theta \exp\left(\lambda\delta z(p)\right)\lambda\delta q_{norm}(p), & n=0 \\ \theta \exp\left(\lambda \sinh^{-1}(\delta z(p)) + \sinh^{-1}(nc\delta)\right)\lambda \frac{1}{\sqrt{1+(\delta z(p))^2}}\delta q_{norm}(p), & n \neq 0 \end{cases}$$

J-QPD-S distribution function

$$F_S(x) = \begin{cases} F_{lnorm}(x-l_b|\ln(\theta), \frac{H-B}{c}), & n=0 \\ \Phi\left(\frac{1}{\delta}\sinh\left(\sinh^{-1}\left(\frac{1}{\lambda}\ln\frac{x-l_b}{\theta}\right) - \sinh^{-1}(nc\delta)\right)\right), & n \neq 0 \end{cases}$$

J-QPD-S probability density function (PDF)

$$f_S(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln x - ln\xi)^2}{2\frac{(H-B)^2}{c^2}}\right), & n=0 \\ \varphi\left(\frac{\sinh(\sinh^{-1}(cn\sigma)-\sinh^{-1}(\frac{1}{\lambda}\ln\frac{x-l_b}{\theta}))}{\delta}\right)\frac{\cosh(\sinh^{-1}(cn\delta)-\sinh^{-1}(\frac{1}{\lambda}\ln\frac{x-l_b}{\theta}))}{(x-l_b)\delta\lambda\sqrt{1+\left(\frac{\ln\frac{x-l_b}{\theta}}{\lambda}\right)^2}}, & n \neq 0 \end{cases}$$

where $\mu = \ln\xi$ and $\sigma = \frac{H-B}{c}$.

Metalog distribution

This section recapitulates ideas and formulas provided in (Keelin 2016) with our own notation and minor reinterpretations.

Metalog distribution is created from the logistic quantile function $Q(p) = \mu + s\text{logit}(p)$, where μ is the mean, s is proportional to the standard deviation such that $\sigma = s\pi/\sqrt{3}$, p is the probability $p \in [0,1]$. The metalog quantile function is built by substitution and series expansion of its parameters μ and s with the polynomial of the form:

$$\mu = a_1 + a_4(p-0.5) + a_5(p-0.5)^2 + a_7(p-0.5)^3 + a_9(p-0.5)^4 + \dots,$$
$$s = a_2 + a_3(p-0.5) + a_6(p-0.5)^2 + a_8(p-0.5)^3 + a_{10}(p-0.5)^4 + \dots,$$

where $a_i,\ \ i \in (1\dots n)$ are real constants. Given a size-m QPT $\{p,q\}_m$, where $p = \{p_1 \dots p_m\}$ and $q = \{q_1 \dots q_m\}$ the vector of coefficients $a = \{a_1 \dots a_m\}$ can be determined through the set of linear equations.

$$
\begin{aligned}
&q_1 = a_1 + a_2 \text{logit}(p_1) + a_3(p_1 - 0.5)\text{logit}(p_1) + a_4(p_1 - 0.5) + \cdots, \\
&q_2 = a_1 + a_2 \text{logit}(p_2) + a_3(p_2 - 0.5)\text{logit}(p_2) + a_4(p_2 - 0.5) + \cdots, \\
&\vdots \\
&q_m = a_1 + a_2 \text{logit}(p_m) + a_3(p_m - 0.5)\text{logit}(p_m) + a_4(p_m - 0.5) + \cdots.
\end{aligned}
$$

In the matrix form, this system of equations is equivalent to $q = \mathbb{P}a$, where q and a are column vectors and $\mathbb{P}$ is a $m \times n$ matrix:

$$
\mathbb{P} = \begin{bmatrix}
1 & \text{logit}(p_1) & (p_1 - 0.5)\text{logit}(p_1) & (p_1 - 0.5) & \cdots \\
1 & \text{logit}(p_2) & (p_2 - 0.5)\text{logit}(p_2) & (p_2 - 0.5) & \cdots \\
 & & \vdots & & \\
1 & \text{logit}(p_m) & (p_m - 0.5)\text{logit}(p_m) & (p_m - 0.5) & \cdots
\end{bmatrix}
$$

If $m = n$ and $\mathbb{P}$ is invertible, then the vector of coefficients a of this *properly parameterized* metalog QPD can be uniquely determined by

$$
a = \mathbb{P}^{-1} q \tag{4}
$$

If $m > n$ and $\mathbb{P}$ has a rank of at least n, then the vector of coefficients a of the *approximated* metalog QPD, can be estimated using

$$
a = [\mathbb{P}^T \mathbb{P}]^{-1} \mathbb{P}^T q
$$

The matrix to be inverted is always $n \times n$ regardless of the size m of QPT used.

