1 Objectives

Calculate mutual information using numerical integration for a complex, linearized signal model $\vec{z}\left(\mu,\vec{k}\right)$.

$$\vec{z}\left(\mu,\vec{k}\right)=\int_{\Omega}M\left(\vec{x}\right)e^{-s(\mu,\vec{x})}e^{-2\pi i\vec{k}\cdot\vec{x}}d\vec{x}+\nu=\mathcal{G}\left(\mu,\vec{k}\right)+\nu \qquad \nu\sim\mathcal{N}\left(\vec{0},\Sigma_{\nu}\right) \qquad \Sigma_{\nu}=\begin{bmatrix}\sigma_{\nu,r}^{2} & 0\\ 0 & \sigma_{\nu,i}^{2}\end{bmatrix} \tag{1}$$

$$s\left(\mu,\vec{x}\right)=\frac{T_E}{T_2^*\left(\vec{x}\right)}+i\left[2\pi\gamma\alpha B_0T_E\Delta u\left(\mu,\vec{x}\right)+T_E\Delta\omega_0\left(\vec{x}\right)\right]$$

Therefore, the probability distribution for $p\left(\mathbf{z}|\mu\right)$ is

$$p\left(\mathbf{z}|\mu\right)=\mathcal{N}\left(\mathcal{G}\left(\mu\right),\Sigma_{\nu}\right) \tag{2}$$

Assume normal distribution for the model parameter, optical attenuation coefficient μ.

$$\mu\sim\mathcal{N}\left(\bar{\mu},\sigma_{\mu}\right) \tag{3}$$

2 Work

2.1 Probability Distribution Definitions

Calculate mutual information using numerical integration for a complex, linearized signal model $\vec{z}\left(\mu,\vec{k}\right)$.

$$\vec{z}\left(\mu,\vec{k}\right)=\int_{\Omega}M\left(\vec{x}\right)e^{-s(\mu,\vec{x})}e^{-2\pi i\vec{k}\cdot\vec{x}}d\vec{x}+\nu=\mathcal{G}\left(\mu,\vec{k}\right)+\nu \qquad \nu\sim\mathcal{N}\left(\vec{0},\Sigma_{\nu}\right) \qquad \Sigma_{\nu}=\begin{bmatrix}\sigma_{\nu,r}^{2} & 0\\ 0 & \sigma_{\nu,i}^{2}\end{bmatrix} \tag{4}$$

$$s\left(\mu,\vec{x}\right)=\frac{T_E}{T_2^*\left(\vec{x}\right)}+i\left[2\pi\gamma\alpha B_0T_E\Delta u\left(\mu,\vec{x}\right)+T_E\Delta\omega_0\left(\vec{x}\right)\right]$$

Therefore, the probability distribution for $p\left(\mathbf{z}|\mu\right)$ is

$$p\left(\mathbf{z}|\mu\right)=\mathcal{N}\left(\mathcal{G}\left(\mu\right),\Sigma_{\nu}\right) \tag{5}$$

Assume normal distribution for the model parameter, optical attenuation coefficient μ.

$$\mu\sim\mathcal{N}\left(\bar{\mu},\sigma_{\mu}\right) \tag{6}$$

Also of note, because the real and imaginary components of z are assumed independent, the covariance matrix Σ_z is diagonal, and the following simplification results.

$$p\left(\mathbf{z}|\mu\right)=\mathcal{N}_{\mathbf{z}}\left(\mathcal{G}\left(\mu\right),\Sigma_{\nu}\right)=\mathcal{N}_{z_r}\left(\mathcal{G}_r\left(\mu\right),\sigma_{\nu}\right)\mathcal{N}_{z_i}\left(\mathcal{G}_i\left(\mu\right),\sigma_{\nu}\right)=p\left(z_r|\mu\right)p\left(z_i|\mu\right) \tag{7}$$

2.2 Problem Statement

The most approachable way to solve mutual information is to begin with the difference of entropies definition.

$$I\left(\mu;z\right)=H\left(z\right)-H\left(z|\mu\right)=\frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot\left|\Sigma_z\right|\right)-\frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot\left|\Sigma_{\nu}\right|\right) \tag{8}$$

