Viscous Flow around a Corner

04/05/16

$Q=1.8 \times 10^{-6}$ m^3/s
Grid Size: Normal

Dmitriy Kats

The flow of air through a micromachine was analyzed for the following narrow duct:

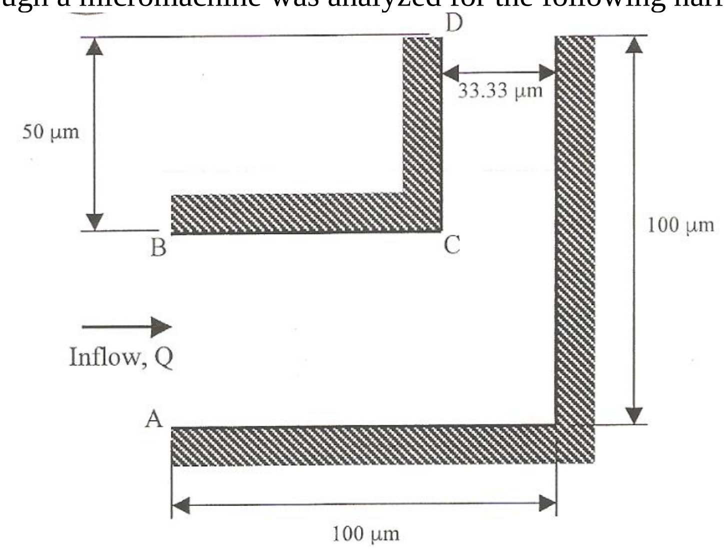

Fig. 1. Duct from project statement[1]

Reynolds number calculation

The flow rate was given in the problem statement as Q=1.8 x 10^{-6} m^3/s and the depth of the duct is 0.01 m. The velocity of the flow is approximated as uniform for the Reynolds number calculation even though it is actually fully developed flow. The velocity in the inlet is just the flow rate divided by the cross sectional area:

$$V = \frac{Q}{wh} = \frac{1.8 * 10^{-6}}{50 * 10^{-6} * 0.01} = 3.6\, m/s$$

The Reynolds number, Re, is calculated as follows:

$$Re = \frac{VD_H}{\nu} = \frac{3.6 * \frac{4 * 50 * 10^{-6} * 0.01}{2(50 * 10^{-6} + 0.01)}}{1.5 * 10^{-5}} = 23.88$$

where D_H is the hydrodynamic diameter [D_H=4*Area/Perimeter] and ν is the kinematic viscosity. Based on this Reynolds number the flow is in the laminar region.

Justification for 2D analysis

The width into the page is 200 times larger than the height of the microchannel for the inlet condition so it is justified to use a 2D analysis of parallel plates for the analysis.

Analytical analysis of the shear force

The coordinate system was positioned at point A in Fig. 1. above with x on the horizontal axis. For steady, laminar flow, between fixed parallel plates the Naiver Stokes equation can be reduced to the following equation when gravity is ignored:

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial \rho}{\partial x}$$

For steady, laminar flow, between fixed parallel plates the velocity profile is of form:

$$u(y) = \frac{1}{2\mu}\frac{\partial \rho}{\partial x}y^2 + c_1 y + c_2$$

[1] Meneveau, Charles. Computer Assignment 2: Viscous Flow Around Corner. Rep. N.p.: n.p., n.d. Print.

Using the boundary conditions that $u(0) = 0, u(h) = 0$ [h is the height at point B] due to the no slip condition the expression becomes:

$$u(y) = \frac{-1}{2\mu}\frac{\partial \rho}{\partial x}(yh - y^2)$$

The flow rate for a unit width in the z direction, q, is found by integrating the velocity:

$$q = \frac{Q}{w} = \int_0^h u(y)dy = -\frac{\partial \rho}{\partial x}\frac{h^3}{12\mu}$$

When this expression is substituted into the velocity equation through the partial derivative of pressure, the velocity equation becomes:

$$u(y) = \frac{6q}{h^3}(yh - y^2) = \frac{6Q}{wh^2}y - \frac{6Q}{wh^3}y^2$$

The viscous along line BC, F_{BC}, was calculated by using the following equation:

$$F_{BC}(y) = -w\int_{x=0}^{x=66.67*10^{-6}} \mu\frac{\partial u}{\partial y}dx$$

where w is the width into the page, μ is the dynamic viscosity, u is the horizontal velocity, and x/y are positions in the coordinate system. Analytically the integral is evaluated as follows:

$$\frac{\partial u(y)}{\partial y} = \frac{6Q}{wh^2} - \frac{12Q}{wh^3}y$$

$$F_{BC}(y) = -\mu\left(\frac{6Q}{h^2} - \frac{12Q}{h^3}y\right)(x_{BC})$$

where $x_{BC} = 66.67 * 10^{-6}m$ which corresponds to the right bound of line BC or in other words the length of line BC. When this is evaluated the viscous force on line BC is found to be:

$$F_{BC}(y = h) = 1.84 * 10^{-5} * 1.8 * 10^{-6}\left(\frac{6}{(50 * 10^{-6})^2}\right)(66.67 * 10^{-6}) = 5.30 * 10^{-6}N$$

