The Connes character formula for locally compact spectral triples

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Dedicated to Alain Connes for his 70-th anniversary. With gratitude and admiration.

Abstract. A fundamental tool in noncommutative geometry is Connes' character formula. This formula is used in an essential way in the applications of noncommutative geometry to index theory and to the spectral characterisation of manifolds.

A non-compact space is modelled in noncommutative geometry by a non-unital spectral triple. Our aim is to establish the Connes character formula for non-unital spectral triples. This is significantly more difficult than in the unital case and we achieve it with the use of recently developed double operator integration techniques. Previously, only partial extensions of Connes' character formula to the non-unital case were known.

In the course of the proof, we establish two more results of importance in noncommutative geometry: an asymptotic for the heat semigroup of a non-unital spectral triple, and the analyticity of the associated ζ-function.

We require certain assumptions on the underlying spectral triple, and we verify these assumptions in the case of spectral triples associated to arbitrary complete Riemannian manifolds and also in the case of Moyal planes.

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CHAPTER 1

Introduction

Acknowledgements

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1.1. Introduction

One of the fundamental tools in noncommutative geometry is the Chern character. The Connes Character Formula (also known as the Hochschild character theorem) provides an expression for the class of the Chern character in Hochschild cohomology, and it is an important tool in the computation of the Chern character. The formula has been applied to many areas of noncommutative geometry and its applications: such as the local index formula [**24**], the spectral characterisation of manifolds [**23**] and recent work in mathematical physics [**16**].

In its original formulation, [**18**], the Character Formula is stated as follows: Let $(\mathcal{A}, H, D)$ be a p-summable compact spectral triple with (possibly trivial) grading Γ (as defined in Section 2.2). By the definition of a spectral triple, for all $a \in \mathcal{A}$ the commutator $[D, a]$ has an extension to a bounded operator $\partial(a)$ on H. Furthermore, if $F = \chi_{(0,\infty)}(D) - \chi_{(-\infty,0)}(D)$ then for all $a \in \mathcal{A}$ the commutator $[F, a]$ is a compact operator in the weak Schatten ideal $\mathcal{L}_{p,\infty}$. For simplicity assume that $\ker(D) = \{0\}$, and now consider the following two linear maps on the algebraic tensor power $\mathcal{A}^{\otimes(p+1)}$, defined on an elementary tensor $c = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$ by,

$$\mathrm{Ch}(c) := \frac{1}{2}\mathrm{Tr}(\Gamma F[F, a_0][F, a_1] \cdots [F, a_p])$$

and

$$\Omega(c) := \Gamma a_0 \partial a_1 \partial a_2 \cdots \partial a_p.$$

Then the Connes Character Formula states that if c is a Hochschild cycle (as defined in Section 2.2.4) then

$$\mathrm{Tr}_\omega(\Omega(c)(1 + D^2)^{-p/2}) = \mathrm{Ch}(c)$$

for every Dixmier trace Tr_ω. In other words, the multilinear maps Ch and $c \mapsto \mathrm{Tr}_\omega(\Omega(c)(1 + D^2)^{-p/2})$ define the same class in Hochschild cohomology.

There has been great interest in generalising the tools and results of noncommutative geometry to the "non-compact"(i.e., non-unital) setting. The definition of a spectral triple associated to a non-unital algebra originates with Connes [**22**], was furthered by the work of Rennie [**54, 55**] and Gayral, Gracia-Bondía, Iochum, Schücker and Varilly [**31**]. Earlier, similar ideas appeared in the work of Baaj and Julg [**1**]. Additional contributions to this area were made by Carey, Gayral, Rennie, and the first named author [**10, 11**]. The conventional definition of a non-compact spectral triple is to replace the condition that $(1+D^2)^{-1/2}$ be compact with the assumption that for all $a \in \mathcal{A}$ the operator $a(1+D^2)^{-1/2}$ is compact.

This raises an important question: is the Connes Character Formula true for locally compact spectral triples?

In this paper we are able to provide an affirmative answer to this question, provided that one assumes certain regularity properties on the spectral triple. There is a substantial difference between the theories of compact and non-compact spectral triples, in particular issues pertaining to summability are more subtle. We achieve our proof of the non-unital Character Formula using recently developed techniques of operator integration.

1.2. The main results

In this paper we prove three key theorems (Theorems 1.2.2, 1.2.3 and 1.2.5) and a new result concerning universal measurability (Theorem 1.2.7).

Essential to our approach is a certain set of assumptions on a spectral triple to be outlined below. The notion of a spectral triple, and all of the corresponding notations are explained fully in Section 2.2. By definition, if $(\mathcal{A}, H, D)$ is a spectral triple then for $a \in \mathcal{A}$, the notation $\partial(a)$ denotes the bounded extension of the commutator $[D, a]$, and for an operator T on H which preserves the domain of D, $\delta(T)$ denotes the bounded extension of $[|D|, T]$ when it exists. The notation $\mathcal{L}_{r,\infty}$, $r \geq 1$, denotes the ideal of compact operators T whose singular value sequence $\{\mu(n,T)\}_{n=0}^{\infty}$ satisfies $\mu(n,T) = O(n^{-1/r})$. The norm $\|\cdot\|_1$ is the trace-class norm.

Our main assumption on $(\mathcal{A}, H, D)$ is as follows:

HYPOTHESIS 1.2.1. *The spectral triple $(\mathcal{A}, H, D)$ satisfies the following conditions:*

(i) $(\mathcal{A}, H, D)$ *is a smooth spectral triple.*

(ii) *There exists $p \in \mathbb{N}$ such that $(\mathcal{A}, H, D)$ is $p-$dimensional, i.e., for every $a \in \mathcal{A}$,*

$$a(D+i)^{-p} \in \mathcal{L}_{1,\infty},$$
$$\partial(a)(D+i)^{-p} \in \mathcal{L}_{1,\infty}.$$

(iii) *for every $a \in \mathcal{A}$ and for all $k \geq 0$, we have*

$$\left\|\delta^k(a)(D+i\lambda)^{-p-1}\right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty,$$
$$\left\|\delta^k(\partial(a))(D+i\lambda)^{-p-1}\right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty.$$

Condition 1.2.1.(i) is well known and widely used in the literature. The notion of "smoothness" that we use here is identical to what is sometimes referred to as QC^∞ (see Definition 2.2.7).

Condition 1.2.1.(ii) is also widely used, but we caution the reader that elsewhere in the literature an alternative definition of dimension is often used: where $(\mathcal{A}, H, D)$

is said to be p-dimensional if for all $a \in \mathcal{A}$ we have $a(D+i)^{-1} \in \mathcal{L}_{p,\infty}$ and $\partial(a)(D+i)^{-1} \in \mathcal{L}_{p,\infty}$. The definition of dimension in 1.2.1.(ii) is strictly stronger, and we discuss this issue in 2.2.3.

Condition 1.2.1.(iii) is new and specific to the locally compact situation. Indeed, if $\mathcal{A}$ is unital then 1.2.1.(iii) is redundant, as it follows from 1.2.1.(ii).

In order to show that Hypothesis 1.2.1.(iii) is reasonable, we prove that it is satisfied for spectral triples associated to the following two classes of examples:

(i) Noncommutative Euclidean spaces, a.k.a. Moyal spaces. (Section 3.3)
(ii) Complete Riemannian manifolds. (Section 3.4).

In deciding on the conditions of Hypothesis 1.2.1, we have avoided the assumption that the spectral dimension of $(\mathcal{A}, H, D)$ is isolated: this is an assumption made in [**34**], [**24**] and in some parts of [**11**].

Our first main result is established in Section 4.5. This result provides an asymptotic estimate of the trace of the heat operator $s \mapsto e^{-s^2D^2}$, and we remark that the following theorem is new even in the compact case.

Theorem 1.2.2. *Let $p \in \mathbb{N}$ and let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a Hochschild cycle, then*

$$\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2D^2}) = \frac{p}{2}\mathrm{Ch}(c)s^{-2} + O(s^{-1}), \quad s \downarrow 0. \tag{1.1}$$

Note that we do not require that the parity of the dimension of p match the parity of the spectral triple (i.e., p can be an odd integer while $(\mathcal{A}, H, D)$ has a nontrivial grading, and similarly p can be even while $(\mathcal{A}, H, D)$ has no grading).

Our second main result proves the the analytic continuation of the $\zeta-$function associated with the operator $(1+D^2)^{-\frac{1}{2}}$. This result recovers all previous results concerning the residue of the ζ function on a Hochschild cycle.

Theorem 1.2.3. *Let $p \in \mathbb{N}$ and let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a Hochschild cycle, then the function*

$$\zeta_{c,D}(z) := \mathrm{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}), \quad \Re(z) > p, \tag{1.2}$$

is holomorphic, and has analytic continuation to the set $\{\Re(z) > p-1\} \setminus \{p\}$. The point $z = p$ is a simple pole of the analytic continuation of $\zeta_{c,D}$, with corresponding residue equal to $p\mathrm{Ch}(c)$.

To prove our analogue of the Character Theorem in the unital setting, we require an additional *locality* assumption on the Hochschild cycle c. The use of locality in noncommutative geometry was pioneered by Rennie in [**55**].

Definition 1.2.4. *A Hochschild cycle $c = \sum_{j=1}^m a_0^j \otimes \cdots \otimes a_p^j \in \mathcal{A}^{\otimes(p+1)}$ is said to be local if there exists a positive element $\phi \in \mathcal{A}$ such that $\phi a_0^j = a_0^j$ for all $1 \le j \le m$.*

For example, if X is a manifold and $\mathcal{A} = C_c^\infty(X)$ is the algebra of smooth compactly supported functions on X, then every Hochschild cycle is local since we may choose ϕ to be smooth and equal to 1 on the union of the supports of $\{a_0^j\}_{j=1}^m$.

Our final result is the Connes Character Formula for locally compact spectral triples. In the compact case, our result recovers all previous results of this type (e.g. [**33**, Theorem 10.32], [**2**, Theorem 6], [**12**, Theorem 10] and [**15**, Theorem 16]).

Theorem 1.2.5. *Let $p \in \mathbb{N}$ and let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a local Hochschild cycle, then*

$$\varphi(\Omega(c)(1+D^2)^{-\frac{p}{2}}) = \mathrm{Ch}(c). \tag{1.3}$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

The notion of a normalised trace on $\mathcal{L}_{1,\infty}$ is recalled in Subsection 2.1.3. The purpose of the Connes Character Formula is to compute the Hochschild class of the Chern character by a "local" formula, here stated in terms of singular traces.

A consequence of Theorem 1.2.5 being stated for arbitrary normalised traces on $\mathcal{L}_{1,\infty}$ is that we can deduce precise behaviour of the distribution of eigenvalues of the operator $\Omega(c)(1+D^2)^{-p/2}$:

Corollary 1.2.6. *Let $(\mathcal{A}, H, D)$ satisfy Hypothesis 1.2.1, and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a local Hochschild cycle. Then the sequence $\{\lambda(k, \Omega(c)(1+D^2)^{-p/2})\}_{k=0}^{\infty}$ of eigenvalues of the operator $\Omega(c)(1+D^2)^{-p/2}$ arranged in non-increasing absolute value satisfies:*

$$\sum_{k=0}^{n} \lambda(k, \Omega(c)(1+D^2)^{-p/2}) = \mathrm{Ch}(c)\log(n) + O(1), \quad n \to \infty.$$

The above corollary is an immediate consequence of Theorem 1.2.5 and Theorem 2.1.5.

The main technical innovation of this paper concerns a certain integral representation for the difference of complex powers of positive operators, which originally appeared in [**25**] and which is reproduced here as Theorem 5.2.1.

An operator $T \in \mathcal{L}_{1,\infty}$ is called universally measurable if all normalised traces on $\mathcal{L}_{1,\infty}$ take the same value on T. A new result of this paper, and a crucial component of our proof of Theorem 1.2.5, is the following:

Theorem 1.2.7. *Let $0 \leq V \in \mathcal{L}_{1,\infty}$ and let $A \in \mathcal{L}_{\infty}$. Define the ζ-function:*

$$\zeta_{A,V}(z) := \mathrm{Tr}(AV^{1+z}), \quad \Re(z) > 0.$$

If there exists $\varepsilon > 0$ such that $\zeta_{A,V}$ admits an analytic continuation to the set $\{z : \Re(z) > -\varepsilon\} \setminus \{0\}$ with a simple pole at 0, then for every normalised trace φ on $\mathcal{L}_{1,\infty}$ we have:

$$\varphi(AV) = \mathrm{Res}_{z=0}\zeta_{A,V}(z).$$

In particular, AV is universally measurable.

Theorem 1.2.7 is a strengthening of an earlier result [**63**, Theorem 4.13], and a complete proof is given in Section 5.5.

1.3. Context of this paper

Connes' Character Formula dates back to Connes' 1995 paper [**18**]. There the character theorem was discovered in order to "compute by a local formula the cyclic cohomology Chern character of $(\mathcal{A}, H, D)$". Connes' work initiated a lengthy and ongoing program to strengthen, generalise and better understand the Character Formula.

Closely linked to the Character Formula is the Local Index Theorem of Connes and Moscovici [**24**], and much of the work in this field was from the point of view of index theory. Among the approaches to generalising Connes character theorem, there is [**2**] by Benamuer and Fack, and [**12**] by Carey, Philips, Rennie and the first named author.

Instead of considering traces on $\mathcal{L}_{1,\infty}$, [**12**] deals with Dixmier traces on the Lorentz space $\mathcal{M}_{1,\infty}$. Due to an error in the statement of Lemma 14 of [**12**] which invalidates the proof in the $p = 1$ case, a followup paper [**15**] was written. In [**15**], the Character Formula is proved in the compact case for arbitrary normalised traces (rather than Dixmier traces).

During the creation of the present manuscript an oversight was located in [**15**]: in that paper the case where D has a nontrivial kernel and $(\mathcal{A}, H, D)$ is even was not handled correctly. It was incorrectly assumed in [**15**, Case 3, page 20] that if $(\mathcal{A}, H, D)$ is an even spectral triple with grading Γ, then so is

$$(\mathcal{A}, H, (\chi_{[0,\infty)}(D) - \chi_{(-\infty,0)}(D))(1 + |D|^2)^{1/2}).$$

This is false if the kernel of D is nontrivial, since then it is not necessarily the case that $\chi_{[0,\infty)}(D) - \chi_{(-\infty,0)}(D)$ anticommutes with Γ. The outcome of this oversight is that the proof of the Character Theorem as given in [**15**] is incomplete. This oversight can be corrected by using the well known "doubling trick" that was already present in [**12**, Definition 6]. The present work supersedes that of [**15**], and so rather than submit an erratum we have decided to instead supply a complete proof here, in a more general setting.

All of the work mentioned so far in this section applies exclusively in the compact case. Adapting the tools of noncommutative geometry to the locally compact case involves substantial difficulties and this task has been heavily studied by multiple authors over the past few decades: as a small sample of this body of work we mention [**54, 55, 31, 32, 10, 11**] and more recently work by Marius Junge and Li Gao concerning noncommutative planes.

In 2000 Professor Nigel Higson published [**34**]: a detailed exposition of the local index theorem, including in the final appendix a claimed proof of the Connes Character Formula in the non-unital setting. Higson's work was a major inspiration for the present paper, since it is now understood and acknowledged by Higson that the claimed proof of the Character Formula [**34**, Theorem C.3] has a gap. This paper arose from our efforts to produce a correct statement and complete proof of the Character Formula in the non-unital setting using recently developed methods of Double Operator Integration theory.

The nature of the gap in [**34**] is subtle, and concerns the relationship between Dixmier traces and zeta-function residues. To be precise: the proof relied on an equality between

$$\lim_{s\downarrow 0} \mathrm{Tr}(Z|D|^{-n-s})$$

and

$$\mathrm{Tr}_\omega(Z|D|^{-n})$$

(in the notation of [**34**]). In the case where $|D|^{-1}$ is compact this result can be attained using existing techniques from [**45**, Theorem 8.6.4, Theorem 8.6.5 and Theorem 9.3.1]. In the case where $|D|^{-1}$ is not compact the situation is less well understood. The present text was motivated by an effort to understand the equality above in the non-compact case.

After circulating a draft of our manuscript Carey and Rennie pointed out that there was a different way to obtain a similar result on the Hochschild class using

[**11**] (which is based on [**14**]). It is proved in these papers that the "resolvent cocycle" introduced there represents the cohomology class of the Chern character. From that point of view one may obtain a different representative of the Hochschild class of the Chern character using residues of zeta functions under weaker hypotheses on the Hochschild chains and substantially stronger summability conditions on the spectral triple. For Hochschild chains satisfying some additional conditions, but not requiring locality as employed here, Carey and Rennie also have a Dixmier trace formula for the Hochschild class of the Chern character evaluated on such Hochschild chains.

1.4. Structure of the paper

This paper is structured as follows:

- Chapter 2 is devoted to preliminary definitions and concepts: we introduce the relevant definitions for operator ideals, traces, spectral triples, operator valued integrals and double operator integrals.
- Chapter 3 provides important technical properties of spectral triples. In Section 3.3 we prove that Hypothesis 1.2.1 is satisfied for the canonical spectral triple associated to noncommutative Euclidean spaces $\mathbb{R}^p_\theta$, and in Section 3.4 we show that the Hypothesis is satisfied for Hodge-Dirac spectral triples associated to arbitrary complete Riemannian manifolds.
- Chapter 4 contains the proof of Theorem 1.2.2.
- Chapter 5 contains the proofs of Theorems 1.2.3, 1.2.7 and 1.2.5.
- Finally, an appendix is included to collect some of the lengthier computations.

CHAPTER 2

Preliminaries

2.1. Operators, ideals and traces

2.1.1. General notation. Fix throughout a separable, infinite dimensional complex Hilbert space H. We denote by $\mathcal{L}_\infty$ the algebra of all bounded operators on H, with operator norm denoted $\|\cdot\|_\infty$. For a compact operator T on H, let $\lambda(T) := \{\lambda(k,T)\}_{k=0}^\infty$ denote the sequence of eigenvalues of T arranged in order of non-increasing magnitude and with multiplicities. Similarly, let $\mu(T) := \{\mu(k,T)\}_{k=0}^\infty$ denote the sequence of singular values of T, also arranged in non-increasing order with multiplicities. The kth singular value may be described equivalently as either $\mu(k,T) := \lambda(k,|T|)$ or

$$\mu(k,T) = \inf\{\|T-R\|_\infty \;:\; \mathrm{rank}(R) \leq k\}.$$

The standard trace on $\mathcal{L}_\infty$ (more precisely on the trace-class ideal) is denoted Tr.

Fix an orthonormal basis $\{e_k\}_{k=0}^\infty$ on H (the particular choice of basis is inessential). We identify the algebra ℓ_∞ of all bounded sequences with the subalgebra of diagonal operators on H with respect to the chosen basis. For a given $\alpha \in \ell_\infty$, we denote the corresponding diagonal operator by $\mathrm{diag}(\alpha)$.

For $A, B \in \mathcal{L}_\infty$, we say that B is submajorized by A in the sense of Hardy-Littlewood, written as $B \prec\prec A$, if

$$\sum_{k=0}^n \mu(k,B) \leq \sum_{k=0}^n \mu(k,A), \quad n \geq 0.$$

We say that B is logarithmically submajorized by A, written as $B \prec\prec_{\log} A$ if

$$\prod_{k=0}^n \mu(k,B) \leq \prod_{k=0}^n \mu(k,A), \quad n \geq 0.$$

An important result concerning logarithmic submajorisation is the Araki-Lieb-Thirring inequality [**39**, Theorem 2], which states that for all positive bounded operators A and B and all $r \geq 1$,

$$|AB|^r \prec\prec_{\log} A^r B^r. \tag{2.1}$$

We make frequent use of the following commutator identity: if A and B are operators with B invertible, then

$$[B^{-1}, A] = -B^{-1}[B,A]B^{-1}. \tag{2.2}$$

We must take care to ensure that (2.2) remains valid when A and B are potentially unbounded. If A is bounded, then it is enough that $A : \mathrm{dom}(B) \to \mathrm{dom}(B)$.

2.1.2. Ideals in $\mathcal{L}_\infty$ and related inequalities. For $p \in (0,\infty)$, we let $\mathcal{L}_p$ denote the Schatten-von Neumann ideal of $\mathcal{L}_\infty$,

$$\mathcal{L}_p := \{T \in \mathcal{L}_\infty \ : \ \mu(T) \in \ell_p\}$$

where ℓ_p is the space of p-summable sequences. As usual, for $p \geq 1$ the ideal $\mathcal{L}_p$ is equipped with the norm

$$\|T\|_p := \left(\sum_{k=0}^{\infty} \mu(k,T)^p\right)^{1/p}.$$

Similarly, given $0 < p < \infty$, we denote by $\mathcal{L}_{p,\infty}$ the ideal in $\mathcal{L}_\infty$ defined by

$$\mathcal{L}_{p,\infty} := \{T \in \mathcal{L}_\infty \ : \ \sup_{k\geq 0} (1+k)^{1/p}\mu(k,T) < \infty\}.$$

Equivalently,

$$\mathcal{L}_{p,\infty} := \{T \in \mathcal{L}_\infty \ : \ \sup_{n\geq 0} n^{-p}\mathrm{Tr}(\chi_{(\frac{1}{n},\infty)}(|T|)) < \infty\}.$$

It is well known that the ideal $\mathcal{L}_{p,\infty}$ may be equipped with a quasi-norm given by the formula

$$\|T\|_{p,\infty} := \sup_{k\geq 0}(k+1)^{1/p}\mu(k,T), \quad T \in \mathcal{L}_{p,\infty}.$$

As is conventional, $\mathcal{L}_{\infty,\infty} := \mathcal{L}_\infty$.

We make use of the following Hölder inequality: let $p, p_1, p_2, \ldots, p_n \in (0,\infty]$ satisfy $\frac{1}{p} = \sum_{k=1}^n \frac{1}{p_k}$. If $A_k \in \mathcal{L}_{p_k,\infty}$ for all $k = 1,\ldots,n$, then $A_1A_2\cdots A_n \in \mathcal{L}_{p,\infty}$, with an inequality of norms:

$$\|A_1A_2\cdots A_n\|_{p,\infty} \leq c_{p_1,p_2,\ldots,p_n}\|A_1\|_{p_1,\infty}\|A_2\|_{p_2,\infty}\cdots\|A_n\|_{p_n,\infty} \tag{2.3}$$

where $c_{p_1,p_2,\ldots,p_n} > 0$.

