1 Objectives

The objective is to solve the mutual information calculation for the complex signal model and N tissue types.

2 Work

2.1 Probability Distribution Definitions

Calculate mutual information using numerical integration for a complex signal model $\mathbf{z}(\boldsymbol{\mu}, \mathbf{k})$.

$$\mathbf{z}(\boldsymbol{\mu}, \mathbf{k}) = \int_{\Omega} M(\mathbf{x})\, e^{-s(\boldsymbol{\mu}, \mathbf{x})} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} d\mathbf{x} + \boldsymbol{\nu} = \mathcal{G}(\boldsymbol{\mu}, \mathbf{k}) + \boldsymbol{\nu} \tag{1}$$

$$s(\boldsymbol{\mu}, \mathbf{x}) = \frac{T_E}{T_2^*(\mathbf{x})} + i\left[2\pi\gamma\alpha B_0 T_E \Delta u(\boldsymbol{\mu}, \mathbf{x}) + T_E \Delta\omega_0(\mathbf{x})\right] \tag{2}$$

$$\boldsymbol{\nu} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\nu) \qquad \boldsymbol{\Sigma}_\nu = \begin{bmatrix} \sigma_{\nu,r}^2 & 0 \\ 0 & \sigma_{\nu,i}^2 \end{bmatrix} \tag{3}$$

Therefore, the probability distribution for $p(\mathbf{z}|\boldsymbol{\mu})$ is

$$p(\mathbf{z}|\boldsymbol{\mu}) = \mathcal{N}(\mathcal{G}(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu) \tag{4}$$

The tissue properties can described by the following piecewise functions.

$$\boldsymbol{\mu}(\mathbf{x}) = \sum_{n=1}^{N} \boldsymbol{\mu}_n U(\mathbf{x} - \Omega_n) \tag{5}$$

$$\bigcup_{n=1}^{N} \Omega_n = \Omega \qquad \Omega_n \cap \Omega_m = \varnothing \tag{6}$$

$$U(\mathbf{x} - \Omega_n) = \begin{cases} 1, & x \in \Omega_n \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Assume normal distribution for the model parameter, optical attenuation coefficient $\boldsymbol{\mu}$.

$$p(\boldsymbol{\mu}) = \mathcal{N}(\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) \tag{8}$$

$$\mathbf{m}_\mu = \begin{bmatrix} m_{\mu_1} \\ \vdots \\ m_{\mu_N} \end{bmatrix} \qquad \boldsymbol{\Sigma}_\mu = \begin{bmatrix} \sigma_{\mu_1} & 0 & \cdots & 0 & 0 \\ 0 & \sigma_{\mu_2} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \sigma_{\mu_{N-1}} & 0 \\ 0 & 0 & \cdots & 0 & \sigma_{\mu_N} \end{bmatrix} \tag{9}$$

Also of note, because the real and imaginary components of $\mathbf{z}$ are assumed independent, the covariance matrix $\boldsymbol{\Sigma}_z$ is diagonal, and the following simplification results.

$$p(\mathbf{z}|\boldsymbol{\mu}) = \mathcal{N}_{\mathbf{z}}(\mathcal{G}(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu) = \mathcal{N}_{z_r}(\mathcal{G}_r(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu)\, \mathcal{N}_{z_i}(\mathcal{G}_i(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu) = p(z_r|\boldsymbol{\mu})\, p(z_i|\boldsymbol{\mu}) \tag{10}$$

Similarly, the various tissue types are independent, and the covariance matrix $\boldsymbol{\Sigma}_\mu$ is also diagonal.

$$p(\boldsymbol{\mu}) = \mathcal{N}_{\boldsymbol{\mu}}(\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) = \prod_{i=1}^{N} \mathcal{N}_{\mu_i}(m_{\mu_i}, \sigma_{\mu_i}) = \prod_{i=1}^{N} p(\mu_i) \tag{11}$$

2.2 Problem Statement

The most approachable way to solve mutual information is to begin with the difference of entropies definition.

