Project Title Asymptotics of non-commutative Laplacians, quantum symmetries and the Chern character.

Project quality and innovation

Brief outline The project belongs to the broad field of Non-commutative Analysis. Shortly, this area appeared from the foundational questions in Quantum Physics. While classically, observables are functions, in quantum world observables are operators. In abstract terms, observables in a classical system form a commutative von Neumann algebra, while the ones in a quantum system form a non-commutative von Neumann algebra. Noncommutative topology and integration theory were developed in 1940-1960. Non-commutative Geometry, the more specific field of this project, takes its origin in 1980's in the works of Alain Connes.

One of the objectives of this project is a Connes Character Formula. In short, it declares the equality between the 2 cocycles: one appears from the (non-commutative) Riemannian geometry, while another appears from the conformal geometry. The formula can be found (without any proof) in the book "Noncommutative geometry" by Connes. A number of authors tried to prove it with various degrees of success. For compact non-commutative manifolds, the final word is due to the CI and co-authors [4].

One of the cocycles in the Connes Character Formula is defined via Dixmier trace. Those traces are very special case of singular traces. The CI co-authored world first monograph on singular traces and their applications and a number of his papers in this direction are published in highly prestigious journals: Journal für die reine und angewandte Mathematik (the oldest mathematical journal still in publishing), Advances in Mathematics, Journal of Functional Analysis, etc.

Another objective in this project is to provide a non-commutative version of Minakshisundaram-Plejel theorem, that is, to find an analogue for the heat kernel expansion. In this expansion, the coefficient in the most important term is a constant, while the one in the second term is a scalar curvature. Higher order terms can be expressed via higher order differential invariants. Heuristically, such an expansion is strongly related with singular traces: the most important term should be equal to the value of the singular trace. This heuristic was recently verified by the CI in a quite general setting (which includes, e.g. compact manifolds).

For the simplest non-flat non-commutative manifold (a conformal deformation of a 2-dimensional noncommutative torus), curvature was computed by Connes and collaborators (Tretkoff, Moscovici, Fathizadeh, Khalkhali). In this project, we propose a new technique to compute the curvature (and higher order coefficients) by using theory of Double Operator Integrals. The theory developing since 1970's recently experienced a breakthrough by the CI and co-authors (to appear in American Journal of Mathematics).

Third objective in this project is to create a spectral triple from a given quantum group. Quantum groups are certain deformations of well known Lie groups. The latter are, by definition, smooth manifolds. It is of crucial importance that Dirac operator on such manifolds commutes with the action of the group on itself. A similar property for quantum groups is called equivariance. A number of attempts were made to construct an equivariant spectral triple for the simplest possible example, the quantum unitary group. These attempts suffer from 2 drawbacks: one is dimension drop phenomenon (i.e. spectral dimension does not equal to the homological one), the other is that the first term in the heat kernel expansion does not recover the Haar state (as it should). The CI proposes to construct an equivariant spectral triple on quantum groups which is free from these drawbacks.

The CI's believes (and his publication list supports this) that all the stated objectives are plausible.

National/international progress in the field of research and its relationship to this Proposal Operator algebras provide a natural edifice to many areas of classical and modern mathematics, and also play a paramount role in the present proposal. A particular role is played by the class of C^*-algebras (uniformly closed $*$-subalgebras in the $*-$algebra $B(H)$ of all bounded operators on a Hilbert space H) and that of von Neumann algebras (weakly closed unital $*$-subalgebras of $B(H)$). The topological requirements imposed on these algebras make them a suitable foundation for developing noncommutative analysis.

The starting point of noncommutative geometry can be traced back to the Gelfand-Naimark theorem which delivers a duality between the category of locally compact topological spaces and that of commutative C^*-algebras. Briefly speaking, it allows us to express numerous topological properties of topological spaces in the language of C^*-algebras. This theorem can be viewed as an anti-equivalence between a category of locally compact Hausdorff spaces and the category of commutative C^*-algebras. The corresponding anti-equivalence is given by the map $X \to C_0(X)$, where $C_0(X)$ is the algebra of all continuous complex valued functions which vanish at infinity. This may be interpreted as the statement that all of the information about a space is actually encoded in the algebra of continuous complex valued functions on that space. Thus one may think of noncommutative C^*-algebras as noncommutative topological spaces and attempts to apply topological methods to understand these algebras are well justified.