The quasi-norm $\|\cdot\|_{1,\infty}$ is not monotone with respect to Hardy-Littlewood submajorisation. It is, however, monotone under logarithmic submajorisation. To be precise, we have that for all $A, B \in \mathcal{L}_{1,\infty}$ if $B \prec\prec_{\log} A$ then

$$\|B\|_{1,\infty} \leq e\|A\|_{1,\infty}. \tag{2.4}$$

Indeed, since the sequence $\{\mu(k,B)\}_{k=0}^\infty$ is nonincreasing, for all $n \geq 0$ we have:

$$\mu(n,B)^{n+1} \leq \prod_{k=0}^n \mu(k,B).$$

So if $B \prec\prec_{\log} A$,

$$\mu(n,B)^{n+1} \leq \prod_{k=0}^n \mu(k,A). \tag{2.5}$$

However by definition, $\mu(k,A) \leq \frac{\|A\|_{1,\infty}}{k+1}$ for all k, so

$$\prod_{k=0}^n \mu(k,A) \leq \frac{\|A\|_{1,\infty}^{n+1}}{(n+1)!}. \tag{2.6}$$

Now combining (2.5) and (2.6),

$$\mu(n,B)^{n+1} \leq \frac{\|A\|_{1,\infty}^{n+1}}{(n+1)!}.$$

Now using the Stirling approximation

$$(n+1)! \geq \big(\frac{n+1}{e}\big)^{n+1},$$

we arrive at

$$\mu(n,B)^{n+1} \leq \Big(\frac{e\|A\|_{1,\infty}}{n+1}\Big)^{n+1}.$$

Hence, for all $n \geq 0$,

$$\mu(n,B) \leq \frac{e\|A\|_{1,\infty}}{n+1}.$$

Multiplying by $n+1$, and then taking the supremum over n yields $\|B\|_{1,\infty} \leq e\|A\|_{1,\infty}$ as desired.

Another ideal to which we will refer is the Schatten-Lorentz ideal $\mathcal{L}_{q,1}$ for $q > 1$, defined by

$$\mathcal{L}_{q,1} := \{T \in \mathcal{L}_\infty \ : \ \sum_{k=0}^\infty \mu(k,T)(k+1)^{\frac{1}{q}-1} < \infty\}$$

and equipped with the quasi-norm

$$\|A\|_{q,1} := \sum_{k=0}^\infty \mu(k,A)(1+k)^{\frac{1}{q}-1}.$$

If $\frac{1}{p}+\frac{1}{q} = 1$, then we have the following Hölder-type inequality:

$$\|AB\|_1 \leq \|A\|_{p,\infty}\|B\|_{q,1}, \quad A \in \mathcal{L}_{p,\infty}, B \in \mathcal{L}_{q,1}. \tag{2.7}$$

2.1.3. Traces on $\mathcal{L}_{1,\infty}$.

Definition 2.1.1. *If $\mathcal{I}$ is an ideal in $\mathcal{L}_\infty$, then a unitarily invariant linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ is said to be a trace.*

Here φ being "unitarily invariant" means that $\varphi(U^*TU) = \varphi(T)$ for all $T \in \mathcal{I}$ and unitary operators U. Equivalently, $\varphi(UT) = \varphi(TU)$ for all unitary operators U and $T \in \mathcal{I}$. Since every bounded linear operator can be written as a linear combination of at most four unitary operators [**53**, Page 209], one may equivalently say that $\varphi(AT) = \varphi(TA)$ for all $A \in \mathcal{L}_\infty$ and $T \in \mathcal{I}$.

The most well-known example of a trace is the classical trace Tr on the ideal $\mathcal{L}_1$, however we will be primarily concerned with traces on the ideal $\mathcal{L}_{1,\infty}$. There exist many traces on $\mathcal{L}_{1,\infty}$, of which the earliest discovered class of examples are the Dixmier traces which we now describe.

Recall that an extended limit is a continuous linear functional $\omega \in \ell_\infty^*$ from the set of bounded sequences ℓ_∞ which extends the limit functional on the subspace c of convergent sequences. Readers who are more familiar with ultrafilters may consider the special case where ω is the limit along a non-principal (free) ultrafilter on $\mathbb{Z}_+$.

Example 2.1.2. *Let ω be an extended limit. Then the functional* Tr_ω *is defined on a positive operator $T \in \mathcal{L}_{1,\infty}$ by*

$$\mathrm{Tr}_\omega(T) := \omega\left(\left\{\frac{1}{\log(2+n)}\sum_{k=0}^n \mu(k,T)\right\}_{n=0}^\infty\right).$$

The functional Tr_ω *is additive on the cone of positive elements of* $\mathcal{L}_{1,\infty}$*, and therefore extends by linearity to a a functional on* $\mathcal{L}_{1,\infty}$*. The thus defined functional* $\mathrm{Tr}_\omega : \mathcal{L}_{1,\infty} \to \mathbb{C}$ *is a trace, and we call such traces Dixmier traces.*

Proof. Let A and B be positive operators. Combining [**45**, Theorem 3.3.3, Theorem 3.3.4], for all $n \geq 0$ we have:

$$\sum_{k=0}^{n} \mu(k, A+B) \leq \sum_{k=0}^{n} \mu(k, A) + \mu(k, B) \leq \sum_{k=0}^{2n+1} \mu(k, A+B).$$

Hence,

$$0 \leq \sum_{k=0}^{n} \mu(k, A) + \mu(k, B) - \mu(k, A+B) \leq \sum_{k=n+1}^{2n+1} \mu(k, A+B).$$

However $A + B \in \mathcal{L}_{1,\infty}$, so there is a constant $C > 0$ such that for all $k \geq 0$ we have $\mu(k, A+B) \leq \frac{C}{k+1}$ and therefore

$$0 \leq \sum_{k=0}^{n} \mu(k, A) + \mu(k, B) - \mu(k, A+B) \leq C, \quad n \geq 0.$$

Dividing by $\log(2+n)$:

$$\begin{aligned} 0 &\leq \frac{1}{\log(2+n)} \sum_{k=0}^{n} \mu(k, A) + \frac{1}{\log(2+n)} \sum_{k=0}^{n} \mu(k, B) - \frac{1}{\log(2+n)} \sum_{k=0}^{n} \mu(k, A+B) \\ &\leq O(\frac{1}{\log(2+n)}), \quad n \to \infty. \end{aligned}$$

Then applying ω:

$$0 \leq \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B) - \mathrm{Tr}_\omega(A+B) \leq 0.$$

So indeed $\mathrm{Tr}_\omega(A+B) = \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$ for any two positive operators A and B. $\square$

Remark 2.1.3. *Dixmier traces were first defined by J. Dixmier in* [**28**]*, albeit with some important differences to* Tr_ω *as given in the above example.*

(i) *Originally Dixmier traces were defined on the ideal* $\mathcal{M}_{1,\infty}$ *which is strictly larger than* $\mathcal{L}_{1,\infty}$*.*
(ii) Tr_ω *was originally shown to be additive only for those extended limits which are translation and dilation invariant.*

For more technical details and historical information we refer the reader to [**45**, Chapter 6]*.*

As the preceding example shows, Tr_ω *is additive on* $\mathcal{L}_{1,\infty}$ *for an arbitrary extended limit.*

An extensive discussion of traces, and more recent developments in the theory, may be found in [**45**] including a discussion of the following facts:

(1) All Dixmier traces on $\mathcal{L}_{1,\infty}$ are positive.
(2) All positive traces on $\mathcal{L}_{1,\infty}$ are continuous in the quasi-norm topology.
(3) Every continuous trace is a linear combination of positive traces.
(4) There exist positive traces on $\mathcal{L}_{1,\infty}$ which are not Dixmier traces (see [**58**]).
(5) There exist traces on $\mathcal{L}_{1,\infty}$ which fail to be continuous (see [**35**]).

(6) Every trace on $\mathcal{L}_{1,\infty}$ vanishes on $\mathcal{L}_1$ (see [**35**]).

We are mostly interested in *normalised traces* $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$, that is, satisfying $\varphi(\mathrm{diag}(\{\frac{1}{k+1}\}_{k\geq 0})) = 1$.

The following definition, extending that originally introduced in [**19**, Definition 2.β.7], plays an important role here.

DEFINITION 2.1.4. *An operator $T \in \mathcal{L}_{1,\infty}$ is called universally measurable if all normalised traces take the same value on T.*

The following result characterises universally measurable operators in terms of their eigenvalues, and a detailed proof may be found in [**45**, Theorem 10.1.3(g)]

THEOREM 2.1.5. *An operator $T \in \mathcal{L}_{1,\infty}$ is universally measurable if and only if there exists a constant c such that*

$$\sum_{k=0}^{n} \lambda(k,T) = c \cdot \log(n) + O(1), \quad n \to \infty$$

In this case, we have $\varphi(T) = c$ for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

2.2. Spectral triples

A spectral triple is an algebraic model for a Riemannian manifold, defined as follows:

DEFINITION 2.2.1. *A spectral triple $(\mathcal{A}, H, D)$ consists of the following data:*
(a) *a separable Hilbert space H.*
(b) *a (possibly unbounded) self-adjoint operator D on H with a dense domain $\mathrm{dom}(D) \subseteq H$.*
(c) *a $*$-subalgebra $\mathcal{A}$ of the algebra of bounded linear operators on H.*

Such that for all $a \in \mathcal{A}$ we have:
(1) $a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D)$,
(2) The commutator $[D, a] : \mathrm{dom}(D) \to H$ extends to a bounded linear operator on H, which we denote $\partial(a)$,
(3) $a(D + i)^{-1}$ is a compact operator.

REMARK 2.2.2. *Definition 2.2.1 should be compared to* [**11**, Definition 3.1]*, of which it is a special case (when the underlying von Neumann algebra is $\mathcal{L}_\infty(H)$). Within the literature there is some variation in the definition of a spectral triple. In many sources (such as* [**33**, Definition 9.16]*) it is assumed that the resolvent $(D + i)^{-1}$ is compact. We will refer to spectral triples where $(D + i)^{-1}$ is compact as compact spectral triples. In particular a spectral triple where $\mathcal{A}$ contains the identity operator is compact. If $(\mathcal{A}, H, D)$ is not necessarily compact, we will say that is it locally compact.*

DEFINITION 2.2.3. *Given a spectral triple $(\mathcal{A}, H, D)$ let F_D denote the partial isometry defined via functional calculus as*

$$F_D := \chi_{(0,\infty)}(D) - \chi_{(-\infty,0)}(D).$$

Where there is no ambiguity, we will frequently denote F_D as F.

If D has trivial kernel, then $F_D^2 = 1$.

We may define the operator $|D| : \mathrm{dom}(D) \to H$ by functional calculus. Since D is self-adjoint, for all $n \geq 1$ we have $|D|^n = |D^n|$, and so $\mathrm{dom}(|D|^n) = \mathrm{dom}(D^n)$. We have $F|D| = D$ as an equality of operators on $\mathrm{dom}(D)$, and on $\mathrm{dom}(D^2)$:

$$|D|D = D|D|.$$

We also have $|D| = FD$.

Note that $F_D^* = F_{D^*} = F_D$. Hence, we also have $D = D^* = |D|^* F^* = |D|F$. Since $|D|F = D$, it follows that $F : \mathrm{dom}(D) \to \mathrm{dom}(D)$.

By similar reasoning, we also have that for all $n \geq 1$ that $D^n F = F D^n$ and hence that $F : \mathrm{dom}(D^n) \to \mathrm{dom}(D^n)$.

Consequently, for $n, m \geq 1$, the operators F, D^n and $|D|^m$ all mutually commute on $\mathrm{dom}(D^{n+m})$.

2.2.1. Properties of spectral triples. Smoothness of a spectral triple is defined in terms of boundedness of commutators with $|D|$ (see Subsection 2.2.2 for discussion of this issue). The following results will be known to the expert reader. The notion of smoothness defined in terms of domains of commutators with $|D|$ originates with Connes [**18**, Section 1] and is also used in [**24**] and [**11**, Section 1.3]. We provide detailed proofs here for convenience.

If T is a bounded operator with $T : \mathrm{dom}(D) \to \mathrm{dom}(D)$, then the commutator $|D|T - T|D| : \mathrm{dom}(D) \to H$ is meaningful. More generally, if there is some n such that for all $0 \leq k \leq n$ we have $T : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$ then we may consider the higher iterated commutator:

$$[|D|, [|D|, [\cdots [|D|, T] \cdots]]] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} |D|^k a |D|^{n-k}. \tag{2.8}$$

This is a well defined operator on $\mathrm{dom}(D^n)$ in the following sense: for each k we have $|D|^{n-k} : \mathrm{dom}(D^n) \to \mathrm{dom}(D^k)$, $a : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$ and $|D|^k : \mathrm{dom}(D^k) \to H$.

We wish to define $\delta^n(T)$ as the bounded extension of the nth iterated commutator $[|D|, [|D|, [\cdots, [|D|, T] \cdots]]]$ when such an extension exists. This motivates the following definition:

Definition 2.2.4. *For $n \geq 1$, we define $\mathrm{dom}(\delta^n)$ to be the set of bounded linear operators T such that for all $0 < k \leq n$ we have $T : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$ and the nth iterated commutator in (2.8) has bounded extension.*

For $T \in \mathrm{dom}(\delta^n)$, we let $\delta^n(T)$ be the bounded extension of the nth iterated commutator (2.8).

The $n = 0$ case is defined by $\mathrm{dom}(\delta^0) := \mathcal{L}_\infty(H)$ and $\delta^0(T) := T$.

We define

$$\mathrm{dom}_\infty(\delta) := \bigcap_{k \geq 0} \mathrm{dom}(\delta^k).$$

Lemma 2.2.5. *The set $\mathrm{dom}_\infty(\delta)$ is closed under multiplication.*

Proof. Let $T, S \in \mathrm{dom}_\infty(\delta)$. Then by definition, for all $k \geq 1$ we have that $T, S : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$, and hence $TS : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$. The kth iterated commutator $\delta^k(TS)|\mathrm{dom}(D^k)$ is given by:

$$\delta^k(TS) = \sum_{j=0}^{k} \binom{k}{j} \delta^{k-j}(T)\delta^j(S).$$

Since for all j we have $\delta^j(S) \in \mathrm{dom}_\infty(\delta)$ and $\delta^{k-j}(T) \in \mathrm{dom}_\infty(\delta)$, the above expression is well defined as an operator on $\mathrm{dom}(D^k)$ and has bounded extension. $\square$

It is clear that if $k \geq 0$ and $T \in \mathrm{dom}(\delta^{k+1})$, then $\delta(T) \in \mathrm{dom}(\delta^k)$ and $\delta^k(T) \in \mathrm{dom}(\delta)$. Moreover $\delta^{k+1}(T) = \delta^k(\delta(T)) = \delta(\delta^k(T))$.

We may also define $\mathrm{dom}(\partial)$ to be the set of bounded operators T such that $T : \mathrm{dom}(D) \to \mathrm{dom}(D)$ and $[D, T] : \mathrm{dom}(D) \to H$ has a bounded extension, which we denote $\partial(T)$.

The relevance of $\mathrm{dom}(\partial)$ is the following:

Lemma 2.2.6. *Suppose that $T \in \mathrm{dom}(\delta) \cap \mathrm{dom}(\partial)$ is such that $\partial(T) \in \mathrm{dom}(\delta)$ and $\delta(T) : \mathrm{dom}(D) \to \mathrm{dom}(D)$. Then $\delta(T) \in \mathrm{dom}(\partial)$ and*

$$\partial(\delta(T)) = \delta(\partial(T)).$$

Proof. Since $T \in \mathrm{dom}(\delta) \cap \mathrm{dom}(\partial)$, we have in particular that $T : \mathrm{dom}(D) \to \mathrm{dom}(D)$. Since $\partial(T) \in \mathrm{dom}(\delta)$, we also have that $\partial(T) : \mathrm{dom}(D) \to \mathrm{dom}(D)$. Let $\xi \in \mathrm{dom}(D^2)$. Then,

$$DT\xi = \partial(T)\xi + TD\xi.$$

Since $T : \mathrm{dom}(D) \to \mathrm{dom}(D)$ and $\partial(T) : \mathrm{dom}(D^2) \to \mathrm{dom}(D)$, it follows that $DT\xi \in \mathrm{dom}(D)$ and therefore $T : \mathrm{dom}(D^2) \to \mathrm{dom}(D^2)$.

Now since the operators D and $|D|$ commute on $\mathrm{dom}(D^2)$, we have that for all $\xi \in \mathrm{dom}(D^2)$:

$$[D, [|D], T]]\xi = [|D|, [D, T]]\xi.$$

Since by assumption $T \in \mathrm{dom}(\delta)$, $\partial T \in \mathrm{dom}(\delta)$ and $\delta(T) : \mathrm{dom}(D) \to \mathrm{dom}(D)$, we may further write:

$$[D, \delta(T)]\xi = \delta(\partial(T))\xi.$$

Since the operator on the right hand side by assumption has bounded extension, and using the fact that $\mathrm{dom}(D^2)$ is dense in H it follows that $[D, \delta(T)]$ has bounded extension and therefore $\delta(T) \in \mathrm{dom}(\partial)$. Thus, $\partial(\delta(T)) = \delta(\partial(T))$. $\square$

Next we define the notion of a smooth spectral triple. Some sources (such as [**11**, Definition 3.18]) use the term "QC^∞ spectral triple", and others (such as [**34**, Definition 4.25] use the term "regular" spectral triple).

Definition 2.2.7. *A spectral triple $(\mathcal{A}, H, D)$ is called smooth if for all $a \in \mathcal{A}$, we have*

$$a, \partial(a) \in \mathrm{dom}_\infty(\delta).$$

If $(\mathcal{A}, H, D)$ is smooth, we let $\mathcal{B}$ be the $*-$subalgebra of $\mathcal{L}_\infty(H)$ generated by all elements of the form $\delta^k(a)$ or $\delta^k(\partial(a))$, $k \geq 0$, $a \in \mathcal{A}$.

By Lemma 2.2.5 and since $\delta^k(a)^* = (-1)^k\delta^k(a^*)$, we automatically have that $\mathcal{B} \subseteq \mathrm{dom}_\infty(\delta)$.

Corollary 2.2.8. *Let $(\mathcal{A}, H, D)$ be smooth, and $a \in \mathcal{A}$. Then for all $k \geq 1$ we have $\delta^k(a) \in \mathrm{dom}(\partial)$ and*

$$\partial(\delta^k(a)) = \delta^k(\partial(a)).$$

Proof. This proof proceeds by induction on k. For $k = 1$, by the definition of smoothness we have $\partial(a) \in \mathrm{dom}(\delta)$ and $a \in \mathrm{dom}(\delta) \cap \mathrm{dom}(\partial)$, and by definition $a : \mathrm{dom}(D) \to \mathrm{dom}(D)$. So by Lemma 2.2.6 it follows that $\delta(a) \in \mathrm{dom}(\partial)$ and

$$\partial(\delta(a)) = \delta(\partial(a)).$$

Now we suppose that the claim is proved for $k-1$, $k \geq 2$ and we prove the claim for k. Since $(\mathcal{A}, H, D)$ is smooth, $\delta^{k-1}(a) : \mathrm{dom}(D) \to \mathrm{dom}(D)$ and by the inductive hypothesis, $\delta^{k-1}(a) \in \mathrm{dom}(\partial)$ and

$$\delta^{k-1}(\partial(a)) = \partial(\delta^{k-1}(a)).$$

However since $\delta^{k-1}(a) \in \mathrm{dom}(\delta) \cap \mathrm{dom}(\partial)$ and $\delta^{k-1}(a) : \mathrm{dom}(D) \to \mathrm{dom}(D)$ we may apply Lemma 2.2.6 with $T = \delta^{k-1}(a)$ to conclude that $\delta^k(a) \in \mathrm{dom}(\partial)$ and

$$\begin{aligned} \delta(\partial(\delta^{k-1}(a))) &= \partial(\delta(\delta^{k-1}(a))) \\ &= \partial(\delta^k(a)). \end{aligned}$$

By the inductive hypothesis, $\delta(\partial(\delta^{k-1}(a))) = \delta(\delta^{k-1}(\partial(a))) = \delta^k(\partial(a))$, and so this proves the result for k. ☐

Definition 2.2.9. *Let $T \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\delta)$. Define*

$$L(T) := \partial(T) - F\delta(T).$$

Note that by definition $L(T)$ is bounded. On $\mathrm{dom}(D)$ we have:

$$L(T) = [F, T]|D|.$$

The boundedness of $L(T)$ on $\mathrm{dom}(D)$ was already implicitly noted in the proof of [**12**, Lemma 2].

Our computations are greatly simplified by introducing a common dense subspace $H_\infty \subseteq H$ on which all powers D^k are defined:

Definition 2.2.10. *Let $H_\infty := \bigcap_{n \geq 0} \mathrm{dom}(D^n)$.*

The subspace H_∞ is a well known object in noncommutative geometry, appearing in [**18**, Section 1] and more recently in [**33**, Equation 10.64] and [**11**, Definition 1.20]. One way to see that H_∞ is dense in H (and in particular non-zero) is to note that $\mathrm{dom}(e^{D^2}) \subseteq H_\infty$. If $T \in \mathrm{dom}_\infty(\delta)$, then $T : H_\infty \to H_\infty$ since by definition if $T \in \mathrm{dom}(\delta^n)$ then $T : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$ for all $0 \leq k \leq n$. Moreover since $F : \mathrm{dom}(D^n) \to \mathrm{dom}(D^n)$ for all n, we also have $F : H_\infty \to H_\infty$. It is useful to note that for each k the unbounded operators D^k and $|D|^k$ map H_∞ to H_∞. This observation is justified in the next lemma.

Lemma 2.2.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function which has "polynomial growth at infinity" in the sense that there exists $n \geq 0$ such that $t \mapsto (1+t^2)^{-n/2} f(t)$ is bounded on $\mathbb{R}$. Then $f(D)$ (defined by Borel functional calculus) maps H_∞ to H_∞.*

Proof. Let $k > n$, and $\xi \in \mathrm{dom}(D^k)$. By assumption $(1 + D^2)^{-n/2} f(D)$ defines a bounded operator,

$$(1 + D^2)^{-\frac{n}{2}} f(D) D^k : \mathrm{dom}(D^k) \to H.$$

However $(1+D^2)^{-n/2}f(D)D^k = D^k(1+D^2)^{-n/2}f(D)$ on a dense domain. Since $D^n(1+D^2)^{-n/2}$ defines a bounded operator, we get that $D^{k-n}f(D) : \mathrm{dom}(D^k) \to H$. Therefore $f(D) : \mathrm{dom}(D^k) \to \mathrm{dom}(D^{k-n})$. Since $k > n$ is arbitrary, we have that $f(D) : H_\infty \to H_\infty$. $\square$

Lemma 2.2.12. *If $T \in \mathrm{dom}(\delta) \cap \mathrm{dom}(\partial)$ is such that $T : H_\infty \to H_\infty$, then $L(T) : H_\infty \to H_\infty$.*

Proof. For $\xi \in H_\infty$,

$$L(T)\xi = [F,T]|D|\xi.$$

However $|D|\xi \in H_\infty$, and $F : H_\infty \to H_\infty$. Thus $[F,T]|D|\xi \in H_\infty$. $\square$

Lemma 2.2.13. *Let $T \in \mathrm{dom}(\delta^2) \cap \mathrm{dom}(\partial)$ be such that $\partial(T) \in \mathrm{dom}(\delta)$. Then $L(T) \in \mathrm{dom}(\delta)$ and*

$$\delta(L(T)) = L(\delta(T)).$$

Proof. Since $\partial(T) \in \mathrm{dom}(\delta)$, we have from Lemma 2.2.6 that $\delta(T) \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\delta)$ and hence $L(\delta(T))$ is defined and bounded.