Assuming $\sigma_{\nu,r}^2=\sigma_{\nu,i}^2$, then

$$\left|\Sigma_{\nu}\right|=\sigma_{\nu}^2\sigma_{\nu}^2-0=\sigma_{\nu}^4 \tag{9}$$

Calculation of Σ_z is less straightforward.

$$\Sigma_z = \begin{bmatrix} \mathrm{E}\left[(z_r - \mathrm{E}[z_r])^2\right] & \mathrm{E}[(z_r - \mathrm{E}[z_r])(z_i - \mathrm{E}[z_i])] \\ \mathrm{E}[(z_i - \mathrm{E}[z_i])(z_r - \mathrm{E}[z_r])] & \mathrm{E}\left[(z_i - \mathrm{E}[z_i])^2\right] \end{bmatrix} \tag{10}$$

$\mathrm{E}[z_r]$, $\mathrm{E}[z_i]$, $\mathrm{E}\left[(z_r - \mathrm{E}[z_r])^2\right]$, $\mathrm{E}\left[(z_i - \mathrm{E}[z_i])^2\right]$, and $\mathrm{E}[(z_r - \mathrm{E}[z_r])(z_i - \mathrm{E}[z_i])]$ all need to be calculated numerically. As one example of this process, take $\mathrm{E}[z_r]$:

$$\mathrm{E}[z_r] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_r p(\mathbf{z})\, dz_r dz_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_r \left[\int_{-\infty}^{\infty} p(\mathbf{z}|\mu)\, p(\mu)\, d\mu\right] dz_r dz_i \tag{11}$$

$$\mathrm{E}[z_r] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_r \left[\int_{-\infty}^{\infty} \mathcal{N}_{z_r}\left(a_r(\mu - \hat{\mu}) + b_r, \sigma_\nu^2\right) \mathcal{N}_{z_i}\left(a_i(\mu - \hat{\mu}) + b_i, \sigma_\nu^2\right) \mathcal{N}_\mu\left(\mu_\mu, \sigma_\mu^2\right) d\mu\right] dz_r dz_i \tag{12}$$

$\int p(z|\mu)p(\mu)$ can probably be calculated analytically using [Bromiley 2003]. The product of three normal distributions is a scaled normal distribution, so after converting N_{z_i} and N_{z_r} to distributions in terms of μ for the integration, the normal distribution integrates to 1, and the scaling factor comes out as a function of z_r and z_i, $S_{fgh}(z_r, z_i)$, for the remaining two integrals. This leaves potentially very difficult numerical integrations at this step.

Gaussian quadrature:

$$\mathrm{E}[z_r] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_r \left[\int_{-\infty}^{\infty} p(\mathbf{z}|\mu)\, p(\mu)\, d\mu\right] dz_r dz_i \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_r \left[\Sigma_{i=1}^{n} w_i p(\mathbf{z}|\mu_i)\, p(\mu_i)\right] dz_r dz_i \tag{13}$$

2.3 Gauss-Hermite Quadrature

Gaussian quadrature:

$$\int_{-\infty}^{\infty} \exp^{-x^2} f(x)\, dx \approx \sum_{i=1}^{N} \omega_i f(x_i) \tag{14}$$

Substitution for normal distributions using Gauss-Hermite quadrature:

$$\mathrm{E}[h(y)] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) h(y)\, dy \tag{15}$$

h is some function of y, and random variable Y is normally distributed.

$$x = \frac{y-\mu}{\sqrt{2}\sigma} \Leftrightarrow y = \sqrt{2}\sigma x + \mu \tag{16}$$

$$\mathrm{E}[h(y)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right) h\left(\sqrt{2}\sigma x + \mu\right) dx \tag{17}$$

$$\mathrm{E}[h(y)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \omega_i h\left(\sqrt{2}\sigma x_i + \mu\right) \tag{18}$$

2.4 Mutual Information Calculation

The signal measurement probability distribution $p(\mathbf{z})$ can be approximated using Gauss-Hermite quadrature. The calculation is performed separately for the real and imaginary components, $p(z_r)$ and $p(z_i)$, because z_r and z_i are assumed independent.