$$I(\boldsymbol{\mu}; \mathbf{z}) = H(\mathbf{z}) - H(\mathbf{z}|\boldsymbol{\mu}) = \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_z|\right) - \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_\nu|\right) \tag{12}$$

Assuming $\sigma^2_{\nu,r} = \sigma^2_{\nu,i}$, then

$$|\boldsymbol{\Sigma}_\nu| = \sigma^2_\nu \sigma^2_\nu - 0 = \sigma^4_\nu \tag{13}$$

Calculation of $\boldsymbol{\Sigma}_z$ is less straightforward.

$$\boldsymbol{\Sigma}_z = \begin{bmatrix} \mathrm{E}\left[(z_r - \mathrm{E}[z_r])^2\right] & \mathrm{E}\left[(z_r - \mathrm{E}[z_r])(z_i - \mathrm{E}[z_i])\right] \\ \mathrm{E}\left[(z_i - \mathrm{E}[z_i])(z_r - \mathrm{E}[z_r])\right] & \mathrm{E}\left[(z_i - \mathrm{E}[z_i])^2\right] \end{bmatrix} \tag{14}$$

$\mathrm{E}[z_r]$, $\mathrm{E}[z_i]$, $\mathrm{E}\left[(z_r - \mathrm{E}[z_r])^2\right]$, $\mathrm{E}\left[(z_i - \mathrm{E}[z_i])^2\right]$, and $\mathrm{E}\left[(z_r - \mathrm{E}[z_r])(z_i - \mathrm{E}[z_i])\right]$ all need to be calculated numerically.

Alternatively, $\int p(z|\mu)p(\mu)$ can probably be calculated analytically using [Bromiley 2003]. The product of three normal distributions is a scaled normal distribution, so after converting N_{z_i} and N_{z_r} to distributions in terms of μ for the integration, the normal distribution integrates to 1, and the scaling factor comes out as a function of z_r and z_i, $S_{fgh}(z_r, z_i)$, for the remaining two integrals. This leaves potentially very difficult numerical integrations at this step.

2.3 Gauss-Hermite Quadrature

Gaussian quadrature:

$$\int_{-\infty}^{\infty} \exp^{-x^2} f(x)\, dx \approx \sum_{i=1}^{N} \omega_i f(x_i) \tag{15}$$

$$\omega_i = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 \left[H_{n-1}(x_i)\right]^2} \tag{16}$$

where n is the number of sample points used, $H_n(x)$ is the physicists' Hermite polynomial, x_i are the roots of the Hermite polynomial, and ω_i are the associated Gauss-Hermite weights.

Substitution for normal distributions using Gauss-Hermite quadrature:

$$\mathrm{E}[h(y)] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) h(y)\, dy \tag{17}$$

h is some function of y, and random variable Y is normally distributed.

$$x = \frac{y-\mu}{\sqrt{2}\sigma} \Leftrightarrow y = \sqrt{2}\sigma x + \mu \tag{18}$$

$$\mathrm{E}[h(y)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right) h\left(\sqrt{2}\sigma x + \mu\right) dx \tag{19}$$

$$\mathrm{E}[h(y)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \omega_i h\left(\sqrt{2}\sigma x_i + \mu\right) \tag{20}$$

2.4 Mutual Information Calculation

$$I(\boldsymbol{\mu};\mathbf{z}) = H(\mathbf{z}) - H(\mathbf{z}|\boldsymbol{\mu}) = \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_z|\right) - \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_\nu|\right) \tag{21}$$

$$|\boldsymbol{\Sigma}_\nu| = \sigma_\nu^2\sigma_\nu^2 - 0 = \sigma_\nu^4 \tag{22}$$