The required identity can be checked on $\mathrm{dom}(D^2)$, since $T : \mathrm{dom}(D^2) \to \mathrm{dom}(D^2)$. For $\xi \in \mathrm{dom}(D^2)$, we have:

$$\begin{aligned}\delta(L(T))\xi &= [|D|,[F,T]|D|]\xi \\ &= [F,[|D|,T]]|D|\xi \\ &= L(\delta(T))\xi.\end{aligned}$$

$\square$

Spectral triples are often classed as *even* or *odd*:

Definition 2.2.14. *A spectral triple $(\mathcal{A}, H, D)$ is said to be*

(a) *even if equipped with $\Gamma \in \mathcal{L}_\infty$ such that $\Gamma = \Gamma^*$, $\Gamma^2 = 1$ and such that $[\Gamma, a] = 0$ for all $a \in \mathcal{A}$, $\{D, \Gamma\} = 0$. Here $\{\cdot,\cdot\}$ denotes anticommutator.*
(b) *odd if not equipped with such Γ. In this case, we set $\Gamma = 1$.*
(c) *$p-$dimensional if for all $a \in \mathcal{A}$ we have $a(D+i)^{-p} \in \mathcal{L}_{1,\infty}$ and $\partial(a)(D+i)^{-p} \in \mathcal{L}_{1,\infty}$, and for all $q < p$ there exists $a_0 \in \mathcal{A}$ such that $a_0(D+i)^{-q} \notin \mathcal{L}_{1,\infty}$.*

For an even spectral triple, we have $D^2\Gamma = \Gamma D^2$. Therefore, $|D|\Gamma = \Gamma|D|$. We furthermore have that $F\Gamma + \Gamma F = 0$.

We follow the convention of [**11**], where we write Γ in all formulae referring to spectral triples, with the understanding that if the spectral triple is odd then $\Gamma = 1$ and the assumption that $\{D, \Gamma\} = 0$ is dropped.

For an arbitrary spectral triple, we have $|D|^k\Gamma = \Gamma|D|^k$ for all k, and therefore $\Gamma : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$ for all k. Hence $\Gamma : H_\infty \to H_\infty$.

The following assertion is well-known in the compact case (see e.g. [**12**] and [**51**]). To the best of our knowledge, no proof has been published in the locally compact case. We supply a proof in Section 3.1, Proposition 3.1.5.

Proposition. *Let $p \in \mathbb{N}$. If $(\mathcal{A}, H, D)$ is a $p-$dimensional spectral triple satisfying Hypothesis 1.2.1, then $[F, a] \in \mathcal{L}_{p,\infty}$ for all $a \in \mathcal{A}$.*

Let $\mathcal{A}^{\otimes(p+1)}$ denote the $(p+1)$-fold algebraic tensor power of $\mathcal{A}$. We now define the two important mappings ch and Ω.

Definition 2.2.15. *Suppose that D has a spectral gap at 0. Define the multilinear mappings* $\mathrm{ch} : \mathcal{A}^{\otimes(p+1)} \to \mathcal{L}_\infty$ *and* $\Omega : \mathcal{A}^{\otimes(p+1)} \to \mathcal{L}_\infty$ *on elementary tensors* $a_0 \otimes a_1 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$ *by*

$$\mathrm{ch}(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \Gamma F \prod_{k=0}^{p} [F, a_k]$$

and

$$\Omega(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \Gamma a_0 \prod_{k=1}^{p} \partial(a_k).$$

If $(\mathcal{A}, H, D)$ is p-dimensional, then it follows from Proposition 3.1.5 and the Hölder inequality (2.3) that $\mathrm{ch}(c) \in \mathcal{L}_{\frac{p}{p+1},\infty} \subset \mathcal{L}_1$ for all $c \in \mathcal{A}^{\otimes(p+1)}$. This permits the following definition:

Definition 2.2.16. *If $(\mathcal{A}, H, D)$ is $p-$dimensional spectral triple satisfying Hypothesis 1.2.1 and if kernel of D is trivial, then Connes-Chern character* $\mathrm{Ch} : \mathcal{A}^{\otimes(p+1)} \to \mathbb{C}$ *is defined by setting*

$$\mathrm{Ch}(c) = \frac{1}{2} \mathrm{Tr}(\mathrm{ch}(c)), \quad c \in \mathcal{A}^{\otimes(p+1)}.$$

In general, however, Chern character cannot be defined in terms of F because $F^2 \neq 1$ when D has non-trivial kernel. In order to ensure that Ch is a cyclic cocycle (in the sense of [**43**, 2.1.4]), we require that $F^2 = 1$. Hence we define the Chern character of a general spectral triple in terms of another F_0 such that $F_0 = F_0^* = F_0^2$. For this purpose, we use a doubling trick.

Definition 2.2.17. *Let $(\mathcal{A}, H, D)$ be a spectral triple with grading Γ, and let P be the projection onto* $\ker(D)$. *Consider the following unitary self-adjoint operators on the Hilbert space $H_0 = \mathbb{C}^2 \otimes H$ defined by:*

$$F_0 := \begin{pmatrix} F & P \\ P & -F \end{pmatrix}$$

$$\Gamma_0 := \begin{pmatrix} \Gamma & 0 \\ 0 & (-1)^{\deg} \Gamma \end{pmatrix}.$$

Here, $\deg = 1$ *for even triples and* $\deg = 0$ *for odd triples. The algebra $\mathcal{A}$ is represented on H_0 by:*

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

For an elementary tensor $a_0 \otimes \cdots a_p \in \mathcal{A}^{\otimes(p+1)}$ we set:

$$\mathrm{ch}_0(a_0 \otimes \cdots \otimes a_p) = \Gamma_0 F_0 \prod_{k=0}^{p} [F_0, \pi(a_k)].$$

Definition 2.2.18. *If $(\mathcal{A}, H, D)$ is $p-$dimensional spectral triple satisfying Hypothesis 1.2.1, then Connes-Chern character* $\mathrm{Ch} : \mathcal{A}^{\otimes(p+1)} \to \mathbb{C}$ *is defined by setting*

$$\mathrm{Ch}(c) = \frac{1}{2} (\mathrm{Tr}_2 \otimes \mathrm{Tr})(\mathrm{ch}_0(c)), \quad c \in \mathcal{A}^{\otimes(p+1)}.$$

Here, Tr_2 *denotes the 2×2 trace on matrices.*

Note that Definition 2.2.17 does not conflict with Definition 2.2.15: if $\ker(D)$ is trivial (i.e., $P = 0$) then both definitions of Ch coincide.

Strictly speaking, the Connes-Chern character is conventionally defined to be the class of Ch in periodic cyclic cohomology. This distinction is not relevant to the results of this paper, so in the sequel we will consider Ch merely as a multilinear functional as above.

REMARK 2.2.19. *We have opted to define the Chern character of a spectral triple* $(\mathcal{A}, H, D)$ *in terms of the "doubled" operator* F_0 *in Definition 2.2.17. This definition is different from earlier work such as* [**12**, Definition 6] *and* [**11**, Definition 2.23]. *In those papers the Chern character of a spectral triple is defined to be the Chern character of any Fredholm module equivalent to the pre-Fredholm module* $(\mathcal{A}, H, F)$. *It is known that the class in periodic cyclic cohomology of the chern character defined in that way is independent of the choice of Fredholm module equivalent to* $(\mathcal{A}, H, F)$ [**17**, Section 5, Lemma 1].

In order to avoid technicalities, we have defined the Chern character in terms of a specific Fredholm module $(\pi(\mathcal{A}), H_0, F_0)$. *This has the advantage of simplicity of presentation, and makes no difference in regards to the character formula. A reader interested in a more refined definition of the Chern character in periodic cyclic cohomology may wish to consult* [**17, 12, 11**].

2.2.2. Discussion of smoothness. It is tempting to define smoothness only in terms of ∂, without reference to δ. One might naively suggest that $(\mathcal{A}, H, D)$ is smooth if for all $n \geq 0$ we have $a \cdot \mathrm{dom}(D^n) \to \mathrm{dom}(D^n)$ and the nth iterated commutator $[D, [D, [\cdots, [D, a]\cdots]$ extends to a bounded operator on H.

However this condition does not hold for even the simplest spectral triples. A standard spectral triple associated to the 2-torus $\mathbb{T}^2$ is $(1\otimes C^\infty(\mathbb{T}^2), L_2(\mathbb{T}^2, \mathbb{C}^2), D)$, where the L_2 space is defined with respect to the Haar measure, the algebra $C^\infty(\mathbb{T}^2)$ of smooth complex valued functions on $\mathbb{T}^2$ acts on $L_2(\mathbb{T}^2, \mathbb{C}^2)$ by pointwise multiplication, and the Dirac operator D is defined by:

$$D = -i\gamma_1 \otimes \partial_1 - i\gamma_2 \otimes \partial_2,$$

where ∂_1 and ∂_2 are differentiation with respect to the first and second coordinates on $\mathbb{T}^2$ and γ_1, γ_2 are 2×2 complex matrices satisfying $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}1,\ j, k = 1, 2$.

Then if $f \in C^\infty(\mathbb{T}^2)$,

$$\begin{aligned}[D, [D, 1 \otimes M_f]] &= -[\gamma_1 \otimes \partial_1 + \gamma_2 \otimes \partial_2, \gamma_1 \otimes M_{\partial_1 f} + \gamma_2 \otimes M_{\partial_2 f}] \\ &= -1 \otimes M_{\partial_1^2 f + \partial_2^2 f} + 2\gamma_1\gamma_2 \otimes (M_{\partial_2 f}\partial_1 + M_{\partial_1 f}\partial_2)\end{aligned}$$

However this operator is typically unbounded: if we choose $f(z_1, z_2) = z_1$, then $[D, [D, 1 \otimes M_f]] = 2\gamma_1\gamma_2 \otimes (\partial_2)$ which is unbounded.

This example breaks the implication: "if $f \in C^\infty(\mathbb{T}^2)$, then $[D, [D, 1 \otimes M_f]]$ extends to a bounded linear operator".

2.2.3. Discussion of dimension. As we have defined it, we say that a spectral triple $(\mathcal{A}, H, D)$ is p-dimensional if for all $a \in \mathcal{A}$ the operators $a(D + i)^{-p}$ and $\partial(a)(D + i)^{-p}$ are in $\mathcal{L}_{1,\infty}$.

An alternative definition, also used in the literature, is to say that $(\mathcal{A}, H, D)$ is p-dimensional if $a(D+i)^{-1}$ and $\partial(a)(D+i)^{-1}$ are in $\mathcal{L}_{p,\infty}$. An example of a definition along these lines is [**31**, Definition 3.1]. Clearly in the case where $\mathcal{A}$ is unital these definitions are equivalent, since $(D+i)^{-1} \in \mathcal{L}_{p,\infty}$ if and only if $(D+i)^{-p} \in \mathcal{L}_{1,\infty}$. However in the non-unital case, the distinction may be important.

2.2.4. Hochschild (co)homology. Hochschild homology and cohomology provide noncommutative generalisations of the notion of differential forms and de Rham currents respectively. A detailed exposition of the theory of Hochschild (co)homology and its relationship with noncommutative geometry may be found in [**52, 43**].

Let A be a (possibly non-unital) algebra. The Hochschild complex is a chain complex:

$$\cdots \xrightarrow{b} A\otimes A\otimes A\otimes A \xrightarrow{b} A\otimes A\otimes A \xrightarrow{b} A\otimes A \xrightarrow{b} A.$$

For $n \geq 1$, the nth entry in the Hochschild chain complex is the nth tensor power $A^{\otimes n}$. The Hochschild boundary operator $b : A^{\otimes(n+1)} \to A^{\otimes n}$ is defined on elementary tensors $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ by:

$$\begin{aligned} b(a_0\otimes a_1\otimes\cdots\otimes a_n) = a_0a_1\otimes a_2\otimes\cdots\otimes a_n + \sum_{k=1}^{n-1}(-1)^k a_0\otimes\cdots\otimes a_ka_{k+1}\otimes\cdots\otimes a_n \\ + (-1)^n a_na_0\otimes a_1\cdots\otimes a_{n-1}. \end{aligned}$$

It can be verified that $b^2 = 0$, so the Hochschild complex is indeed a chain complex. An element $c \in A^{\otimes(n+1)}$ such that $bc = 0$ is called a Hochschild cycle. For example, when $n = 1$, an elementary tensor $a_0 \otimes a_1$ is a Hochschild cycle if and only if $b(a_0 \otimes a_1) = a_0a_1 - a_1a_0 = 0$, i.e. if a_0 and a_1 commute.

The Hochschild cochain complex is defined in a similar way: Let $C_n(A)$ denote the space of continuous multilinear functionals from $A^{\otimes n} \to \mathbb{C}$. The Hochschild cochain complex is,

$$C_1(A) \xrightarrow{b} C_2(A) \xrightarrow{b} C_3(A) \xrightarrow{b} \cdots$$

where the Hochschild coboundary operator b is defined as follows: if $\theta : A^{\otimes n} \to \mathbb{C}$, then $b\theta : A^{\otimes(n+1)} \to \mathbb{C}$ is given on an elementary tensor $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ by

$$\begin{aligned} (b\theta)(a_0\otimes\cdots\otimes a_n) &= \theta(a_0a_1\otimes a_2\otimes\cdots\otimes a_n) \\ &\quad + \sum_{k=1}^{n-1}(-1)^k\theta(a_0\otimes a_1\otimes\cdots\otimes a_{k-1}\otimes a_ka_{k+1}\otimes a_{k+2}\otimes\cdots\otimes a_n) \\ &\quad + (-1)^n\theta(a_na_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}). \end{aligned}$$

Put simply, for $c \in A^{\otimes(n+1)}$ and $\theta \in C_n(A)$, the Hochschild boundary and coboundary operators are linked by,

$$(b\theta)(c) = \theta(bc). \tag{2.9}$$

A cochain $\theta \in C^n(A)$ is called a Hochschild cocycle if $b\theta = 0$. Due to (2.9), a Hochschild coboundary vanishes on every Hochschild cycle.

REMARK 2.2.20. *One must distinguish between Hochschild (co)homology as we have just defined it, and the analogous continuous Hochschild homology* [**33**, Section 8.5]*,* [**49**]. *Continuous Hochschild (co)homology is defined with topological tensor products in place of algebraic tensor products. In this text we are only concerned with algebraic tensor products.*

2.3. Weak integrals and double operator integrals

2.3.1. Weak integration in $\mathcal{L}_\infty$. This section concerns the theory of "weak operator topology integrals" of operator valued functions. The following definitions, and the subsequent construction of weak integrals, are folklore. We provide suitable references whenever they exist, otherwise we supply a proof. For example, one can look at [**57**, Definition 3.26], and consider the example where the topological vector space X there is $\mathcal{L}_\infty$ equipped with the strong operator topology. Every continuous linear functional on X can be written as a linear combination of $x \to \langle x\xi, \eta\rangle$, $\xi, \eta \in H$.

DEFINITION 2.3.1. *A function $f : \mathbb{R} \to \mathcal{L}_\infty$ is measurable in the weak operator topology if, for every pair of vectors $\xi, \eta \in H$, the function*

$$s \to \langle f(s)\xi, \eta\rangle, \quad s \in \mathbb{R},$$

is (Lebesgue) measurable.

For a function f, measurable in the weak operator topology, there is a notion of "pointwise norm". Namely, the scalar-valued mapping

$$s \mapsto \|f(s)\|_\infty := \sup_{\|\xi\|,\|\eta\|\le 1} |\langle f(s)\xi, \eta\rangle|, \quad s \in \mathbb{R}$$

is Lebesgue measurable. Here it is crucial that we work with separable Hilbert spaces, as otherwise it is not clear whether the function $s \mapsto \|f(s)\|_\infty$ is measurable.

Suppose that a function $f : \mathbb{R} \to \mathcal{L}_\infty$ is measurable in the weak operator topology. We say that f is integrable in the weak operator topology if

$$\int_{\mathbb{R}} \|f(s)\|_\infty ds < \infty. \tag{2.10}$$

In particular, for all $\xi, \eta \in H$, we have

$$\int_{\mathbb{R}} |\langle f(s)\xi, \eta\rangle|\, ds < \infty.$$

Hence for a function f satisfying (2.10), we may therefore define the sesquilinear form

$$(\xi, \eta)_f := \int_{\mathbb{R}} \langle f(s)\xi, \eta\rangle\, ds, \quad \xi, \eta \in H.$$

It then follows that:

$$\begin{aligned}|(\xi, \eta)_f| &\le \int_{\mathbb{R}} |\langle f(s)\xi, \eta\rangle|\, ds \\ &\le \int_{\mathbb{R}} \|f(s)\|_\infty \|\xi\|\|\eta\|\, ds \\ &= \left(\int_{\mathbb{R}} \|f(s)\|_\infty\, ds\right) \|\xi\|\|\eta\|.\end{aligned}$$

Thus for a fixed $\xi \in H$, the mapping $\eta \mapsto (\xi, \eta)_f$ defines a bounded linear functional on H. Hence there is a unique $x_\xi \in H$ such that $(\xi, \eta)_f = \langle x_\xi, \eta\rangle$ for all $\eta \in H$.

One can easily verify that the map $\xi \mapsto x_\xi$ is linear, and furthermore

$$\begin{aligned}\|x_\xi\|^2 &= \langle x_\xi, x_\xi\rangle \\ &= |(\xi, x_\xi)_f| \\ &\leq \left(\int_{\mathbb{R}} \|f(s)\|_\infty \, ds\right) \|\xi\|\|x_\xi\|.\end{aligned}$$

So the mapping $\xi \to x_\xi$ is bounded. Let T be the unique bounded linear operator such that $x_\xi = T\xi$, we now define

$$\int_{\mathbb{R}} f(s)\, ds := T. \tag{2.11}$$

Due to the above computation, we have that

$$\left\|\int_{\mathbb{R}} f(s)\, ds\right\|_\infty \leq \int_{\mathbb{R}} \|f(s)\|_\infty \, ds.$$

Furthermore, we have that if $A \in \mathcal{L}_\infty$, and f is integrable in the weak operator topology, then $s \mapsto Af(s)$ is also integrable in the weak operator topology, and

$$\int_{\mathbb{R}} Af(s)\, ds = A\int_{\mathbb{R}} f(s)\, ds.$$

Closely related to the weak integral is the Bochner integral: indeed, if $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a normed ideal in $\mathcal{L}_\infty$ and $f : \mathbb{R} \to \mathcal{E}$ is Bochner integrable, then it is integrable in the weak operator topology and the weak integrals and Bochner integrals coincide, since if f is weakly $\mathcal{E}$-valued measurable, then it is weak operator topology measurable, and if $\|f\|_{\mathcal{E}}$ is integrable then $\|f\|_\infty$ is integrable.

2.3.2. Properties of the weak integral. The authors thank Professor Peter Dodds for his assistance with the arguments in this subsection.

LEMMA 2.3.2. *Let $s \to a(s)$, $s \in \mathbb{R}$, be continuous in the weak operator topology. If $a(s) \in \mathcal{L}_1$ for every $s \in \mathbb{R}$ and if*

$$\int_{\mathbb{R}} \|a(s)\|_1 ds < \infty,$$

then $a(s)$ is integrable in the weak operator topology, $\int_{\mathbb{R}} a(s)ds \in \mathcal{L}_1$ and

$$\Big\|\int_{\mathbb{R}} a(s)ds\Big\|_1 \leq \int_{\mathbb{R}} \|a(s)\|_1 ds, \quad \mathrm{Tr}\Big(\int_{\mathbb{R}} a(s)ds\Big) = \int_{\mathbb{R}} \mathrm{Tr}(a(s))ds.$$

PROOF. Since $\|a(s)\|_\infty \leq \|a(s)\|_1$ for all $s \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{R}} \|a(s)\|_\infty \, ds &\leq \int_{\mathbb{R}} \|a(s)\|_1 \, ds \\ &< \infty\end{aligned}$$

so that condition (2.10) holds, so $s \mapsto a(s)$ is integrable in the weak operator topology. Thus, let A be the bounded linear operator on H given by

$$A := \int_{\mathbb{R}} a(s)ds$$

in the sense of (2.11). Next, we shall show that $A \in \mathcal{L}_1$.

Let $A = U|A|$ be a polar decomposition of A. For an arbitrary finite rank projection p, we have

$$p|A|p = \int_{\mathbb{R}} pU^*a(s)pds$$

Since p is finite rank, the algebra $p\mathcal{L}_\infty p$, is finite dimensional, and so here the weak operator topology coincides with the norm topology. Hence, the mapping $s \mapsto pU^*a(s)p$ is continuous in the norm topology. Since on the algebra $p\mathcal{L}_\infty p$ the classical trace is a continuous functional with respect to the uniform norm, it follows that

$$\mathrm{Tr}(p|A|p) = \int_{\mathbb{R}} \mathrm{Tr}(pU^*a(s)p)ds.$$

Thus,

$$\begin{aligned}\mathrm{Tr}(p|A|p) &\leq \int_{\mathbb{R}} |\mathrm{Tr}(pU^*a(s)p)|ds \\ &\leq \int_{\mathbb{R}} \|a(s)\|_1 ds.\end{aligned}$$

Taking the supremum over all finite rank projections p, we arrive at

$$\|A\|_1 \leq \int_{\mathbb{R}} \|a(s)\|_1 ds.$$

This proves the first assertion.

Choose now a sequence $\{p_n\}_{n\geq 1}$ of finite rank projections such that $p_n \uparrow 1$. We have

$$\mathrm{Tr}(p_n A p_n) = \int_{\mathbb{R}} \mathrm{Tr}(p_n a(s) p_n)ds.$$

Clearly, $\mathrm{Tr}(p_n a(s) p_n) \to \mathrm{Tr}(a(s))$ as $n \to \infty$ for every $s \in \mathbb{R}$. Since the function $s \mapsto \mathrm{Tr}(a(s))$ is integrable, we can apply the dominated convergence theorem to obtain

$$\int_{\mathbb{R}} \mathrm{Tr}(p_n a(s) p_n)ds \to \int_{\mathbb{R}} \mathrm{Tr}(a(s))ds.$$

On the other hand, we have $\mathrm{Tr}(p_n A p_n) \to \mathrm{Tr}(A)$ as $n \to \infty$. This proves the second assertion. $\square$

According to the preceding lemma, if $a : \mathbb{R} \to \mathcal{L}_1$ is continuous and $\int_{\mathbb{R}} \|a(s)\|_1\, ds < \infty$, then we have that $\int_{\mathbb{R}} a(s)\, ds \in \mathcal{L}_1$. The following lemma shows that the same implication holds when $\mathcal{L}_1$ is replaced by $\mathcal{L}_{r,\infty}$ for any $r > 1$.