The coordinate system is now shifted parallel to the first coordinate system to point C in Fig. 1 and called (x', y'). The same analysis holds true as above when the symbols are replaced appropriately for the vertical flow between the parallel plates:

$$v(x') = \frac{1}{2\mu}\frac{\partial \rho}{\partial y'}x'^2 + c_1x' + c_2$$

$$v(0) = 0, v(h') = 0\ [h' is\ the\ line\ at\ x = 100\mu m\ or\ x' = 33.33\mu m]$$

$$v(x') = \frac{-1}{2\mu}\frac{\partial \rho}{\partial y'}(x'h' - x'^2)$$

$$q = \frac{Q}{w} = \int_0^{h'} v(x')dx' = -\frac{\partial \rho}{\partial y'}\frac{h'^3}{12\mu}$$

$$v(x') = \frac{6Q}{wh'^2}x' - \frac{6Q}{wh'^3}x'^2$$

The viscous along line CD, F_{CD}, was calculated analytically:

$$F_{CD}(x') = w\int_{x=0}^{x=33.33*10^{-6}} \mu\frac{\partial v}{\partial x}dy$$

$$F_{CD}(x') = \mu\left(\frac{6Q}{h'^2} - \frac{12Q}{h'^3}x'\right)(y_{CD})$$

where w is the width into the page, μ is the dynamic viscosity, v is the vertical velocity, x'/y' are positions in the second coordinate system, and $y_{CD} = 50 * 10^{-6}\ m$ which corresponds to the length of line CD.

When this is evaluated the viscous force on line CD is found to be:

$$F_{CD}(x' = 0) = 1.84 * 10^{-5} * 1.8 * 10^{-6}\left(\frac{6}{33 * 10^{-6^2}}\right)(50 * 10^{-6}) = 9.12 * 10^{-6}\, N$$

<u>Numerical Analysis of the viscous forces</u>

The viscous force on line BC can also be computed numerically as well. A finite difference approach was taken to evaluate the velocity difference:

$$\left.\frac{\partial u}{\partial y}\right|_i \approx \frac{u_{i+1} - u_i}{y_{i+1} - y_i}$$

The shear force along line BC due to the viscous stress was evaluated as a summation over all the nodes along line BC to approximate the integral as:

$$F_{BC} = w\sum_{i=1}^{N_x} \mu \frac{u_{i+1} - u_i}{y_{i+1} - y_i}(x_{i+1} - x_i)$$

A similar approach was to get the shear force along line CD:

$$F_{CD} = w\int_{y=50*10^{-6}}^{x=100*10^{-6}} \mu \frac{\partial v}{\partial x} dy$$

A finite difference approach was taken to evaluate the velocity difference:

$$\left.\frac{\partial v}{\partial x}\right|_i \approx \frac{v_{i+1} - v_i}{x_{i+1} - x_i}$$

The shear force along line CD due to the viscous stress was evaluated as a summation over all the nodes along line CD to approximate the integral as:

$$F_{CD} = w\sum_{i=1}^{N_y} \mu \frac{v_{i+1} - v_i}{x_{i+1} - x_i}(y_{i+1} - y_i)$$

The matlab code is attached to show these calculations were evaluated.

<u>COMSOL Analysis and Results:</u>

A COMSOL analysis was performed for laminar flow in the narrow duct for air at 20^{o}C assuming it is fully developed at the inlet. The no slip wall condition was used and the figures below show the setup and results.