$$\boldsymbol{\Sigma}_z = \begin{bmatrix} \mathrm{E}\left[(z_r - \mathrm{E}[z_r])^2\right] & \mathrm{E}\left[(z_r - \mathrm{E}[z_r])(z_i - \mathrm{E}[z_i])\right] \\ \mathrm{E}\left[(z_i - \mathrm{E}[z_i])(z_r - \mathrm{E}[z_r])\right] & \mathrm{E}\left[(z_i - \mathrm{E}[z_i])^2\right] \end{bmatrix} \tag{23}$$

$$p(\mathbf{z}) = \int p(\mathbf{z}|\boldsymbol{\mu})\, p(\boldsymbol{\mu})\, d\boldsymbol{\mu} \tag{24}$$

$$p(\mathbf{z}|\boldsymbol{\mu}) = \mathcal{N}_{\mathbf{z}}(\mathcal{G}(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu) = \mathcal{N}_{z_r}(\mathcal{G}_r(\boldsymbol{\mu}), \sigma_\nu)\, \mathcal{N}_{z_i}(\mathcal{G}_i(\boldsymbol{\mu}), \sigma_\nu) = p(z_r|\boldsymbol{\mu})\, p(z_i|\boldsymbol{\mu}) \tag{25}$$

$$p(\boldsymbol{\mu}) = \mathcal{N}_{\boldsymbol{\mu}}(\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) = \prod_{i=1}^{N} \mathcal{N}_{\mu_i}(m_{\mu_i}, \sigma_{\mu_i}) = \prod_{i=1}^{N} p(\mu_i) \tag{26}$$

The signal measurement probability distribution $p(\mathbf{z})$ can be approximated using Gauss-Hermite quadrature. The calculation is performed separately for the real and imaginary components, $p(z_r)$ and $p(z_i)$, because z_r and z_i are assumed independent.

$$\begin{aligned}
p(z_r) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(z_r|\mu_1, \cdots, \mu_N) \prod_{i=1}^{N} p(\mu_i)\, d\mu_1 \cdots d\mu_N && (27)\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_{\mu_i}} \exp\left(-\frac{(\mu_i - m_{\mu_i})^2}{2\sigma_{\mu_i}^2}\right) p(z_r|\mu_1, \cdots, \mu_N)\, d\mu_1 \cdots d\mu_N && (28)\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{\sqrt{\pi}} \exp\left(-x_i^2\right) p\left(z_r \middle| \left(\sqrt{2}\sigma_{\mu_1} x_1 + m_{\mu_1}, \cdots, \sqrt{2}\sigma_{\mu_N} x_N + m_{\mu_N}\right)\right) dx_1 \cdots dx_N && (29)\\
&\approx \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} p\left(z_r \middle| \left(\sqrt{2}\sigma_{\mu_1} x_1 + m_{\mu_1}, \cdots, \sqrt{2}\sigma_{\mu_N} x_N + m_{\mu_N}\right)\right) && (30)\\
&= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} p(z_r|\mu_{1,q}, \cdots, \mu_{N,q}) && (31)
\end{aligned}$$

$$\begin{cases} \mu_{1,q} = \sqrt{2}\sigma_{\mu_1} x_{1,q} + m_{\mu_1} \\ \vdots \\ \mu_{N,q} = \sqrt{2}\sigma_{\mu_N} x_{N,q} + m_{\mu_N} \end{cases} \tag{32}$$

An identical calculation for the imaginary component results in

$$p(z_i) = \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} p(z_i|\mu_{1,q}, \cdots, \mu_{N,q}) \tag{33}$$

$$\begin{cases} \mu_{1,q} = \sqrt{2}\sigma_{\mu_1} x_{1,q} + m_{\mu_1} \\ \vdots \\ \mu_{N,q} = \sqrt{2}\sigma_{\mu_N} x_{N,q} + m_{\mu_N} \end{cases} \tag{34}$$

Calculate expectation values for real and imaginary components by definition and approximating $p(\mathbf{z})$ with Gauss-Hermite quadrature as shown above.

