Introduction to simulations and Monte Carlo methods 20.04.2022

Project nr 2

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DEADLINE: TBA (in Moodle)

1 Brownian motion and geometric Brownian motion

DISCLAIMER: By any means this is no full introduction to Brownian motion. It is a type of a *minimalist* introduction for needs of this project.

1.1 Brownian motion

Roughly speaking, the stochastic process $\mathbf{B} = (B(t))_{t\leq T}$ is a **Brownian motion** if for any $t_0 = 0 \leq t_1 < \ldots \leq t_n \leq T$ we have $B(t_0) = 0$ and $(B(t_1), \ldots, B(t_n))$ is a zero-mean multivariate normal $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ random variable with the covariance matrix

$$\mathbf{\Sigma}(i, j) = \mathrm{Cov}\left(B(t_i), B(t_j)\right) = \min(t_i, t_j), \quad i, j = 1, \ldots, n$$

In this project we consider $T = 1$ and equally-distanced time points $(t_1, t_2, \ldots, t_n) = (1/n, 2/n, \ldots, 1)$.

1.2 Stratified sampling of multivariate normal $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ random variable.

Suppose we want to sample $\mathbf{B} = (B_1, \ldots, B_n)^T \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ random variable using m strata. Let $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ be a multivariate standard normal random variable. Strata will be defined by ascending rings $A^1, \ldots, A^m$ defined by balls $A_i^{'}$ with the center at $(0, \ldots, 0)$ and suitable radius for which $\mathbb{P}(\mathbf{Z} \in A^i) = 1/m$. Thus let

- A_1' be an n-dimensional ball, such that $P(\mathbf{Z} \in A_1') = 1/m$;
- A_2' is a ball such that $P(\mathbf{Z} \in A_2' \setminus A_1') = 1/m$
- etc.

Set $A^1 = A_1'$, $A^2 = A_2' \setminus A_1'$, $\ldots$, $A^m = A_m' \setminus A_{m-1}'$.

Let $\mathbf{A}$ be such that $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$ (Cholesky decomposition).

Define the i-th stratum by $S^i = \{\mathbf{A}\mathbf{z} : \mathbf{z} \in A^i\}$.

Assume that $\mathbf{Z}^i \stackrel{D}{=} (\mathbf{Z}|\mathbf{Z} \in A^i)$. Then $\mathbf{B}^i = \mathbf{A}\mathbf{Z}^i$ is from stratum S^i.

It remains to show how to sample $\mathbf{Z}^i \stackrel{D}{=} (\mathbf{Z}|\mathbf{Z} \in A^i)$. For $n = 2$ and $m = 1$ the method was presented in lecture (which *de facto* is a Box-Muller method). For general $n \geq 2$ let

$\xi_1, \ldots, \xi_n$ be i.i.d. standard normal $\mathcal{N}(0,1)$ random variables. Denote $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T$. Let $D > 0$. Then the vector

$$\left(D\frac{\xi_1}{||\boldsymbol{\xi}||}, \ldots, D\frac{\xi_n}{||\boldsymbol{\xi}||} \right)^T$$

has a uniform distribution on a sphere with radius D. We have a following propoposition

Proposition 1 *Let $\mathbf{Z} = (Z_1, \ldots, Z_n)$ be a standard multivariate normal random variable. Then the square of length of $\mathbf{Z}$ is $D^2 = Z_1^2 + \ldots + Z_n^2$ and has χ_n^2 distribution (χ^2 with n degrees of freedom).*

Recall that the density and c.d.f of χ_n^2 are following

$$f_{\chi_n^2}(r) = \frac{1}{2^{n/2}\Gamma(n/2)} r^{n/2-1} e^{-r/2}, \quad F_{\chi_n^2}(r) = \frac{1}{\Gamma(n/2)} \gamma_{n/2}(r/2),$$

where Γ is a gamma function, and γ is the so-called *incomplete gamma function*.[1] For $n = 2$ random variable D has so-called Rayleigh distributoin. Admittedly, there is no explicit formula for the inverse function of $F_{\chi_n^2}(r)$ for general n, but numerically this inverse is available in several libraries. [2]
Summing up, sampling $\mathbf{B}^i \stackrel{D}{=} (\mathbf{B}|\mathbf{B} \in A^i)$ is following:

1. Perform Cholesky decomposition: $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$.
2. Sample $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T,\ \xi), \sim \mathcal{N}(0,1)$ i.i.d.. Set
$$\mathbf{X} = (X_1, \ldots, X_n)^T = \left(\frac{\xi_1}{||\boldsymbol{\xi}||}, \ldots, \frac{\xi_n}{||\boldsymbol{\xi}||} \right)^T.$$
3. Sample $U \sim \mathcal{U}(0,1)$. Set
$$D^2 = F_{\chi_n^2}^{-1}(r) \left(\frac{i}{m} + \frac{1}{m} U \right).$$
4. Set $\mathbf{Z} = (Z_1, \ldots, Z_n) = (DX_1, \ldots, DX_n)$.
5. Return $\mathbf{B}^i = \mathbf{A}\mathbf{Z}$.

