Natural Construction of Homography Transformation

Dejan Milosavljevic

dmilos@gmail.com

TOC

Description

Computing the plane to plane homography without fancy thing such as inversion of big matrix or eugen vectors values.
In here it will be present method in step-by-step manner where every step has some geometric meaning.

Nomenclature

2D Euclid vector

Ordered pair of two real numbers.
$\mathbf{p} = (x, y) = [x,y]^T; x,y \in \mathbb{R}$

2D Homography point

Ordered triplet of three real numbers.
$\mathbf{p} = (x, y, z) = [x,y,z]^T; x, y, z \in \mathbb{R}$

Homography

Definition: project (first) plane to (second) plane.

Matrix form Algebraic form

$$\left[\begin{array}{cc|c} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{22} \\ \hline h_{31} & h_{32} & h_{33} \end{array}\right]$$

$$x' = \frac{x \cdot h_{11} + x \cdot h_{12} + h_{13}}{x \cdot h_{31} + x \cdot h_{32} + h_{33}}$$

$$y' = \frac{x \cdot h_{21} + x \cdot h_{22} + h_{23}}{x \cdot h_{31} + x \cdot h_{32} + h_{33}}$$

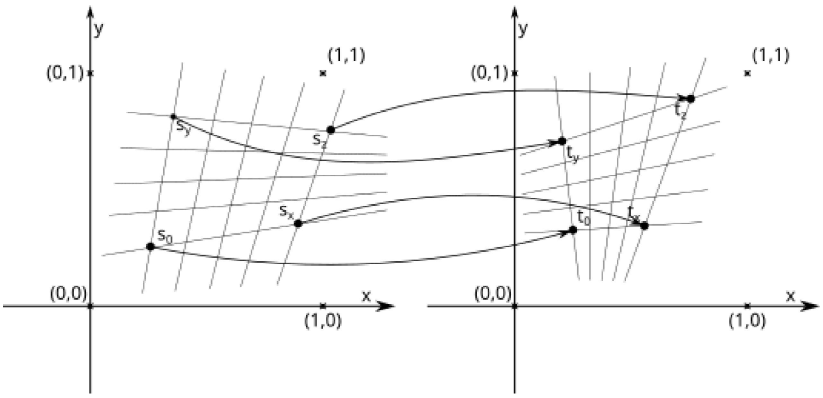
Problem

What we have:

- $s_0, s_x, s_y, s_z \in \mathbb{R}^2$
- $t_0, t_x, t_y, t_z \in \mathbb{R}^2$
- o=(0,0)
- $area(s_i, s_j, s_k) \neq 0$; $i \neq j \neq k$; $i, j, k \in \{ 0, x, y, z \}$, this condition can be relaxed to only $area(s_0, s_x, s_y) \neq 0$;
- $area(t_i, t_j, t_k) \neq 0$; $i \neq j \neq k$; $i, j, k \in \{ 0, x, y, z \}$
- Nice to have:
  $| area(t_i, t_j, t_k) | \leq area(t_0, t_x, t_y)$; $i, j, k \in \{ 0, x, y, z \}$
  $| area(s_i, s_j, s_k) | \leq area(s_0, s_x, s_y)$; $i, j, k \in \{ 0, x, y, z \}$

What we want:

- **H** is homography.
- $t_i = \mathbf{H} (s_i)$; $i \in \{ 0, x, y, z \}$



Existing Solutions

||h||=1

For simplicity in here $x_1 = s_{0_x}$, $y_1 = s_{0_y}$, ..., $x'_1 = t_{0_x}$, $y'_1 = t_{0_y}$, ...

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1 x'_1 & -y_1 x'_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1 y'_1 & -y_1 y'_1 & -y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2 x'_2 & -y_2 x'_2 & -x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2 y'_2 & -y_2 y'_2 & -y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3 x'_3 & -y_3 x'_3 & -x'_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3 y'_3 & -y_3 y'_3 & -y'_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4 x'_4 & -y_4 x'_4 & -x'_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4 y'_4 & -y_4 y'_4 & -y'_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Ah=0 and $||h|| = h_{11}^2 + h_{12}^2 + \dots + h_{33}^2 = 1$

...

$A^T A\, h = \lambda\, h$

Eigenvector h with smallest eigenvalue λ of matrix $A^T A$.

h_{33} = 1

For simplicity in here $x_1 = s_{0_x}$, $y_1 = s_{0_y}$, ..., $x'_1 = t_{0_x}$, $y'_1 = t_{0_y}$, ...

And we have to solve :

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1 x'_1 & -y_1 x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1 y'_1 & -y_1 y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2 x'_2 & -y_2 x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2 y'_2 & -y_2 y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3 x'_3 & -y_3 x'_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3 y'_3 & -y_3 y'_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4 x'_4 & -y_4 x'_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4 y'_4 & -y_4 y'_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ y_4 \\ x_4 \end{bmatrix}$$

Main disadvantaged here is that h_{33} can be near zero and possibly produce numerical instabilities.

