Natural Construction of Homography Transformation

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TOC

Description

Computing the plane to plane homography without fancy thing such as inversion of big matrix or eugen vectors values.
In here it will be present method in step-by-step manner where every step has some geometric meaning.

Nomenclature

2D Euclid vector

Ordered pair of two real numbers.
$\mathbf{p} = (x, y) = [x,y]^T; x,y \in \mathbb{R}$

2D Homography vector

Ordered triplet of three real numbers.
$\mathbf{p} = (x, y, z) = [x,y,z]^T \; x, y, z \in \mathbb{R}$

2D Euclid matrix

2 by 2 table of real number

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

2D Homography matrix

$$\left[\begin{array}{cc|c} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{22} \\ \hline h_{23} & h_{32} & h_{33} \end{array}\right]$$

Problem

What we have

- $s_0, s_x, s_y, s_z \in \mathbb{R}^2$
- $t_0, t_x, t_y, t_z \in \mathbb{R}^2$
- o=(0,0)
- det(s_i, s_j, s_k) $\neq 0$; $i \neq j \neq k$; i,j, k $\in$ { 0, x, y, z }, this condition can be relaxed to only det(s_0, s_x, s_y) $\neq 0$;
- det(t_i, t_j, t_k) $\neq 0$; $i \neq j \neq k$; i,j, k $\in$ { 0, x, y, z }
- Nice to have:
  | det(t_i, t_j, t_k) | $\leq$ det(t_0, t_x, t_y)
  | det(s_i, s_j, s_k) | $\leq$ det(s_0, s_x, s_y)

What we want

- **H** is homography matrix.
- t_i = **H** (s_i); i $\in$ { 0, x, y, z }

Existing Solution

TODO

TODO

TODO

TODO

$h_{33} = 1$

For simplicity in here $x_1 = s_{0_x}$, $y_1 = s_{0_y}$, ..., $x'_1 = t_{0_x}$, $y'_1 = t_{0_y}$, ...

And we have to solve :

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1 x'_1 & -y_1 x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1 y'_1 & -y_1 y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2 x'_2 & -y_2 x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2 y'_2 & -y_2 y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3 x'_3 & -y_3 x'_3 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \end{bmatrix}$$

$$
\begin{array}{cccccccccc}
0 & 0 & 0 & x_3 & y_3 & 1 & -x_3\,y'_3 & -y_3\,y'_3 & h_{23} & y_3 \\
x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4\,x'_4 & -y_4\,x'_4 & h_{31} & y_4 \\
0 & 0 & 0 & x_4 & y_4 & 1 & -x_4\,y'_4 & -y_4\,y'_4 & h_{32} & x_4
\end{array}
$$

Main disadvantaged here is that h_{33} can be near zero and possibly produce numerical instabilities.

Bricks

... or what we need to assemble to get our matrix.

Translation

Move for some vector.

(x+X,y+Y)=**P**(x,y)

Homography matrix:

$$
\left[\begin{array}{cc|c}
1 & 0 & X \\
0 & 1 & Y \\
\hline
0 & 0 & 1
\end{array}\right]
$$

Linear

If **M** is linear function then:
$\mathbf{M}(\alpha\,\mathbf{x} + \beta\,\mathbf{y}) = \alpha\,\mathbf{M}(\mathbf{x}) + \beta * \mathbf{M}(\mathbf{y})$

Represented using homography matrix:

$$
\left[\begin{array}{cc|c}
a_{11} & a_{12} & 0 \\
a_{12} & a_{22} & 0 \\
\hline
0 & 0 & 1
\end{array}\right]
$$

Affine

Represented using homography matrix:

$$
\left[\begin{array}{cc|c}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
\hline
0 & 0 & 1
\end{array}\right]
$$

Where: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is linear part.

$[\,a_{13}, a_{23}]^T$ is translation part.

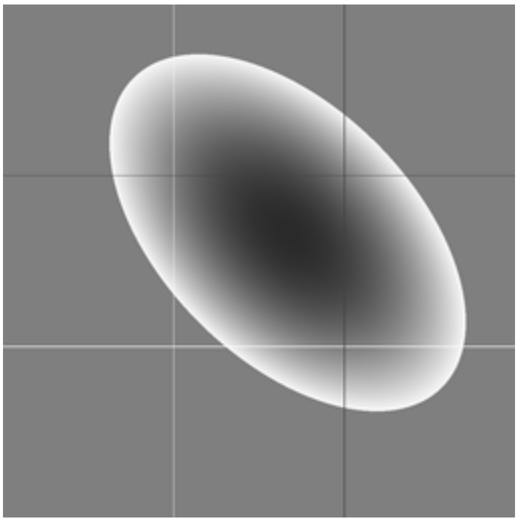
Simple Homography

(0,0) = **P**(0,0);
(1,0) = **P**(1,0);
(0,1) = **P**(0,1);
(X,Y) = **P**(1,1);

This can be represented by homography matrix:

$$\left[\begin{array}{cc|c} X & 0 & 0 \\ 0 & Y & 0 \\ \hline 1-Y & 1-X & X+Y-1 \end{array}\right]$$

$0 \neq (1-Y)^2 + (1-X)^2 + (X+Y-1)^2$
Always $\neq 0$.
Always exists.



