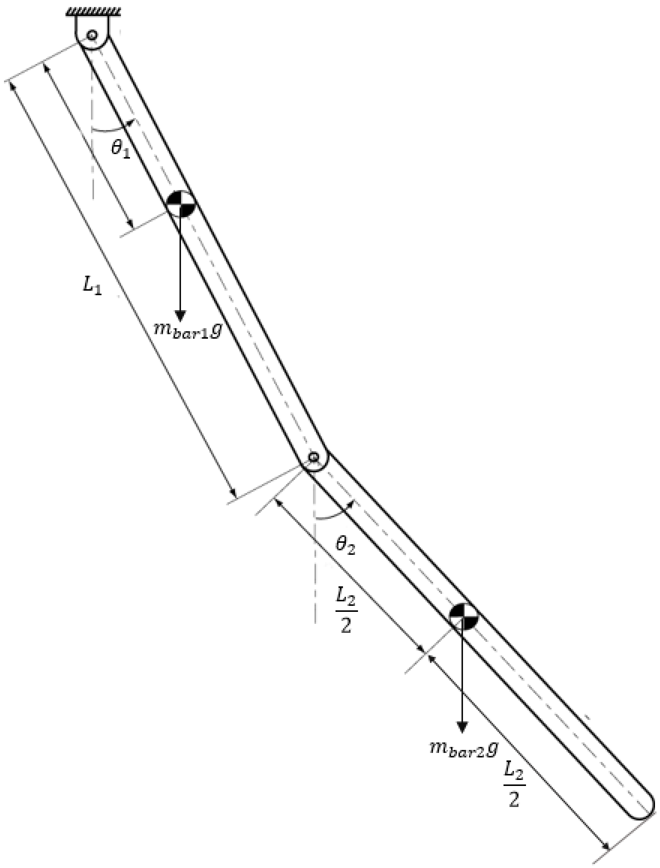
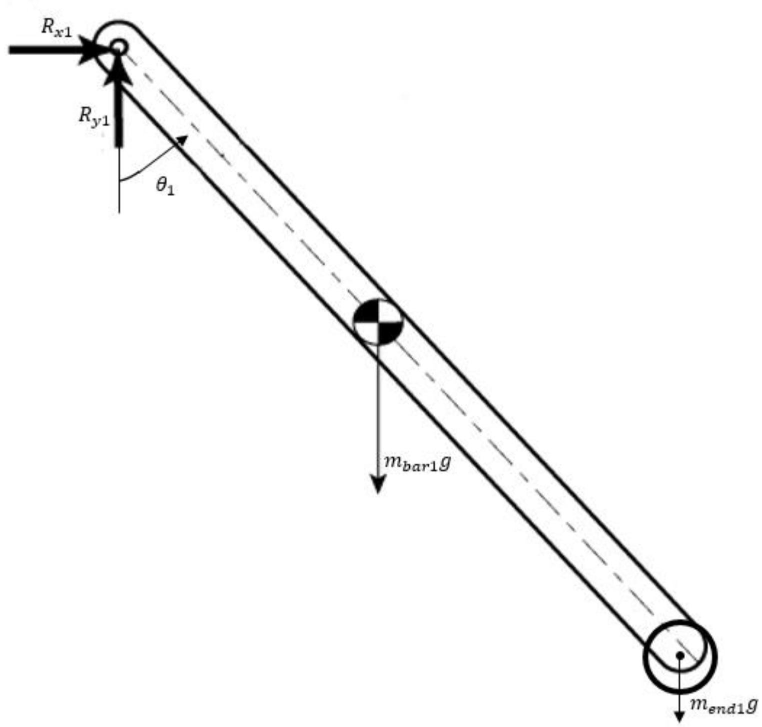
Dmitriy Kats
Simulation of Double Pendulum 10/25/15

The double pendulum used for this problem is drawn here:



Looking at bar 1 and simplifying the second bar as an end mass



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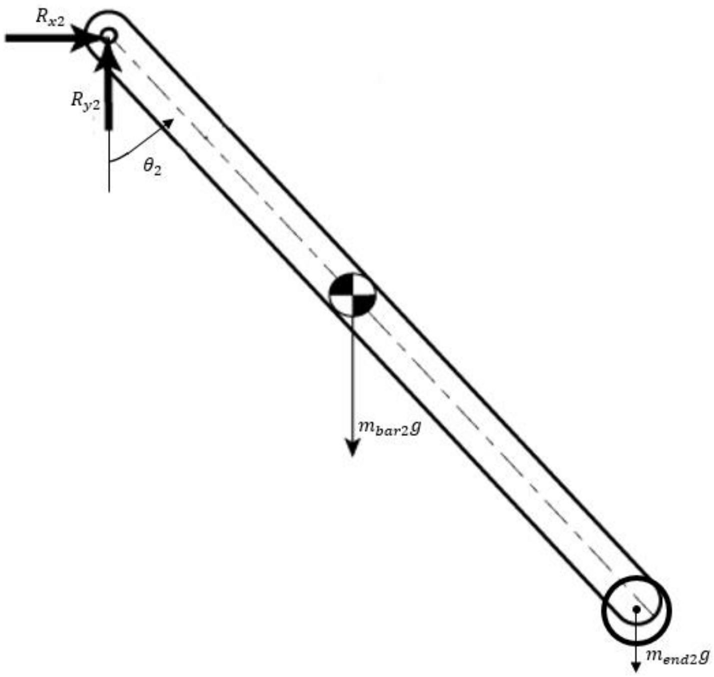
For the second bar, the pin joint reaction forces are accounted for and there is also an end lumped mass



The potential energy of the entire system is as follows:

$$PE_{system} = m_{bar_1} g\left(\frac{L_1}{2} - \frac{L_1}{2}\cos\theta_1\right) + m_{end_1} g(L_1 - L_1\cos\theta_1) + m_{bar_2} g\left(L_1 - L_1 cos\theta_1 + \frac{L_2}{2} - \frac{L_2}{2} cos\theta_2\right) + m_{end_2} g(L_1 - L_1 cos\theta_1 + L_2 - L_2 cos\theta_2)$$

The kinetic energy is the sum of KE_1 and KE_2 (the kinetic energies of the top and bottom bar).

The kinetic energy of the top bar:

$$KE_1 = \frac{1}{2} m_{bar_1}\left(\frac{L_1}{2}\dot{\theta}_1\right)^2 + \frac{1}{2} m_{end_1}\left(L_1\dot{\theta}_1\right)^2 + 1/2 J_{bar_{1CM}}\left(\dot{\theta}_1\right)^2$$

where

$$J_{bar_{1CM}} = \frac{1}{12} m_{bar_1} L_1^2$$

and simplifying

$$KE_1 = \left(\dot{\theta}_1\right)^2 L_1^2 \left(\frac{1}{6} m_{bar_1} + \frac{1}{2} m_{end_1}\right)$$

While the kinetic energy of the bottom bar is more involved beginning with relative velocities:

$$\vec{v}_{bar_{2(CM)}} = \vec{v}_{end_1} + \vec{v}_{bar_2/end_1}$$

$$\vec{v}_{bar_{2(CM)}} = L_1\dot{\theta}_1(\cos\theta_1\hat{\imath} + \sin\theta_1\hat{\jmath}) + \frac{L_2\dot{\theta}_2}{2}(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath})$$

$$\vec{v}_{bar_{2(CM)}} = \left(L_1\dot{\theta}_1\cos\theta_1 + \frac{L_2\dot{\theta}_2}{2}\cos\theta_2\right)\hat{\imath} + \left(L_1\dot{\theta}_1\sin\theta_1 + \frac{L_2\dot{\theta}_2}{2}\sin\theta_2\right)\hat{\jmath}$$

$$\vec{v}_{bar_{2(end)}} = L_1\dot{\theta}_1(\cos\theta_1\hat{\imath} + \sin\theta_1\hat{\jmath}) + L_2\dot{\theta}_2(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath})$$

$$\vec{v}_{bar_{2(end)}} = \left(L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2\right)\hat{\imath} + \left(L_1\dot{\theta}_1\sin\theta_1 + L_2\dot{\theta}_2\sin\theta_2\right)\hat{\jmath}$$

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Kinetic energy is a function of the velocities squared, hence:

$$\left(\vec{v}_{bar_{2(CM)}}\right)^2 = L_1^2\dot{\theta}_1^2\cos^2\theta_1 + L_1\dot{\theta}_1 L_2\dot{\theta}_2\cos\theta_1\cos\theta_2 + \frac{L_2^2\dot{\theta}_2^2}{4}\cos^2\theta_2 +$$
$$L_1^2\dot{\theta}_1^2\sin^2\theta_1 + L_1\dot{\theta}_1 L_2\dot{\theta}_2\sin\theta_1\sin\theta_2 + \frac{L_2^2\dot{\theta}_2^2}{4}\sin^2\theta_2$$
$$\left(\vec{v}_{bar_{2(end)}}\right)^2 = L_1^2\dot{\theta}_1^2\cos^2\theta_1 + 2L_1\dot{\theta}_1 L_2\dot{\theta}_2\cos\theta_1\cos\theta_2 + L_2^2\dot{\theta}_2^2\cos^2\theta_2 +$$
$$L_1^2\dot{\theta}_1^2\sin^2\theta_1 + 2L_1\dot{\theta}_1 L_2\dot{\theta}_2\sin\theta_1\sin\theta_2 + L_2^2\dot{\theta}_2^2\sin^2\theta_2$$