Paul Heckbert

Same idea same principle, only difference Paul Heckbert assuem that h_{33} = 1. In here value h_{33} has no constraint.

Elements

... or what we need to make matrix.

Translation

Move for some vector.

$(x+X,y+Y)=\mathbf{P}(x,y)$

Homography matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & X \\ 0 & 1 & Y \\ \hline 0 & 0 & 1 \end{array}\right]$$

Linear

If **L** is linear function then:
$\mathbf{L}(\alpha \cdot \mathbf{x} + \beta \cdot \mathbf{y}) = \alpha \cdot \mathbf{L}(\mathbf{x}) + \beta \cdot \mathbf{L}(\mathbf{y})$

Represented using homography matrix (with one of many possible decompositions):

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} s_x & 0 & 0 \\ 0 & s_y & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} 1 & \tan(\beta) & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$\alpha, s_x, s_y, \beta \in \mathbb{R}$

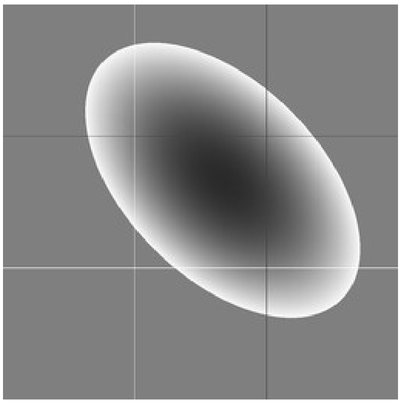
Simple Homography

$(0,0) = \mathbf{P}(0,0)$;
$(1,0) = \mathbf{P}(1,0)$;
$(0,1) = \mathbf{P}(0,1)$;
$(X,Y) = \mathbf{P}(1,1)$;

Homography matrix:

$$\left[\begin{array}{cc|c} X & 0 & 0 \\ 0 & Y & 0 \\ \hline 1-Y & 1-X & X+Y-1 \end{array}\right]$$

$0 \neq (1-Y)^2 + (1-X)^2 + (X+Y-1)^2$
Always $\neq 0$.
Always exists.



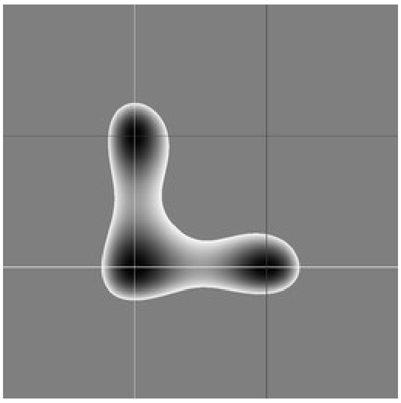
... and vice versa

$(0,0) = \mathbf{P}(0,0)$;
$(1,0) = \mathbf{P}(1,0)$;
$(0,1) = \mathbf{P}(0,1)$;
$(1,1) = \mathbf{P}(X,Y)$;

Homography matrix:

$$\left[\begin{array}{cc|c} Y \cdot (X + Y - 1) & 0 & 0 \\ 0 & X \cdot (X + Y - 1) & 0 \\ \hline Y \cdot (Y - 1) & X \cdot (X - 1) & X \cdot Y \end{array}\right]$$

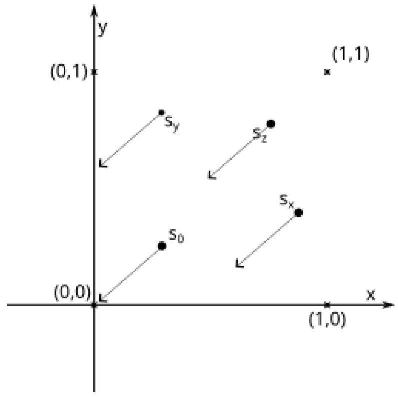
$0 \neq (Y \cdot (Y - 1))^2 + (X \cdot (X - 1))^2 + (X \cdot Y)^2$
$(X,Y) \neq (0,0)$
$(X,Y) \neq (1,0)$
$(X,Y) \neq (0,1)$



Building

Elements

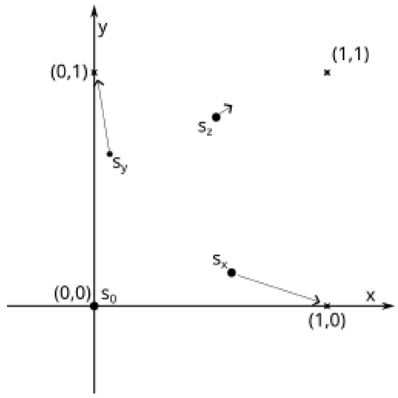
$\mathbf{T_s}$: translation matrix, translate from s_0 to $o=(0,0)$



$$\mathbf{T_s} = \left[\begin{array}{cc|c} 1 & 0 & -s_{0_x} \\ 0 & 1 & -s_{0_y} \\ \hline 0 & 0 & 1 \end{array}\right]$$

