Accelerating Options Pricing via Fourier Transforms

In the previous post, I introduced stock options and an algorithm for pricing them known as the Binomial Options Pricing Model. However, as we saw from a simple Python implementation, the computations for this algorithm are expensive, and it takes $O(N^2)$ operations to compute the options price for N timesteps away from expiration. However, it turns out there is a very clever way to accelerate this algorithm using the dicrete Fourier transform.

Discrete Fourier Transforms

Before using discrete Fourier transforms to accelerate the Binomial Options Pricing Model algorithm, we will first take some time to establish the definitions of the Fourier transform, its inverse, and several other useful mathematical tools. Begin by considering a sequence of values $x_0, x_1, \ldots, x_{n-1}$. In that case, we define the discrete Fourier transform (DFT) as a sequence $X_0, X_1, \ldots, X_{n-1}$ where

$$X_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j e^{i \frac{2\pi}{n} jk}.$$

We often write the Fourier transform of a sequence as $X = \mathcal{F}(x)$.

This equation can be understood in several ways. For instance, if you consider that the exponentials are simply basis vectors for the vector space of functions, the Fourier transform immediately reduces to simply a projection of the sequence $\{x_j\}$ onto the vector space, since each product $x_j e^{i\frac{2\pi}{n}jk}$ is simply the projection onto a single basis vector. The Fourier transform can also be viewed as a matrix multiplication, since it is simply a linear transformation from the vector of x_j observations into the frequency observations X_k.

Viewing the Fourier transform as a linear transformation (via a matrix multiplication by the $\vec{x}$ vector) explains the scaling factor of $\frac{1}{\sqrt{n}}$, because that is the scaling factor necessary to make the Fourier matrix a unitary transformation. Unitary matrices have the incredibly useful property that for a unitary matrix A, the inverse A^{-1} is equal to the adjoint operator (conjugate transpose) A^*. With this in mind, we can define the inverse Fourier transform easily by noting that the Fourier transform is symmetric and thus the conjugate transpose is simply the conjugate of each term, and thus we simply need to change the sign in the exponential. Thus, we obtain the expression

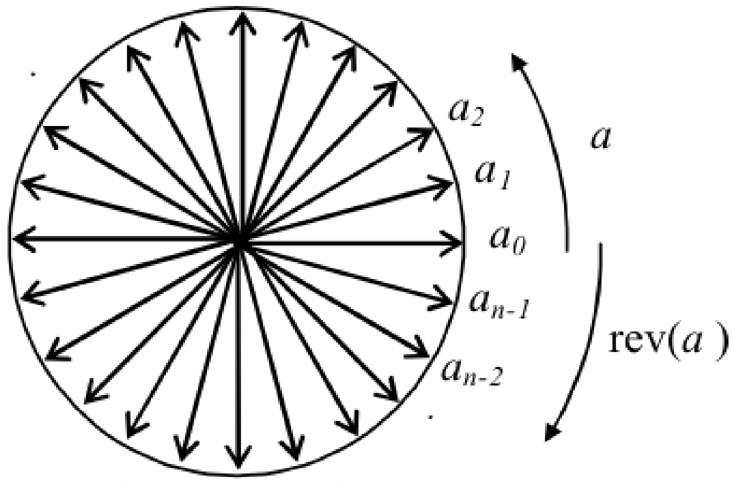
$$x_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} X_j e^{-i \frac{2\pi}{n} jk}.$$

The inverse discrete Fourier transform is often written as $x = \mathcal{F}^{-1}(X)$.