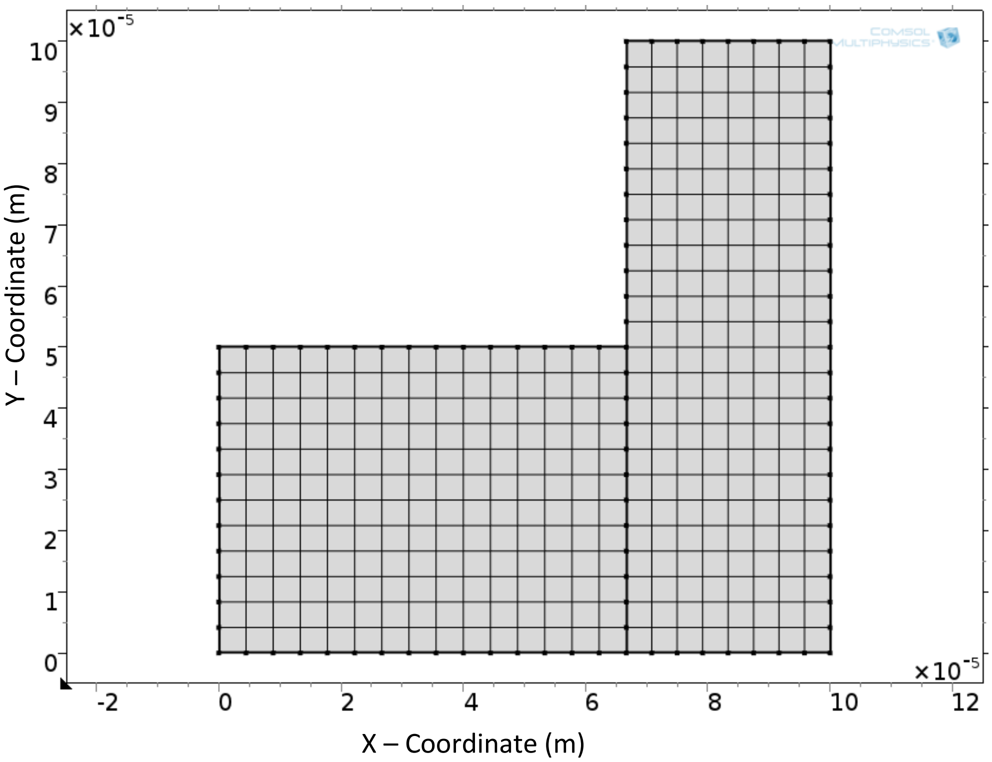

Fig. 2. Normal grid size to discretize the geometry

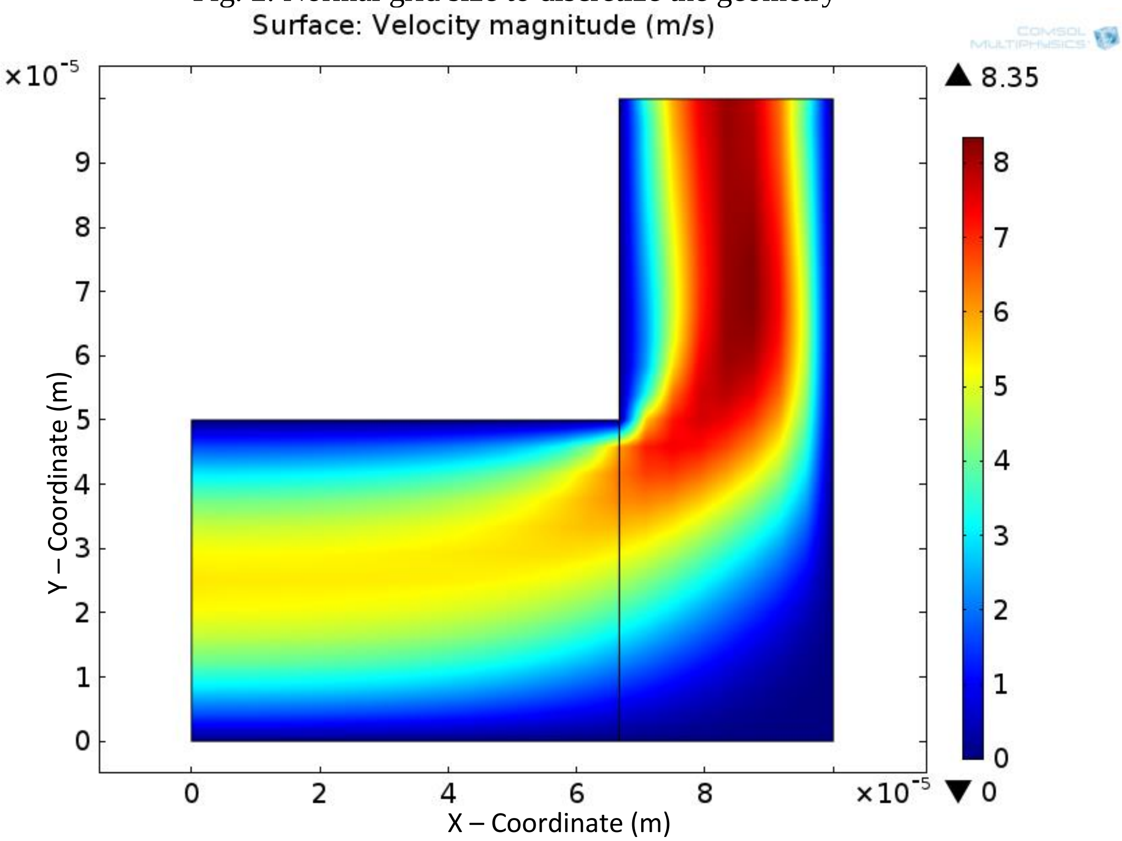

Fig. 3. Velocity magnitudes of the channel

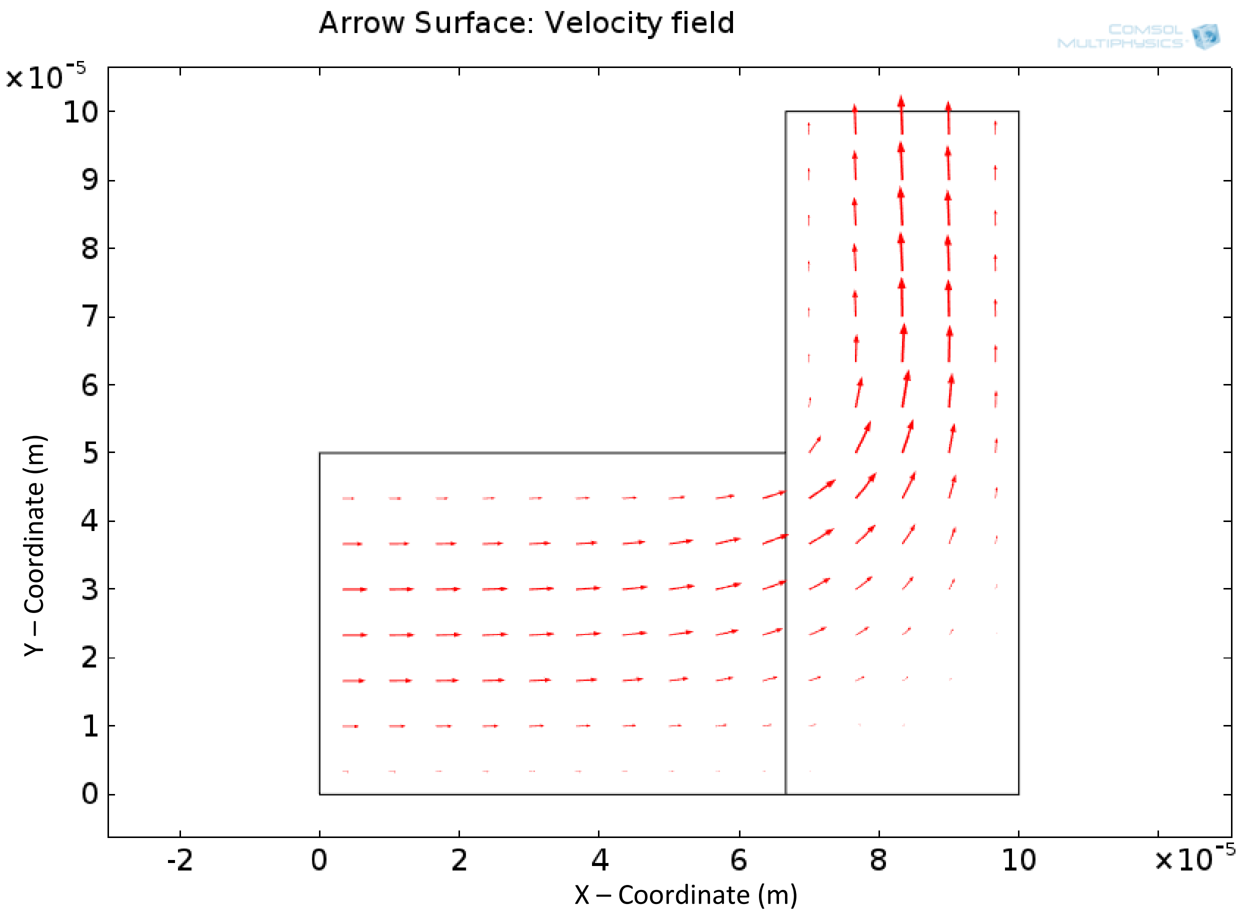

Fig. 4. Velocity field inside the channel

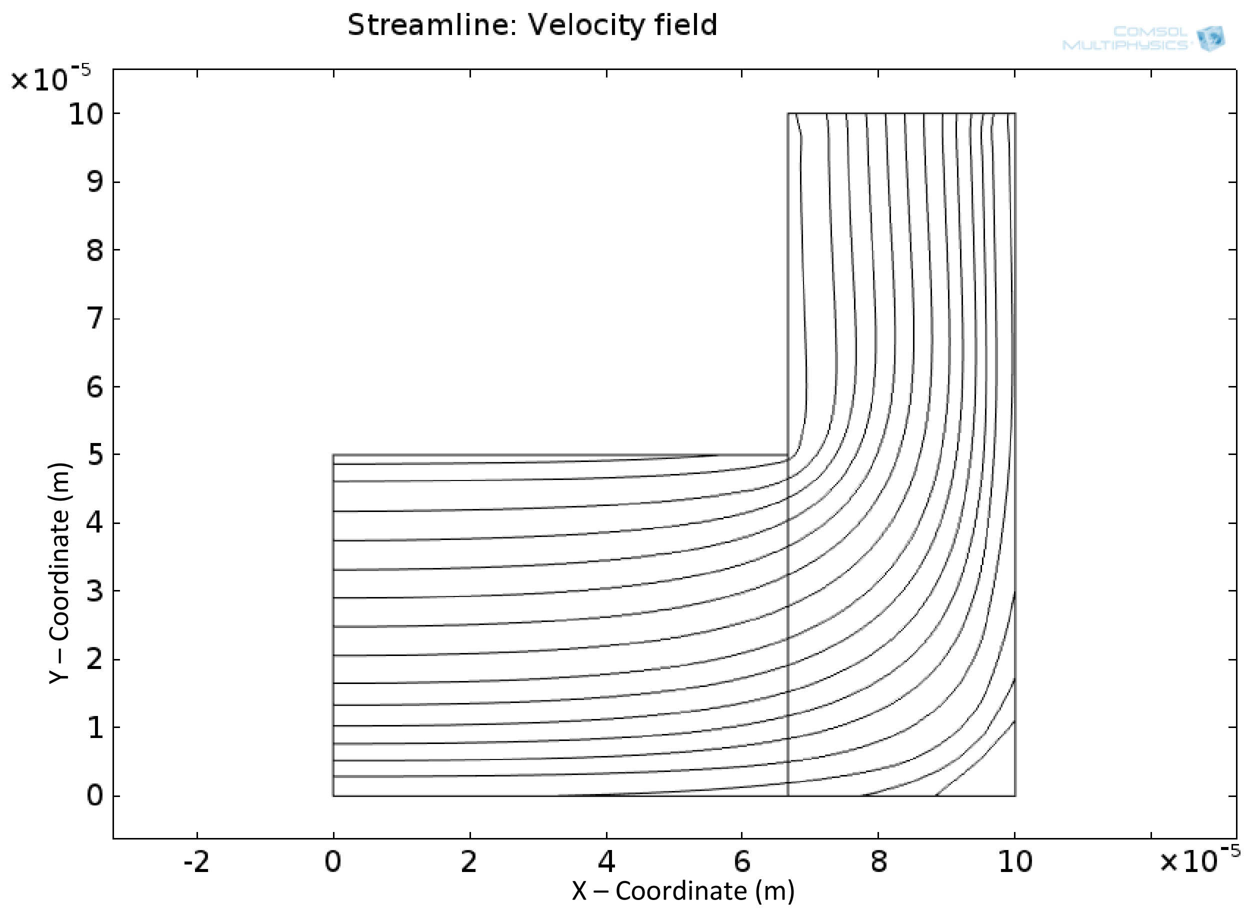

Fig. 5. Streamline velocities inside the channel

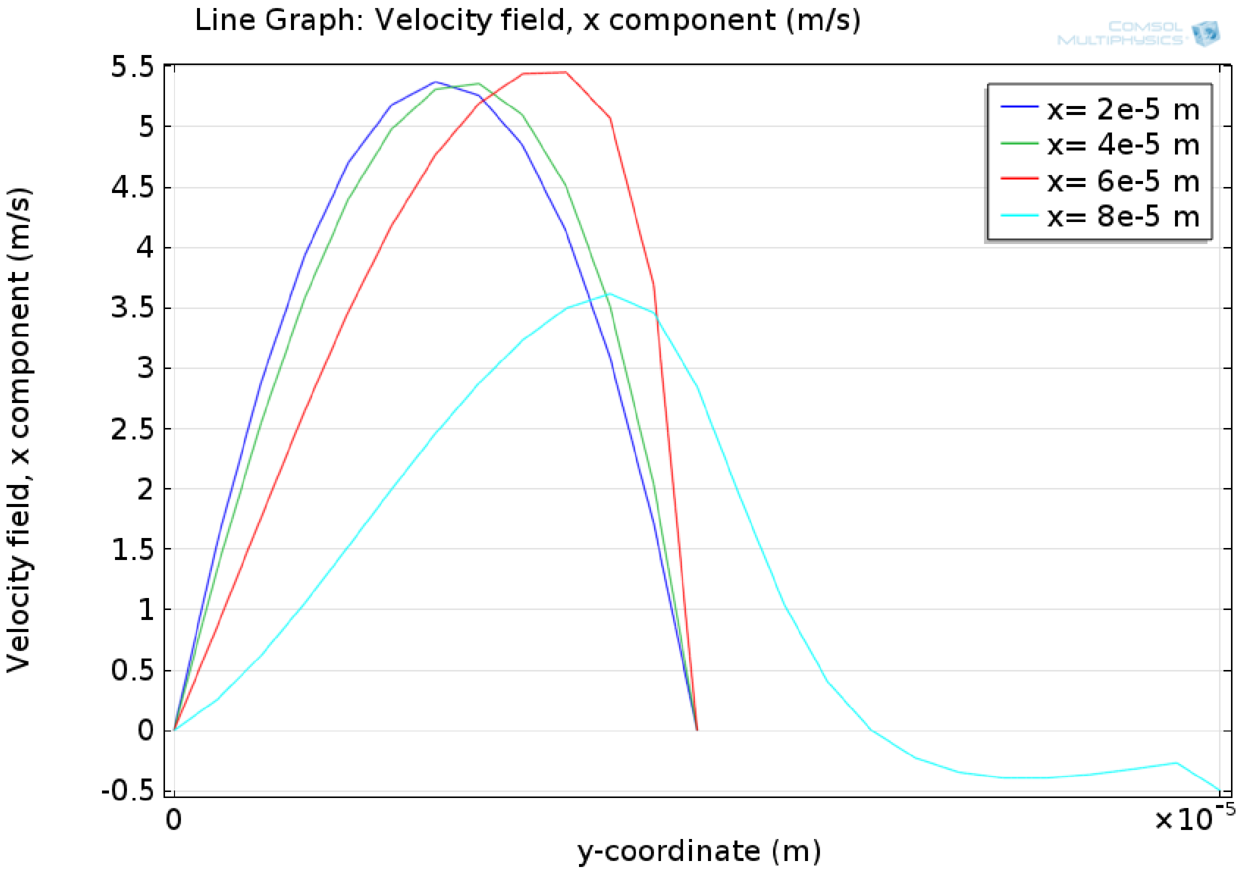

Fig. 6. U-velocity component in the channel at various x distances

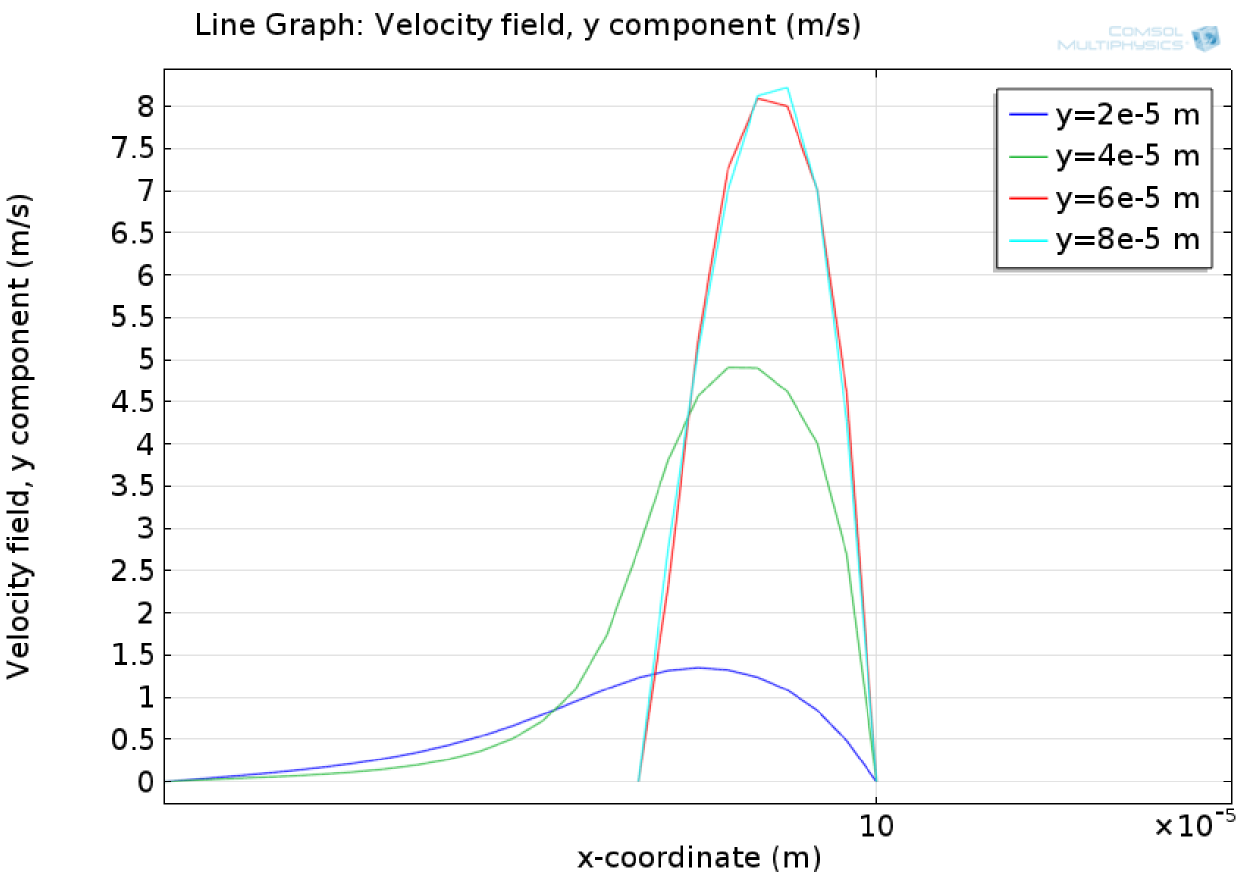