PROPOSITION 2.3.3. *Let $s \to a(s)$, $s \in \mathbb{R}$, be continuous in the weak operator topology. Fix $r > 1$, and suppose that for all s we have $a(s) \in \mathcal{L}_{r,\infty}$. If $\int_{\mathbb{R}} \|a(s)\|_{r,\infty}\, ds < \infty$ then $\int_{\mathbb{R}} a(s)\, ds \in \mathcal{L}_{r,\infty}$, where the integral is understood in a weak sense, and we have a bound:*

$$\left\| \int_{\mathbb{R}} a(s)\, ds \right\|_{r,\infty} \leq \frac{r}{r-1} \int_{\mathbb{R}} \|a(s)\|_{r,\infty}\, ds.$$

PROOF. Similar to the $\mathcal{L}_1$ case, since $\|a(s)\|_\infty \leq \|a(s)\|_{r,\infty}$, we have that $\int_{\mathbb{R}} \|a(s)\|_\infty\, ds < \infty$, and so condition (2.10) holds. Hence, $s \mapsto a(s)$ is integrable in the weak operator topology. Let $A := \int_{\mathbb{R}} a(s)\, ds$ in the sense of (2.11). Let

$A = U|A|$ be a polar decomposition of A, and let p be a rank n projection, $n \geq 1$. Then,

$$p|A|p = \int_{\mathbb{R}} pU^*a(s)p\,ds.$$

Thus,

$$\begin{aligned}\mathrm{Tr}(p|A|p) &\leq \int_{\mathbb{R}} |\mathrm{Tr}(pU^*a(s)p)|\,ds \\ &\leq \int_{\mathbb{R}} \|pU^*a(s)p\|_1\,ds.\end{aligned}$$

The latter integral converges because $\|pU^*a(s)p\|_1 \leq n\|a(s)\|_\infty$. Now,

$$\begin{aligned}\|pU^*a(s)p\|_1 &\leq \sum_{k=0}^{n-1}\mu(k,a(s)) \\ &\leq \|a(s)\|_{r,\infty}\sum_{k=0}^{n-1}(k+1)^{-1/r}.\end{aligned}$$

The right hand side depends only on n and not p, so we may take the supremum over all projections of rank n to obtain:

$$\sum_{k=0}^{n-1}\mu(k,A) \leq \int_{\mathbb{R}}\|a(s)\|_{r,\infty}\,ds\cdot\sum_{k=0}^{n-1}(k+1)^{-1/r}.$$

We can bound the latter sum as:

$$\begin{aligned}\sum_{k=0}^{n-1}(k+1)^{-1/r} &\leq 1+\int_1^n t^{-1/r}\,dt \\ &\leq \frac{r}{r-1}n^{1-\frac{1}{r}}.\end{aligned}$$

Therefore:

$$\sum_{k=0}^{n-1}\mu(k,A) \leq \int_{\mathbb{R}}\|a(s)\|_{r,\infty}\,ds\frac{r}{r-1}n^{1-\frac{1}{r}}.$$

Hence,

$$n\mu(n-1,A) \leq \frac{r}{r-1}n^{1-\frac{1}{r}}\int_{\mathbb{R}}\|a(s)\|_{r,\infty}\,ds.$$

Multiplying through by $n^{\frac{1}{r}-1}$, and taking the supremum over n, it follows that

$$\sup_{n\geq 1} n^{\frac{1}{r}}\mu(n-1,A) \leq \frac{r}{r-1}\int_{\mathbb{R}}\|a(s)\|_{r,\infty}\,ds.$$

So by the definition of the quasinorm on $\mathcal{L}_{r,\infty}$, the assertion follows. $\square$

2.4. Double operator integrals

Here, we state the definition and basic properties of double operator integrals. This theory was initiated by the work of Birman and Solomyak [**4, 5, 6**], and more recent summaries of the theory may be found in [**7, 48**].

Heuristically, given self-adjoint operators X and Y with spectra $\sigma(X)$ and $\sigma(Y)$, spectral resolutions E_X and E_Y and a bounded measurable function ϕ on

$\sigma(X) \times \sigma(Y)$, the double operator integral $T_\phi^{X,Y}$ applied to an operator $A \in \mathcal{L}_\infty$ is given by the formula:

$$T_\phi^{X,Y}(A) = \iint_{\sigma(X)\times\sigma(Y)} \phi(\lambda,\mu) dE_X(\lambda) A dE_Y(\mu).$$

The formal expression for $T_\phi^{X,Y}$ is well defined as a bounded operator on the Hilbert-Schmidt class $\mathcal{L}_2$. The theory of double operator integrals is primarily concerned with defining $T_\phi^{X,Y}$ on other ideals. This is not possible for arbitrary bounded measurable functions ϕ, so we must restrict attention to the following class of "good" functions.

That is, we assume that ϕ admits a representation

$$\phi(\lambda,\mu) = \int_\Omega a(\lambda,s)b(\mu,s)\,d\kappa(s), \quad \lambda\in\sigma(X),\mu\in\sigma(Y) \tag{2.12}$$

where (Ω,κ) is a measure space, and where

$$\int_\Omega \sup_{\lambda\in\sigma(X)} |a(\lambda,s)| \sup_{\mu\in\sigma(Y)} |b(\mu,s)|\,d\kappa(s) < \infty. \tag{2.13}$$

For such functions ϕ, we may define

$$T_\phi^{X,Y}(A) := \int_\Omega a(X,s)Ab(Y,s)\,d\kappa(s) \tag{2.14}$$

where the operators $a(X,s)$ and $b(Y,s)$ are defined by Borel functional calculus, and the integral can be understood in the weak operator topology.

The following is proved in [**51**, Theorem 4]:

THEOREM 2.4.1. *If ϕ admits a decomposition as in* (2.12)*, then the operator $T_\phi^{X,Y}$ is a bounded linear map from:*
(a) *$\mathcal{L}_\infty$ to $\mathcal{L}_\infty$;*
(b) *$\mathcal{L}_1$ to $\mathcal{L}_1$;*
(c) *$\mathcal{L}_r$ to $\mathcal{L}_r$, for all $r\in(1,\infty)$;*
(d) *$\mathcal{L}_{r,\infty}$ to $\mathcal{L}_{r,\infty}$ for all $r\in(1,\infty)$.*

One of the key properties of double operator integrals is that they respect algebraic operations (see e.g.[**46**, Proposition 2.8]). Namely,

$$T_{\phi_1+\phi_2}^{X,Y} = T_{\phi_1}^{X,Y} + T_{\phi_2}^{X,Y}, \quad T_{\phi_1\cdot\phi_2}^{X,Y} = T_{\phi_1}^{X,Y}\circ T_{\phi_2}^{X,Y}. \tag{2.15}$$

If, in (2.12) we take Ω to be a one-point set, then $\phi(\lambda,\mu) = a(\lambda)b(\mu)$ and

$$T_\phi^{X,Y}(A) = a(X)Ab(Y). \tag{2.16}$$

2.5. Fourier transform conventions

We follow the convention that the Fourier transform of a function $g\in L_1(\mathbb{R})$ is defined by the formula

$$\mathcal{F}(g)(t) := (2\pi)^{-1/2}\int_{\mathbb{R}} g(s)e^{-its}\,ds$$

So that the inverse Fourier transform is given for $h\in L_1(\mathbb{R})$ by,

$$\mathcal{F}^{-1}(h)(s) := (2\pi)^{-1/2}\int_{\mathbb{R}} h(t)e^{its}\,dt$$

and so that $\mathcal{F}$ extends to a unitary operator on $L_2(\mathbb{R})$.

We often make use of the fact that if $g \in L_1(\mathbb{R})$ satisfies $g, g' \in L_2(\mathbb{R})$ then $\mathcal{F}g \in L_1(\mathbb{R})$ [**51**, Lemma 7]. Here, the derivative g' may be defined in a distributional sense.

Everywhere in the text, the symbol $\hat{g}$ denotes $(2\pi)^{-1/2}\mathcal{F}(g)$. This allows us to write for $g \in L_1(\mathbb{R})$ with $\mathcal{F}(g) \in L_1(\mathbb{R})$:

$$g(t) = \int_{\mathbb{R}} \hat{g}(s) e^{its}\, ds.$$

We caution the reader that $\hat{g}$ does not denote the Fourier transform, but its rescaling by $(2\pi)^{-1/2}$.

CHAPTER 3

Spectral Triples: Basic properties and examples

This chapter is primarily concerned with Hypothesis 1.2.1. We study the consequences of this hypothesis, and also show that it is satisfied for two important classes of examples.

We begin with the proof of Proposition 3.1.5, an important prerequisite to the definition of the Chern character (Definition 2.2.18). Next, we show that Hypothesis 1.2.1 is equivalent to a modified set of assumptions, Hypothesis 3.2.7. Hypothesis 3.2.7 is stated in terms of an operator Λ (given in Definition 3.2.1) rather than δ. This has the advantage of making Hypothesis 3.2.7 more easily verified in the classes of examples studied in this chapter.

The remainder of the chapter is devoted to demonstrating that the assumptions made in Hypothesis 1.2.1 are satisfied for spectral triples associated to the following classes of examples:

(a) Complete Riemannian manifolds.
(b) Noncommutative Euclidean spaces (also known as Moyal planes or Moyal-Groenwald planes in the 2-dimensional case).

We re-emphasise that Hypothesis 1.2.1 is automatically satisfied for smooth p-dimensional unital spectral triples, and therefore we concern ourselves with showing that it is satisfied for non-unital algebras.

The first class of examples (a) is purely commutative. For the Dirac operator in these examples, we use the Hodge-Dirac operator (see [**38**] or [**56**]). In [**11**], spectral triples for noncompact Riemannian manifolds were studied under the significant restriction that they have bounded geometry: this is a global geometric property which we are able to avoid by working in local coordinates. Earlier, Rennie had studied noncompact Riemannian spin manifolds which are not necessarily of bounded geometry by similar methods [**55**, Section 5]. It is hoped that by including such a wide class of manifolds we may demonstrate the applicability of noncommutative methods in "classical" (commutative) geometry.

The second example (b) is one of the most heavily studied classes of non-unital and strictly noncommutative spectral triples. A detailed exposition of the noncommutative Euclidean spaces may be found in [**31**].

3.1. A spectral triple defines a Fredholm module

This section is devoted to the proof of Proposition 3.1.5. We prove this in several steps, initially working with the assumption that D has a spectral gap at 0 (i.e., that D has bounded inverse). We later show how this assumption can be removed.

Note that if D has a spectral gap at 0, then $F = D|D|^{-1} = |D|^{-1}D$.

REMARK 3.1.1. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. Suppose D has a spectral gap at 0. For every $a \in \mathcal{A}$, and all $k \geq 1$ we have the following four inclusions:*

$$a|D|^{-p} \in \mathcal{L}_{1,\infty}, \quad \partial(a)|D|^{-p} \in \mathcal{L}_{1,\infty},$$
$$\delta^k(a)|D|^{-p-1} \in \mathcal{L}_1, \quad \partial(\delta^k(a))|D|^{-p-1} \in \mathcal{L}_1.$$

PROOF. All four inclusions follow from the observation that since by assumption $|D|$ is invertible, the operator $\frac{D+i}{|D|} : \mathrm{dom}(D) \to H$ has bounded extension. Since $(\mathcal{A}, H, D)$ is p-dimensional, we have

$$a(D+i)^{-p},\ \partial(a)(D+i)^{-p} \in \mathcal{L}_{1,\infty}$$

and multiplying by (the bounded extension of) $\left(\frac{D+i}{|D|}\right)^p$ yields the first two inclusions.

The second pair of inclusions follow from Hypothesis 1.2.1.(iii): we have $\delta^k(a)(D+i)^{-p-1} \in \mathcal{L}_1$ and $\partial(\delta^k(a))(D+i)^{-p-1} \in \mathcal{L}_1$. Then simply multiplying by $\left(\frac{D+i}{|D|}\right)^{p+1}$ again yields the result. □

Note that the preceding lemma showed that $\partial(a)|D|^{-p} \in \mathcal{L}_{1,\infty}$. We require a little more effort to show that $\delta(a)|D|^{-p} \in \mathcal{L}_{1,\infty}$.

LEMMA 3.1.2. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. Suppose D has a spectral gap at 0. Then for all $a \in \mathcal{A}$, we have $\delta(a)|D|^{-p} \in \mathcal{L}_{1,\infty}$.*

PROOF. Using (2.2), we have the following equality of operators on H_∞:

$$\begin{aligned} [|D|^{-1}, \partial(a)] &= -|D|^{-1}[|D|, \partial(a)]|D|^{-1} \\ &= -|D|^{-1}\partial(\delta(a))|D|^{-1}. \end{aligned} \tag{3.1}$$

Where the last equality uses the fact that δ and ∂ commute (from Lemma 2.2.6). Similarly,

$$\begin{aligned} [|D|^{-1}, \partial(\delta(a))] &= -|D|^{-1}[|D|, \partial(\delta(a))]|D|^{-1} \\ &= -|D|^{-1}\partial(\delta^2(a))|D|^{-1}. \end{aligned} \tag{3.2}$$

Additionally, working with operators on H_∞:

$$\begin{aligned} |D|^{-1}[D^2, a] &= |D|^{-1} \cdot (D\partial(a) + \partial(a)D) \\ &= F\partial(a) + |D|^{-1}\partial(a)D \\ &= F\partial(a) + \partial(a)|D|^{-1}D + [|D|^{-1}, \partial(a)]D. \end{aligned}$$

Now applying (3.1):

$$\begin{aligned} |D|^{-1}[D^2, a] &= F\partial(a) + \partial(a)F - |D|^{-1}\partial(\delta(a))F \\ &= F\partial(a) + \partial(a)F - \partial(\delta(a))|D|^{-1}F - [|D|^{-1}, \partial(\delta(a))]F \end{aligned}$$

then using (3.2):

$$|D|^{-1}[D^2, a] = F\partial(a) + \partial(a)F - \partial(\delta(a))D^{-1} + |D|^{-1}\partial(\delta^2(a))D^{-1}.$$

So multiplying on the right by $|D|^{-p}$:

$$|D|^{-1}[D^2,a]|D|^{-p} = \Big(F\cdot\partial(a)|D|^{-p} + \partial(a)|D|^{-p}\cdot F\Big) + \Big(-\partial(\delta(a))|D|^{-p-1}\cdot F + |D|^{-1}\cdot\partial(\delta^2(a))|D|^{-p-1}\cdot F\Big).$$

From Remark 3.1.1, the first summand extends to an operator in $\mathcal{L}_{1,\infty}$ and the second summand extends to an operator in $\mathcal{L}_1$. Hence, the operator $|D|^{-1}[D^2,a]|D|^{-p}$ has extension to an operator in $\mathcal{L}_{1,\infty}$.

On the other hand since $|D|^2 = D^2$, we have (again, as operators on H_∞)

$$\begin{aligned}|D|^{-1}[D^2,a] &= |D|^{-1}[|D|^2,a]\\ &= |D|^{-1}\cdot(|D|\delta(a)+\delta(a)|D|)\\ &= \delta(a)+|D|^{-1}\delta(a)|D|\\ &= \delta(a)+\delta(a)|D|^{-1}|D| + [|D|^{-1},\delta(a)]|D|\\ &= 2\delta(a) - |D|^{-1}\delta^2(a)\\ &= 2\delta(a)-\delta^2(a)|D|^{-1} - [|D|^{-1},\delta^2(a)]\\ &= 2\delta(a)-\delta^2(a)|D|^{-1}+|D|^{-1}\delta^3(a)|D|^{-1}.\end{aligned}$$

So multiplying by $|D|^{-p}$:

$$|D|^{-1}[D^2,a]|D|^{-p} = 2\delta(a)|D|^{-p} - \delta^2(a)|D|^{-p-1} + |D|^{-1}\delta^3(a)|D|^{-p-1}.$$

By Remark 3.1.1, the operators $\delta^2(a)|D|^{-p-1}$ and $\delta^3(a)|D|^{-p-1}$ are in $\mathcal{L}_1$. Since $|D|^{-1}[D^2,a]|D|^{-p}$ has extension to an operator in $\mathcal{L}_{1,\infty}$, it follows that $2\delta(a)|D|^{-p}\in\mathcal{L}_{1,\infty}$. ☐

Still working with the assumption that D has a spectral gap at 0, the following lemma is a refinement of the $\mathcal{L}_{1,\infty}$ inclusions in Remark 3.1.1 and the result of Lemma 3.1.2. The following result should be compared with [**11**, Lemma 1.37], which is of a similar nature but is stated in terms of Schatten ideals rather than weak Schatten ideals. There is a substantial difference between Schatten ideals and weak Schatten ideals, necessitating the introduction of new tools: here we use logarithmic submajorisation and the Araki-Lieb-Thirring inequality.

A related antecedent to the following lemma is also [**55**, Proposition 10], which proved a result similar to the first assertion in the setting of local spectral triples.

Lemma 3.1.3. *Let $(\mathcal{A},H,D)$ be a smooth $p-$dimensional spectral triple satisfying Hypothesis 1.2.1. Suppose D has a spectral gap at* 0*. For every $a\in\mathcal{A}$ and for every $0<s\le p$, we have $a|D|^{-s}\in\mathcal{L}_{\frac{p}{s},\infty}$, $\partial(a)|D|^{-s}\in\mathcal{L}_{\frac{p}{s},\infty}$, and $\delta(a)|D|^{-s}\in\mathcal{L}_{\frac{p}{s},\infty}$.*

Proof. We prove here only the third statement: that $\delta(a)|D|^{-s}\in\mathcal{L}_{p/s,\infty}$, the other results can be proved similarly.

Let $r=\frac{p}{s}\ge 1$. By the Araki-Lieb-Thirring inequality (2.1),

$$|\delta(a)|D|^{-s}|^r \prec\prec_{\log} |\delta(a)|^r|D|^{-p}.$$

Due to (2.4) (the $\mathcal{L}_{1,\infty}$ quasi-norm is monotone with respect to logarithmic submajorisation)

$$\begin{aligned}\||\delta(a)|D|^{-s}|^r\|_{1,\infty} &\le e\||\delta(a)|^r|D|^{-p}\|_{1,\infty}\\ &\le e\|\delta(a)\|_\infty^{r-1}\|\delta(a)|D|^{-p}\|_{1,\infty}.\end{aligned}$$

Hence,

$$\|\delta(a)|D|^{-s}\|_{r,\infty}^{r} \leq e\|\delta(a)\|_{\infty}^{r-1}\|\delta(a)|D|^{-p}\|_{1,\infty}.$$

By Lemma 3.1.2, the right hand side is finite, and so $\delta(a)|D|^{-s} \in \mathcal{L}_{\frac{p}{s},\infty}$.

To prove the first two statements, one applies the same proof but with Remark 3.1.1 in place of Lemma 3.1.2. □

So far the results of this section have been stated with the assumption that D is invertible. The following proposition shows how we can apply these results to a spectral triple where D may not have a spectral gap at zero, by finding a spectral triple with very similar properties but where the corresponding operator D is invertible. A similar proposition appeared in the Remark following Definition 2.2 of [**13**].

PROPOSITION 3.1.4. *Let $(\mathcal{A}, H, D)$ be a spectral triple, and define $D_0 := F(1+D^2)^{1/2}$ with $\mathrm{dom}(D_0) = \mathrm{dom}(D)$. Then $(\mathcal{A}, H, D_0)$ is spectral triple, and:*

(i) *$(\mathcal{A}, H, D_0)$ is p-dimensional if and only if $(\mathcal{A}, H, D)$ is p-dimensional;*
(ii) *Let δ_0 denote the bounded extension of $[|D_0|, T]$, and define $\mathrm{dom}_\infty(\delta_0)$ identically to $\mathrm{dom}_\infty(\delta)$ with D_0 in place of D. Then we have $\mathrm{dom}_\infty(\delta_0) = \mathrm{dom}_\infty(\delta)$.*
(iii) *$(\mathcal{A}, H, D_0)$ is smooth if and only if $(\mathcal{A}, H, D)$ is smooth;*
(iv) *$(\mathcal{A}, H, D_0)$ satisfies Hypothesis 1.2.1 if and only if $(\mathcal{A}, H, D)$ does.*

Moreover, D_0 has a spectral gap at 0.

PROOF. First, note that $\mathrm{dom}(D_0^n) = \mathrm{dom}(D^n)$ for all $n \geq 1$ and therefore that the space H_∞ is identical for D_0 and for D. It is clear that D_0 has a spectral gap at 0, since $|D_0| = (1+D^2)^{1/2} \geq 1$. Since $D_0^2 = 1 + D^2$, we have $|D_0| = (1+D^2)^{1/2}$, and $F|D_0| = D_0$. As $|D_0| \geq 1$, the operator $|D_0| + |D|$ is invertible. Furthermore, since $|D_0|^2 = |D|^2 + 1$, we have:

$$\frac{1}{|D_0| + |D|} = |D_0| - |D|.$$

Multiplying by F we obtain

$$\frac{F}{|D_0| + |D|} = D_0 - D.$$

So both $|D_0| - |D|$ and $D_0 - D$ extend to bounded operators. Moreover, since for all $k \geq 1$,

$$\frac{1}{|D_0| + |D|} : \mathrm{dom}(D^k) \to \mathrm{dom}(D^{k+1}) \subseteq \mathrm{dom}(D^k)$$

we have that the bounded extensions of $D_0 - D$ and $|D_0| - |D|$ map $\mathrm{dom}(D^k)$ to $\mathrm{dom}(D^k)$ for all $k \geq 1$.

For $T \in \mathcal{L}_\infty(H)$, let $\partial_1(T)$ denote the commutator of the bounded extension of $D_0 - D$ with T, $\partial_1(T) := [\frac{F}{|D_0|+|D|}, T]$. Similarly, let $\delta_1(T)$ denote the commutator of the bounded extension of $|D_0| - |D|$ with T, $\delta_1(T) := [\frac{1}{|D_0|+|D|}, T]$.

Then we have the following identity on H_∞:

$$[D_0, a] = \partial_1(a) + \partial(a).$$

Since $\partial(a)$ and $\partial_1(a)$ are bounded, it follows that $[D_0, a]$ extends to a bounded linear operator, which we denote $\partial_0(a)$.

Since $D_0^2 = D^2 + 1$, we have that the operator $(D+i)(D_0+i)^{-1}$ has bounded extension. Hence, for all $a \in \mathcal{A}$ we have that $a(D_0+i)^{-1}$ is compact. This completes the proof that $(\mathcal{A}, H, D_0)$ is a spectral triple.

