Elastic Solid Simulation

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December 03, 2020

1 Overview

A tutorial on finite-element schemes for simulating elastic solids. Largely drawn from [Sifakis and Barbic, 2012]

2 Definitions

2.1 Reference configuration

$\Omega \subset \mathbb{R}^3$ is the volumetric domain occupied by an object.

$\vec{X} \in \Omega$ is a point in this undeformed domain.

Note

Capital letters $\vec{X}$ are used for *undeformed* points. Lower case letters $\vec{x}$ will be used for *deformed* points.

2.2 Deformation function

When an object deforms, each point $\vec{X}$ is displaced to $\vec{x}$.

This transform is the *deformation function*.

$$\vec{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$$
$$\vec{x} = \vec{\phi}(\vec{X})$$

2.3 Deformation gradient tensor

The *deformation gradient* tensor $\mathbf{F} \in \mathbb{R}^{3\times 3}$ is the Jacobian of $\vec{\phi}$.

Examples

Translation

$$\vec{\phi}(\vec{X}) = \vec{X} + \vec{t}, \quad \mathbf{F} = \frac{\partial \vec{\phi}(\vec{X})}{\partial \vec{X}} = \mathbf{I}$$

Uniform Scaling

$$\vec{\phi}(\vec{X}) = \gamma \vec{X}, \quad \mathbf{F} = \gamma \mathbf{I}$$

Rotation

$$\vec{\phi}(\vec{X}) = \mathbf{R}_\theta \vec{X}, \quad F = \mathbf{R}_\theta \mathbf{I}$$

2.4 Strain energy

Elasticity: Deformation causes potential energy, called *strain energy* $E[\vec{\phi}]$.

For *hyperelastic* materials, energy depends only on *final* deformed shape.

Locally, there is an *energy density function* $\Psi[\vec{\phi};\vec{X}]$ such that

$$E[\vec{\phi}] = \int_{\Omega} \Psi[\vec{\phi};\vec{X}] d\vec{X}$$

We will approximate strain energy density $\Psi[\vec{\phi}; \vec{X}_*]$ at a particular point $\vec{X}_*$.

$$\begin{aligned}\vec{\phi}(\vec{X}) &\approx \vec{\phi}(\vec{X}_*) + \left.\frac{\partial \vec{\phi}}{\partial \vec{X}}\right|_{\vec{X}_*} (\vec{X} - \vec{X}_*) \\ &= \mathbf{F}(\vec{X}_*)\vec{X} + \underbrace{\vec{x}_* - \mathbf{F}(\vec{X}_*)\vec{X}_*}_{\text{constant translation}}\end{aligned}$$

Translation alone doesn't effect strain energy, so $\Psi[\vec{\phi}; \vec{X}] \approx \Psi(\mathbf{F})$.

Different Ψ will result in different material properties, we will revisit this.

2.5 Forces and traction

What elastic forces are caused by deformation?

Continuous objects have a *force density function* $\vec{f}(\vec{X})$

$$\vec{f}_{\text{aggregate}}(A) = \int_{A \subset \Omega} \vec{f}(\vec{X}) d\vec{X}$$

Along the surface, this is replaced with *traction* $\vec{\tau}(\vec{X})$. This distinction disappears with discretization.

$$\vec{f}_{\text{aggregate}}(B) = \oint_{B \subset \partial\Omega} \vec{\tau}(\vec{X}) dS$$

2.6 First Piola-Kirchoff stress tensor

A unifying concept for force and traction is the *stress tensor*.