-()	+	-	·
2	0	0	0

$\mathbf{L_s}$:



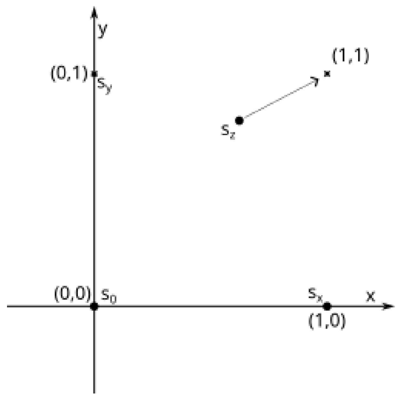
$(0, 0) = \mathbf{L_s}((0, 0))$,
$(1, 0) = \mathbf{L_s}(\mathbf{T_s} (s_x))$,
$(0, 1) = \mathbf{L_s}(\mathbf{T_s} (s_y))$

$a_x = \mathbf{T_s} (s_x)$
$a_y = \mathbf{T_s} (s_y)$

$$\mathbf{L_s} = \left[\begin{array}{cc|c} a_{y_y} & -a_{y_x} & 0 \\ -a_{x_y} & a_{x_x} & 0 \\ \hline 0 & 0 & a_{x_x} \cdot a_{y_y} - a_{x_y} \cdot a_{y_x} \end{array}\right]$$

-()	+	-	·
0+2	0+0	4+1	0+2

$\mathbf{P_s}$:



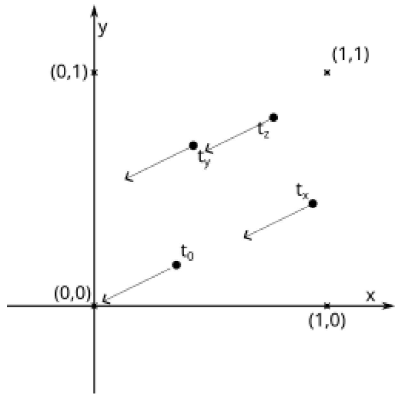
$(0,0) = \mathbf{P_s}((0,0))$,
$(1,0) = \mathbf{P_s}((1,0))$,
$(0,1) = \mathbf{P_s}((0,1))$,
$(1,1) = \mathbf{P_s}(\mathbf{L_s}\,\mathbf{T_s}(s_z))$

$a_z = \mathbf{L_s}\,\mathbf{T_s}(s_z)$
$X = a_{z\,x}$
$Y = a_{z\,y}$

$$\mathbf{P_s} = \left[\begin{array}{cc|c} Y \cdot (X + Y - 1) & 0 & 0 \\ 0 & X \cdot (X + Y - 1) & 0 \\ \hline Y \cdot (Y - 1) & X \cdot (X - 1) & X \cdot Y \end{array}\right]$$

-()	+	-	·
0+0+0	0+2+2	2+0+4	0+4+4

$\mathbf{T_t}$: translation matrix, translate from t_0 to o



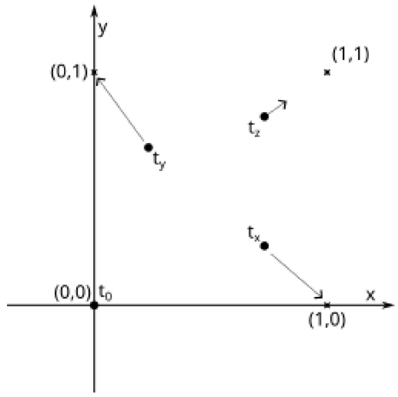
$$\mathbf{T_t} = \left[\begin{array}{cc|c} 1 & 0 & -t_{0_x} \\ 0 & 1 & -t_{0_y} \\ \hline 0 & 0 & 1 \end{array}\right] \qquad \mathbf{T_t}^{-1} = \left[\begin{array}{cc|c} 1 & 0 & +t_{0_x} \\ 0 & 1 & +t_{0_y} \\ \hline 0 & 0 & 1 \end{array}\right]$$

$\mathbf{T_t}^{-1}$:

-()	+	-	·
0	0	0	0

$\mathbf{L_t}$:

$(0, 0) = \mathbf{L_t}((0, 0)),$
$(1, 0) = \mathbf{L_t}(\mathbf{T_t} (t_x)),$
$(0, 1) = \mathbf{L_t}(\mathbf{T_t} (t_y))$