One may similarly prove (i): to see that $(\mathcal{A}, H, D_0)$ is p-dimensional if $(\mathcal{A}, H, D)$ is p-dimensional, we write:

$$a(D_0+i)^{-p} = a(D+i)^{-p} \cdot \left(\frac{D+i}{D_0+i}\right)^p$$

which is in $\mathcal{L}_{1,\infty}$, because $(D+i)(D_0+i)^{-1}$ has bounded extension. Next,

$$\partial_0(a)(D_0+i)^{-p} = \left(\partial(a)(D+i)^{-p} + \partial_1(a)(D+i)^{-p}\right)\left(\frac{D+i}{D_0+i}\right)^p.$$

and $\partial_1(a)(D+i)^{-p} = (D_0-D)a(D+i)^{-p} - a(D+i)^{-p}(D_0-D)$, hence $\partial_0(a)(D_0+i)^{-p} \in \mathcal{L}_{1,\infty}$ and so $(\mathcal{A}, H, D_0)$ is p-dimensional. The reverse implication may be established by an identical argument using the fact that $(D_0+i)(D+i)^{-1}$ has bounded extension.

Next we prove (ii). We have already shown that $|D_0| - |D|$ is an operator with bounded extension and which maps $\mathrm{dom}(D^k)$ to $\mathrm{dom}(D^k)$, for all $k \geq 1$. By verifying the identity on H_∞, we have:

$$\delta^k(\delta_1(T)) = \delta_1(\delta^k(T)).$$

Hence if $T \in \mathrm{dom}(\delta^k)$, then $\delta_1(T) \in \mathrm{dom}(\delta^k)$.

If $T \in \mathrm{dom}(\delta)$, then $[|D_0|, T] = \delta_1(T) + \delta(T)$ on H_∞. So if $T \in \mathrm{dom}_\infty(\delta)$ we can compute the kth iterated commutator of T with $|D_0|$ as:

$$\begin{aligned}[|D_0|, [|D_0|, [\cdots, [|D_0|, T]\cdots]]] &= (\delta+\delta_1)^k(T) \\ &= \sum_{j=0}^{k} \binom{k}{j} \delta_1^{k-j}(\delta^j(T))\end{aligned} \tag{3.3}$$

Thus the kth iterated commutator of $|D_0|$ and T has bounded extension, so $T \in \mathrm{dom}_\infty(\delta_0)$. Repeating the proof using the identity $[|D|, T] = \delta_0(T) - \delta_1(T)$, we also have that $\mathrm{dom}_\infty(\delta_0) \subseteq \mathrm{dom}_\infty(\delta)$. This completes the proof of (ii).

Now we prove (iii). Note that if $T \in \mathrm{dom}_\infty(\delta_0)$, then $\partial_1(T) \in \mathrm{dom}_\infty(\delta_0)$. Hence, if

$$a,\, \partial(a) \in \mathrm{dom}_\infty(\delta) = \mathrm{dom}_\infty(\delta_0)$$

then

$$a,\, \partial(a) + \partial_1(a) \in \mathrm{dom}_\infty(\delta_0).$$

Since $\partial_0(a) = \partial(a) + \partial_1(a)$, this completes the proof that if $(\mathcal{A}, H, D)$ is smooth then $(\mathcal{A}, H, D_0)$ is smooth. For the converse, we use $\partial_0(a) = \partial(a) - \partial_1(a)$.

It now remains to show (iv). Assume that $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1. From (3.3), we have that

$$\begin{aligned}\delta_0^k(a)(D_0+i\lambda)^{-p-1} &= \delta_0^k(a)(D+i\lambda)^{-p-1}\left(\frac{D+i\lambda}{D_0+i\lambda}\right)^{p+1} \\ &= \left(\sum_{l=0}^{k}\binom{k}{l}\delta_1^{k-l}(\delta^l(a))(D+i\lambda)^{-p-1}\right)\left(\frac{D+i\lambda}{D_0+i\lambda}\right)^{p+1}.\end{aligned}$$

However since $|D_0| - |D|$ commutes with functions of D,

$$\delta_0^k(a)(D_0 + i\lambda)^{-p-1} = \left(\sum_{l=0}^{k} \binom{k}{l} \delta_1^{k-l}(\delta^l(a)(D + i\lambda)^{-p-1})\right) \left(\frac{D + i\lambda}{D_0 + i\lambda}\right)^{p+1}.$$

Now since the operator $\frac{D+i\lambda}{D_0+i\lambda}$ is bounded, and δ_1 is a commutator with a bounded operator, and since $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1,

$$\|\delta^l(a)(D + i\lambda)^{-p-1}\|_1 = O(\lambda^{-1}), \quad \lambda > 0$$

it follows that

$$\|\delta_0^l(a)(D_0 + i\lambda)^{-p-1}\|_1 = O(\lambda^{-1}), \quad \lambda > 0.$$

Similarly, by writing $\partial_0 = \partial_1 + \partial$, we also obtain:

$$\|\partial_0(\delta_0^k(a))(D_0 + i\lambda)^{-p-1}\|_1 = O(\lambda^{-1}).$$

To prove the converse, we write $\delta(a) = \delta_0 - \delta_1$ and repeat the same argument. □

We are now able to prove Proposition 3.1.5 – without any assumptions on the invertibility of D. Similar results are well known in the unital case (see e.g. [**12**, Lemma 1], [**33**, Lemma 10.18] and [**2**, Lemma 5]). In the non-unital setting, a related result is [**11**, Proposition 2.14] which instead proves that $[F, a] \in \mathcal{L}_{p+1}$. To the best of our knowledge, no complete proof of the following result has been published in the non-unital setting.

Proposition 3.1.5. *If $(\mathcal{A}, H, D)$ is a $p-$dimensional spectral triple satisfying Hypothesis 1.2.1, then $[F, a] \in \mathcal{L}_{p,\infty}$ for all $a \in \mathcal{A}$.*

Proof. Let $D_0 = F(1 + D^2)^{1/2}$, so that by Proposition 3.1.4, the spectral triple $(\mathcal{A}, H, D_0)$ satisfies Hypothesis 1.2.1. As an equality of operators on H_∞, we have:

$$\begin{aligned}[F, a] &= [D_0|D_0|^{-1}, a] \\ &= [D_0, a]|D_0|^{-1} + D_0[|D_0|^{-1}, a].\end{aligned}$$

Using (2.2),

$$[F, a] = [D_0, a]|D_0|^{-1} - F[|D_0|, a]|D_0|^{-1}.$$

Since the spectral triple $(\mathcal{A}, H, D_0)$ satisfies Hypothesis 1.2.1 and has a spectral gap at 0, we may apply Lemma 3.1.3 with $s = p$ to conclude that the operators $[D_0, a]|D_0|^{-1}$ and $[|D_0|, a]|D_0|^{-1}$ have extension to operators in $\mathcal{L}_{p,\infty}$. Thus, $[F, a] \in \mathcal{L}_{p,\infty}$. □

3.2. Restatement of Hypothesis 1.2.1

In this section, we introduce the operator Λ, formally defined by:

$$\Lambda(T) = (1 + D^2)^{-\frac{1}{2}}[D^2, T].$$

Strictly speaking, $\Lambda(T)$ will be defined to be the bounded extension of the above operator. What is here denoted Λ appeared in the unital settings of [**24**, Appendix B] (there denoted L), [**13**, Definition 6.5] (there denoted L_1) and [**33**, Equation 10.66] (there denoted L). The mapping Λ was also used in the non-unital setting of [**11**, Definition 1.20] (there called L). We undertake a self-contained development of these ideas, since our assumptions are different to those used in previous work. There is not any substantial conceptual difference between the proofs for the unital and nonunital cases, however there are small technical obstacles which require care to be taken when computing repeated integrals of operator valued functions (see the proof of Lemma 3.2.5). An expert reader familiar with this theory could skip to Hypothesis 3.2.7.

We must take care to ensure that $\Lambda(T)$ is well defined, as well as that higher powers $\Lambda^k(T)$ are defined. For this purpose we introduce the spaces $\mathrm{dom}(\Lambda^k)$.

DEFINITION 3.2.1. *Let $k \geq 1$. We define $\mathrm{dom}(\Lambda^k)$ to be the set of bounded linear operators T such that for all $1 \leq j \leq k$, we have $T : \mathrm{dom}(D^{2j}) \to \mathrm{dom}(D^{2j})$ and such that the kth iterated commutator,*

$$(1+D^2)^{-1/2}[D^2,(1+D^2)^{-1/2}[D^2,\cdots,T]] : \mathrm{dom}(D^{2k}) \to H$$

has bounded extension, which we denote $\Lambda^k(T)$.

Define

$$\mathrm{dom}_\infty(\Lambda) := \bigcap_{k\geq 0} \mathrm{dom}(\Lambda^k).$$

The mapping Λ can be thought of as a replacement for δ, and we introduce it since it is easier to work with Λ rather than δ in the examples covered in this chapter.

DEFINITION 3.2.2. *A spectral triple $(\mathcal{A}, H, D)$ is called Λ-smooth if for all $a \in \mathcal{A}$ we have,*

$$a,\, \partial(a) \in \mathrm{dom}_\infty(\Lambda).$$

We will show that $\mathrm{dom}_\infty(\Lambda) = \mathrm{dom}_\infty(\delta_0)$, and so in view of Theorem 3.1.4.(ii) the notion of Λ-smoothness is identical to smoothness. This fact is well known in the unital setting, similar results having appeared in [**24**, Appendix B] and [**13**, Proposition 6.5]. We provide a full proof here since to the best of our knowledge no published proof exists in the non-unital setting.

The easiest direction to establish is that $\mathrm{dom}_\infty(\delta_0) \subseteq \mathrm{dom}_\infty(\Lambda)$, as the following Lemma shows:

LEMMA 3.2.3. *We have $\mathrm{dom}_\infty(\delta_0) \subseteq \mathrm{dom}_\infty(\Lambda)$.*

PROOF. Let $T \in \mathrm{dom}_\infty(\delta_0)$. We have $T : \mathrm{dom}(D^k) \to \mathrm{dom}(D^k)$ for all $k \geq 1$, and so working on H_∞, we can write,

$$\begin{aligned}(1+D^2)^{-1/2}[D^2,T] &= |D_0|^{-1}[|D_0|^2,T] \\ &= 2[|D_0|,T] - |D_0|^{-1}[|D_0|,[|D_0|,T]] \\ &= 2\delta_0(T) - |D_0|^{-1}\delta_0^2(T).\end{aligned}$$

By assumption $T \in \mathrm{dom}_\infty(\delta)$, hence, $\Lambda(T)$ has bounded extension and so $T \in \mathrm{dom}(\Lambda)$, and on all H we have:

$$\Lambda(T) = 2\delta_0(T) - |D_0|^{-1}\delta_0^2(T).$$

However since $\delta_0(T), \delta_0^2(T)$ and $|D_0|^{-1}$ are in $\mathrm{dom}_\infty(\delta_0)$, it follows that $\Lambda(T) \in \mathrm{dom}_\infty(\delta_0)$.

Hence, $\Lambda(T) \in \mathrm{dom}(\Lambda)$, and continuing by induction we get that $T \in \mathrm{dom}(\Lambda^k)$ for all $k \geq 1$. □

It takes some more work to prove that $\mathrm{dom}_\infty(\Lambda) \subseteq \mathrm{dom}_\infty(\delta_0)$. We achieve this by an integral representation of $\delta_0(T)$ in terms of $\Lambda(T)$ and $\Lambda^2(T)$. We make use of the dense subspace H_∞ from Definition 2.2.10. The following Lemma should be compared with the proof of [**24**, Lemma B2].

LEMMA 3.2.4. *Let $T \in \mathrm{dom}_\infty(\Lambda)$. Then for all $\xi \in H_\infty$ we have:*

$$[|D_0|, T]\xi = \frac{1}{2}\Lambda(T)\xi + \frac{1}{\pi}\int_0^\infty \lambda^{1/2}\frac{D_0^2}{(\lambda + D_0^2)^2}\Lambda^2(T)\frac{1}{\lambda + D_0^2}\xi \, d\lambda.$$

The integral above may be understood as a weak operator topology integral.

PROOF. This is essentially a combination of the following two well known integral formulae:

$$(1 + D^2)^{-1/2} = \frac{1}{\pi}\int_0^\infty \frac{1}{1 + \lambda + D^2}\frac{d\lambda}{\lambda^{1/2}} \tag{3.4}$$

and

$$(1 + D^2)^{-1/2} = \frac{2}{\pi}\int_0^\infty \frac{\lambda^{1/2}}{(1 + \lambda + D^2)^2} d\lambda \tag{3.5}$$

which can both be understood as integrals in the weak operator topology, since

$$\left\| \frac{1}{1 + \lambda + D^2} \right\|_\infty \leq \frac{1}{1 + \lambda}, \quad \lambda > 0$$

and

$$\left\| \frac{\lambda^{1/2}}{(1 + \lambda + D^2)^2} \right\|_\infty \leq \frac{\lambda^{1/2}}{(1 + \lambda)^2}, \quad \lambda > 0.$$

Let $\xi \in H_\infty$. Multiplying (3.4) by $(1 + D^2)\xi$, we get:

$$(1 + D^2)^{1/2}\xi = \frac{1}{\pi}\int_0^\infty \frac{1 + D^2}{1 + \lambda + D^2}\xi\frac{d\lambda}{\lambda^{1/2}}. \tag{3.6}$$

The above is a convergent Bochner integral in H, since

$$\left\| \frac{1 + D^2}{1 + \lambda + D^2}\xi \right\|_H \leq \frac{1}{1 + \lambda}\|(1 + D^2)\xi\|_H.$$

Now, replacing $1 + D^2 = D_0^2$ by (2.2):

$$\begin{aligned}\left[\frac{1}{\lambda + D_0^2}, T\right] &= -(\lambda + D_0^2)^{-1}[D^2, T](\lambda + D_0^2)^{-1} \\ &= -\frac{D_0^2}{\lambda + D_0^2}\Lambda(T)(\lambda + D_0^2)^{-1}.\end{aligned} \tag{3.7}$$

Hence,

$$\begin{aligned}\left[\frac{D_0^2}{\lambda + D_0^2}, T\right] &= \left[1 - \frac{\lambda}{\lambda + D_0^2}, T\right] \\ &= \frac{\lambda(D_0^2)}{\lambda + D_0^2}\Lambda(T)(\lambda + D_0^2)^{-1} \\ &= \frac{\lambda(D_0^2)}{\lambda + D_0^2}([\Lambda(T), (\lambda + D_0^2)^{-1}] + (\lambda + D_0^2)^{-1}\Lambda(T)).\end{aligned}$$

Applying (3.7) a second time:

$$\begin{aligned}\left[\frac{D_0^2}{\lambda + D_0^2}, T\right] &= \frac{\lambda D_0^2}{\lambda + D_0^2}\Big(\frac{D_0^2}{\lambda + D_0^2}\Lambda^2(T)(\lambda + D_0^2)^{-1} + (\lambda + D_0^2)^{-1}\Lambda(T)\Big)\\ &= \frac{\lambda|D_0|}{(\lambda + D_0^2)^2}\Lambda(T) + \frac{\lambda D_0^2}{(\lambda + D_0^2)^2}\Lambda^2(T)(\lambda + D_0^2)^{-1}.\end{aligned}$$

Now we apply the integral formula (3.6) to obtain:

$$\begin{aligned}[|D_0|, T] &= \frac{1}{\pi}\int_0^\infty \left[\frac{D_0^2}{\lambda + D_0^2}, T\right]\frac{d\lambda}{\lambda^{1/2}}\\ &= \frac{1}{\pi}\int_0^\infty \frac{\lambda|D_0|}{(\lambda + D_0^2)^2}\Lambda(T)\frac{d\lambda}{\lambda^{1/2}}\\ &\quad + \frac{1}{\pi}\int_0^\infty \frac{\lambda D_0^2}{(\lambda + D_0^2)^2}\Lambda^2(T)(\lambda + D_0^2)^{-1}\frac{d\lambda}{\lambda^{1/2}}.\end{aligned}$$

Now applying (3.5) again, we have:

$$\int_0^\infty \lambda^{1/2}\frac{|D_0|}{(\lambda + D_0^2)^2}d\lambda = \frac{\pi}{2}.$$

Hence,

$$[|D_0|, T]\xi = \frac{1}{2}\Lambda(T) + \frac{1}{\pi}\int_0^\infty \lambda^{1/2}\frac{D_0^2}{(\lambda + D_0^2)^2}\Lambda^2(T)\frac{1}{\lambda + D_0^2}\xi\, d\lambda.$$

□

The following lemma provides an integral representation of the nth iterated commutator $\delta_0^n(T)$. This will allow us to relate $\mathrm{dom}_\infty(\delta_0)$ to $\mathrm{dom}_\infty(\Lambda)$. We need to take care to ensure that the relevant version of a Fubini's theorem applies.

Lemma 3.2.5. *For all $m \geq 1$, and $T \in \mathrm{dom}_\infty(\Lambda)$. Then for all $\xi \in H_\infty$ the mth iterated commutator of $|D_0|$*

$$\begin{aligned}[|D_0|, [|D_0|, [\cdots[|D_0|, T]\cdots]]]\xi = 2^{-m}\sum_{k=0}^{m}\binom{m}{k}\Big(\frac{2}{\pi}\Big)^k \int_{\mathbb{R}_+^k}\prod_{l=1}^{k}\frac{\lambda_l^{1/2}D_0^2}{(\lambda_l + D_0^2)^2}\\ \cdot\Lambda^{m+k}(T)\prod_{l=1}^{k}\frac{1}{\lambda_l + D_0^2}\xi d\lambda_1 d\lambda_2\cdots d\lambda_k.\end{aligned}$$

Proof. Let

$$\Theta(T) := \int_0^\infty \lambda^{1/2}\frac{D_0^2}{(\lambda + D_0^2)^2}\Lambda^2(T)\frac{1}{\lambda + D_0^2}d\lambda$$

so that Lemma 3.2.4 states that $\delta_0 = \frac{1}{2}\Lambda + \frac{1}{\pi}\Theta$.

Since Λ commutes with $\frac{D_0^2}{(\lambda + D_0^2)^2}$ and $\frac{1}{\lambda + D_0^2}$, we have $\Theta \circ \Lambda = \Lambda \circ \Theta$. Hence,

$$\delta_0^m = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{2}{\pi}\right)^k \Theta^k \circ \Lambda^{m-k}. \tag{3.8}$$

By the Fubini theorem for Hilbert space valued functions (see [**29**, Theorem III.11.13]), for all $\xi \in H_\infty$ we have:

$$\Theta^k(T)\xi = \int_{[0,\infty)^k} \prod_{l=1}^{k} \frac{\lambda_l^{1/2} D_0^2}{(\lambda_l + D_0^2)^2} \Lambda^{2k}(T) \prod_{l=1}^{k} \frac{1}{\lambda_l + D_0^2} \xi d\lambda_1 d\lambda_2 \cdots d\lambda_k.$$

Therefore,

$$\Theta^k(\Lambda^{m-k}(T))\xi = \int_{[0,\infty)^k} \prod_{l=1}^{k} \frac{\lambda_l^{1/2} D_0^2}{(\lambda_l + D_0^2)^2} \Lambda^{m+k}(T) \prod_{l=1}^{k} \frac{1}{\lambda_l + D_0^2} \xi d\lambda_1 d\lambda_2 \cdots d\lambda_k.$$

Substituting into (3.8) yields the result. □

The following corollary was already noted in the unital settings of [**24**, Appendix B], [**13**, Proposition 6.5] and in the non-unital setting of [**11**, Equation 1.12].

Corollary 3.2.6. *We have* $\mathrm{dom}_\infty(\Lambda) = \mathrm{dom}_\infty(\delta_0)$*, and* Λ*-smoothness of a spectral triple is equivalent to smoothness as stated in Definition 2.2.7*

Proof. From Lemma 3.2.3 we already know that $\mathrm{dom}_\infty(\delta_0) \subseteq \mathrm{dom}_\infty(\Lambda)$, and so we concentrate on the reverse inclusion.

If $T \in \mathrm{dom}_\infty(\Lambda)$, then for each $k \geq 1$ the operator $\Lambda^k(T)$ on H_∞ has bounded extension. Hence the integral in Lemma 3.2.5 converges as a Bochner integral, and so the mth iterated commutator $\delta_0^m(T)$ is bounded, for all $m \geq 0$. Thus $T \in \mathrm{dom}_\infty(\delta_0)$, and this completes the proof. □

The remainder of this section is devoted to showing that Hypothesis 1.2.1 is equivalent to the following:

Hypothesis 3.2.7. *The spectral triple* $(\mathcal{A}, H, D)$ *satisfies the following conditions:*

(i) $(\mathcal{A}, H, D)$ *is a* Λ*-smooth spectral triple.*
(ii) $(\mathcal{A}, H, D)$ *is* $p-$*dimensional, i.e., for every* $a \in \mathcal{A}$,

$$a(D+i)^{-p} \in \mathcal{L}_{1,\infty}, \quad \partial(a)(D+i)^{-p} \in \mathcal{L}_{1,\infty}.$$

(iii) *For every* $a \in \mathcal{A}$ *and for all* $k \geq 0$*, we have*

$$\left\| \Lambda^k(a)(D+i\lambda)^{-p-1} \right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty,$$

$$\left\| \partial(\Lambda^k(a))(D+i\lambda)^{-p-1} \right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty.$$

Hypothesis 3.2.7 is precisely Hypothesis 1.2.1, but with smoothness replaced by Λ-smoothness, and the occurances of δ replaced with Λ.

For the next two lemmas, we borrow techniques that were developed in [**11**]. The next Lemma shows that if $(\mathcal{A}, H, D)$ satisfies Hypothesis 3.2.7 then $(\mathcal{A}, H, D_0)$ satisfies Hypothesis 1.2.1.

Lemma 3.2.8. *Let* $(\mathcal{A}, H, D)$ *be a spectral triple satisfying Hypothesis 3.2.7. For every* $a \in \mathcal{A}$ *and for all* $m \geq 0$*, we have*

$$\left\| \delta_0^m(a)(D_0+i\lambda)^{-p-1} \right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty,$$

$$\left\| \partial_0(\delta_0^m(a))(D_0+i\lambda)^{-p-1} \right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty.$$

Proof. We prove only the first assertion. The proof of the second assertion is similar.