Note that there is a very clear relationship between the Fourier transform and the inverse Fourier transform. Consider the vector defined by the complex number $e^{i\theta} = \cos\theta + i \sin\theta$. As θ increases, this vector traces out the unit circle, rotating around it. Since the exponentials can be viewed as unit vectors rotating along a circle, the Fourier transform and the inverse transform really only differ in that one of them has exponentials rotating in the counter clockwise direction ($e^{-i\theta}$) while the other has them rotating in the clockwise direction ($e^{i\theta}$). Thus, they only differ by which elements of the sequence get multiplied by which exponentials. In the Fourier transform, the elements are chosen "counter-clockwise", while in the inverse Fourier transform, the elements are chosen "clockwise". Mathematically speaking, we can define a reversal operation

$$\text{rev}(a_0, a_1, a_2, \ldots, a_{n-1}) = (a_0, a_{n-1}, a_{n-2}, \ldots, a_1).$$

Note that this is not a simple reversal of the elements, since the first element a_0 stays in the same place. Instead, this can be viewed as a reversal of direction, with the original sequence going "counter clockwise" and the reversal going "clockwise".



With this in mind, it is intuitively clear that since in the forward transform the exponentials are rotating counter clockwise and in the inverse transform the exponentials are rotating clockwise, the Fourier transform of a sequence will be equal to the inverse Fourier transform of the reversal:

$$\mathcal{F}(x) = \mathcal{F}^{-1}(\mathrm{rev}(x))$$
$$\mathcal{F}^{-1}(x) = \mathcal{F}(\mathrm{rev}(x))$$

Discrete Convolutions

Now that we have established the theory behind the Discrete Fourier Transform, let us define yet another mathematical construct known as the discrete convolution. The discrete convolution of two n-dimensional vectors a and b with components a_i and b_i is a vector with components

$$(a * b)_j = \sum_{i=0}^{n-1} a_{j-i} b_i$$

where if $j - i < 0$ use instead the value $j - i \mod n$. This can be viewed as aligning the two sequences with one going in the opposite direction and starting at an offset of j, doing a pointwise multiplication, and then summing up the results and letting that be the jth component of the convolution.

Although this construct may seem esoteric, it turns out it is immensely useful because of the relation that convolution holds to the Fourier transform. Consider the kth component of the Fourier transform of $a * b$:

$$\mathcal{F}(a * b)_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \omega_{-kj} (a * b)_j,$$

where we write the exponential $\omega_x = e^{i\frac{2\pi}{n}x}$ for convenience. Expanding this via the definition of the discrete convolution, we find that

$$\mathcal{F}(a * b)_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \omega_{-kj} \sum_{i=0}^{n-1} a_{j-i} b_i$$

We know that since exponents add during multiplication,

$$\omega_{-kj} = e^{-i\frac{2\pi}{n}kj} = e^{-i\frac{2\pi}{n}k(j-i)}e^{-i\frac{2\pi}{n}ki} = \omega_{-k(j-i)}\omega_{-ki}$$

Using this fact in addition to the fact that the summation can be moved (due to the distributive property), we can rewrite our Fourier transform component as

$$\begin{aligned}
\mathcal{F}(a * b)_k &= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\omega_{-k(j-i)}\omega_{-ki}\right) \sum_{i=0}^{n-1} a_{j-i}b_i \\
&= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \omega_{-k(j-i)}\omega_{-ki}a_{j-i}b_i \\
&= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \left(\omega_{-k(j-i)}a_{j-i}\right)\left(\omega_{-ki}b_i\right)
\end{aligned}$$

If we let $m = j - l$, we can note that m goes from zero to $n - 1$ independently, and thus we can rearrange the sums again to yield our final result that

$$\begin{aligned}
\mathcal{F}(a * b)_k &= \frac{1}{\sqrt{n}} \left(\sum_{i=0}^{n-1} \omega_{-km}a_m\right)\left(\sum_{j=0}^{n-1} \omega_{-ki}b_i\right) \\
&= \sqrt{n}\left(\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \omega_{-km}a_m\right)\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \omega_{-ki}b_i\right)
\end{aligned}$$

In other words, the Fourier transform of a convolution is the scaled product of the Fourier transforms of the vectors being convoluted:

$$\mathcal{F}(a * b)_k = \sqrt{n}\mathcal{F}(a)_k\mathcal{F}(b)_k$$

This property is absolutely essential to our use of the Fourier transform. Note that element-wise multiplication of two vectors in the Fourier domain is *much* faster than computing the convolution; while we only need N multiplications to compute the element-wise product, computing the convolution the regular way would require N multiplications and a sum for *each* component, and since there are N components, there would be N^2 multiplications. In algorithmic terms, the element-wise multiplication is bounded from above by $O(N)$, while the convolution is bounded by $O(N^2)$.