Fig. 6. V-velocity component in the channel at various y distances

Table 1. COMSOL exported values of velocity

Parallel to line BC at y= 46 μm			Parallel to line CD at x= 70.67 μm	
x(μm)	u(m/s)		y (μm)	v(m/s)
0.00	1.58		50.00	4.46
4.62	1.56		54.00	3.12
8.89	1.58		54.17	3.06
9.07	1.56		58.17	2.33
13.33	1.58		58.33	2.30
13.51	1.58		62.33	2.15
17.78	1.60		62.50	2.14
17.96	1.61		66.50	2.19
22.22	1.63		66.67	2.19
22.40	1.63		70.67	2.31
26.67	1.67		70.83	2.32
26.85	1.67		74.83	2.47
31.11	1.71		75.00	2.47
31.29	1.72		79.00	2.63
35.56	1.77		79.17	2.63
35.74	1.78		83.17	2.78
40.00	1.85		83.33	2.79
40.18	1.86		87.33	2.92
44.45	1.96		87.50	2.92
44.62	1.96		91.50	3.03
48.89	2.11		91.67	3.03
49.07	2.11		95.67	3.06
53.34	2.31		95.83	3.07
53.51	2.33		99.83	3.37
57.78	2.62		100.00	3.38
57.96	2.64			
62.23	3.12			
62.40	3.15			
66.67	3.97			

Note: The velocities along both walls BC and CD are zero.

Table 2. COMSOL exported partial derivatives

Along line BC		Along line CD	