By the spectral theorem,

$$\begin{aligned}\left\|\prod_{l=1}^{k}\frac{\lambda_l^{1/2}(1+D^2)}{(1+\lambda_l+D^2)^2}\right\|_\infty &\leq \prod_{l=1}^{k}\left\|\frac{\lambda_l^{1/2}(1+D^2)}{(1+\lambda_l+D^2)^2}\right\|_\infty\\ &\leq \prod_{l=1}^{k}\sup_{t_l\geq 1}\frac{\lambda_l^{1/2}t_l}{(\lambda_l+t_l)^2}\\ &\leq \prod_{l=1}^{k}\lambda_l^{-1/2}\end{aligned}$$

and also

$$\left\|\prod_{l=1}^{k}\frac{1}{1+\lambda_l+D^2}\right\|_\infty \leq \prod_{l=1}^{k}\frac{1}{1+\lambda_l}.$$

Hence, for all $a\in\mathcal{A}$, since $(D+i\lambda)^{-1}$ and $(1+\lambda_l+D^2)^{-1}$ commute,

$$\begin{aligned}&\left\|\left(\prod_{l=1}^{k}\frac{\lambda_l^{1/2}(1+D^2)}{(1+\lambda_l+D^2)^2}\right)\Lambda^{m+k}(a)\left(\prod_{l=1}^{k}\frac{1}{1+\lambda_l+D^2}\right)(D+i\lambda)^{-p-1}\right\|_1\\ &\qquad\leq \|\Lambda^{m+k}(a)(D+i\lambda)^{-p-1}\|_1\prod_{l=1}^{k}\frac{1}{\lambda_l^{1/2}(1+\lambda_l)}.\end{aligned}$$

Now applying Lemma 3.2.5 with Lemma 2.3.2,

$$\begin{aligned}&\|\delta_0^m(a)(D+i\lambda)^{-p-1}\|_1\\ &\leq 2^{-m}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{2}{\pi}\right)^k\|\Lambda^{m+k}(a)(D+i\lambda)^{-p-1}\|_1\int_{[0,\infty)^k}\prod_{l=1}^{k}\frac{d\lambda_l}{\lambda_l^{1/2}(1+\lambda_l)}.\end{aligned}$$

Since $\int_0^\infty \frac{1}{\lambda^{1/2}(1+\lambda)}\,d\lambda=\pi$, we arrive at

$$\|\delta_0^m(a)(D+i\lambda)^{-p-1}\|_1\leq 2^{-m}\sum_{k=0}^{m}\binom{m}{k}2^k\|\Lambda^{m+k}(a)(D+i\lambda)^{-p-1}\|_1\pi^k.$$

By Hypothesis 3.2.7, each summand above is $O(\lambda^{-1})$. Hence $\|\delta_0^m(a)(D+i\lambda)^{-p-1}\|_1 = O(\lambda^{-1})$.

Now using the fact that the operator $\left(\frac{D+i\lambda}{D_0+i\lambda}\right)^{p+1}$ has bounded extension, and

$$\begin{aligned}\left\|\left(\frac{D+i\lambda}{D_0+i\lambda}\right)^{p+1}\right\|_\infty &\leq \sup_{t\in\mathbb{R}}\left(\frac{t^2+\lambda^2}{1+t^2+\lambda^2}\right)^{\frac{p+1}{2}}\\ &\leq 1,\end{aligned}$$

we get:

$$\begin{aligned}\|\delta_0^m(a)(D_0+i\lambda)^{-p-1}\|_1 &\leq \|\delta_0(a)(D+i\lambda)^{-p-1}\|_1\left\|\left(\frac{D+i\lambda}{D_0+i\lambda}\right)^{p+1}\right\|_\infty\\ &= O(\lambda^{-1})\end{aligned}$$

as desired.

□

We can now conclude the proof of the main result of this subsection:

THEOREM 3.2.9. *A spectral triple $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1 if and only if it satisfies Hypothesis 3.2.7.*

PROOF. We have already proved that $(\mathcal{A}, H, D)$ satisfies 3.2.7.(i) if and only if it satisfies 1.2.1.(i), and (3.2.7).(ii) is identical to (1.2.1).(ii). We now focus on (3.2.7).(iii).

Suppose that $(\mathcal{A}, H, D)$ satisfies Hypothesis 3.2.7. By Lemma 3.2.8, we have that $(\mathcal{A}, H, D_0)$ satisfies Hypothesis 1.2.1. Then by Proposition 3.1.4 $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1.

Now we prove the converse. Suppose that $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1. For $T \in \mathrm{dom}_\infty(\delta)$, we define $\alpha(T)$ and $\beta(T)$ by:

$$\alpha(T) := \frac{|D|}{(D^2+1)^{1/2}}\delta(T),$$
$$\beta(T) := \frac{1}{(D^2+1)^{1/2}}\delta^2(T).$$

We can express Λ in terms of α and β, by applying the Leibniz rule as follows:

$$\begin{aligned}\Lambda(T) &= (1+D^2)^{-1/2}[|D|^2, T]\\ &= \frac{|D|}{(1+D^2)^{1/2}}\delta(T) + (1+D^2)^{-1/2}\delta(T)|D|\\ &= 2\alpha(T) - (1+D^2)^{-1/2}\delta^2(T)\\ &= 2\alpha(T) - \beta(T).\end{aligned}$$

Since $\alpha \circ \beta = \beta \circ \alpha$,

$$\Lambda^m = \sum_{k=0}^{m} \binom{m}{k} (-1)^k 2^{m-k} \beta^k \circ \alpha^{m-k}.$$

For every $k = 0, \dots, m$ and $T \in \mathrm{dom}_\infty(\delta)$, we have:

$$\beta^k(\alpha^{m-k}(T)) = \frac{|D|^{m-k}}{(D^2+1)^{m/2}}\delta^{m+k}(T).$$

So for $a \in \mathcal{A}$ and $m \geq 1$,

$$\Lambda^m(a) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k 2^{m-k} \frac{|D|^{m-k}}{(D^2+1)^{m/2}}\delta^{m+k}(a).$$

Noting that the operator $\frac{|D|^{m-k}}{(D^2+1)^{m/2}}$ is bounded, there exists a constant C_m such that

$$\|\Lambda^m(a)(D+i\lambda)^{-p-1}\|_1 \leq C_m \sum_{k=0}^{m} \|\delta^{m+k}(a)(D+i\lambda)^{-p-1}\|_1$$

Since we are assuming that $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1, it follows that $\|\Lambda^m(a)(D+i\lambda)^{-p-1}\|_1 = O(\lambda^{-1})$.

We may similarly deal with $\|\partial(\Lambda^m(a))(D+i\lambda)^{-p-1}\|_1$: since ∂ commutes with functions of D:

$$\partial(\Lambda^m(a)) = \sum_{k=0}^{m}(-1)^k 2^{m-k}\frac{|D|^{m-k}}{(D^2+1)^{m/2}}\partial(\delta^{m+k}(a)).$$

Thus by the same argument, we have $\|\partial(\Lambda^m(a))(D+i\lambda)^{-p-1}\|_1 = O(\lambda^{-1})$, and so $(\mathcal{A}, H, D)$ satisfies Hypothesis 3.2.7. □

Thanks to Theorem 3.2.9, we can work assuming Hypothesis 3.2.7 rather than Hypothesis 1.2.1.

3.3. Example: Noncommutative Euclidean space

We now discuss the most heavily studied example of a non-unital spectral triple: noncommutative Euclidean space. Subsection 3.3.1 covers the definitions of noncommutative Euclidean spaces and their associated spectral triples. Subsection 3.3.2 is devoted to the proof that these spectral triples satisfy Hypothesis 1.2.1.

Noncommutative Euclidean spaces can be found in the literature under various names, such as canonical commutation relation (CCR) algebras (as in [**9**, Section 5.2.2.2]), and in the 2-dimensional case are called Moyal planes or Moyal-Groenwald planes (as in [**31**]).

3.3.1. Definitions for Noncommutative Euclidean spaces. Our approach to noncommutative Euclidean space is to proceed from the Weyl commutation relations, in line with [**9**, Section 5.2.2.2] and [**42**]. An alternative approach is to use the Moyal product, as in [**31**] and [**11**, Section 5.2]. We caution the reader that the approach considered here is the "Fourier dual" of the approach in [**31**]. We briefly cite the required facts needed for this section, and refer the reader to [**42**] for detailed exposition and proofs.

Let θ be an antisymmetric real $p\times p$ matrix. Abstractly, the von Neumann algebra $L_\infty(\mathbb{R}^p_\theta)$ is generated by a strongly continuous family $\{U(t)\}_{t\in\mathbb{R}^p}$ satisfying

$$U(t+s) = \exp\left(\frac{1}{2}i(t,\theta s)\right)U(t)U(s), \quad t,s\in\mathbb{R}^p. \tag{3.9}$$

Here we avoid technicalities by defining $L_\infty(\mathbb{R}^p_\theta)$ to be a von Neumann algebra generated by a specific family of unitary operators on $L_2(\mathbb{R}^p)$.

Definition 3.3.1. *Let θ be an antisymmetric real matrix. For $t\in\mathbb{R}^p$, let $U(t)$ be the linear operator on $L_2(\mathbb{R}^p)$ given by:*

$$(U(t)\xi)(r) = \exp\left(-i(t,\theta r)\right)\xi(r-t), \quad r\in\mathbb{R}^p,\, \xi\in L_2(\mathbb{R}^p).$$

Define $L_\infty(\mathbb{R}^p_\theta)$ to be the von Neumann subalgebra of $\mathcal{L}_\infty(L_2(\mathbb{R}^p))$ generated by the family $\{U(t)\}_{t\in\mathbb{R}^p}$.

Remark 3.3.2. *It can easily be shown that $U(t)$ satisfies* (3.9)*. Since $U(t)$ is a composition of a translation, and pointwise multiplication by $\exp\left(-i\frac{1}{2}(t,\theta s)\right)$, it is clear that each $U(t)$ is unitary, and that $t\mapsto U(t)$ is strongly continuous. Since θ is antisymmetric, $U(-t) = U(t)^{-1} = U(t)^*$.*

The map $t\mapsto U(t)$ is a twisted left-regular representation of $\mathbb{R}^p$ on $L_2(\mathbb{R}^p)$, in the sense of [**36**].

Note that if $\theta = 0$, then the family $\{U(t)\}_{t\in\mathbb{R}^p}$ is simply the semigroup of translations on $\mathbb{R}^p$, and so generates the von Neumann algebra $L_\infty(\mathbb{R}^p)$.

If θ is nondegenerate (that is, $\det(\theta) \neq 0$) then p is even and the algebra $L_\infty(\mathbb{R}^p_\theta)$ is isomorphic to $\mathcal{L}_\infty(L_2(\mathbb{R}^{p/2}))$. This is proved in [**42**], where a spatial isomorphism is constructed.

THEOREM 3.3.3. *If $\det(\theta) \neq 0$, then there is a spatial isomorphism*

$$L_\infty(\mathbb{R}^p_\theta) \cong \mathcal{L}_\infty(L_2(\mathbb{R}^{p/2})).$$

We now focus exclusively on the case where $\det(\theta) \neq 0$.

DEFINITION 3.3.4. *The semifinite trace τ_θ on $L_\infty(\mathbb{R}^p)$ is defined via the isomorphism in Theorem 3.3.3 to be simply the classical trace* Tr *on $\mathcal{L}_\infty(L_2(\mathbb{R}^{p/2}))$. For $r \in [1, \infty)$, the space $L_r(\mathbb{R}^p_\theta)$ is defined by:*

$$L_r(\mathbb{R}^p_\theta) := \{x \in L_\infty(\mathbb{R}^p_\theta) \ : \ \tau_\theta(|x|^r) < \infty\}.$$

The space $L_r(\mathbb{R}^p_\theta)$ is equipped with the norm $\|x\|_{L_r} = \tau_\theta(|x|^r)^{1/r}$.

Note that $L_r(\mathbb{R}^p_\theta)$ is identical to the Schatten-von Neumann class $\mathcal{L}_r(L_2(\mathbb{R}^{p/2}))$, since τ_θ is simply the classical trace.

DEFINITION 3.3.5. *For $k = 1, \ldots, p$, we define the operator D^θ_k on $L_2(\mathbb{R}^p)$ by*

$$(D^\theta_k \xi)(t) = t_k \xi(t), \quad t \in \mathbb{R}^p.$$

The Dirac operator D^θ is defined on the Hilbert space $L_2(\mathbb{R}^d, \mathbb{C}^{2^{p/2}})$ by $D^\theta = \gamma_1 \otimes D^\theta_1 + \cdots + \gamma_p \otimes D^\theta_p$, where $\gamma_1, \gamma_2, \ldots \gamma_p$ are complex $2^{p/2} \times 2^{p/2}$ matrices satisfying $\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{j,k}$ and $\gamma_j = \gamma_j^*$ for $1 \leq j, k \leq p$.

Evidently, the operators D_j are unbounded, but may be initially defined on the dense subspace of compactly supported functions.

It follows readily from the definitions of D^θ_k and $U(t)$ that

$$[D^\theta_k, U(t)] = t_k U(t), \quad t \in \mathbb{R}^p.$$

Since the operators $D^\theta_1, \ldots D^\theta_p$ form a family of mutually commuting self-adjoint operators, we may apply functional calculus to define $e^{i(s,\nabla)}$, $s \in \mathbb{R}^p$. where $\nabla^\theta = (D^\theta_1, D^\theta_2, \ldots, D^\theta_p)$, given by:

$$(e^{i(s,\nabla^\theta)}\xi)(r) = \exp(i(s,r))\xi(r), \quad r \in \mathbb{R}^p.$$

Hence,

$$e^{i(s,\nabla^\theta)} U(s) e^{-i(s,\nabla^\theta)} = e^{i(s,t)} U(s), \quad s, t \in \mathbb{R}^p.$$

For convenience we also introduce the notation $\Delta^\theta := \sum_{k=1}^p (D^\theta_k)^2$.

The following is [**42**, Proposition 6.12]:

LEMMA 3.3.6. *Let $k = 1, \ldots, p$. If $x \in L_\infty(\mathbb{R}^p_\theta)$, and the operator $[D^\theta_k, x]$ has bounded extension, then its extension is an element of $L_\infty(\mathbb{R}^d_\theta)$.*

DEFINITION 3.3.7. *If $x \in L_\infty(\mathbb{R}^d_\theta)$ is such that $[D^\theta_k, x]$ has bounded extension, then we denote $\partial_k x$ for the extension.*

We denote $\partial^0_j x := x$, for all $x \in L_\infty(\mathbb{R}^p_\theta)$ and j.

Generally, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ be a multi-index. If for all $1 \leq j \leq p$ the operator

$$\partial_j^{\alpha_j}(\partial_{j+1}^{\alpha_{j+1}}(\cdots(\partial_p^{\alpha_p}(x))\cdots))$$

has bounded extension, then the mixed partial derivative $\partial^\alpha x$ is defined as the operator:

$$\partial_1^{\alpha_1}(\partial_2^{\alpha_2}(\cdots(\partial_p^{\alpha_p}(x))\cdots)).$$

By Lemma 3.3.6, we always have $\partial^\alpha x \in L_\infty(\mathbb{R}^p_\theta)$ if it is well defined.

DEFINITION 3.3.8. *Let $m \geq 0$ and $r \geq 1$. The space $W^{m,r}(\mathbb{R}^p_\theta)$ is defined to be the set of $x \in L_\infty(\mathbb{R}^p_\theta)$ such that $\partial^\alpha x \in L_r(\mathbb{R}^p_\theta)$ for every $|\alpha| \leq m$, equipped with the norm:*

$$\|x\|_{W^{m,r}} := \sum_{|\alpha|\leq m} \|\partial^\alpha x\|_r.$$

We define $W^{\infty,r}(\mathbb{R}^p_\theta) := \bigcap_{m\geq 0} W^{m,r}(\mathbb{R}^p_\theta)$.

As suggested by the notation, the spaces $W^{m,r}(\mathbb{R}^p_\theta)$ are the analogues of Sobolev spaces for noncommutative Euclidean spaces. The space $W^{\infty,1}(\mathbb{R}^p_\theta)$ is important because it forms a part of our spectral triple for noncommutative Euclidean space.

The remainder of this section is devoted to showing that the triple

$$(1_{2^{p/2}} \otimes W^{\infty,1}(\mathbb{R}^p_\theta), L_2(\mathbb{R}^p, \mathbb{C}^{2^{p/2}}), D^\theta) \tag{3.10}$$

is a spectral triple satisfying Hypothesis 1.2.1.

We wish to verify Hypothesis 1.2.1 for noncommutative spaces in order to support our claim that 1.2.1 is a reasonable assumption to make. However, in the nondegenerate case $\det(\theta) \neq 0$, the Character Theorem 1.2.5 is trivial for at least a dense subalgebra of $W^{\infty,1}(\mathbb{R}^d_\theta)$.

The reason for this is that due to [**31**, Proposition 2.5], there is a dense subalgebra of $W^{\infty,1}(\mathbb{R}^d_\theta)$ isomorphic to the algebra $M_\infty(\mathbb{C})$ of finitely supported infinite matrices. However due to [**43**, Theorem 1.4.14], if $n \geq 0$ then the nth Hochschild homology of $M_\infty(\mathbb{C})$ is computed by:

$$HH_n(M_\infty(\mathbb{C})) = HH_n(\mathbb{C}).$$

For $n > 0$, the nth Hochschild homology of $\mathbb{C}$ is trivial [**33**, Lemma 8.9]. Hence for $n > 0$, the nth Hochschild homology of $M_\infty(\mathbb{C})$ is trivial.

This entails that every degree $p+1$ Hochschild cycle of $M_\infty(\mathbb{C})$ is a Hochschild boundary. However the left and right hand sides of the Character Theorem, $c \mapsto \mathrm{Ch}(c)$ and $c \mapsto \varphi(\Omega(c)(1+D^2)^{-p/2})$, are both Hochschild cocycles and hence vanish on any Hochschild boundary.

3.3.2. Verification of Hypothesis 1.2.1 for Noncommutative Euclidean spaces. Now we prove that the triple (3.10) is a spectral triple satisfying Hypothesis 1.2.1. In fact it is easier to use Hypothesis 3.2.7.

Our main reference for this section is [**42**, Section 7]. As in that reference, the spaces $\ell_1(L_\infty)$ and $\ell_{1,\infty}(L_\infty)$ are defined as follows: Let $K = [0,1]^p$ be the unit p-cube. Then $\ell_1(L_\infty)$ and $\ell_{1,\infty}(L_\infty)$ are the subspaces of $L_\infty(\mathbb{R}^d)$ such that the following norms are finite:

$$\|g\|_{\ell_1(L_\infty)} := \|\{\|g\|_{L_\infty(m+K)}\}_{m\in\mathbb{Z}^p}\|_{\ell_1(\mathbb{Z}^p)},$$
$$\|g\|_{\ell_{1,\infty}(L_\infty)} := \|\{\|g\|_{L_\infty(m+K)}\}_{m\in\mathbb{Z}^p}\|_{\ell_{1,\infty}(\mathbb{Z}^p)}.$$

The following is a special case of [**42**, Theorem 7.6, Theorem 7.7]:

THEOREM 3.3.9. *Let $p \geq 1$. There are constants $c_p > 0$ and $c'_p > 0$ such that for all $x \in W^{p,1}(\mathbb{R}^p_\theta)$ we have:*

(a) *If $g \in \ell_1(L_\infty)$, then $xg(\nabla^\theta) \in \mathcal{L}_1$, and*

$$\|xg(\nabla^\theta)\|_1 \leq c_p \|x\|_{W^{p,1}} \|g\|_{\ell_1(L_\infty)}.$$

(b) *If $g \in \ell_{1,\infty}(L_\infty)$, then $xg(\nabla^\theta) \in \mathcal{L}_{1,\infty}$ and*

$$\|xg(\nabla^\theta)\|_{1,\infty} \leq c_p' \|x\|_{W^{p,1}} \|g\|_{\ell_{1,\infty}(L_\infty)}.$$

With Theorem 3.3.9 at hand we can prove the following:

THEOREM 3.3.10. *Let $p \geq 1$. Then there exist constants $c_p > 0$ and $c_p' > 0$ such that for all $x \in W^{p,1}(\mathbb{R}^p_\theta)$ we have:*

(a) $(1 \otimes x)(D^\theta + i\lambda)^{-p-1} \in \mathcal{L}_1$ *and*

$$\|(1 \otimes x)(D^\theta + i\lambda)^{-p-1}\|_1 \leq c_p \frac{\|x\|_{W^{p,1}}}{\lambda},$$

(b) $(1 \otimes x)(D^\theta + i)^{-p} \in \mathcal{L}_{1,\infty}$ *and*

$$\|(1 \otimes x)(D^\theta + i)^{-p}\|_{1,\infty} \leq c_p' \|x\|_{W^{p,1}}.$$

PROOF. Let $g(t) := (\lambda^2 + \sum_{k=1}^p t_k^2)^{-(p+1)/2}$. Since $\|ab\|_1 = \|a|b^*|\|_1$,

$$\begin{aligned} \|(1 \otimes x)(D^\theta + /i\lambda)^{-p-1}\|_1 &= \|(1 \otimes x)|(D^\theta - i\lambda)^{-p-1}|\|_1 \\ &= \|(1 \otimes x)((D^\theta)^2 + \lambda^2)^{-\frac{p+1}{2}}\|_1. \end{aligned}$$

So we have

$$\|(1 \otimes x)(D^\theta + i\lambda)^{-p-1}\|_1 = c_p \|xg(\nabla^\theta)\|_1.$$

It can be directly verified that $\|g\|_{\ell_1(L_\infty)} = O(\lambda^{-1})$, we can immediately apply Theorem 3.3.9 to obtain (a).

To obtain (b), we instead consider the function $g(t) = (1 + \sum_{k=1}^p t_k^2)^{-p/2}$ and apply Theorem 3.3.9.(b). □

Recall the operator Λ from Section 3.2, defined formally as $\Lambda(T) = (1 + D^2)^{-1/2}[D^2, T]$.

LEMMA 3.3.11. *If $x \in W^{\infty,1}(\mathbb{R}^p_\theta)$, then for all $m \geq 0$:*

$$\left\|\Lambda^m(1 \otimes x)(D^\theta + i\lambda)^{-p-1}\right\|_1 = O(\lambda^{-1}), \quad \lambda \to \infty.$$

PROOF. We prove the assertion by induction on m. Since by definition Λ^0 is the identity, the $m = 0$ case is handled by Theorem 3.3.10.(a).

Now suppose that $m \geq 1$ and the assertion holds for $m - 1$. Since $(D^\theta)^2 = 1 \otimes \sum_{k=1}^p (D_k^\theta)^2$, we have

$$\Lambda(1 \otimes x) = (1 + (D^\theta)^2)^{-\frac{1}{2}} \cdot (\sum_{k=1}^p 1 \otimes [(D_k^\theta)^2, x]).$$

Applying the Leibniz rule,

$$\begin{aligned} [(D_k^\theta)^2, x] &= [D_k^\theta, x]D_k^\theta + D_k^\theta[D_k^\theta, x] \\ &= 2D_k^\theta[D_k^\theta, x] - [D_k^\theta, [D_k^\theta, x]]. \end{aligned}$$

By assumption, (the bounded extensions of) $[D_k^\theta, x]$ and $[D_k^\theta, [D_k^\theta, x]]$ are in $W^{m,1}(\mathbb{R}^p_\theta)$ for all $m \geq 0$.