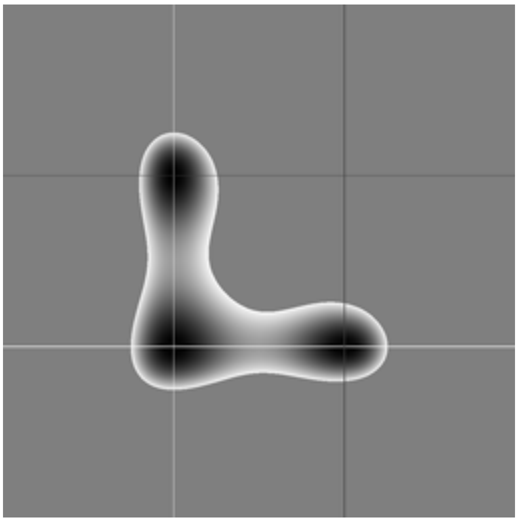
... and vice versa

(0,0) = **P**(0,0);
(1,0) = **P**(1,0);
(0,1) = **P**(0,1);
(1,1) = **P**(X,Y);

Homography matrix:

$$\left[\begin{array}{cc|c} Y * (X + Y - 1) & 0 & 0 \\ 0 & X * (X + Y - 1) & 0 \\ \hline Y*(Y - 1) & X*(X - 1) & X*Y \end{array}\right]$$

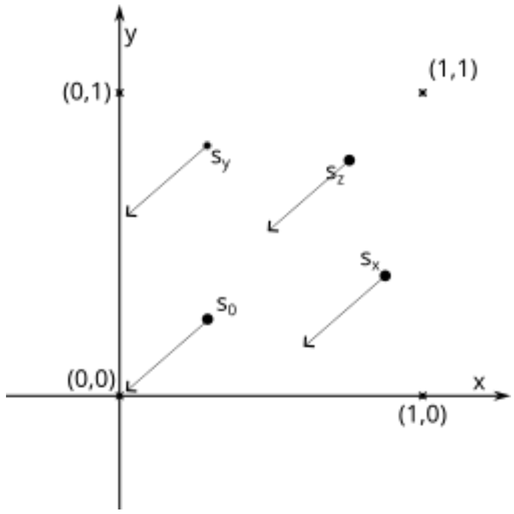
$0 \neq (Y*(Y - 1))^2 + (X*(X - 1))^2 + (X*Y)^2$
$(X,Y) \neq (0,0)$
$(X,Y) \neq (0,1)$
$(X,Y) \neq (0,0)$



Building

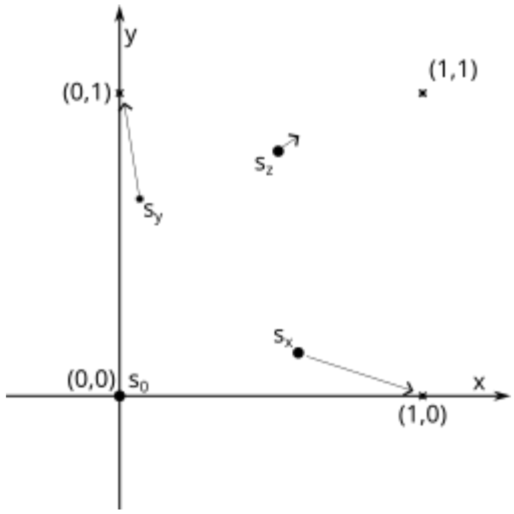
Elements

- **$\mathbf{T_s}$**: translation matrix, translate from s_0 to *o=(0,0)*



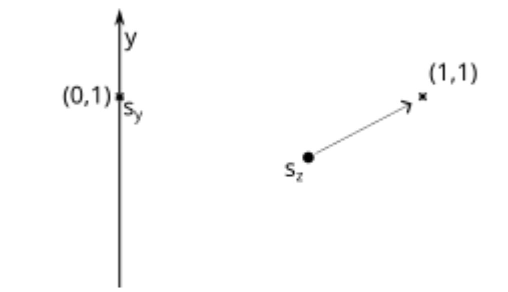
- **$\mathbf{M_s}$**:

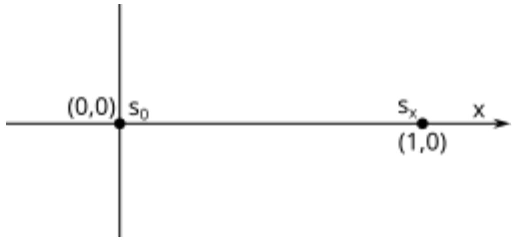
  $(1, 0) = \mathbf{M_s}(\mathbf{T_s} (s_x))$,
  $(0, 1) = \mathbf{M_s}(\mathbf{T_s} (s_y))$



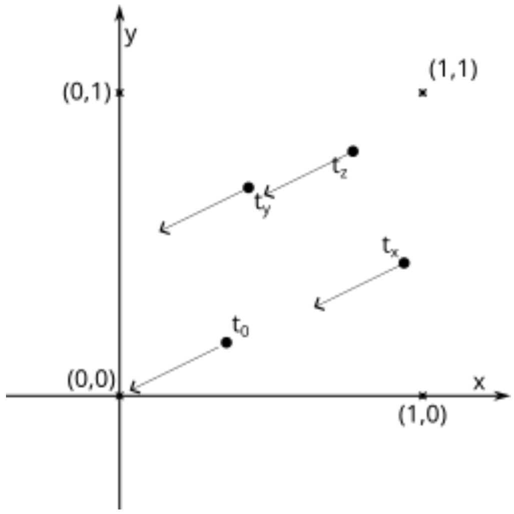
- **$\mathbf{P_s}$**:

  $(0,0) = \mathbf{P_s}((0,0))$,
  $(1,0) = \mathbf{P_s}((1,0))$,
  $(0,1) = \mathbf{P_s}((0,1))$,
  $(1,1) = \mathbf{P_s}(\mathbf{M_s}\ \mathbf{T_s}(s_z))$





T_t: translation matrix, translate from t_0 to *o*



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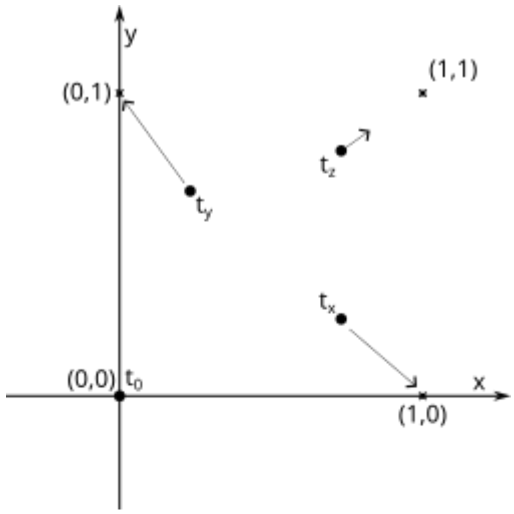
M_t:

(1, 0) = **M_t**(**T_t** (t_x)),
(0, 1) = **M_t**(**T_t** (t_y))



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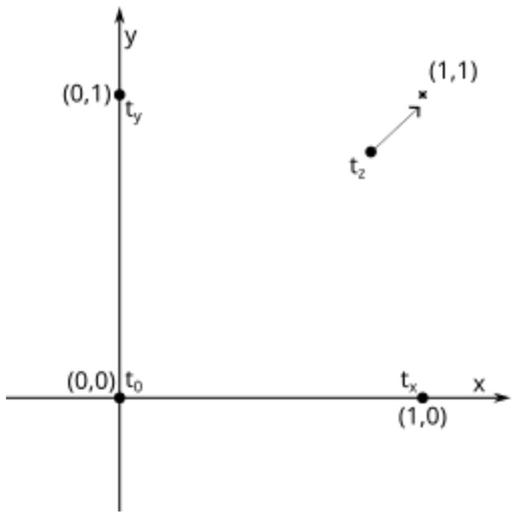
P_t:

(0,0) = **P_t**((0,0)),
(1,0) = **P_t**((1,0)),
(0,1) = **P_t**((0,1)),
(1,1) = **P_t**(**M_t** **T_t**(t_z))
M_t **T_t**(t_z) $\notin$ { (0,0), (1,0), (0,1) }



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Assembling

$$\mathbf{H} = (\mathbf{P_t}\ \mathbf{M_t}\ \mathbf{T_t})^{-1}\ \mathbf{P_s}\ \mathbf{M_s}\ \mathbf{T_s} = \mathbf{T_t}^{-1}\ \mathbf{M_t}^{-1}\ \mathbf{P_t}^{-1}\ \mathbf{P_s}\ \mathbf{M_s}\ \mathbf{T_s}$$

Miscellaneous

Source code

github.com/dmilos/math/linar/homography/construct2.hpp

Comment

- This can be easily extend to higher dimensions
- With appropriate effort it can be make in close forme.