x(μm)	$\frac{\partial u}{\partial y}$ (1/s)	y (μm)	$\frac{\partial v}{\partial x}$ (1/s)
0.00	3.96E+05	50.00	1115657.53
4.44	3.91E+05	54.17	765485.33
8.89	3.96E+05	58.33	574457.97
13.33	4.01E+05	62.50	535138.94
17.78	4.08E+05	66.67	548201.80
22.22	4.17E+05	70.83	579268.13
26.67	4.29E+05	75.00	617820.57
31.11	4.44E+05	79.17	658093.99
35.56	4.63E+05	83.33	696630.00
40.00	4.90E+05	87.50	730911.66
44.45	5.26E+05	91.67	758168.71
48.89	5.78E+05	95.83	766542.91
53.34	6.56E+05	100.00	845459.17
57.78	7.80E+05		
62.23	9.92E+05		
66.67	-1.06E-09		

Note: In order to check that slope calculation is correct, I did an example calculation at point B:

At point B the velocity at the wall is 0 due to the no slip condition and when the velocities from Table 1 are substituted:

$$\left.\frac{\partial u}{\partial y}\right|_i \approx \frac{u_{i+1} - u_i}{y_{i+1} - y_i} = \frac{1.584}{4 * 10^{-6}} = 3.96 * 10^5 \ 1/s$$

This evaluation agrees with the result at point B from Table 2.

I also plotted the COMSOL extracted slopes against the calculated slopes.