$$\mathrm{E}[z_r] = \int z_r p(z_r)\, dz_r \approx \int z_r \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} p(z_r | \mu_{1,q}, \cdots, \mu_{N,q})\, dz_r \tag{35}$$

$$= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \int z_r p(z_r | \mu_{1,q}, \cdots, \mu_{N,q})\, dz_r \tag{36}$$

$$= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathrm{E}[z_r | \mu_{1,q}, \cdots, \mu_{N,q}] \tag{37}$$

$$= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathcal{G}_r(\mu_{1,q}, \cdots, \mu_{N,q}) \tag{38}$$

Similarly for the imaginary component,

$$\mathrm{E}[z_i] \approx \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathrm{E}[z_i | \mu_{1,q}, \cdots, \mu_{N,q}] = \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathcal{G}_i(\mu_{1,q}, \cdots, \mu_{N,q}) \tag{39}$$

The variance can be calculated as follows:

$$\mathrm{E}\left[(z_r - \mathrm{E}[z_r])^2\right] = \mathrm{E}\left[z_r^2\right] - (\mathrm{E}[z_r])^2 = \int z_r^2 p(z_r)\, dz_r - (\mathrm{E}[z_r])^2 \tag{40}$$

$$\approx \int z_r^2 \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} p(z_r | \mu_{1,q}, \cdots, \mu_{N,q})\, dz_r - (\mathrm{E}[z_r])^2 \tag{41}$$

$$= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \int z_r^2 p(z_r | \mu_{1,q}, \cdots, \mu_{N,q})\, dz_r - (\mathrm{E}[z_r])^2 \tag{42}$$

$$= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \int z_r^2 \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{(z_r - \mathcal{G}_r(\mu_{1,q}, \cdots, \mu_{N,q}))^2}{2\sigma_\nu^2}\right) dz_r - (\mathrm{E}[z_r])^2 \tag{43}$$

$$= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \int \left(x^2\sigma_\nu^2 + 2x\sigma_\nu \mathcal{G}_r(\mu_{1,q}, \cdots, \mu_{N,q}) + \mathcal{G}_r^2(\mu_{1,q}, \cdots, \mu_{N,q})\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx - (\mathrm{E}[z_r])^2 \tag{44}$$

$$= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \left(\sigma_\nu^2 + \mathcal{G}_r^2(\mu_{1,q}, \cdots, \mu_{N,q})\right) - (\mathrm{E}[z_r])^2 \tag{45}$$

$$= \pi^{-N/2}\sigma_\nu^2 + \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathcal{G}_r^2(\mu_{1,q}, \cdots, \mu_{N,q}) - (\mathrm{E}[z_r])^2 \tag{46}$$

$$\begin{cases} \mu_{1,q} = \sqrt{2}\sigma_{\mu_1} x_{1,q} + m_{\mu_1} \\ \vdots \\ \mu_{N,q} = \sqrt{2}\sigma_{\mu_N} x_{N,q} + m_{\mu_N} \end{cases} \tag{47}$$

Again, an identical calculation for the imaginary component yields

$$\mathrm{E}\left[(z_i - \mathrm{E}[z_i])^2\right] = \pi^{-N/2}\sigma_\nu^2 + \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathcal{G}_i^2(\mu_{1,q}, \cdots, \mu_{N,q}) - (\mathrm{E}[z_i])^2 \tag{48}$$

$$\begin{cases} \mu_{1,q} = \sqrt{2}\sigma_{\mu_1} x_{1,q} + m_{\mu_1} \\ \vdots \\ \mu_{N,q} = \sqrt{2}\sigma_{\mu_N} x_{N,q} + m_{\mu_N} \end{cases} \tag{49}$$

The off-diagonal elements of Σ_z are equal.