Hence since Λ commutes with ∂,

$$\Lambda^m(1\otimes x) = \sum_{k=1}^{p}\Big(1\otimes \frac{2D_k^\theta}{(1-\Delta^\theta)^{\frac{1}{2}}}\Big)\cdot\Lambda^{m-1}(1\otimes[D_k^\theta,x]) \\ - \sum_{k=1}^{p}\Big(1\otimes \frac{1}{(1-\Delta^\theta)^{\frac{1}{2}}}\Big)\cdot\Lambda^{m-1}(1\otimes[D_k^\theta,[D_k^\theta,x]]).$$

So by the triangle inequality, we have

$$\big\|\Lambda^m(1\otimes x)(D^\theta+i\lambda)^{-p-1}\big\|_1 \le 2\sum_{k=1}^{p}\big\|\Lambda^{m-1}(1\otimes[D_k^\theta,x])(D^\theta+i\lambda)^{-p-1}\big\|_1 \\ + \sum_{k=1}^{p}\|\Lambda^{m-1}(1\otimes[D_k^\theta,[D_k^\theta,x]])(D^\theta+i\lambda)^{-p-1}\Big\|_1.$$

The right hand side is $O(\lambda^{-1})$ as $\lambda\to\infty$ by the inductive assumption. Hence, so is the left hand side. □

We can now conclude with the main result of this subsection:

THEOREM 3.3.12. *The triple*

$$(1\otimes W^{\infty,1}(\mathbb{R}^p,\theta), L_2(\mathbb{R}^d,\mathbb{C}^{2^{p/2}}), D^\theta)$$

is a spectral triple satisfying Hypothesis 1.2.1.

PROOF. We establish Hypothesis 3.2.7 instead, as permitted by Theorem 3.2.9. First we prove that we indeed have a spectral triple.

By the definition of $W^{\infty,1}(\mathbb{R}^p)$, if $x\in W^{\infty,1}(\mathbb{R}^p)$ then $[D^\theta,1\otimes x]$ has bounded extension, and therefore $1\otimes x:\mathrm{dom}(D^\theta)\to\mathrm{dom}(D^\theta)$.

If $x\in W^{\infty,1}(\mathbb{R}^p_\theta)$ then:

$$\partial(1\otimes x) = \sum_{j=1}^{p}\gamma_j\otimes(\partial_j x)$$

and this is bounded, by the definition of $W^{\infty,1}$.

Now we show that Hypothesis 3.2.7.(i) holds. If $x\in W^{\infty,1}(\mathbb{R}^p_\theta)$, we show that $x,\partial(x)\in\mathrm{dom}(\Lambda^m)$ for all $m\ge 0$ by induction. We have automatically that $x,\partial(x)\in\mathrm{dom}(\Lambda^0)$. Now if we assume that $x,\partial(x)\in\mathrm{dom}(\Lambda^{m-1})$, for $m\ge 1$, we apply the Leibniz rule to obtain:

$$\Lambda^m(1\otimes x) = \sum_{k=1}^{p}\Big(1\otimes \frac{2D_k^\theta}{(1-\Delta^\theta)^{\frac{1}{2}}}\Big)\cdot\Lambda^{m-1}(1\otimes[D_k^\theta,x]) \\ - \sum_{k=1}^{p}\Big(1\otimes \frac{1}{(1-\Delta^\theta)^{\frac{1}{2}}}\Big)\cdot\Lambda^{m-1}(1\otimes[D_k^\theta,[D_k^\theta,x]]).$$

By the definition of $W^{\infty,1}(\mathbb{R}^p_\theta)$, the operators $[D_k^\theta,x]$ and $[D_k^\theta,[D_k^\theta,x]]$ have bounded extension, and by Lemma 3.3.6, the extensions of $[D_k^\theta,x]$ and $[D_k^\theta,[D_k^\theta,x]]$ are elements of $W^{\infty,1}(\mathbb{R}^p_\theta)$, and therefore by the inductive hypothesis are in $\mathrm{dom}(\Lambda^{m-1})$. Hence, $1\otimes x\in\mathrm{dom}(\Lambda^m)$ and so by induction $1\otimes x\in\mathrm{dom}_\infty(\Lambda)$. Applying an identical argument to $\partial(1\otimes x)$ yields $\partial(1\otimes x)\in\mathrm{dom}_\infty(\Lambda)$, and so $(1\otimes W^{\infty,1}(\mathbb{R}^p_\theta), L_2(\mathbb{R}^d,\mathbb{C}^{2^{p/2}}), D^\theta)$ is Λ-smooth.

We now show that Hypothesis 3.2.7.(ii) holds. Let $x \in W^{\infty,1}(\mathbb{R}^p_\theta)$. By Lemma 3.3.10.(a), the first inclusion in Hypothesis 3.2.7.(ii) follows.

To see the second inclusion in Hypothesis 3.2.7.(ii), write

$$\partial(1\otimes x)(D^\theta+i)^{-p} = \sum_{k=1}^p (\gamma_k\otimes 1)\cdot\left((1\otimes[D^\theta_k,x])(D^\theta+i)^{-p}\right).$$

Using the quasi-triangle inequality for $\mathcal{L}_{1,\infty}$, there is a constant C_p such that

$$\|\partial(1\otimes x)(D^\theta+i)^{-p}\|_{1,\infty} \le C_p\sum_{k=1}^p \|(1\otimes\partial_k x)(D^\theta+i)^{-p}\|_{1,\infty}.$$

By the definition of $W^{\infty,1}(\mathbb{R}^p_\theta)$, for all $1\le k\le p$ we have $\partial_k x\in W^{\infty,1}(\mathbb{R}^p_\theta)$, so we may apply Theorem 3.3.10.(b) to each summand to deduce the second inclusion in Hypothesis 3.2.7.(ii).

Now we discuss Hypothesis 3.2.7.(iii). By Lemma 3.3.11, the first inequality in Hypothesis 3.2.7.(iii) holds.

To deduce the second inequality, we may commute ∂ with Λ^m to obtain:

$$\partial(\Lambda^m(1\otimes x))(D^\theta+i\lambda)^{-p-1} = \sum_{k=1}^p (\gamma_k\otimes 1)\cdot\left(\Lambda^m(1\otimes[D^\theta_k,x]\right)(D^\theta+i\lambda)^{-p-1}.$$

Note that here $\partial(T)$ denotes $[D^\theta,T]$. Using the $\mathcal{L}_1$-norm triangle inequality,

$$\|\partial(\Lambda^m(1\otimes x))(D^\theta+i\lambda)^{-p-1}\|_1 \le C_p\sum_{k=1}^p \|\Lambda^m(1\otimes\partial_k x)(D^\theta+i\lambda)^{-p-1}\|_1.$$

By assumption, each $\partial_k x$ is in $W^{\infty,1}(\mathbb{R}^p_\theta)$, and so by Lemma 3.3.11, each summand above is $O(\lambda^{-1})$ as $\lambda\to\infty$. □

REMARK 3.3.13. *We have worked exclusively with the case that* $\det(\theta)\neq 0$. *Of course this excludes the fundamental* $\theta=0$ *case of Euclidean space* $\mathbb{R}^d$. *One may verify directly that the standard spectral triple for* $\mathbb{R}^d$ *satisfies Hypothesis 1.2.1, by using classical Cwikel theory, or alternatively* $\mathbb{R}^d$ *may be considered as a special case of the complete Riemannian manifolds considered in the following section.*

3.4. Example: Riemannian manifolds

The authors wish to thank Professor Yuri Kordyukov for significant contributions to this section, including providing many of the proofs.

3.4.1. Basic notions about manifolds. We briefly recall the relevant definitions for Riemannian manifolds. The material in this subsection is standard, and may be found in for example [**56**, Chapter 2] or [**41**]. Let X be a second countable p-dimensional complete smooth Riemannian manifold with metric tensor g. Recall that g defines a canonical measure ν_g on X. The notation $L_r(X,g)$ denotes $L_r(X,\nu_g)$. The assumption that X is second countable ensures that $L_2(X,g)$ is separable.

We denote the space of smooth compactly supported differential k-forms as $\Omega^k_c(X)$, and define $\Omega_c(X) := \bigoplus_{k=0}^p \Omega^k_c(X)$. Associated to the metric g is an inner product $(\cdot,\cdot)_g$ defined on $\Omega_c(X)$. Let H_k denote the completion of $\Omega^k_c(X)$ with respect to this inner product, and define $L_2\Omega(X,g) := \bigoplus_{k=0}^p H_k$. There is a grading on $L_2\Omega(X,g)$ with grading operator Γ defined by $\Gamma|_{H_k} = (-1)^k$.

For $f \in C_c^\infty(X)$, let M_f denote the operator of pointwise multiplication by f on $L_2\Omega(X,g)$.

The exterior differential d is a linear map $d : \Omega_c(X) \to \Omega_c(X)$ such that for all $k = 0, 1, \ldots, p-1$ we have $d : \Omega_c^k(X) \to \Omega_c^{k+1}(X)$, and $d|\Omega_c^p(X) = 0$. The linear operator d has a formal adjoint d^* with respect to the inner product on $\Omega_c(X)$.

The Hodge-Dirac operator D_g is defined by $D_g := d + d^*$. Since X is complete, the operator D_g uniquely extends to a self-adjoint unbounded operator on $L_2\Omega(X,g)$ (see [**26**]). The Hodge-Laplace operator is defined as $\Delta_g := -D_g^2 = -dd^* - d^*d$, and each subspace H_k is invariant under Δ_g. The restriction of Δ_g to $H_0 = L_2(X,g)$ coincides with the Laplace-Beltrami operator.

The main focus of this section is the following:

THEOREM 3.4.1. *Let (X,g) be a second countable p-dimensional complete Riemannian manifold. The algebra $C_c^\infty(X)$ acts on $L_2\Omega(X,g)$ by pointwise multiplication, and D_g denotes the Hodge-Dirac operator. Then*

$$(C_c^\infty(X), L_2\Omega(X,g), D_g)$$

is an even spectral triple satisfying Hypothesis 1.2.1, where the grading Γ is defined by $\Gamma|_{H_k} = (-1)^k$.

Note that the spectral triple is always even regardless of p. The use of the Hodge-Dirac operator to define spectral triples for arbitrary Riemannian manifolds has previously been studied in [**44**], and for related work see [**30**].

To the best of our knowledge, the main results of this paper, Theorems 1.2.2, 1.2.3 and 1.2.5 are new in the setting of the Hodge-Dirac operator on arbitrary complete manifolds. Most previous work on geometric applications of noncommutative geometry, such as [**55**] and [**11**] are applied to a spin Dirac operator.

The Cwikel-type estimates we establish in this section: Lemma 3.4.8 and 3.4.9, are of interest in their own right. A predecessor to this work may be found in [**55**].

3.4.2. Proof of Theorem 3.4.1. The proof proceeds by showing the required Cwikel-type estimates for the case of a torus: $X = \mathbb{T}^p$ with the flat metric. We then deduce the general case by an argument involving local coordinates.

We define the p-torus as $\mathbb{T}^p := \mathbb{R}^p/\mathbb{Z}^p$. The space $\mathbb{T}^p$ is a smooth p-dimensional manifold, and we may select local coordinates $x_1, \ldots, x_p \in (0,1]$ defined by considering the image of $(x_1, \ldots, x_d)$ in $\mathbb{R}^p/\mathbb{Z}^p$. We equip $\mathbb{T}^p$ with the flat metric g_0 defined locally by $g_0 = dx_1^2 + \cdots + dx_p^2$.

First, we describe the Cwikel-type estimates for $\mathbb{T}^p$.

LEMMA 3.4.2. *Let g_0 denote the flat metric on $\mathbb{T}^p$, with corresponding Hodge-Dirac operator denoted D_0. Then:*

(i) *We have*
$$(D_0 + i)^{-1} \in \mathcal{L}_{p,\infty}(L_2\Omega(\mathbb{T}^p, g_0)).$$
(ii) *For $\lambda \to \infty$, we have:*
$$\|(D_0 + i\lambda)^{-1}\|_{\mathcal{L}_{p+1}(L_2\Omega(\mathbb{T}^p, g_0))} = O(\lambda^{-\frac{1}{p+1}}).$$

Since the manifold $\mathbb{T}^p$ is compact, there is essentially no difference between the spaces $L_2\Omega(\mathbb{T}^p, g)$ for different metrics g as the following lemma shows:

LEMMA 3.4.3. *Let g_0 be the flat metric on $\mathbb{T}^p$, and let g be an arbitrary metric on $\mathbb{T}^p$. Then the Hilbert spaces $L_2\Omega(\mathbb{T}^p, g)$ and $L_2\Omega(\mathbb{T}^p, g_0)$ coincide with an* *equivalence of norms. To be precise, there exist constants* $0 < c_g < C_g < \infty$ *with* $v \in L_2\Omega(\mathbb{T}^p, g)$ *we have:*

$$c_g\|v\|_{L_2\Omega(\mathbb{T}^p,g_0)} \leq \|v\|_{L_2\Omega(\mathbb{T}^p,g)} \leq C_g\|v\|_{L_2\Omega(\mathbb{T}^p,g_0)}.$$

PROOF. The metric ν_g corresponding to g has Radon-Nikodym derivative $\sqrt{|\det(g)|}$ with respect to ν_{g_0}. Since $\mathbb{T}^p$ is compact and g is positive definite, the Radon-Nikodym derivative $\sqrt{|\det(g)|}$ is bounded above and bounded away from zero. Hence, the L_2-norms corresponding to ν_g and ν_{g_0} are equivalent. □

Sobolev spaces on $\mathbb{T}^p$ – and more generally on a compact Riemannian manifold – are defined following [**41**]:

DEFINITION 3.4.4. *Let g be a metric on $\mathbb{T}^p$, and for $j = 1, \ldots, p$ we let $\frac{\partial}{\partial x_j}$ denote the differentiation with respect to the jth coordinate of $\mathbb{T}^p$. The Sobolev space $H^1\Omega(\mathbb{T}^p)$ is defined to be the set of $v \in L_2\Omega(\mathbb{T}^p, g)$ such that the Sobolev norm:*

$$\|v\|^2_{H^1\Omega(\mathbb{T}^p,g)} := \|v\|^2_{L_2\Omega(\mathbb{T}^p,g)} + \sum_{j=1}^{p} \left\|\frac{\partial v}{\partial x_j}\right\|^2_{L_2\Omega(\mathbb{T}^p,g)}$$

is finite.

Lemma 3.4.3 shows that the space $H^1\Omega(\mathbb{T}^p)$ is independent of the choice of metric used to define the Sobolev norm, since different metrics will define equivalent norms.

The following result is the well known Gårding's inequality. A proof for general compact manifolds may be found in [**56**, Theorem 2.44] and for general elliptic operators on compact manifolds in [**41**, Theorem 5.2].

LEMMA 3.4.5. *Let g be a metric on $\mathbb{T}^p$, and let D_g denote the corresponding Dirac operator. Then there is a constant $C_g > 0$ such that for all $v \in L_2\Omega(\mathbb{T}^p, g)$ such that $D_g v \in L_2\Omega(\mathbb{T}_p, g)$, we have*

$$\|v\|_{H^1\Omega(\mathbb{T}^p,g)} \leq C_g(\|v\|_{L_2\Omega(\mathbb{T}^p)} + \|D_g v\|_{L_2\Omega(\mathbb{T}^p,g)}).$$

The following lemma is the essential technical result allowing us to transfer the Cwikel-type estimates for $\mathbb{T}^p$ with the flat metric to $\mathbb{T}^p$ with an arbitrary metric.

LEMMA 3.4.6. *Let g be an arbitrary metric on $\mathbb{T}^p$, and let g_0 be the flat metric on $\mathbb{T}^p$. Denote by D_g and D_0 the Hodge-Dirac operators corresponding to g and g_0 respectively. Then the operator*

$$D_0(D_g + i)^{-1}$$

defined initially on $\Omega(\mathbb{T}^p)$ has bounded extension to $L_2\Omega(\mathbb{T}^p, g)$ (or equivalently $L_2\Omega(\mathbb{T}^p, g_0)$).

PROOF. By the definition of the Sobolev norm $\|\cdot\|_{H^1\Omega(\mathbb{T}^p,g)}$, for all $v \in H^1\Omega(\mathbb{T}^p, g)$ we have:

$$\|D_0 v\|^2_{L_2\Omega(\mathbb{T}^p,g)} \leq \|v\|^2_{H^1\Omega(\mathbb{T}^p,g)}.$$

Thus by Lemma 3.4.5, there is a constant C such that,

$$\|D_0 v\|_{L_2\Omega(\mathbb{T}^p,g)} \leq C(\|D_g v\|_{L_2\Omega(\mathbb{T}^p,g)} + \|v\|_{L_2\Omega(\mathbb{T}^p,g)}).$$

Hence, if $v \in \operatorname{dom}(D_g)$ then $v \in \operatorname{dom}(D_0)$, and so if $u \in L_2\Omega(\mathbb{T}^p, g)$ and $v = (D_g + i)^{-1}u$ then $v \in \operatorname{dom}(D_0)$. So substituting $v = (D_g + i)^{-1}u$ we obtain:

$$\|D_0(D_g + i)^{-1}u\|_{L_2\Omega(\mathbb{T}^p,g)} \leq C(1 + \|(D_g + i)^{-1}\|_{\mathcal{L}_\infty(L_2\Omega(\mathbb{T}^p,g))})\|u\|_{L_2\Omega(\mathbb{T}^p,g)}.$$

However since $(D_g+i)^{-1}$ is bounded, we get:

$$\|D_0(D_g+i)^{-1}u\|_{L_2\Omega(\mathbb{T}^p,g)} \leq C\|u\|_{L_2\Omega(\mathbb{T}^p,g)}.$$

□

As a consequence of Lemmas 3.4.2 and 3.4.6, we get:

Corollary 3.4.7. *Let g be an arbitrary metric on $\mathbb{T}^p$ and let D_g be the corresponding Hodge-Dirac operator. Then,*

(i) *We have,*

$$(D_g+i)^{-1} \in \mathcal{L}_{p,\infty}(L_2\Omega(\mathbb{T}^p,g)).$$

(ii) *For $\lambda \to \infty$, we have*

$$\|(D_g+i\lambda)^{-1}\|_{\mathcal{L}_{p+1}(L_2\Omega(\mathbb{T}^p,g))} = O(\lambda^{-\frac{1}{p+1}}).$$

Proof. Part (i) follows immediately from Lemma 3.4.2 and Lemma 3.4.6.i.

Now we prove (ii). First we compute $(D_0+i\lambda)(D_g+i\lambda)^{-1}$ (working on the dense domain $\Omega(\mathbb{T}^p)$):

$$\begin{aligned}(D_0+i\lambda)(D_g+i\lambda)^{-1} &= (D_0-D_g+D_g+i\lambda)(D_g+i\lambda)^{-1}\\ &= 1+(D_0-D_g)(D_g+i\lambda)^{-1}\\ &= 1+D_0(D_g+i)^{-1}\frac{D_g+i}{D_g+i\lambda}-\frac{D_g}{D_g+i\lambda}.\end{aligned}$$

By Lemma 3.4.6, the operator $D_0(D_g+i)^{-1}$ has bounded extension. Moreover, by functional calculus the operators $\frac{D_g+i}{D_g+i\lambda}$ and $\frac{D_g}{D_g+i\lambda}$ have bounded extension with norm bounded by a constant independent of λ. Thus,

$$\|(D_0+i\lambda)(D_g+i\lambda)^{-1}\|_{\mathcal{L}_\infty(L_2\Omega(\mathbb{T}^p,g))} = O(1).$$

Now Lemma 3.4.2.(ii) yields the result. □

With the above results in hand we are able to establish our first Cwikel-type result for (X,g). A similar result to Lemma 3.4.8 can also be found in [**55**, Proposition 13]. The result in [**55**] is very similar in nature (despite applying to the somewhat different situation of Riemannian spin manifolds). The method of proof here is different: we use reduce the problem to $\mathbb{T}^p$ by using local coordinates, rather than a doubling construction as employed in [**55**].

Lemma 3.4.8. *Let $f \in C_c^\infty(X)$. We have*

$$M_f(D_g+i)^{-1} \in \mathcal{L}_{p,\infty}(L_2\Omega(X,g)).$$

Proof. By using a partition of unity if necessary, we may assume without loss of generality that f is supported in a single chart (U,h) where $h : U \to \mathbb{R}^p$ is a homeomorphism onto its image, and since f has compact support we may further assume without loss of generality that $h(U)$ is bounded. Since $h(U)$ is bounded, there is a sufficiently large box $[-N,N]^p$ with $h(U)$ in the interior of $[-N,N]^p$. By applying a translation and dilation if necessary, we may assume without loss of generality that $h(U)$ is contained within the interior of the box $[0,1]^p$. By identifying the edges of $[0,1]^p$, we may view h as a continuous function $h : U \to \mathbb{T}^p$.

We define three smooth "cut-off" functions ϕ_1, ϕ_2, ϕ_3 compactly supported in $h(U)$, defined so that for each $j = 1,2,3$ we have $0 \leq \phi_j \leq 1$, and

(a) for all $x \in U$, $\phi_1(h(x))f(x) = f(x)$,

(b) we have $\phi_2\phi_1 = \phi_1$,
(c) we have $\phi_3\phi_2 = \phi_2$.

In other words, on $\mathrm{supp}(f \circ h^{-1})$ we have $\phi_1 = 1$, and on $\mathrm{supp}(\phi_1)$ we have $\phi_2 = 1$ and on $\mathrm{supp}(\phi_2)$ we have $\phi_3 = 1$.

For $j = 1, 2, 3$, we also define the function ψ_j by pulling back ϕ_j to X:

$$\psi_j(x) = \begin{cases} (\phi_j \circ h)(x), & x \in U. \\ 0, & x \notin U. \end{cases}$$

Since ϕ_j is compactly supported in $h(U)$, the function ψ_j is smooth and compactly supported in U.

Let g_0 denote the flat metric on $\mathbb{T}^p$. The metric g can be pushed forward by h to a metric h^*g on $h(U)$. We then define a new metric g_1 on $\mathbb{T}^p$ by:

$$g_1 := (h^*g)\phi_3 + g_0(1 - \phi_3).$$

Since ϕ_3 is compactly supported in $h(U)$, the metric g_1 is well defined. Moreover, on $\mathrm{supp}(\phi_2)$ we have $\phi_3 = 1$ so on $\mathrm{supp}(\phi_2)$ the metric g_1 is identical to h^*g.

We define a partial isometry $V : L_2\Omega(X, g) \to L_2\Omega(\mathbb{T}^p, g_1)$ on $\xi \in L_2\Omega(X, g)$, $z \in \mathbb{T}^p$ by:

$$V\xi(z) = \begin{cases} \xi \circ h^{-1}(z), & z \in \mathrm{supp}(\phi_2), \\ 0, & z \notin \mathrm{supp}(\phi_2). \end{cases}$$

By construction, V induces an isometry from $L_2\Omega(\mathrm{supp}(\psi_2), g) \to L_2\Omega(\mathrm{supp}(\phi_2), g_1)$, and

$$VV^* = M_{\chi_{\mathrm{supp}\phi_2}},$$
$$V^*V = M_{\chi_{\mathrm{supp}\psi_2}}.$$

We also have that if $j = 1, 2$ then

$$M_{\phi_j}V = VM_{\psi_j}.$$

We use the important fact that for $j = 1, 2$, we have an equality on $\Omega_c(X)$:

$$V^*D_{g_1}M_{\phi_j} = D_gM_{\psi_j}V^*.$$

Next we consider the following two operators on $L_2\Omega(\mathbb{T}^p, g_1)$.

$$P := M_{\phi_1}(D_{g_1} + i)^{-1}M_{\phi_1}$$
$$Q := M_{\phi_2}(D_{g_1} + i)^{-1}M_{\phi_2}.$$

By Lemma 3.4.7, the operators P and Q are in $\mathcal{L}_{p,\infty}(L_2\Omega(\mathbb{T}^p, g_1))$.