The only problem is that the computation via the Fourier transform requires us to compute the Fourier transform first; we will discuss the efficiency of computing the Fourier transform in the last section. Although the naïve algorithm for computing the discrete Fourier transform turns out to be $O(N^2)$, there are some tricks we can utilize - known together as the Fast Fourier Transform (FFT) - which will permit us to compute the transform in $O(N \lg N)$ time instead.

Accelerating the Binomial Model

In this section, we will demonstrate that the Binomial Options Pricing Model algorithm described in the first section can be accelerated dramatically by utilizing the Discrete Fourier Transform. First, consider the option prices calculated for a time $t = k\Delta t$ in the future. As discussed in the first section, this is a $k+1$-dimensional vector, which we will call $\vec{C}_j$. Recall that in order to compute the first element of the vector $\vec{C}_{j-1}$ for the previous timestep, we must compute the expression

$$C_{j-1,1} = e^{-r\Delta t}\left(pC_{j,1} + (1-p)C_{j,2}\right)$$

where notationally $C_{a,b}$ is the bth component of the vector for time $t = a\Delta t$. More generally, to compute $C_{j-1,k}$, we must evaluate

$$C_{j-1,k} = e^{-r\Delta t}\left(pC_{j,k} + (1-p)C_{j,k+1}\right).$$

Due to the commutative property of multiplication, we can add in some zeros to this sum and rewrite this as

$$\begin{aligned} C_{j-1,k} &= e^{-r\Delta t}\left(C_{j,k}p + C_{j,k+1}(1-p)\right) \\ &= e^{-r\Delta t}\left(C_{j,k}p + C_{j,k+1}(1-p) + C_{j,k+2}\cdot 0 + \cdots\right). \end{aligned}$$

However, if we define a vector $\vec{q} = (p, 1-p, 0, \ldots)$ containing the growth and decay probabilities and padded with as many zeros as necessary to make it an n-dimensional vector, then we see that the above expression simply reduces to the discrete convolution

$$C_{j-1} = e^{-r\Delta t}\left(C_j * \text{rev }(q)\right) = C_j * \left(e^{-r\Delta t}\text{rev }(q)\right).$$

By adding the extra padding components, we have make it so that every vector is of dimension $N+1$, since that is the dimension of the leaf vector C_N. However, only the first j components interest us, and the others are irrelevant. Extrapolating the preceeding expression from the leaf vector C_N to the time $t = k\Delta t$, this yields the expression

$$C_k = C_N * \underbrace{e^{-r\Delta t}\text{rev }(q) * e^{-r\Delta t}\cdots * e^{-r\Delta t}\text{rev }(q)}_{\text{N-k times}}.$$

Taking the inverse Fourier transform of both sides of the equation, we find that

$$\mathcal{F}^{-1}(C_k) = \mathcal{F}^{-1}(C_N)\left(\sqrt{N+1}e^{-r\Delta t}\mathcal{F}^{-1}(\text{rev } q)\right)^{N-k}$$

However, recall that $\mathcal{F}^{-1}(\text{rev } a) = \mathcal{F}(a)$, so this just becomes

$$\mathcal{F}^{-1}(C_k) = \mathcal{F}^{-1}(C_N)\left(\sqrt{N+1}e^{-r\Delta t}\mathcal{F}(q)\right)^{N-k},$$

where the exponentiation is just repeated element-wise multiplication of the vector components. With this equation in mind, we can compute the price of the option at time $t = 0$ by substituting for $k = 0$ and taking the Fourier transform of both sides to obtain

$$C_0 = \mathcal{F}\left(\mathcal{F}^{-1}(C_N)\left(\sqrt{N+1}e^{-r\Delta t}\mathcal{F}(q)\right)^N\right).$$