1.2.1 Stratified sampling of a Brownian motion

We simply may use the procedure described in Section 1.2. Recall, $\mathbf{B} = (B(1/n), B(2/n), \ldots, B(1))$ is a multivariate normal random variable $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ with the covariance matrix

$$\mathbf{\Sigma}(i,j) = \frac{1}{n} \min(i,j).$$

[1] https://en.wikipedia.org/wiki/Incomplete_gamma_function
[2] E.g. https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.chi2.html in `Python` or https://www.mathworks.com/help/stats/chi2inv.html in `Matlab`

We can decompose (Cholesky decomposition) $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$, where

$$\mathbf{A}(i,j) = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

In Figure 1 5000 points within 4 strata were simulated using the above method.

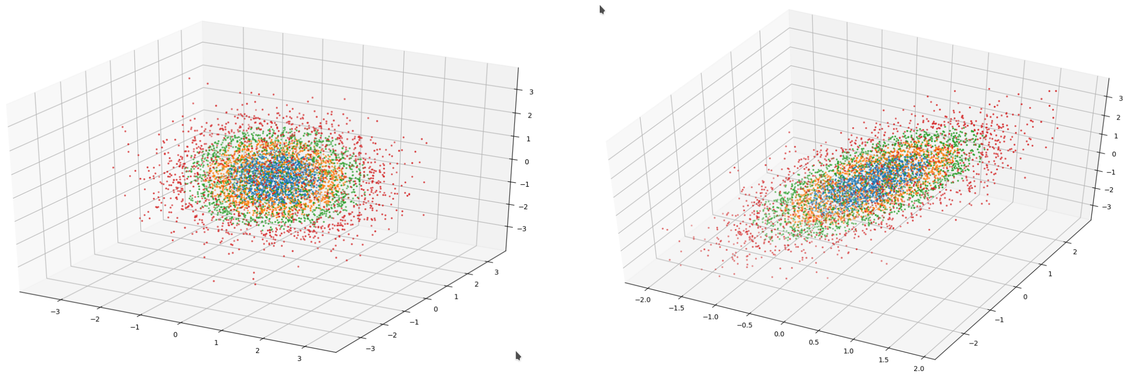

Figure 1: 5000 points from 3-dimensional standard normal distribution obtained using stratified (4 strata) sampling (left). Points from 3-dimensional normal distribution with covariance matrix $\mathbf{\Sigma}(i,j) = \min(i,j)/3$ (right).

1.3 Geometric Brownian motion

Evolutions of the stocks (assets) are often modelled as geometric Brownian motion – $\text{GBM}(\mu, \sigma)$ – which is defined by

$$S(t) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B(t)\right), \qquad 0 \leq t \leq T, \tag{1.1}$$

where $B(t)$ $(0 \leq t \leq T)$ is the Brownian motion. In computing the options prices, often the interest rate r and volatility σ are known, we then make computations for $\text{GBM}(r, \sigma)$. Denote $\mu^* = r - \sigma^2/2$. Then we have

$$S(t) = S(0) \exp\left(\mu^* t + \sigma B(t)\right), \qquad 0 \leq t \leq T, \tag{1.2}$$

2 European and Asian call options

We are interested in estimating the following (called *option*, with discounted payoff at time 1) with price given by formula

$$I = e^{-r} E(A_n - K)_+, \tag{2.3}$$

where

$$A_n = \frac{1}{n} \sum_{i=1}^{n} S(i/n)$$

and $S(t)$ is given in (1.2).

In case $n = 1$ this is called an **European call option**, otherwise it is called an **Asian call option**

2.1 Black-Scholes formula

In case $n = 1$ (i.e., European call option), the exact value of $E(A_1 - K) = E(S(1) - K)_+$ is provided by (Φ is a c.d.f. of $\mathcal{N}(0, 1)$).

$$E(S(1) - K)_+ = S(0)\Phi(d_1) - Ke^{-r}\Phi(d_2), \tag{2.4}$$

where

$$d_1 = \frac{1}{\sigma}\left[\log\left(\frac{S(0)}{K}\right) + r + \sigma^2/2\right]$$

and

$$d_2 = d_1 - \sigma.$$

3 TASK

Fix the parameters: $r = 0.05, \sigma = 0.25$ (thus $\mu^* = r - \sigma^2/2 = -0.0125$), $S(0) = 100$ and $K = 100$.

Estimate the I given in (2.3) using

- a) Crude monte carlo estimator
- b) Stratified estimator. Consider separately $n = 1$ and $n \geq 2$.
- c) For $n = 1$: Antithetic estimator. You may take (Z_{2i-1}, Z_{2i}) with $Z_{2i} = -Z_{2i-1}$, where $Z_{2i-1}, i = 1, \ldots, R/2$ are i.i.d standard normal $N(0, 1)$.
- d) For $n = 1$: Control variate estimator. As a control variate you may take $X = B(1)$.

Compare the results. For case $n = 1$ compare estimations with the exact value using Black-Scholes formula (2.4). For stratified estimators consider proportional and optimal allocation schemes. Provide a report in `.pdf` file and working implementation you used.