Another fundamental result due to von Neumann and Segal provides a duality between a certain category of measure spaces and that of commutative von Neumann algebras. This fact provided impetus to view classical measure theory as a part of von Neumann algebra theory led to the development of noncommutative integration theory.

Conventional wisdom suggests further analogies between the classical and quantum worlds. While topology and measure theory were supplied with natural quantum counterparts in the 1950s, a development of further geometric concepts were lacking until the 1980s. In fact, starting from von Neumann himself, a number of researchers attempted to find a quantum analogue of geometry (that is, to construct a functor from useful geometric categories to treatable quantum categories), however the success of their attempts was questionable.

Let us now briefly review the foundational features of classical differential geometry. We start with the notion of a Riemannian manifold which is its central concept. Usually, a manifold is defined as a topological space in which every point admits a neighborhood homeomorphic to Euclidean space. A manifold (denoted further by X) equipped with a smooth metric tensor (denoted further by g) is called Riemannian.

So far, the most successful attempt to quantise the notion of Riemannian manifold is due to Connes who introduced the notion of a spectral triple and promoted it as a non-commutative analogue of a Riemannian manifold and further as a convenient vehicle for a new (non-commutative) differential geometry encompassing closed Riemannian manifolds as a very special example.

Let H be a (separable) Hilbert space, and let $\mathscr{A}$ be a $*$-algebra. We say that $(\mathscr{A},H,D)$ is a (compact) spectral triple if: (a) there is a representation $\pi:\mathscr{A}\to B(H)$ of the $*$-algebra $\mathscr{A}$ on the Hilbert space H; (b) D is a self-adjoint unbounded operator on H; (c) for every $a\in\mathscr{A}$, the commutator $[D,a]$ is has bounded extension; (d) the operator $(D+i)^{-1}$ is compact. If the singular values of the operator $(D+i)^{-1}$ decay like the sequence $k^{-\frac{1}{d}}$, $k\geq 1$ the spectral triple is called $d-$dimensional.

Associated to every (compact) Riemannian manifold X, there is a spectral triple $(\mathscr{A},H,D)$ defined as follows: (i) $\mathscr{A}$ is the algebra $C^\infty(X)$ of all smooth functions on X; (ii) H is the Hilbert space $L_2\Omega(X)$ of all square-integrable forms on X and π is the representation of $\mathscr{A}$ on H by pointwise multiplication; (iii) D is the Hodge-Dirac operator (see e.g. [2]) on $L_2\Omega(X)$. It is folklore that the just constructed spectral triple $(\mathscr{A},H,D)$ is d-dimensional, where $d=\dim(X)$.

As we already stated earlier, the celebrated Reconstruction Theorem due to Connes [10] stipulates that every spectral triple with commutative $\mathscr{A}$ (satisfying a few natural conditions) comes from a d-dimensional Riemannian manifold X. Thus it makes good sense to think of spectral triples as noncommutative manifolds. We wholeheartedly adopt this point of view in this proposal and shall develop it much further.

A Hodge-Dirac operator D is a convenient tool to identify and describe the isometries of the manifold X. If $\gamma:X\to X$ is an isometry and if $U:L_2\Omega(X)\to L_2\Omega(X)$ is the operator of composition with γ, then (i) U is unitary on $L_2\Omega(X)$; (ii) U preserves $\pi(\mathscr{A})$ (that is, $U^{-1}\pi(\mathscr{A})U=\pi(\mathscr{A})$); (iii) U commutes with D. Conversely, every diffeomorphism $\gamma:X\to X$ which commutes with D is an isometry. This simple but revealing result can be found in [23].

It is typical in applications to harmonic analysis [23] and mathematical physics [20] that manifolds are equipped with isometric action of Lie groups. Moreover, most examples of manifolds which are of serious interest and importance in applications are actually homogeneous spaces of some Lie group (e.g., the sphere $\mathbb{S}^{n-1}$ is a homogeneous space of the Lie group $SO(n)$). In such a situation, it does not make sense to consider the manifold alone, but rather only a manifold equipped with an isometric group action. We propose to investigate spectral triples which arise from an action of a Lie group (or some other group-like object, such as a quantum group) on a manifold (with a particular focus on generalisations to non-commutative manifolds).