We consider the *first Piola-Kirchoff stress tensor* $\mathbf{P} \in \mathbb{R}^{3\times 3}$ with properties

- Traction at $\vec{X} \in \partial\Omega$ with (reference) surface normal $\vec{N}$ is

$$\vec{\tau}(\vec{X}) = -\mathbf{P} \cdot \vec{N}$$

- Internal force density is

$$\vec{f}(\vec{X}) = \boldsymbol{\nabla} \cdot \mathbf{P}(\vec{X}) \quad \text{i.e.} \quad f_i = \frac{\partial P_{i1}}{\partial X_1} + \frac{\partial P_{i2}}{\partial X_2} + \frac{\partial P_{i3}}{\partial X_3}$$

For hyperelastic matrials,

$$\mathbf{P}(F) = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}}$$

Example

Consider $\Psi(\mathbf{F}) = (k/2)\left\|\mathbf{F} - \mathbf{I}\right\|_{\mathbf{F}}^2$.

$$\delta\Psi(\mathbf{F}) = (k/2)\delta[(\mathbf{F} - \mathbf{I}) : (\mathbf{F} - \mathbf{I})] = k(\mathbf{F} - \mathbf{I}) : \delta\mathbf{F} = \frac{\partial \Psi}{\partial \mathbf{F}} : \delta\mathbf{F}$$

and so $P(\mathbf{F}) = k(\mathbf{F} - \mathbf{I})$ and internal forces are $\vec{f} = \boldsymbol{\nabla} \cdot P = k\Delta\vec{\phi}$

2.7 Lamé parameters μ, λ

Parameters used to parameterize materials (e.g. rubber vs steel).

Related to the Young's modulus k of elasticity and Poisson ratio ν of incompressibility.

$$\mu = \frac{k}{2(1+\nu)}$$

$$\lambda = \frac{k\nu}{(1+\nu)(1-2\nu)}$$

3 Material models

A material can be described by Piola stress $\mathbf{P}(\mathbf{F})$ or $\Psi(\mathbf{F})$

3.1 Strain measures

$\mathbf{F}$ is unintuitive to reason about. Instead we use *strain measures*, which

- Quantify severity of deformation
- Discard irrelevant information (like rotations)

3.2 Green strain tensor

The *Green strain tensor* $\mathbf{E} \in \mathbb{R}^{3\times 3}$ is

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I})$$

and has several desirable properties

1. At rest $\vec{\phi}(\vec{X}) = \vec{X}$, $\mathbf{F} = \mathbf{I}$, and $\mathbf{E} = 0$
2. For rigid motion $\vec{\phi}(\vec{X}) = \mathbf{R}\vec{X} + \vec{t}$, $\mathbf{F} = \mathbf{R}$, and $\mathbf{E} = \mathbf{R}^\top \mathbf{R} - \mathbf{I} = 0$
3. Generally $\mathbf{F} = \mathbf{RS}$ and $\mathbf{E} = \frac{1}{2}(\mathbf{S}^2 - I)$, discarding rotations

3.3 Small strain tensor

The *small strain tensor* ϵ is a linearization of $\mathbf{E}$

$$\epsilon = \frac{1}{2}(\mathbf{F} + \mathbf{F}^\top) - \mathbf{I}$$

3.4 Linear elasticity

The simplest constitutive model, which is valid for small relative motions

$$\Psi(\mathbf{F}) = \mu\epsilon : \epsilon + \frac{\lambda}{2}\mathrm{tr}^2(\epsilon)$$

$$\mathbf{P}(\mathbf{F}) = 2\mu\epsilon + \lambda\mathrm{tr}(\epsilon)\mathbf{I}$$

3.5 St. Venant-Kirchhoff model

Improve the approximation by using $\mathbf{E}$ instead of ϵ

$$\Psi(\mathbf{F}) = \mu \mathbf{E} : \mathbf{E} + \frac{\lambda}{2} \mathrm{tr}^2(\mathbf{E})$$

$$\mathbf{P}(\mathbf{F}) = \mathbf{F}[2\mu \mathbf{E} + \lambda \mathrm{tr}(\mathbf{E})\mathbf{I}]$$

An recall that the Green strain $\mathbf{E}$ is

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I})$$

Rotationally invariant, but poor resistance to forceful compression.