$$p(z_r) = \int p(z_r|\mu)\, p(\mu)\, d\mu = \int \frac{1}{\sqrt{2\pi}\sigma_\mu} \exp\left(-\frac{(\mu - \bar{\mu})^2}{2\sigma_\mu^2}\right) p(z_r|\mu)\, d\mu \tag{19}$$

$$= \int \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right) p\left(z_r | \left(\sqrt{2}\sigma_\mu x + \bar{\mu}\right)\right) dx \approx \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n p\left(z_r | \sqrt{2}\sigma_\mu x_n + \bar{\mu}\right) \tag{20}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n p\left(z_r | \mu_n\right) \tag{21}$$

An identical calculation for the imaginary component results in

$$p\left(z_i\right) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n p\left(z_i | \mu_n\right) \tag{22}$$

Calculate expectation values for real and imaginary components by definition and approximating $p\left(\mathbf{z}\right)$ with Gauss-Hermite quadrature as shown above.

$$\mathrm{E}\left[z_r\right] = \int z_r p\left(z_r\right) dz_r \approx \int z_r \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n p\left(z_r | \mu_n\right) dz_r = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \int z_r p\left(z_r | \mu_n\right) dz_r \tag{23}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \mathrm{E}\left[z_r | \mu_n\right] = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \mathcal{G}_r\left(\mu_n\right) \tag{24}$$

Similarly for the imaginary component,

$$\mathrm{E}\left[z_i\right] \approx \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \mathrm{E}\left[z_i | \mu_n\right] = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \mathcal{G}_i\left(\mu_n\right) \tag{25}$$

The variance can be calculated as follows:

$$\mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r\right]\right)^2\right] = \mathrm{E}\left[z_r^2\right] - \left(\mathrm{E}\left[z_r\right]\right)^2 = \int z_r^2 p\left(z_r\right) dz_r - \left(\mathrm{E}\left[z_r\right]\right)^2 \tag{26}$$

$$\approx \int z_r^2 \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n p\left(z_r | \mu_n\right) dz_r - \left(\mathrm{E}\left[z_r\right]\right)^2 = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \int z_r^2 p\left(z_r | \mu_n\right) dz_r - \left(\mathrm{E}\left[z_r\right]\right)^2 \tag{27}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \int z_r^2 \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{\left(z_r - \mathcal{G}_r\left(\mu_n\right)\right)^2}{2\sigma_\nu^2}\right) dz_r - \left(\mathrm{E}\left[z_r\right]\right)^2 \tag{28}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \int \left(x^2\sigma_\nu^2 + 2x\sigma_\nu \mathcal{G}_r\left(\mu_n\right) + \mathcal{G}_r^2\left(\mu_n\right)\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx - \left(\mathrm{E}\left[z_r\right]\right)^2 \tag{29}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \left(\sigma_\nu^2 + \mathcal{G}_r^2\left(\mu_n\right)\right) - \left(\mathrm{E}\left[z_r\right]\right)^2 = \frac{\sigma_\nu^2}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \mathcal{G}_r^2\left(\mu_n\right) - \left(\mathrm{E}\left[z_r\right]\right)^2 \tag{30}$$

Again, an identical calculation for the imaginary component yields

$$\mathrm{E}\left[\left(z_i - \mathrm{E}\left[z_i\right]\right)^2\right] = \frac{\sigma_\nu^2}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \omega_n \mathcal{G}_i^2\left(\mu_n\right) - \left(\mathrm{E}\left[z_i\right]\right)^2 \tag{31}$$

The off-diagonal elements of Σ_z are equal.

$$\mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right)\right] = \mathrm{E}\left[\left(z_i - \mathrm{E}\left[z_i\right]\right)\left(z_r - \mathrm{E}\left[z_r\right]\right)\right] \tag{32}$$

$$\mathrm{E}\left[\left(z_r-\mathrm{E}\left[z_r\right]\right)\left(z_i-\mathrm{E}\left[z_i\right]\right)\right]=\int\left(z_r-\mathrm{E}\left[z_r\right]\right)\left(z_i-\mathrm{E}\left[z_i\right]\right)p\left(\mathbf{z}\right)d\mathbf{z}=\int\left(z_r-\mathrm{E}\left[z_r\right]\right)\left(z_i-\mathrm{E}\left[z_i\right]\right)\int p\left(\mathbf{z}|\mu\right)p\left(\mu\right)d\mathbf{z} \tag{33}$$