The class of group-like objects which is most suitable for this task are quantum groups, whose theory has been under development since the early 1980's. The theory of quantum groups has its origin in attempts to find a good duality theorem, analogous to Pontryagin duality theorem, for general locally compact non-Abelian groups. In late 1980's, Woronowicz developed a general theory of compact quantum groups and developed a Peter-Weyl theory for them. One of the main examples in Woronowicz's theory is the quantum group $SU_q(n)$ (a natural $q-$deformation of the compact Lie group $SU(n)$).

The rich interplay between Lie groups and differential geometry naturally raises the question of understanding the interaction between quantum groups and non-commutative geometry. Papers [5], [28] (among many others) attempt to put quantum groups within the framework of non-commutative Geometry. These attempts drew the attention of Alain Connes (see [6]) who developed further results by Chakraborty and Pal [5]. Our proposal is in fact partly motivated by these developments.

In this project, we aim to construct spectral triples for certain quantum groups (e.g. compact quantum groups like $SU_q(n)$ and $SO_q(n)$ as well as non-compact quantum groups like $SL_q(n)$) and their homogeneous spaces, to investigate the validity of major results in Non-commutative Geometry (such as Connes' Character Formula) for these examples and to compute numerically those topological invariants (such as their $K-$theory) which follow from these results.

We complete this section by explaining another important notion in this area, which is that a spectral triple $(\mathscr{A},H,D)$ is equipped with a natural (semi)-group action, the heat semigroup. In the particular case where (X,g) is a d-dimensional Riemannian manifold, then D^2 is the Hodge-Laplace operator (denoted further by Δ_g) and its component acting on 0 order forms being the Laplace-Beltrami operator (also denoted by Δ_g) [31]. The heat semi-group is now defined by the formula

$$t \to e^{-t\Delta_g}, \quad t > 0.$$

It happens that $e^{-t\Delta_g}$ belongs to the trace class for $t > 0$.

In his seminal work [35], Weyl proved that, for a compact manifold,

$$\lim_{t\downarrow 0}(4\pi t)^{\frac{d}{2}}\mathrm{Tr}(e^{t\Delta_g}) = \mathrm{Vol}(X), \quad t \downarrow 0. \tag{1}$$

Following Weyl's work, it became an established custom to measure various geometric (and often topological) quantities associated with a Riemannian manifold X in terms of its heat semi-group expansion $t \to e^{t\Delta_g}, \quad t > 0$. The mere existence of such expansion is a famous theorem of Minakshisundaram and Plejel (among all approaches to that theorem, a particularly detailed account is given in [31]; even though Theorem 3.24 there concerns only a special case $f = 1$, the proof of the formula stated below in the general case is very similar).

For every $f \in C^\infty(X)$, the Minakshisundaram-Plejel theorem asserts an existence of an asymptotic expansion

$$\mathrm{Tr}(M_f e^{t\Delta_g}) \approx (4\pi t)^{-\frac{d}{2}} \cdot \sum_{n\geq 0} a_n(f)t^n, \quad t \downarrow 0. \tag{2}$$

Here, d is the dimension of X and $M_f : L_2(X) \to L_2(X)$ is the operator of pointwise multiplication by f. Moreover, there exist functions $A_k \in C^\infty(X)$ such that

$$a_k(f) = \int_X A_k \cdot f d\mathrm{vol}_g, \tag{3}$$

where vol_g is the Riemannian volume on X.

As follows from (1), $A_0 = 1$. Further computations (see e.g. Proposition 3.29 in [31]) show that $A_1 = \frac{1}{6}R$, where R is the scalar curvature. In particular, $a_1(1)$ is the Einstein-Hilbert action (see e.g. [8]).