Using the following trig identity that says:

$$\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2) = \cos(\theta_2 - \theta_1)$$
$$\cos^2\theta_i + \sin^2\theta_i = 1$$

$$\left(\vec{v}_{bar_{2(CM)}}\right)^2 = L_1^2\dot{\theta}_1^2 + L_1\dot{\theta}_1 L_2\dot{\theta}_2\cos(\theta_2 - \theta_1) + \frac{L_2^2\dot{\theta}_2^2}{4}$$
$$\left(\vec{v}_{bar_{2(end)}}\right)^2 = L_1^2\dot{\theta}_1^2 + 2L_1\dot{\theta}_1 L_2\dot{\theta}_2\cos(\theta_2 - \theta_1) + L_2^2\dot{\theta}_2^2$$

These velocities are inputted into:

$$KE_2 = \frac{1}{2}m_{bar_2}\left(\vec{v}_{bar_{2(CM)}}\right)^2 + \frac{1}{2}m_{end_2}\left(\vec{v}_{bar_{2(end)}}\right)^2 + 1/2 J_{bar_{2CM}}\left(\dot{\theta}_2\right)^2$$

where

$$J_{bar_{2CM}} = \frac{1}{12}m_{bar_2}L_2^2$$

When the velocities are substituted into kinetic energy equation of the second bar:

$$KE_2 = \frac{1}{2}m_{bar_2}\left(L_1^2\dot{\theta}_1^2 + L_1\dot{\theta}_1 L_2\dot{\theta}_2\cos(\theta_2 - \theta_1) + \frac{L_2^2\dot{\theta}_2^2}{4}\right) + \frac{1}{2}m_{end_2}\left(L_1^2\dot{\theta}_1^2 + 2L_1\dot{\theta}_1 L_2\dot{\theta}_2\cos(\theta_2 - \theta_1) + L_2^2\dot{\theta}_2^2\right)$$
$$+ \frac{1}{24}m_{bar_2}L_2^2\left(\dot{\theta}_2\right)^2$$

The Lagrange method is

$$L \equiv KE - PE$$
$$\frac{d}{dt}\frac{\partial L}{\partial x_i} - \frac{\partial L}{\partial x_i} = 0$$

For x_i as θ_1 and using Mathematica (attached) to evaluate the equation of motion with the constant values and solving for $\ddot{\theta}_1$:

L_1	0.176 m
L_2	0.135 m
m_{End1}	0.135 kg
m_{End2}	0.085 kg
m_{Bar1}	0.043 kg
m_{Bar2}	0.021 kg
g	9.81 m/s^2

No damping

$$\ddot{\theta}_1 = -0.2868910218371707\dot{\theta}_2^2\sin(\theta_1 - \theta_2) - 57.30310200569665\sin\theta_1$$

For x_i as θ_2 and using Mathematica to evaluate the equation of motion with the same constant values and solving for $\ddot{\theta}_2$:

No damping

$$\ddot{\theta}_2 = 1.3533011272141706\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 75.43115942028986\sin\theta_2$$

