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The successful implementation of QEC protocol for silicon spin three-qubit system was demonstrated by Kenta Takeda et al. Scientists' theory was that phase-flip errors can be fixed by Toffoli gate. Phase-flip errors occur when the phase of the qubit is altered. Since the phase of the qubit is determined by angle φ ($|\psi\rangle = cos\frac{\theta}{2}|0\rangle + sin\frac{\theta}{2}e^{-i\varphi}|1\rangle$), this type of an error is equivalent to applying RZ gate with a certain phase shift. Taking advantage of this fact, Kenta Takeda et al. simulated the phase-flip error by using RZ gate with a known angle from $-\pi$ to π (5, p. 684, fig. 3 a, b).

Building quantum circuits in Qiskit, we can understand better what the scientists did in their experiment. Figure 7 shows the circuit (with phase flip of $\pi/3$) that Kenta Takeda et al. used to simulate and correct the phase-flip error. First, let's look at the probability distribution after measuring the data qubit's (q2) state before applying Toffoli gate[1]. From figure 8a, the probability of measuring q2 in $|0\rangle$ is ~78%. Mathematically analyzing the operations on q2, we obtain:

2nd Hadamard — RZ($\pi/3$) error — 1st Hadamard — Initial $|0\rangle$ state of q2

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^{-i\frac{\pi}{6}} & 0\\ 0 & e^{i\frac{\pi}{6}}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix}e^{-i\frac{\pi}{6}} + e^{i\frac{\pi}{6}}\\ e^{-i\frac{\pi}{6}} - e^{i\frac{\pi}{6}}\end{pmatrix} = \frac{1}{2}(e^{-i\frac{\pi}{6}} + e^{i\frac{\pi}{6}})|0\rangle + \frac{1}{2}(e^{-i\frac{\pi}{6}} - e^{i\frac{\pi}{6}})|1\rangle$$

After calculating the magnitudes of complex coefficients, we see that the probability of measuring $|0\rangle$ is 0.75 which corresponds to the distribution in figure 8a. Without RZ gate two Hadamard gates applied in the row would combine into an Identity gate that does not alter

[1] Although all three qubits are entangled, we can study the operations on q2 individually because it plays the role of a control qubit in encoding and decoding processes

the qubit's state. Therefore, the error due to the phase flip of θ is $sin^2(\frac{\theta}{2})$ as discussed on page 684 of the article.

If we measure data qubit's state after applying the Toffoli gate, the probability of finding q2 in $|0\rangle$ state is 1, and the introduced RZ(θ) error is corrected. Thus, our simulation corresponds with Kenta Takeda et al. results, and the phase-flip error can be fixed with Toffoli gate.