Note that a_0 extends to a normal state h on $L_\infty(X)$ by the obvious formula

$$h(f) = \int_X f d\mathrm{vol}_g, \quad f \in L_\infty(X).$$

Equation (3) can be re-written as

$$a_k(f) = h(A_k \cdot f), \quad f \in C^\infty(X).$$

One of the *primary targets* of this project is to find suitable extensions of the Minakshisundaram-Plejel theorem (and, consequently, of the Weyl theorem — see formula (1)) for a vast class of non-commutative manifolds (such as non-commutative tori with generic, non-flat, metric tensor). This grand program began in [17] (published only in 2011, but the main concepts and techniques were developed yet in the 1990's), where special $2-$dimensional non-commutative manifolds (conformal deformations of a flat non-commutative torus) were considered. The authors of [17] proved that Euler characteristic of such manifold is 0 by means of Gauss-Bonnet theorem (recall that the classical Gauss-Bonnet theorem asserts that Euler characterstic of the $2-$dimensional Riemannian manifold equals to the average of its scalar curvature). Subsequently, the scalar curvature (for the conformal deformation of the $2-$dimensional non-commutative torus) was explicitly computed in [15] and [19] and, later, the term a_2 (the first place where the Riemann curvature tensor manifests itself beyond the scalar curvature) was further computed in [12] (intermediate computations include about a million terms!). Below, we briefly restate the whole programme as it can be surmised from [17].

For a given d-dimensional spectral triple $(\mathscr{A}, H, D)$, the steps needed to be done in the program are as follows (for brevity we assume $\mathscr{A}$ to be unital and omit π from the formulae below):

1. to verify that the following strong generalisation of (2) holds

$$\mathrm{Tr}(xe^{-tD^2}) \approx (4\pi t)^{-\frac{d}{2}} \sum_{n\geq 0} a_n(x)t^n, \quad t \downarrow 0, \quad x \in \mathscr{A}''.$$

2. to verify the normality of coefficients as in (3) with respect to the volume state, or, more precisely, to show that

$$a_k(x) = h(xA_k), \quad x \in \mathscr{A}'', \text{ for some } A_k \in \mathscr{A}'', \quad k \geq 0.$$

3. to compute A_k explicitly.

When this mission is accomplished, one can *define* a scalar curvature of a non-commutative manifold by setting $R = 6A_1$.

We acknowledge that this program sets reasonable objectives but we claim that the tools based on on pseudo-differential calculus in various guises applied up-to-date to accomplish them have been inadequate. Our main technical innovation which we bring here is based on the novel Double Operator Integration techniques developed by the CI recently in close collaboration with Alain Connes and Fedor Sukochev. The particular integral representations (see Approach to Aims 1,2,3 below) arose in [16] when geometric measures on limit sets of Quasi-Fuchsian groups were recovered by means of singular traces. They form the core contribution of UNSW team (including the CI) to [16].

One of the fundamental tools in noncommutative geometry is the Chern character. The Connes Character Formula provides an expression for the class of the Chern character in Hochschild cohomology, and it is an important tool in the computation of the Chern character. The formula has been applied to many areas of noncommutative geometry and its applications: such as the local index formula [14], the spectral characterisation of manifolds [10] and recent work in mathematical physics [11]. We point out to its applications in [10] as particularly relevant to the theme of this application.

In its original formulation, [7], the Character Formula is stated as follows: Let $(\mathscr{A}, H, D)$ be a p-dimensional compact spectral triple with (possibly trivial) grading Γ. By the definition of a spectral triple, for all $a \in \mathscr{A}$ the commutator $[D,a]$ has an extension to a bounded operator $\partial(a)$ on H. Assume for simplicity that $\ker(D) = \{0\}$ and set $F = \mathrm{sgn}(D)$. For all $a \in \mathscr{A}$ the commutator $[F,a]$ is a compact operator in the weak Schatten ideal $\mathscr{L}_{p,\infty}$ (see e.g. [8, 26]).

Consider the following two linear maps on the algebraic tensor power $\mathscr{A}^{\otimes(p+1)}$, defined on an elementary tensor $c = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in \mathscr{A}^{\otimes(p+1)}$ by setting

$$\mathrm{Ch}(c) := \frac{1}{2}\mathrm{Tr}(\Gamma F[F,a_0][F,a_1]\cdots[F,a_p]), \quad \Omega(c) := \Gamma a_0 \partial a_1 \partial a_2 \cdots \partial a_p.$$

Then the Connes Character Formula states that if c is a Hochschild cycle then

$$\mathrm{Tr}_\omega(\Omega(c)(1+D^2)^{-p/2}) = \mathrm{Ch}(c)$$

for every Dixmier trace Tr_ω. In other words, the multilinear maps Ch and $c \mapsto \mathrm{Tr}_\omega(\Omega(c)(1+D^2)^{-p/2})$ define the same class in Hochschild cohomology. We mention, in passing, that the generality of the result just stated was achieved fairly recently by the CI and his co-authors, see [4].