Since $\mathcal{F}$ and $\mathcal{F}^{-1}$ play symmetric roles, we could also have written this as

$$C_0 = \mathcal{F}^{-1}\left(\mathcal{F}(C_N)\left(\sqrt{N+1}e^{-r\Delta t}\mathcal{F}^{-1}(q)\right)^N\right).$$

Although this result is certainly interesting and impressive, it may not be immediately obvious why this is crucial and how this speeds up the computation of the options price. Suppose that computing the Fourier transform or inverse Fourier transform of an N-length sequence takes at most some time $f(N)$. For this algorithm, we must compute the transform of q and the transform of C_N, and at the end compute the inverse transform of the entire expression above, so we will ultimately spend $3f(N)$ units of time computing Fourier transforms. In addition, we will need to raise the components of $\mathcal{F}(q)$ to the Nth power, which is a constant time operation for each component, and will thus take cN time (since we need to do it for each of the N components). Thus, the total time spent will be $cN + 3f(N)$; therefore, if we can devise some way to compute the Fourier transform faster than $O(N^2)$, we will have an algorithm that is provably faster than the naïve Binomial Options Pricing Model and even the optimized dynamic programming algorithm (which runs in $O(N^2)$). Luckily, algorithms for computing the discrete Fourier transforms have been well studied, and there exists an algorithm known as the Fast Fourier Transform (FFT) which can compute the transform in time upper bounded by $O(N \lg N)$.

Fast Fourier Transforms

Due to the importance of the discrete Fourier transform algorithm, many different fast Fourier transform algorithms have been developed over the years. I would like to demonstrate a single relatively simple algorithm known as the Cooley-Tukey radix-2 algorithm, simply to demonstrate that the discrete Fourier transform *can* be computed efficiently. For the simple version of this algorithm, we must assume that the length of the sequence N is equal to some power of two; however, although this is a significant obstacle, there are several ways to overcome it described extensively in the literature (which we will not describe).

Suppose that you have a sequence x_k. Recall that the discrete Fourier transform is defined as

$$X_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j e^{i\frac{2\pi}{n}jk}.$$

Now, consider the two subsequences x_{2k} and x_{2k+1} corresponding to the even and odd terms in the sequence. We can then define the two Fourier transforms $X_{2k} = \mathcal{F}(x_{2k})$ and $X_{2k+1} = \mathcal{F}(x_{2k+1})$. Suppose that the length of those subsequences is $M = \frac{N}{2}$. Assuming we have the values of these two transforms defined by

$$E_k = X_{2k} = \frac{1}{\sqrt{n}} \sum_{j=0}^{M-1} x_{2j} e^{i\frac{2\pi}{n}jk}$$

$$O_k = X_{2k+1} = \frac{1}{\sqrt{n}} \sum_{j=0}^{M-1} x_{2j+1} e^{i\frac{2\pi}{n}jk}$$

we can manipulate the terms of X_k and show that

$$X_k = \begin{cases} E_k + e^{-i\frac{2\pi}{N}k} O_k & \text{if } k < M \\ E_{k-M} - e^{-i\frac{2\pi}{N}(k-M)} O_{k-M} & \text{if } k \geq M \end{cases}$$

Therefore, we have shown that we can take a sequence of length N and express its Fourier transform as a function of two Fourier transforms of length $\frac{N}{2}$. In addition, it is clear that for small enough sequences (say, $N = 3$), the amount of time necessary to compute the discrete Fourier transform via the original formula is negligible and can be assumed to be constant; this small case will form the base case for recursion. We will now verify that this algorithm is efficient by computing the running time.

This algorithm can be broken down into several steps, as follows:

1. Given the original sequence x_k, examine its length N.

2. If N is less than some specified small threshold, such as three, compute the Fourier transform via the naïve approach (by just computing the sums for each component). Then, return the Fourier transformed sequence and do not continue the algorithm.