$b_x = \mathbf{T_t} (t_x)$
$b_y = \mathbf{T_t} (t_y)$

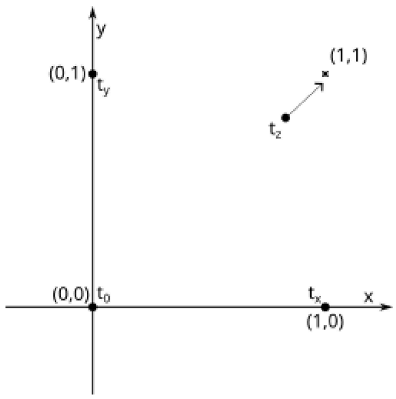
$$\mathbf{L_t} = \left[\begin{array}{cc|c} b_{y_y} & -b_{y_x} & 0 \\ -b_{x_y} & b_{x_x} & 0 \\ \hline 0 & 0 & b_{x_x} \cdot b_{y_y} - b_{x_y} \cdot b_{y_x} \end{array}\right] \quad \mathbf{L_t}^{-1} = \left[\begin{array}{cc|c} b_{x_x} & b_{y_x} & 0 \\ b_{x_y} & b_{y_y} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$\mathbf{L_t}^{-1}$:

-()	+	-	·
0+0	0+0	4+0	0

$\mathbf{P_t}$:

$(0,0) = \mathbf{P_t}((0,0)),$
$(1,0) = \mathbf{P_t}((1,0)),$
$(0,1) = \mathbf{P_t}((0,1)),$
$(1,1) = \mathbf{P_t}(\mathbf{L_t}\, \mathbf{T_t}(t_z))$



$b_z = \mathbf{L_t}\, \mathbf{T_t} (t_z)$
$X = b_{z_x}$
$Y = b_{z_y}$

$$\mathbf{P_t}^{-1} = \left[\begin{array}{cc|c} X & 0 & 0 \\ 0 & Y & 0 \\ \hline 1-Y & 1-X & X+Y-1 \end{array}\right] \quad \mathbf{P_t} = \left[\begin{array}{cc|c} Y \cdot (X + Y - 1) & 0 & 0 \\ 0 & X \cdot (X + Y - 1) & 0 \\ \hline Y \cdot (Y - 1) & X \cdot (X - 1) & X \cdot Y \end{array}\right]$$

$\mathbf{P_t}^{-1}$:

-()	+	-	·
0+0+3	0+2+1	2+0+3	0+4+0

Homographies, Assemble

$$\mathbf{H} = (\mathbf{P_t}\ \mathbf{L_t}\ \mathbf{T_t})^{-1}\ \mathbf{P_s}\ \mathbf{L_s}\ \mathbf{T_s} = \mathbf{T_t^{-1}}\ \mathbf{L_t^{-1}}\ \mathbf{P_t^{-1}}\ \mathbf{P_s}\ \mathbf{L_s}\ \mathbf{T_s}$$

f	-()	+	-	·
$\mathbf{T_s}$	2	0	0	0
$\mathbf{L_s}$	0+2	0+0	4+1	0+2
$\mathbf{P_s}$	0+0+0	0+2+2	2+0+4	0+4+4
$\mathbf{T_t^{-1}}$	0+0+0	0+0+0	0+0+0	0+0+0
$\mathbf{L_t^{-1}}$	0+0+0	0+0+0	0+4+0	0+0+0
$\mathbf{P_t^{-1}}$	0+0+3	0+2+1	2+0+3	0+4+0
H	5 · 0	5 · 4	5 · 0	5 · (4 · 2)
∑	7	27	20	54

Miscellaneous

Source code

github.com/dmilos/math/linar/homography/construct2.hpp

Remarks

- This can be easily extend to higher dimensions
- With appropriate effort it can be make in close form.

Off topic

Homography matrix can be multiplied by some non zero factor for better utilization.
Here are several proposed values to do that:

$\lambda = 1/h_{33}$
: This is one of the most common way to do.

$\lambda = s/\|h_{*1}\|, s \in \mathbb{R}\ s \neq 0;$
: Observe first column as 3D vector. *s* i usually equal to 1. First column can be seen as first basis vector of first plane.

$\lambda = 1/\|h_{*1} \times h_{*2}\|$
: $h_{*1} \times h_{*2}$ is normal of source plane.

$v = h_{*1} \times h_{*2};$
$\lambda = 1/\| v_x \cdot v_x + v_y \cdot v_y \|$
: This cross product also give equation of horizon or vanish line on target plane(z=1).
First and second coordinate will present direction of line in form: $x \cdot \cos(\alpha) + y \cdot \sin(\alpha) - r = 0$

Links

- https://en.wikipedia.org/wiki/Homography
- Projective Mappings for Image Warping, Paul Heckbert, 15-869, Image-Based Modeling and Rendering, 13 Sept 1999.
- Fundamentals of Texture Mapping and Image Warping, Paul S. Heckbert, 17 June 1989.