There has been great interest in generalising the tools and results of noncommutative geometry to the "non-compact"(i.e., non-unital) setting. The definition of a spectral triple associated to a non-unital algebra originates with Connes [9], was furthered by the work of Rennie [30] and Gayral, Gracia-Bondía, Iochum, Schücker and Varilly [21]. Earlier, similar ideas appeared in the work of Baaj and Julg [1]. Additional contributions to this area were made by Carey, Gayral, Rennie and Sukochev [3]. The conventional definition of a non-compact spectral triple is to replace the condition that $(D+i)^{-1}$ be compact with the assumption that for all $a \in \mathscr{A}$ the operator $a(D+i)^{-1}$ is compact. This raises an important question: is the Connes Character Formula true for locally compact spectral triples? This question was suggested to the CI by Professor Connes himself during Shanghai conference celebrating the 70-th aniversary of A. Connes (held at Fudan University). During lengthy discussions covering a substantial range of topics of importance in non-commutative geometry, Professor Connes, in particular had emphasized to the CI that such an extension could be an excellent starting point for several new directions in non-commutative geometry. Here are two such directions. Firstly, the Connes Character Formula plays an important role in the reconstruction theorem for closed Riemannian manifolds, and it is natural to expect that its extension would play a similar role for locally compact Rimannian manifolds. Secondly, developing suitable new techniques needed for a self-contained theory of locally compact spectral triples should also open an avenue for treating the case of noncommutative manifolds with boundary or even incomplete (e.g. punctured) manifolds. This suggestion by Professor Connes has been taken seriously by the CI and preliminary work in this direction has already brought substantial fruits. The ground-breaking manuscript co-authored by Professor Connes and the CI (and a number of collaborators from UNSW) the new approach to spectral triples involving *symmetric, non-self-adjoint* operators has been proposed [13]. These are precisely the required tools allowing the possibility to develop a new theory for non-commutative Riemannian manifolds with boundary. This development indicates the importance and timeliness of the present proposal which has already achieved substantial progress and leads to a unified theory of locally compact non-commutative manifolds with boundary and incomplete manifolds. We emphasize that the progress achieved involves joint work with luminaries like Alain Connes, and local top experts in noncommutative analysis and geometry like Alan Carey, Adam Rennie and Fedor Sukochev.

Significance of the project This project intertwines the small scale geometry (studies in terms of calculus and differential equations) with global geometry (which aims to comprehend the shape of a manifold). The significance of the specific Aims proposed below is to establish links between different fields of mathematics (in particular, between Operator Algebras and Differential Geometry).

Heat semi-group in the classical Differential Geometry provides a solution of a parabolic PDE. For non-commutative manifold, heat semi-group essentially delivers the time evolution of an irreversible quantum system whose Hamiltonian is a (non-commutative) Laplace-Beltrami operator.

The project will concentrate on non-commutative geometries with high degree of symmetry provided by quantum groups, the feature which is paramount both in Mathematics and Physics. The research impact of this project will be enhancement of the Australian profile in the crucial area of Quantised Calculus and its applications to Non-commutative Geometry. The project will maintain and foster international collaborative links already built by the CI (Connes, Junge, Higson, Dykema) simultaneously contributing to training the next generation of Australian mathematicians. Certain parts of this proposal were discussed in the past 3 years with each of just listed collaborators in order to ensure that Aims below are cutting edge research.

Generally speaking, the development of noncommutative geometry and its applications is hindered by the paucity of non-trivial examples demonstrating its richness and ability to *compute* quantities of geometric significance for genuinely non-commutative manifolds. The project will generate an original approach to this computational task which will interact with and improve on the work of top class practitioners of Non-commutative Analysis/Geometry such as Marius Junge (University of Illinois) and Nigel Higson (Penn State University). The CI is in regular contact with both (they both are to visit UNSW in 2019) and this collaboration is to continue unabated.

The project will have scholarly impact on the crucial parts of Non-commutative Differential Geometry with links to several other areas of Mathematics.