$$\mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right)\right] = \mathrm{E}\left[\left(z_i - \mathrm{E}\left[z_i\right]\right)\left(z_r - \mathrm{E}\left[z_r\right]\right)\right] \tag{50}$$

$$\begin{aligned}
\mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right)\right] &= \int \left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right) p\left(\mathbf{z}\right) d\mathbf{z} &(51)\\
&= \int \left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right) \int p\left(\mathbf{z}|\boldsymbol{\mu}\right) p\left(\boldsymbol{\mu}\right) d\mathbf{z} &(52)\\
&\approx \int \left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right) \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} p\left(\mathbf{z}|\mu_{1,q}, \cdots, \mu_{N,q}\right) d\mathbf{z} &(53)\\
&= \int \left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right) \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} p\left(z_r|\mu_{1,q}, \cdots, \mu_{N,q}\right) p\left(z_i|\mu_{1,q}, \cdots, \mu_{N,q}\right) d\mathbf{z} &(54)\\
&= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \int \left(z_r - \mathrm{E}\left[z_r\right]\right) p\left(z_r|\mu_{1,q}, \cdots, \mu_{N,q}\right) dz_r \\
&\qquad \cdot \int \left(z_i - \mathrm{E}\left[z_i\right]\right) p\left(z_i|\mu_{1,q}, \cdots, \mu_{N,q}\right) dz_i &(55)\\
&= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \int \left(z_r - \mathrm{E}\left[z_r\right]\right) \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{\left(z_r - \mathcal{G}_r\left(\mu_{1,q}, \cdots, \mu_{N,q}\right)\right)^2}{2\sigma_\nu^2}\right) dz_r \\
&\qquad \cdot \int \left(z_i - \mathrm{E}\left[z_i\right]\right) \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{\left(z_i - \mathcal{G}_i\left(\mu_{1,q}, \cdots, \mu_{N,q}\right)\right)^2}{2\sigma_\nu^2}\right) dz_i &(56)\\
&= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \int \left(\sigma_\nu x + \mathcal{G}_r\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \mathrm{E}\left[z_r\right]\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\
&\qquad \cdot \int \left(\sigma_\nu y + \mathcal{G}_i\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \mathrm{E}\left[z_i\right]\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy &(57)\\
&= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \left(\left(\mathcal{G}_r\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \mathrm{E}\left[z_r\right]\right)\left(\mathcal{G}_i\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \mathrm{E}\left[z_i\right]\right)\right) &(58)\\
&= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \left(\mathcal{G}_r\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) \mathcal{G}_i\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \mathcal{G}_r\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) \mathrm{E}\left[z_i\right]\right. \\
&\qquad \left. - \mathcal{G}_i\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) \mathrm{E}\left[z_r\right] + \mathrm{E}\left[z_r\right] \mathrm{E}\left[z_i\right]\right) &(59)\\
&= \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathcal{G}_r\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) \mathcal{G}_i\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \mathrm{E}\left[z_r\right] \mathrm{E}\left[z_i\right] &(60)
\end{aligned}$$

Thus, the numerical approximation to the covariance matrix $\mathbf{\Sigma}_z$ is then

$$\Sigma_z = \left[\begin{array}{c} \pi^{-N/2}\sigma_\nu^2 + \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathcal{G}_r^2\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \left(\mathrm{E}\left[z_r\right]\right)^2 \\ \pi^{-N/2} \sum_{q_1=1}^{Q_1} \omega_{q_1} \cdots \sum_{q_N=1}^{Q_N} \omega_{q_N} \mathcal{G}_r\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) \mathcal{G}_i\left(\mu_{1,q}, \cdots, \mu_{N,q}\right) - \mathrm{E}\left[z_r\right] \mathrm{E}\left[z_i\right] \end{array} \right.$$

$$\left.\begin{matrix} \pi^{-N/2}\sum_{q_1=1}^{Q_1}\omega_{q_1}\cdots\sum_{q_N=1}^{Q_N}\omega_{q_N}\mathcal{G}_r\left(\mu_{1,q},\cdots,\mu_{N,q}\right)\mathcal{G}_i\left(\mu_{1,q},\cdots,\mu_{N,q}\right)-\mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right] \\ \pi^{-N/2}\sigma_\nu^2+\pi^{-N/2}\sum_{q_1=1}^{Q_1}\omega_{q_1}\cdots\sum_{q_N=1}^{Q_N}\omega_{q_N}\mathcal{G}_i^2\left(\mu_{1,q},\cdots,\mu_{N,q}\right)-\left(\mathrm{E}\left[z_i\right]\right)^2 \end{matrix}\right] \tag{61}$$