Metalog *quantile function* (QF) with n terms $Q_{M_n}(u|a)$ can be expressed as

$$
Q_{M_n}(u|a) = \begin{cases}
a_1 + a_2 \text{logit}(u), \text{ for } n = 2, \\
a_1 + a_2 \text{logit}(u) + a_3(u - 0.5)\text{logit}(u), \text{ for } n = 3, \\
a_1 + a_2 \text{logit}(u) + a_3(u - 0.5)\text{logit}(u) + a_4(u - 0.5), \text{ for } n = 4, \\
Q_{M_{n-1}} + a_n(u - 0.5)^{(n-1)/2}, \text{ for odd } n \geq 5, \\
Q_{M_{n-1}} + a_n(u - 0.5)^{n/2-1}\text{logit}(u), \text{ for even } n \geq 6,
\end{cases} \tag{5}
$$

where $u \in [0, 1]$ is the cumulative probability and a is the size-n parameter vector of real constants $a = \{a_1 \ldots a_n\}$.

The metalog *quantile density function* (QDF) can be found by differentiating the Equation 5 with respect to u:

$$q_{M_n}(u|a) = \begin{cases} a_2\mathcal{J}(u), \text{ for } n=2, \\ a_2\mathcal{J}(u) + a_3\left((u-0.5)\mathcal{J}(u) + \text{logit}(u)\right), \text{ for } n=3, \\ a_2\mathcal{J}(u) + a_3\left((u-0.5)\mathcal{J}(u) + \text{logit}(u)\right) + a_4, \text{ for } n=4, \\ q_{M_{n-1}} + 0.5a_n(n-1)(u-0.5)^{(n-3)/2}, \text{ for odd } n \geq 5, \\ q_{M_{n-1}} + a_n((u-0.5)^{n/2-1}\mathcal{J}(u) + \\ (0.5n-1)(u-0.5)^{n/2-2}\text{logit}(u)), \text{ for even } n \geq 6, \end{cases} \tag{6}$$

where $\mathcal{J}(u) = [u(1-u)]^{-1}$. The constants a are feasible iif $q_{M_n}(u|a) > 0, \ \forall u \in [0,1]$.

Metalog *density quantile function* (DQF), referred to as the "metalog pdf" in (Keelin 2016) can be obtained by $f(Q_{M_n}(u|a)) = [q_{M_n}(u|a)]^{-1}$.

Metalog *cumulative distribution function* (CDF) $F_{M_n}(x|a)$ does not have an explicit form because $Q_{M_n}(u|a)$ is not invertible (Keelin 2016). It is, however, possible to approximate $\widehat{Q}^{-1}_{M_n}(x|a)$ using approximation.

Metalog distribution is defined for all $x \in \mathbb{R}$ on the real line. (Keelin 2016) provides semi-bounded *log-metalog*, and the bounded *logit-metalog* variations of the metalog distribution. As the names suggest, this is achieved through the variable substitution with $z = \ln(x-b_l)$ or $z = -\ln(b_u-x)$ for the semi-bounded case, and $z = \ln((x-b_l)/(b_u-x))$ for the bounded case, where z is metalog-distributed and b_l, b_u are the lower and upper limits, respectively. Substituting one of the transformations into the QF and QDF functions above, yields semi-bounded or bounded metalog distribution. For the exact formulae of the log-metalog and logit-metalog refer to (Keelin 2016).

CSW GLD

Quantile density function for the CSW GLD is provided in (Chalabi, Scott, and Wuertz 2012)

$$q(u|\tilde{\sigma}, \chi, \xi) = \frac{\tilde{\sigma}}{S(0.75|\chi,\xi) - S(0.25|\chi,\xi)} s(u|\chi,\xi)$$
$$s(u|\chi,\xi) = \frac{d}{du}S(u|\chi,\xi) = u^{\alpha+\beta-1} + (1-u)^{\alpha-\beta-1}$$

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