Aims of the project There are 7 specific aims.

Aim 1: Investigate when Chern character provides an asymptotic expansion for the heat semi-group (arising from a locally compact spectral triple). More precisely, when

$$\mathrm{Tr}(\Omega(c)e^{-s^2D^2}) = \mathrm{Ch}(c)s^{-p} + O(s^{1-p}), \quad s \downarrow 0, \tag{4}$$

for every Hochschild cycle $c \in \mathscr{A}^{\otimes(p+1)}$? It seems plausible that Hochschild cochain on the left hand side is cohomologous to the one in the right hand side (modulo $O(s^{1-p})$). We expect to achieve Aim 1 during Year 1.

Aim 2: Aim 1 above is closely related (albeit, not equivalent) to a question concerning analyticity of a suitable $\zeta-$function. The latter is an analytic function defined by the formula

$$z \to \mathrm{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}), \quad \Re(z) > p.$$

We aim to find an analytic extension to the half-plane $\Re(z) > p-1$ so that

$$\lim_{z\to p}(z-p)\mathrm{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}) = p\mathrm{Ch}(c).$$

Aim 3: To compute the Hochschild class of the Chern character by a "local" formula, which is customarily stated in terms of singular traces on the ideal $\mathscr{L}_{1,\infty}$ ($\mathscr{L}_{1,\infty}$ is the principal ideal in $B(H)$ generated by an operator with singular values $(\frac{1}{k})_{k\geq 1}$). Here, trace $\varphi : \mathscr{L}_{1,\infty} \to \mathbb{C}$ is a unitarily invariant linear functional; it can be seen from CI's results in [26] that such a functional is automatically singular. Our third aim is to show that

$$\varphi(\Omega(c)(1+D^2)^{-\frac{p}{2}}) = \mathrm{Ch}(c). \tag{5}$$

for every (normalised) trace φ on $\mathscr{L}_{1,\infty}$ and for every Hochschild cycle $c \in \mathscr{A}^{\otimes(p+1)}$.

Aims 2 and 3 are intimately connected with Aim 1, however, the amount of analytical complications which arise when one navigates between formulas stated in all three aims is enormous and requires a very careful treatment and certainly warrants a separation of these aims.

We rely on the the deep theory of singular traces and its connections with operator $\zeta-$functions which the CI (with various collaborators) has been developing since 2009 [26,32]. The CI is in the unique position to apply numerous and well-developed techniques from that theory in order to achieve this aim. We expect to achieve Aim 2 during Year 2.

Aim 4: Introduce a generic (i.e., not necessarily a conformal deformation of a flat one) Laplace-Beltrami operator

$$\Delta_g x = M_{G^{-\frac{1}{2}}} \sum_{i,j=1}^{d} D_i M_{G^{\frac{1}{4}}(g^{-1})_{ij}G^{\frac{1}{4}}} D_j \text{ where } G^{-\frac{1}{2}} = (\det(g_{ij}))^{-\frac{1}{2}} \overset{def}{=} \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\sum_{i,j} g_{ij}t_it_j} dt,$$

on the non-commutative torus and non-commutative Euclidean space. Prove a non-commutative version of Minakshisundaram-Plejel theorem (for these manifolds) as outlined above.

Aim 4, in turn, depends on Aims 1 and 2, or rather on the technical instruments which are to be developed to deal with those aims in full generality.

Aim 5: Compute explicitly the curvature for a generic Riemannian metric on the non-commutative torus and non-commutative Euclidean space.

By using Double Operator Integration technique as developed in [29], we aim to prove analyticity of the right hand side and, hence, of the left hand side. The application of these techniques is our trump card in the joint work with Professor Connes [16], in which we have completed the work started by such giants as Connes and Sullivan, and which left dormant for more than 20 years, until our new techniques were brought to bear on that problem.

Aim 6: Construct a spectral triple (or possibly, a twisted spectral triple) on quantum groups (like $\mathrm{SU}_q(n)$) and on their homogeneous spaces (like Podleś sphere). Previous attempts [5] suffer from a "dimension drop" pathology (that is, where spectral dimension differs from cohomological dimension). We aim to have an equivariance property for the Dirac operator in a way that prevents "dimension drop" pathology.