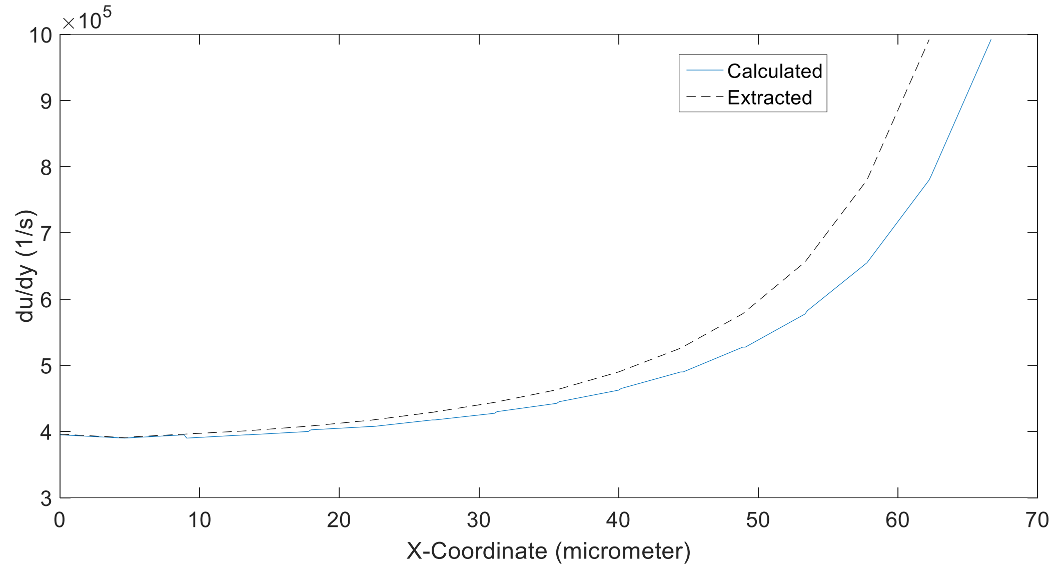

Fig. 6. $\frac{\partial u}{\partial y}$ from COMSOL using Tables 1 and 2

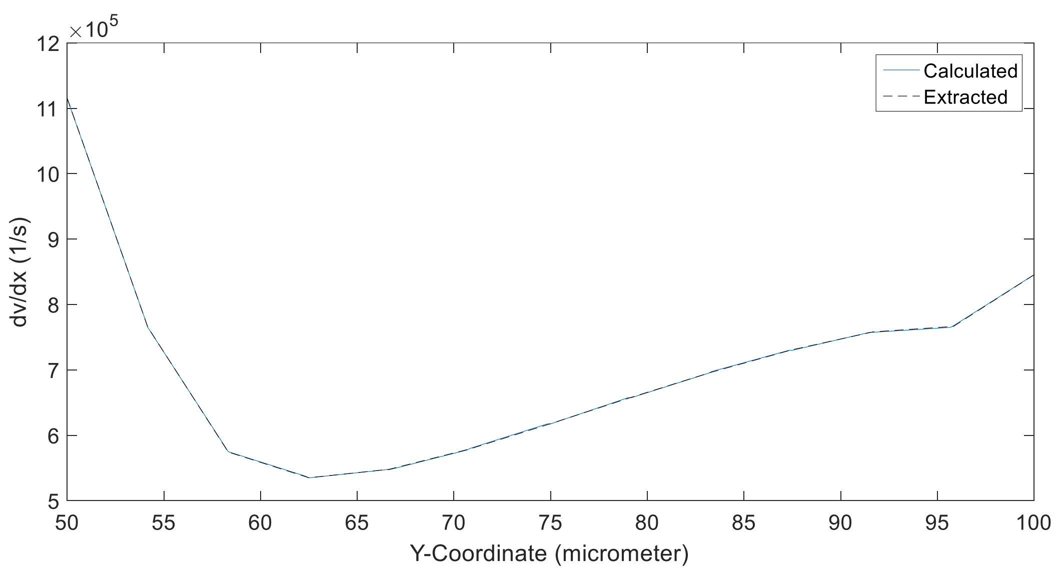

Fig. 7. $\frac{\partial v}{\partial x}$ from COMSOL using Tables 1 and 2

Note: In Fig. 7. the slopes are right on top of each other, while in Fig. 6. they are a little different which can be accounted for with the discrepancy of the nodes positions from the calculated and extracted slopes from Tables 1 and 2.

<u>Comparison of shear force calculation</u>

The shear force was calculated analytically with the assumption of fully developed flow (Method 1), using the summation of shear forces from the calculated velocity gradients from velocity values given by COMSOL (Method 2), and by using summation of the shear forces from the exported velocity gradients from COMSOL. This is summarized in Table 3.

Table 3. Shear force along walls BC and CD

Method	F_{BC} (μN)	F_{CD} (μN)
1	5.30	9.12
2	6.42	6.43
3	6.39	6.44

Methods 2 and 3 are almost the same calculation so it is expected that they are the same or almost the same. The only difference in their calculation is that the gradients are calculated in Method 2, while they are exported directly from COMSOL for Method 3.

Method 1 differs significantly from the other methods primarily because the fully developed assumption is not upheld for the flow around the corner. A fully developed profile implies that the peak velocity is on the centerline of the two parallel plates and the velocity is symmetric about that centerline. From the COMSOL results in Fig. 3. and Fig. 4. it is clear that the faster part of flow is near the corner rather than the symmetric line. This is not characteristic of a symmetric flow so when Method 1 uses this assumption to evaluate the shear force it becomes significantly different than the numerical calculations of the shear force. For the COMSOL simulation, it was only assumed that the entrance of the flow is fully developed. The velocity profile becomes skewed toward the corner for the solution of the laminar, viscous flow around a corner. The velocity magnitude becomes higher towards wall BC as seen in Fig. 3. so this is why Methods 2 and Method 3 have a larger calculated shear force than the theoretical shear force from Method 1. This same reasoning can be used for wall CD where the flow shifts away from it and the numeric Methods 2 and 3 under predict the shear force calculated theoretically using Method 1.

Another source of error could be the grid size not being fine enough. I used a normal grid size and it is clear from Fig. 2. and Fig. 3. that a finer grid is necessary to better capture the velocity profiles especially around the corner. More accurate data would likely make the velocities more accurate for the numerical calculations. The fully developed assumption would still make the calculated shear forces still incorrect though.

Matlab Code for shear force calculation

```
%Method 2
mu=1.85e-5;
w=0.01;
load('MatDataFluidsProjectII.mat'); %velocity + coordinates

F_BC=0;
F_CD=0;

for i=1:28
    F_BC=F_BC+w*mu*(BC(i,2))/4e-6*(BC(i+1,1)-BC(i,1));
end

for i=1:24
    F_CD=F_CD+w*mu*(CD(i,2))/4e-6*(CD(i+1,1)-CD(i,1));
end

%Method 3
load('MatDataCOMSOLFluidsProjectII.mat'); %coordinates + slopes
F_BCs=0;
F_CDs=0;

for i=1:15
    F_BCs=F_BCs+w*mu*(BCs(i,2))*(BCs(i+1,1)-BCs(i,1));
end

for i=1:12
    F_CDs=F_CDs+w*mu*(CDs(i,2))*(CDs(i+1,1)-CDs(i,1));
end
```