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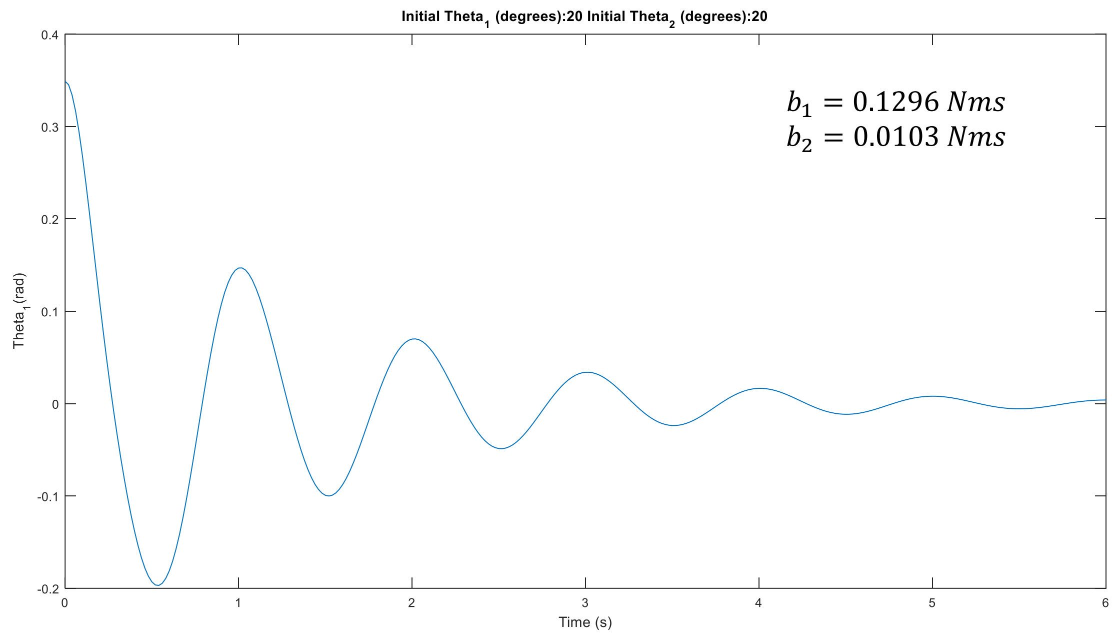
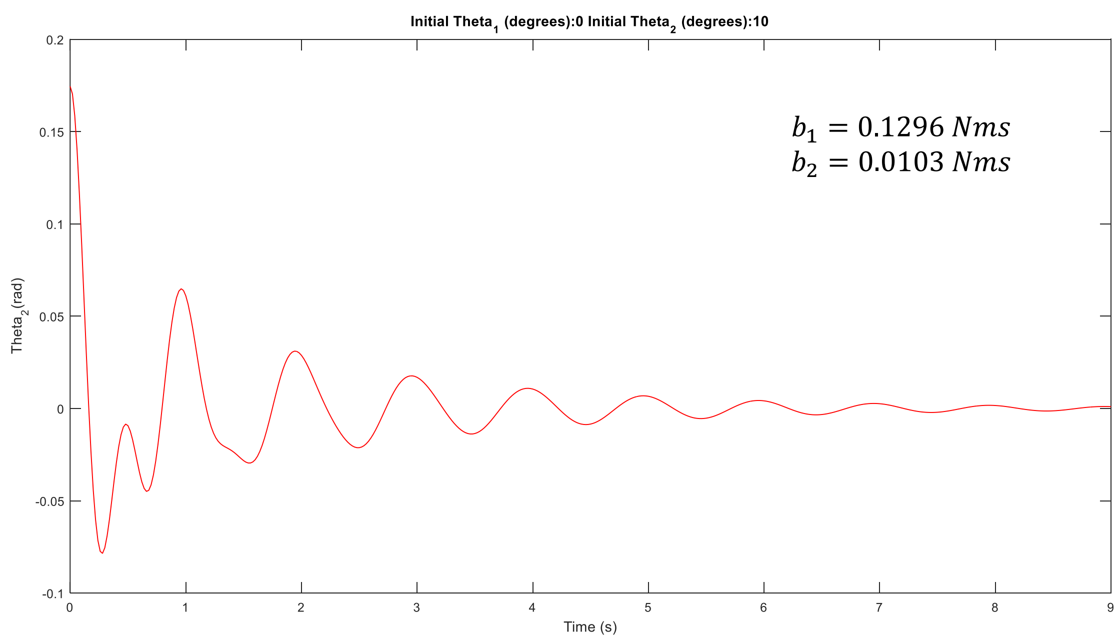
Refer to the attached Mathematica document in regards to the damping. In class, the top bar was raised to 20° with the bottom bar held in line to find that it takes 4 periods to make $\theta_1$to become small when it starts at 20°. The initial conditions were set and the damping coefficients were found to create that response:



In class, the bottom bar was raised to 10° with the top bar in a vertical position to find that it takes 8 periods to make $\theta_2$to become small when it starts at 10°. The initial conditions were set and the damping coefficients were found to create that response:



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The homework asks to elevate the first bar to 45° with the vertical plane and to run the simulation with damping for two seconds. This is the plot of the angles of that simulation. The video did not display anything that violates Newton's second law, so the results seem reasonable.