3. If N is large, separate x_k into two sequences that contain the even terms (x_{2k}) and the odd terms (x_{2k+1}).

4. Recursively compute the discrete Fourier transform of each of those sequences.

5. Compute the total Fourier transform via the piecewise equation above. Return the Fourier transformed sequence.

Each of these steps can be analyzed in order to determine the amount of time they take. The first step, computing the length, takes a constant amount of time, since the data is stored in an array of known length. The second step will either take a constant amount of time (if we're computing the transform for a small sequence), or will take no time at all if we skip it. The third step will take an amount of time proportional to the size of the sequence, since we must iterate over the sequence and place each element into a new array. The last step will also take a time proportional to the size of the sequence, since we must compute a simple mathematical expression which takes constant time for every element in the sequence. Thus, we have found that the total time necessary for all the steps excluding the recursion is linear in the sequence, and thus each recursion incurs $O(N)$ overhead.

Binomial Options Pricing Model: Fast Fourier Transform Accelerated (download)

```
#!/usr/bin/env python
from math import exp
from numpy import array
from numpy.fft import fft, ifft

# Input stock parameters
dt = input("Enter the timestep: ")
S = input("Enter the initial asset price: ")
r = input("Enter the risk-free discount rate: ")
K = input("Enter the option strike price: ")
p = input("Enter the asset growth probability p: ")
u = input("Enter the asset growth factor u: ")
N = input("Enter the number of timesteps until expiration: ")

# Input whether this is a call or a put option
call = raw_input("Is this a call or put option? (C/P) ").upper().startswith("C")

def price(k, us):
    """ Compute the stock price after 'us' growths and 'k - us' decays. """
    return S * (u ** (2 * us - k))

def leaves(k):
    """ Compute the leaves of the tree for a depth of k timesteps. """
    values = []
    for i in xrange(k + 1):
        if call: values.append(max(0, price(k, i) - K))
        else:    values.append(max(0, K - price(k, i)))
    return values

def bopm(k):
    """
    Compute the option price for an option expiring in 'k' timesteps.
    """
    # Obtain the leaf prices as a NumPy array and create the q vector
    leafValues = array(list(reversed(leaves(k))))
    q = array([p, 1-p] + [0] * (k - 1))

    # Compute the options price via the fast Fourier transform
    C = ifft(fft(leafValues) * ((k + 1) * exp(-r * dt) * ifft(q)) ** k)

    # Return the first component, which is the only important one
    return C[0]

print 'Computed option price: $%.2f' % bopm(N).real
```

In order to compute the total running time, imagine the computation tree generated by this algorithm. The top level of the tree incurs N work. The next level of the tree has two nodes and each one incurs $N/2$ work, so in total they incur N work. The next level has four nodes, each of which incurs $N/4$ work, so once more the total at that level is N overhead. Since each next level reduces the sequence lengths by a factor of two, there must be no more than $\lg N$ levels, and thus the total amount of work is bounded by the number of levels times the overhead at each level, which is $O(N \lg N)$. Thus, we have shown that the discrete Fourier transform can be computed quickly in $O(N \lg N)$ time. Recall that at the end of the previous section we showed that if the fast Fourier transform worked in $O(N \lg N)$ time, then we could compute the Binomial Options Pricing Model options price in $O(N \lg N)$ time, which is much faster than the original $O(N^2)$ approach.

Conclusion

In these two blog posts, we derived the Binomial Options Pricing Model, an alternative to the commonly used Black-Scholes model for computing option prices. The binomial model is superior to Black-Scholes because it allows for the possibility of exercising the option prematurely, thus allowing for computing prices of American and Bermudan style options in addition to European style options. However, the computational complexity of the binomial model is a significant disadvantage, especially since the complexity grows quickly as the time until the option expiration date increases. We define several mathematical tools such as discrete Fourier transforms and discrete convolutions, which we then use to rederive the binomial model using Fourier transforms. Finally, we implemented both the naïve $O(2^N)$ algorithm (which can be accelerated to $O(N^2)$ with dynamic programming) and the Fourier transform based $O(N \lg N)$ algorithm, and verified that they gave the same results for the same inputs. Thus, the use of Fourier transforms significantly increases the applicability of the binomial options pricing model, as it speeds up the algorithm by a factor of $\frac{N}{\lg N}$, allowing accurate pricing of American style options at high speeds.