The spectral triple constructed in [5] is equivariant and its spectral dimension is 3. However, every $3-$cocycle is cohomologous to 0 and, therefore, the homological dimension is strictly less than 3.

We expect that the reason for this phenomenon is the wrong choice of $q-$deformed left regular representation. Our aim is to find a suitable $q-$deformation which permit the spectral triple to be equivariant, $3-$dimensional and, at the same time to allow certain $3-$cocycles to be non-trivial. A natural candidate for such a $3-$cocycle is the Hochschild class of the Chern Character. Successful resolution of Aims 1,2,3 will deliver the required technical tools to determine the "correct representation" and such cocycles.

Aim 7: Design a version of Connes Character Formula for the above spectral triples which recovers a non-trivial cocycle.

It is extremely probable that ordinary spectral triples in this setting should be replaced by twisted ones (where commutators $[D,a]$ may be unbounded, but a certain "twisted" commutator is bounded). At the moment, no satisfactory theory is available which allows the derivation of a Connes Character formula for such triples. The only attempt is made in [18]. However, we expect that a combination of approaches from [18] and [4] would yield (at least a germ of) such a theory.

Benefit

The CI expects the project to produce significant results and to publish them in the most reputed journals. In addition, the CI expects the project to be beneficial in the following ways.

a. Australia has strong research profile in Non-Commutative Geometry and Operator Algebras. This area develops rapidly and the CI enthusiastically contributes to this development. The outcomes of this project will therefore be of interest to a number of research groups in Australia (Wollongong, Adelaide, Sydney and Canberra). The project will broaden existing strengths and introduce new directions in a highly active and internationally competitive area of research endeavour.

b. The project will enhance international collaboration in research. The CI is collaborating with highly-distinguished international experts (Alain Connes, Kenneth Dykema, Nigel Higson, Marius Junge). This collaboration already resulted in a number of papers in prestigious journals. Involvement of mathematicians of such a calibre would be beneficial not only for this project, but also to other directions of mathematical research in UNSW.

It should be pointed out that Professor Connes is very enthusiastic about the suggested direction of research and this strong endorsement from the world leading expert and his on-going commitment to the joint research efforts is a strong acknowledgement of the depth and importance of the current research proposal.

c. Improve the international competitiveness of Australian research. The area of Quantised Calculus is the cutting edge research toolkit in the modern analysis. This proposal is on par with efforts of top world experts in the area and thus will strengthen the Australian leadership in this area.

d. Due to the breadth and depth of the proposal which involves groundbreaking research in several mathematical disciplines, the CI expects to attract the top Australian students for postgraduate studies who otherwise might be heading overseas.

Feasibility and Strategic Alignment The CI has already demonstrated his capacity to make significant, original and innovative contributions to wide range of Non-commutative Analysis and Non-commutative Geometry. He has co-authored the very first monograph on singular traces [26], which has albeit in embrionic form, some necessary components to attack Aim 3. The project presents a realistic timeframe as seen from above. Confidence in the feasibility is enhanced by the following:

1. The CI has already demonstrated his research credentials in the field of Noncommutative Analysis. His numerous papers in this area are published in the highly ranked journals like Crelle's Journal, Advances in Mathematics, Journal of Functional Analysis.

2. The CI wrote a number of publications in the field of Mathematical Physics. For example, paper [34] is published in prestigious journal Communications in Mathematical Physics.

3. The CI has co-authored a number of publications in the field of Classical Analysis.

4. The CI has authored a research monograph [26] jointly written with S. Lord and F. Sukochev. The substantial portion of this monograph describes CI's contribution to the field of Non-commutative Geometry and related parts of Quantised Calculus. At the moment, a second edition of the book is in preparation.

5. The CI can rely on support and expert advice from the members of the Noncommutative Analysis group at UNSW (M. Cowling, I. Doust, D. Potapov, F. Sukochev).

It should also be pointed out that some parts of the current proposal have been thoroughly discussed with Professor Alain Connes (Fields medalist) who is the originator of (and leading expert in) Non-commutative Geometry. Professor Connes strongly and enthusiastically endorsed the ideas and methods underlying this proposal and the approach.

Communication of results Results of this project will be published in peer-reviewed journals (as well as on arxiv). CI and collaborators will also deliver the results in thematic international conferences.

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