3.6 Other models

Please refer to [Sifakis and Barbic, 2012] for

- Corotated linear elasticity
- Isotropic materials
- The Neohookean model

4 Discretization

Compute $\vec{\phi}(\vec{X})$ on a finite number of nodes $\mathbf{x} = \{\vec{X}_i\}$.

Compute strain energy on small elements e, defined by the points in $\mathbf{x}$.

$$\mathbf{E}[\phi] := \int_\Omega \Psi(\mathbf{F}) d\vec{X}$$

becomes

$$\sum_e \int_{\Omega_e} \Psi\left(\hat{\mathbf{F}}(\vec{X}; \mathbf{x})\right) d\vec{X}$$

4.1 Nodal forces

Compute force $\vec{f}_i$ per node i, summed from elements in its neighborhood $\mathcal{N}_i$

$$\vec{f}_i(\mathbf{x}) = \sum_{e \in \mathcal{N}_i} \vec{f}_i^e(\mathbf{x}), \quad \text{where} \quad \vec{f}_i^e(\mathbf{x}) = -\frac{\partial E^e(\mathbf{x})}{\partial \vec{x}_i}$$

4.2 Linear tetrahedral elements

Build elements from four points, and linearize ϕ per element.

$$\hat{\phi}(\vec{X}) = \mathbf{F}\vec{X} + \vec{b}$$

It is possible to recover **F** from the tetrahedron vertices.

$$\left[\vec{x}_1 - \vec{x}_4 \quad \vec{x}_2 - \vec{x}_4 \quad \vec{x}_3 - \vec{x}_4\right] = \mathbf{F}\left[\vec{X}_1 - \vec{X}_4 \quad \vec{X}_2 - \vec{X}_4 \quad \vec{X}_3 - \vec{X}_4\right]$$
$$\mathbf{D}_s = \mathbf{F}\mathbf{D}_m$$
$$\mathbf{F}(x) = \mathbf{D}_s(\mathbf{x})\mathbf{D}_m^{-1}$$

Where $\mathbf{D}_s$ is called the *deformed shape matrix* and $\mathbf{D}_m$ is called the *reference shape matrix.*

Since **F** is constant over the element, strain energy reduces to

$$E_i = \int_{\mathcal{T}_i} \Psi(\mathbf{F}) d\vec{X} = \Psi(\mathbf{F}_i) \int_{\mathcal{T}_i} d\vec{X} = W \cdot \Psi(\mathbf{F}_i)$$

4.3 Discretized algorithm

$f = 0;$
foreach $\mathcal{T}_e = (i, j, k, l) \in \mathcal{M}$ **do**
$\quad \mathbf{D}_s = \begin{bmatrix} \vec{x}_1 - \vec{x}_4 & \vec{x}_2 - \vec{x}_4 & \vec{x}_3 - \vec{x}_4 \end{bmatrix};$
$\quad \mathbf{F} = \mathbf{D}_s \mathbf{D}_m^{e\ -1};$
$\quad \mathbf{P} = \mathbf{P}(F);$
$\quad \mathbf{H} = -W\mathbf{P}(\mathbf{D}_m^{e\ -1})^\top;$
$\quad \vec{f}_i += \vec{h}_1, \vec{f}_j += \vec{h}_2, \vec{f}_k += \vec{h}_3;$
$\quad \vec{f}_l += -(\vec{h}_1, \vec{h}_2, \vec{h}_3);$
end

Where $\mathbf{H} = \begin{bmatrix} \vec{h_1} & \vec{h_2} & \vec{h_3} \end{bmatrix}$ and $W = \frac{1}{6}\det(\mathbf{D}_m)$

References

[Sifakis and Barbic, 2012] Sifakis, E. and Barbic, J. (2012). Fem simulation of 3d deformable solids: A practitioner's guide to theory, discretization and model reduction. In *ACM SIGGRAPH 2012 Courses*, SIGGRAPH '12, New York, NY, USA. Association for Computing Machinery.