The determinant of the covariance matrix is easily calculated.

$$\begin{aligned} |\boldsymbol{\Sigma}_z| = &\left(\pi^{-N/2}\sigma_\nu^2+\pi^{-N/2}\sum_{q_1=1}^{Q_1}\omega_{q_1}\cdots\sum_{q_N=1}^{Q_N}\omega_{q_N}\mathcal{G}_r^2\left(\mu_{1,q},\cdots,\mu_{N,q}\right)-\left(\mathrm{E}\left[z_r\right]\right)^2\right) \\ &\cdot\left(\pi^{-N/2}\sigma_\nu^2+\pi^{-N/2}\sum_{q_1=1}^{Q_1}\omega_{q_1}\cdots\sum_{q_N=1}^{Q_N}\omega_{q_N}\mathcal{G}_i^2\left(\mu_{1,q},\cdots,\mu_{N,q}\right)-\left(\mathrm{E}\left[z_i\right]\right)^2\right) \\ &-\left(\pi^{-N/2}\sum_{q_1=1}^{Q_1}\omega_{q_1}\cdots\sum_{q_N=1}^{Q_N}\omega_{q_N}\mathcal{G}_r\left(\mu_{1,q},\cdots,\mu_{N,q}\right)\mathcal{G}_i\left(\mu_{1,q},\cdots,\mu_{N,q}\right)-\mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right]\right)^2 \end{aligned} \tag{62}$$

The mutual information calculation is straightforward once the value of $|\boldsymbol{\Sigma}_z|$ is known.

$$I\left(\boldsymbol{\mu};\mathbf{z}\right)=H\left(\mathbf{z}\right)-H\left(\mathbf{z}|\boldsymbol{\mu}\right)=\frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot|\boldsymbol{\Sigma}_z|\right)-\frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot|\boldsymbol{\Sigma}_\nu|\right) \tag{63}$$

2.5 The Nonlinear Signal Model Case

$z=\mathcal{G}\left(\boldsymbol{\mu}\right)+\boldsymbol{\nu}$ where $\mathcal{G}\left(\boldsymbol{\mu}\right)$ is nonlinear.

$$H\left(\mathbf{z}\right)=\int_{\mathbf{z}}p\left(\mathbf{z}\right)\ln\left(p\left(\mathbf{z}\right)\right)d\mathbf{z}=\int_{\mathbf{z}}\left(\int_{\boldsymbol{\mu}}p\left(\mathbf{z}|\boldsymbol{\mu}\right)p\left(\boldsymbol{\mu}\right)d\boldsymbol{\mu}\right)\ln\left(p\left(\mathbf{z}\right)\right)d\mathbf{z} \tag{64}$$

$$=\int_{\mathbf{z}}\sum_q\omega_q p\left(\mathbf{z}\middle|\boldsymbol{\mu}_q\right)\ln\left(p\left(\mathbf{z}\right)\right)d\mathbf{z}=\sum_q\omega_q\int_{\mathbf{z}}p\left(\mathbf{z}\middle|\boldsymbol{\mu}_q\right)\ln\left(p\left(\mathbf{z}\right)\right)d\mathbf{z} \tag{65}$$

$$=\sum_q^{N_\mu}\omega_q\sum_k^{N_z}\omega_k\ln\left(\sum_q\omega_q p\left(\mathbf{z}_k\middle|\boldsymbol{\mu}_q\right)\right) \tag{66}$$

$$N_z=\text{number of quadrature points over } z \tag{67}$$

$$N_\mu=\text{number of quadrature points over } \mu \tag{68}$$