$$\approx\int\left(z_r-\mathrm{E}\left[z_r\right]\right)\left(z_i-\mathrm{E}\left[z_i\right]\right)\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n p\left(\mathbf{z}|\mu_n\right)d\mathbf{z}=\int\left(z_r-\mathrm{E}\left[z_r\right]\right)\left(z_i-\mathrm{E}\left[z_i\right]\right)\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n p\left(z_r|\mu_n\right)p\left(z_i|\mu_n\right)d\mathbf{z} \tag{34}$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\int\left(z_r-\mathrm{E}\left[z_r\right]\right)p\left(z_r|\mu_n\right)dz_r\int\left(z_i-\mathrm{E}\left[z_i\right]\right)p\left(z_i|\mu_n\right)dz_i \tag{35}$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\int\left(z_r-\mathrm{E}\left[z_r\right]\right)\frac{1}{\sqrt{2\pi}\sigma_\nu}\exp\left(-\frac{\left(z_r-\mathcal{G}_r\left(\mu_n\right)\right)^2}{2\sigma_\nu^2}\right)dz_r\int\left(z_i-\mathrm{E}\left[z_i\right]\right)\frac{1}{\sqrt{2\pi}\sigma_\nu}\exp\left(-\frac{\left(z_i-\mathcal{G}_i\left(\mu_n\right)\right)^2}{2\sigma_\nu^2}\right)dz_i \tag{36}$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\int\left(\sigma_\nu x+\mathcal{G}_r\left(\mu_n\right)-\mathrm{E}\left[z_r\right]\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx\int\left(\sigma_\nu y+\mathcal{G}_i\left(\mu_n\right)-\mathrm{E}\left[z_i\right]\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \tag{37}$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\left(\left(\mathcal{G}_r\left(\mu_n\right)-\mathrm{E}\left[z_r\right]\right)\left(\mathcal{G}_i\left(\mu_n\right)-\mathrm{E}\left[z_i\right]\right)\right)=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\left(\mathcal{G}_r\left(\mu_n\right)\mathcal{G}_i\left(\mu_n\right)-\mathcal{G}_r\left(\mu_n\right)\mathrm{E}\left[z_i\right]-\mathcal{G}_i\left(\mu_n\right)\mathrm{E}\left[z_r\right]+\mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right]\right) \tag{38}$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_r\left(\mu_n\right)\mathcal{G}_i\left(\mu_n\right)-\mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right] \tag{39}$$

$$\Sigma_z=\begin{bmatrix}\frac{\sigma_\nu^2}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_r^2\left(\mu_n\right)-\left(\mathrm{E}\left[z_r\right]\right)^2 & \frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_r\left(\mu_n\right)\mathcal{G}_i\left(\mu_n\right)-\mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right]\\ \frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_r\left(\mu_n\right)\mathcal{G}_i\left(\mu_n\right)-\mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right] & \frac{\sigma_\nu^2}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_i^2\left(\mu_n\right)-\left(\mathrm{E}\left[z_i\right]\right)^2\end{bmatrix} \tag{40}$$

$$\begin{aligned}\left|\Sigma_z\right|=&\left(\frac{\sigma_\nu^2}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_r^2\left(\mu_n\right)-\left(\mathrm{E}\left[z_r\right]\right)^2\right)\left(\frac{\sigma_\nu^2}{\sqrt{\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_i^2\left(\mu_n\right)-\left(\mathrm{E}\left[z_i\right]\right)^2\right)\\&-\left(\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\omega_n\mathcal{G}_r\left(\mu_n\right)\mathcal{G}_i\left(\mu_n\right)-\mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right]\right)^2\end{aligned} \tag{41}$$

$$I\left(\mu;z\right)=H\left(z\right)-H\left(z|\mu\right)=\frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot\left|\Sigma_z\right|\right)-\frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot\left|\Sigma_\nu\right|\right) \tag{42}$$

3 Derivations

4 Directions