We now consider the operator $(D_g+i)M_{\psi_2}V^*PV$. We note that this operator is well defined on $\Omega_c(X)$, since if $u \in \Omega_c(X)$, then $M_{\psi_2}V^*PV$ is smooth and supported in $\mathrm{supp}\psi_2$. Hence (working on $\Omega_c(X)$):

$$\begin{aligned}(D_g + i)M_{\psi_2}V^*PV &= V^*(D_{g_1} + i)M_{\phi_2}PV \\ &= V^*(D_{g_1} + i)M_{\phi_1}(D_{g_1} + i)^{-1}M_{\phi_1}V \\ &= V^*([D_{g_1}, M_{\phi_1}](D_{g_1} + i)^{-1}M_{\phi_1} + M_{\phi_1}^2)V.\end{aligned}$$

Now recalling that D_{g_1} is a local operator, we have $[D_{g_1}, M_{\phi_1}] = [D_{g_1}, M_{\phi_1}]M_{\phi_2}$.

Moreover, $V^*M_{\phi_1}^2V = M_{\psi_1}^2$ and so

$$(D_g + i)M_{\psi_2}V^*PV = V^*[D_{g_1}, M_{\phi_1}]QM_{\phi_1}V + M_{\psi_1}^2.$$

Now multiplying on the left by $(D_g+i)^{-1}$, we arrive at:

$$M_{\psi_2}V^*PV = (D_g+i)^{-1}V^*[D_{g_1},M_{\phi_1}]QM_{\phi_1}V + (D_g+i)^{-1}M_{\psi_1}^2.$$

The final step is to use the fact that $\psi_2 = 1$ on the support of f, so we may use $M_{\psi_2}^2 M_f = M_f$, and multiply on the right by M_f to obtain:

$$M_{\psi_2}V^*PVM_f = (D_g+i)^{-1}V^*[D_1,M_{\phi_1}]QM_{\phi_1}VM_f + (D_g+i)^{-1}M_f.$$

Since both P and Q are in $\mathcal{L}_{p,\infty}(L_2\Omega(\mathbb{T}^p,g_1))$, we finally obtain that $(D_g+i)^{-1}M_f \in \mathcal{L}_{p,\infty}(L_2\Omega(X,g))$. □

LEMMA 3.4.9. *Let* $f \in C_c^\infty(X)$. *Then:*

$$\|M_f(D_g+i\lambda)^{-1}\|_{\mathcal{L}_{p+1}(L_2\Omega(X,g))} = O(\lambda^{-\frac{1}{p+1}}), \quad \lambda\to\infty.$$

PROOF. This proof proceeds along similar lines to Lemma 3.4.8. We again assume without loss of generality that f is supported in a single chart (U,h), and construct the metric g_1 on $\mathbb{T}^p$ and the partial isometry V identically to the proof of Lemma 3.4.8. We also use the same cut-off functions ϕ_1, ϕ_2 and ϕ_3, and ψ_1,ψ_2,ψ_3.

In place of the operators P and Q, we introduce P_λ and Q_λ given by:

$$P_\lambda = M_{\phi_1}(D_{g_1}+i\lambda)^{-1}M_{\phi_1}$$
$$Q_\lambda = M_{\phi_2}(D_{g_1}+i\lambda)^{-1}M_{\phi_2}.$$

Following the argument of Lemma 3.4.8 with P_λ and Q_λ in place of P and Q, we arrive at:

$$M_{\psi_2}V^*P_\lambda VM_f = (D_g+i\lambda)^{-1}V^*[D_1,M_{\phi_1}]Q_\lambda M_{\phi_1}VM_f + (D_g+i\lambda)^{-1}M_f.$$

Due to Lemma 3.4.7.(i), we have $\|P_\lambda\|_{p+1} = O(\lambda^{-\frac{1}{p+1}})$ and similarly for Q_λ. Thus,

$$\|(D_g+i\lambda)^{-1}\|_{\mathcal{L}_{p+1}(L_2\Omega(X,g))} = O(\lambda^{-\frac{1}{p+1}}).$$

□

We now finally have the results necessary to prove Theorem 3.4.1.

PROOF OF THEOREM 3.4.1. We will instead work with Hypothesis 3.2.7, as justified by Theorem 3.2.9.

First, we show that 3.2.7.(ii) holds for the triple $(C_c^\infty(X), L_2\Omega(X,g), D_g)$.

To this end let $f\in C_c^\infty(X)$. We will prove by induction that for all $j\geq 1$, we have

$$M_f(D_g+i)^{-j} \in \mathcal{L}_{\frac{p}{j},\infty}(L_2\Omega(X,g)). \tag{3.11}$$

The case $j=1$ is already established by Lemma 3.4.8.

Suppose now that (3.11) holds for $j\geq 1$. Choose $\phi\in C_c^\infty(X)$ such that $f\phi = f$. Then,

$$\begin{aligned} M_f(D_g+i)^{-j-1} &= M_\phi M_f(D_g+i)^{-1}\cdot(D_g+i)^{-j}\\ &= -M_\phi[(D_g+i)^{-1},M_f](D_g+i)^{-j} + M_\phi(D_g+i)^{-1}M_f(D_g+i)^{-j}\\ &= M_\phi(D_g+i)^{-1}[D,M_f](D_g+i)^{-j-1} + M_\phi(D_g+i)^{-1}M_f(D_g+i)^{-j}\\ &= M_\phi(D_g+i)^{-1}[D,M_f]M_\phi(D_g+i)^{-j-1} + M_\phi(D_g+i)^{-1}M_f(D_g+i)^{-j}.\end{aligned}$$

Due to Lemma 3.4.8, we have $M_\phi(D_g+i)^{-1}\in\mathcal{L}_{p,\infty}$, and by the inductive assumption we also have $M_\phi(D_g+i)^{-j}\in\mathcal{L}_{\frac{p}{j},\infty}$. Then $M_f(D_g+i)^{-j-1}\in\mathcal{L}_{p,\infty}\cdot\mathcal{L}_{\frac{p}{j},\infty}$, so applying the Hölder inequality we arrive at $M_f(D_g+i)^{-j-1} \in \mathcal{L}_{\frac{p}{j+1},\infty}$. Taking $j = p$, we get that $M_f(D_g+i)^{-p} \in \mathcal{L}_{1,\infty}(L_2\Omega(X,g))$.

Similarly, since D_g is a local operator, we have that $[D_g, M_f] = [D_g, M_f]M_\phi$. Hence, $[D_g, M_f](D_g+i)^{-p} \in \mathcal{L}_{1,\infty}(L_2\Omega(X,g))$. This completes the proof of Hypothesis 3.2.7.(ii) in the case $k = 0$.

What remains is to show that Hypothesis 3.2.7.(iii) holds. We will first deal with the $k = 0$ case. To that end, we will show by induction that for all $j \geq 1$:

$$\|M_f(D_g+i\lambda)^{-j}\|_{\frac{p+1}{j}} = O(\lambda^{-\frac{j}{p+1}}), \lambda \to \infty. \tag{3.12}$$

The base case $j = 1$ is the result of Lemma 3.4.9, and the case $j = p+1$ is what is required for Hypothesis 3.2.7.

Suppose now that (3.12) holds for $j \geq 1$, and again choose $\phi \in C_c^\infty(X)$ such that $f\phi = f$. Then,

$$\begin{aligned}
M_f(D_g+i\lambda)^{-j-1} &= M_\phi \cdot M_f(D_g+i\lambda)^{-1}(D_g+i\lambda)^{-j} \\
&= -M_\phi[(D_g+i\lambda)^{-1}, M_f](D_g+i\lambda)^{-j} \\
&\quad + M_\phi(D_g+i\lambda)^{-1}M_f(D_g+i\lambda)^{-j} \\
&= M_\phi(D_g+i\lambda)^{-1}[D, M_f]M_\phi(D_g+i\lambda)^{-j-1} \\
&\quad + M_\phi(D_g+i\lambda)^{-1}M_f(D_g+i\lambda)^{-j}.
\end{aligned}$$

By Lemma 3.4.9, we have $\|M_\phi(D_g+i\lambda)^{-1}\|_{p+1} = O(\lambda^{-\frac{1}{p+1}})$, and by the inductive assumption we also have $\|M_\phi(D_g+i\lambda)^{-k}\|_{\frac{p+1}{j}} = O(\lambda^{-\frac{j}{p+1}})$. Then by the Holder inequality,

$$\|M_f(D_g+i\lambda)^{-j-1}\|_{\frac{p+1}{j+1}} \leq O(\lambda^{-\frac{j}{p+1}}) \cdot O(\lambda^{-\frac{1}{p+1}}).$$

So $\|M_f(D_g+i\lambda)^{-j-1}\|_{\frac{p+1}{j+1}} = O(\lambda^{-\frac{j+1}{p+1}})$. To conclude the same for $\partial(f)$ in place of f, we once more use the fact that $[D_g, M_f]M_\phi = [D_g, M_f]$. This completes the proof of the $k = 0$ case of Hypothesis 3.2.7.(iii).

For $k > 0$, if $\phi f = f$, then we have

$$\Lambda^k(M_f) = \Lambda^k(M_f)M_\phi$$

so we may apply the $k = 0$ case to ϕ to deduce the result.

That $(C_c^\infty(X), L_2\Omega(X,g), D_g)$ satisfies 3.2.7.(i) follows from similar reasoning. $\square$

CHAPTER 4

Asymptotic of the heat trace

In this chapter we complete the proof of Theorem 1.2.2. This will require some delicate computations exploiting Hochschild homology.

For the remainder of this chapter, we assume that $(\mathcal{A}, H, D)$ is a spectral triple satisfying Hypothesis 1.2.1. Furthermore we will need the following auxiliary assumption:

HYPOTHESIS 4.0.1. *The spectral triple $(\mathcal{A}, H, D)$ satisfies the following:*

(i) *$(\mathcal{A}, H, D)$ has the same parity as p: this means that $(\mathcal{A}, H, D)$ is even when p is even (with grading Γ) and odd when p is odd.*

(ii) *D has a spectral gap at 0.*

We will show at the end of this chapter how Hypothesis 4.0.1 can be removed.

Recall from Definition 2.2.15 the fundamental mappings ch and Ω, given by:

$$\mathrm{ch}(a_0 \otimes a_1 \otimes \cdots \otimes a_p) := \Gamma F[F, a_0][F, a_1] \cdots [F, a_p],$$
$$\Omega(a_0 \otimes a_1 \otimes \cdots \otimes a_p) := \Gamma a_0 \partial(a_1) \partial(a_2) \cdots \partial(a_p)$$

and that Theorem 1.2.2 states that for all Hochschild cycles $c \in \mathcal{A}^{\otimes(p+1)}$,

$$\mathrm{Tr}(\Omega(c)(1 + D^2)^{1-\frac{p}{2}} e^{-s^2 D^2}) = \frac{p}{2} \mathrm{Tr}(\mathrm{ch}(c)) s^{-2} + O(s^{-1}), \quad s \to 0.$$

The computations in Sections 4.1 and 4.2 are inspired by those in [**12, 15**]. Since we do not assume that the algebra $\mathcal{A}$ is unital, the computations are more delicate than those of [**12, 15**].

4.1. Combinatorial expression for $\mathrm{Tr}(\Omega(c)|D|^{2-p} e^{-s^2 D^2})$

We begin this section with the introduction of a new set of multilinear maps:

DEFINITION 4.1.1. *Let $\mathscr{A} \subseteq \{1, 2, \ldots, p\}$. We define the multilinear map $\mathcal{W}_{\mathscr{A}} : \mathcal{A}^{\otimes(p+1)} \to \mathcal{L}_\infty$ by:*

$$\mathcal{W}_{\mathscr{A}}(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \Gamma a_0 \prod_{k=1}^{p} b_k(a_k)$$

where for $a \in \mathcal{A}$ and each k we define:

$$b_k(a) = \begin{cases} \delta(a), \ k \in \mathscr{A}, \\ [F, a], \ k \notin \mathscr{A}. \end{cases}$$

In the case where $\mathscr{A} = \{m\}$, a single number, $1 \leq m \leq p$, then write $\mathcal{W}_m$ for $\mathcal{W}_{\{m\}}$.

Since by assumption $(\mathcal{A}, H, D)$ is smooth, the operators $\delta(a_k)$ are defined and bounded, so that $\mathcal{W}_{\mathscr{A}}$ is well defined as a bounded operator.

The two extreme cases, $\mathcal{W}_\emptyset$ and $\mathcal{W}_{\{1,2,\dots,p\}}$ are easily described as:

$$\mathcal{W}_\emptyset(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \Gamma a_0 \prod_{k=1}^{p} [F, a_k]$$

It will be important to observe that if $(\mathcal{A}, H, D)$ has the same parity as p, then $\mathrm{ch}(c) = \mathcal{W}_\emptyset(c) + F\mathcal{W}_\emptyset(c)F$ (this is Lemma 4.3.6).

On the other extreme.

$$\mathcal{W}_{\{1,2,\dots,p\}}(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \Gamma a_0 \prod_{k=1}^{p} \delta(a_k).$$

Associated to a subset $\mathscr{A} \subseteq \{1, 2, \dots, p\}$ we have the number,

$$n_{\mathscr{A}} = |\{(j,k) \in \{1,2,\dots,p\}^2 \ : \ j<k \text{ and } j \in \mathscr{A}, k \neq \mathscr{A}\}|,$$

where $|\cdot|$ denotes the cardinality of a set.

Theorem 4.1.7 (to be stated below) is the main result of this section. Roughly speaking, it shows that one can replace $\Omega(c)$ in $\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2})$ with a sum of $\mathcal{W}_{\mathscr{A}}(c)$ over all subsets $\mathscr{A} \subseteq \{1, \dots, p\}$. However first we need a Lemma which constitutes the core of the proof of Theorem 4.1.7. Most of this section is devoted to the proof of the following lemma, which is split into various parts.

Lemma 4.1.2. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple where D has a spectral gap at* 0. *For all $c \in (\mathcal{A} + \mathbb{C})^{\otimes(p+1)}$, the operator*

$$\left(\Omega(c)|D|^{2-p} - \sum_{\mathscr{A} \subseteq \{1,\cdots,p\}} (-1)^{n_{\mathscr{A}}} \mathcal{W}_{\mathscr{A}}(c) D^{2-|\mathscr{A}|}\right) \cdot |D|^{p-1}.$$

has bounded extension.

Before proving Lemma 4.1.2 above we initially need:

Lemma 4.1.3. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple, where D has a spectral gap at* 0. *For all $c \in (\mathcal{A} + \mathbb{C})^{\otimes(p+1)}$, the operator*

$$\mathcal{W}_{\mathscr{A}}(c) \cdot |D|^{p-|\mathscr{A}|}$$

has bounded extension.

Proof. By linearity it suffices to prove the assertion for elementary tensors. Let $c = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in (\mathcal{A}+\mathbb{C})^{\otimes(p+1)}$ and let $c' = a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \in (\mathcal{A}+\mathbb{C})^{\otimes p}$.

We will prove this assertion by induction on p. If $p = 1$ and $\mathscr{A} = \emptyset$, then

$$\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|} = \Gamma a_0 [F, a_p]|D|$$

and recall that $[F, a_1]|D|$ has a bounded extension $L(a_p)$. On the other hand, if $p = 1$ and $\mathscr{A} = \{1\}$, then

$$\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|} = \Gamma a_0 \delta(a_1) \in \mathcal{L}_\infty.$$

This proves the base of induction.

Next, let $p \geq 2$ and assume that the statement is true for $p-1$. To this end, let $\mathscr{B} \subseteq \{1, \cdots, p-1\}$ be defined by the formula $\mathscr{B} = \mathscr{A} \setminus \{p\}$. There are two distinct cases: when $p \in \mathscr{A}$ and $p \notin \mathscr{A}$.

First suppose $p \in \mathscr{A}$. Here we have $|\mathscr{B}| = |\mathscr{A}| - 1$, and

$$\begin{aligned}\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|} &= \mathcal{W}_{\mathscr{B}}(c')\delta(a_p)|D|^{p-1-|\mathscr{B}|}\\ &= \left(\mathcal{W}_{\mathscr{B}}(c')|D|^{p-1-|\mathscr{B}|}\right)\left(|D|^{-p+1+|\mathscr{B}|}\delta(a_p)|D|^{p-1-|\mathscr{B}|}\right)\end{aligned}$$

The first factor in the right hand side has bounded extension by the inductive assumption. Moreover, the second factor on the right hand side has bounded extension by Lemma A.1.2. This proves step of induction for the case $p \in \mathscr{A}$.

Now we deal with the case where $p \notin \mathscr{A}$. Then $\mathscr{B} = \mathscr{A}$, and

$$\begin{aligned}\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|} &= \mathcal{W}_{\mathscr{B}}(c')L(a_p)|D|^{p-1-|\mathscr{B}|}\\ &= \left(\mathcal{W}_{\mathscr{B}}(c')|D|^{p-1-|\mathscr{B}|}\right)\left(|D|^{-p+1+|\mathscr{B}|}L(a_p)|D|^{p-1-|\mathscr{B}|}\right)\end{aligned}$$

The first factor in the above right hand side is bounded by the inductive assumption. For the second factor, we use the expression $L(a_p) = \partial(a_p) - F\delta(a_p)$. Since $\partial(a_p), \delta(a_p) \in \mathcal{B}$ then it follows from Lemma A.1.2 that the second factor on the right hand side has bounded extension. Hence, $\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|}$ extends to a bounded linear operator.

□

In order to prove Lemma 4.1.2, we introduce two more classes of multilinear functionals.

Definition 4.1.4. *Let $\mathscr{A} \subseteq \{1, 2, \dots, p\}$. We define the multilinear map $\mathcal{R}_{\mathscr{A}} : \mathcal{A}^{\otimes(p+1)} \to \mathcal{L}_\infty$ by*

$$\mathcal{R}_{\mathscr{A}}(a_0 \otimes \cdots \otimes a_p) := \Gamma a_0 \prod_{k=1}^{p} x_k(a_k),$$

where for each $1 \le k \le p$ and $a \in \mathcal{A}$,

$$x_k(a) := \begin{cases} F\delta(a),\ k \in \mathscr{A}, \\ L(a), k \notin \mathscr{A}. \end{cases}$$

We also define the multilinear map $\mathcal{P}_{\mathscr{A}} : \mathcal{A}^{\otimes(p+1)} \to \mathcal{L}_\infty$ by

$$\mathcal{P}_{\mathscr{A}}(a_0 \otimes \cdots \otimes a_p) := \Gamma a_0 \prod_{k=1}^{p} y_k(a_k)$$

where for each $1 \le k \le p$ and $a \in \mathcal{A}$,

$$y_k(a) := \begin{cases} \delta(a),\ k \in \mathscr{A}, \\ L(a),\ k \notin \mathscr{A}. \end{cases}$$

Lemma 4.1.5. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple, where D has a spectral gap at 0. For all $c \in (\mathcal{A} + \mathbb{C})^{\otimes(p+1)}$ and all $\mathscr{A} \subseteq \{1, 2, \dots, p\}$, the operator*

$$\left(\mathcal{R}_{\mathscr{A}}(c) - (-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{A}}(c) \cdot F^{|\mathscr{A}|}\right) \cdot |D|$$

has bounded extension.

Proof. This proof is similar to that of 4.1.3. Once again it suffices to prove the result for an elementary tensor $c = a_0 \otimes a_1 \otimes \cdots \otimes a_p$, and we prove the statement by induction on p. Denote, for brevity, $c' = a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \in (\mathcal{A} + \mathbb{C})^{\otimes p}$.

For the base of induction, when $p = 1$, we deal with the two possibilities $\mathscr{A} = \{1\}$ and $\mathscr{A} = \emptyset$. If $\mathscr{A} = \{1\}$, then

$$\mathcal{R}_{\mathscr{A}}(c) = \Gamma a_0 F\delta(a_1)$$
$$\mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|} = \Gamma a_0\delta(a_1)F.$$

So,

$$\begin{aligned}\left(\mathcal{R}_{\mathscr{A}}(c) - (-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{A}}(c)\cdot F^{|\mathscr{A}|}\right)\cdot|D| &= \Gamma a_0(F\delta(a_1) - \delta(a_1)F)|D| \\ &= \Gamma a_0[F,\delta(a_1)]|D| \\ &= \Gamma a_0 L(\delta(a_1)).\end{aligned}$$

since $L(\delta(a_1))$ has bounded extension, so does the above left hand side.

Now in the case that $p = 1$ and $\mathscr{A} = \emptyset$,

$$\mathcal{R}_{\mathscr{A}}(c) = \Gamma a_0 L(a_1)$$
$$\mathcal{P}_{\mathscr{A}}(c) = \Gamma a_0 L(a_1)$$

and $|\mathscr{A}| = 0$, so $\mathcal{R}_{\mathscr{A}}(c) - \mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|} = 0$. This proves the $p = 1$ case.

Now suppose that $p > 1$ and the assertion is proved for $p-1$. For this purpose, let $\mathscr{B} \subset \{1, \cdots, p-1\}$ be defined by $\mathscr{B} = \mathscr{A}\setminus\{p\}$.

If $p \in \mathscr{A}$, then $n_{\mathscr{A}} = n_{\mathscr{B}}$, and if $p \notin \mathscr{A}$ then $n_{\mathscr{A}} = n_{\mathscr{B}} + |\mathscr{B}|$.

Now we consider separately the cases $p \in \mathscr{A}$ and $p \notin \mathscr{A}$.

First, if $p \in \mathscr{A}$, then,

$$\mathcal{R}_{\mathscr{A}}(c) = \mathcal{R}_{\mathscr{B}}(c')F\delta(a_p),$$
$$\mathcal{P}_{\mathscr{A}}(c) = \mathcal{P}_{\mathscr{B}}(c')\delta(a_p).$$

Hence, since $|\mathscr{A}| = |\mathscr{B}| + 1$ and $n_{\mathscr{A}} = n_{\mathscr{B}}$ in this case:

$$\begin{aligned}(\mathcal{R}_{\mathscr{A}}(c) - (-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|})|D| &= (\mathcal{R}_{\mathscr{B}}(c')F\delta(a_p) - (-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{B}}(c')\delta(a_p)F^{|\mathscr{B}|+1})|D| = \\ &= \mathcal{R}_{\mathscr{B}}(c')[F,\delta(a_p)]|D| + (\mathcal{R}_{\mathscr{B}}(c')\delta(a_p)F - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')\delta(a_p)F\cdot F^{|\mathscr{B}|})|D|. \\ &= \mathcal{R}_{\mathscr{B}}(c')L(a_p) + (\mathcal{R}_{\mathscr{B}}(c')\delta(a_p) - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')\delta(a_p)\cdot F^{|\mathscr{B}|})D.\end{aligned}$$

In the case that $|\mathscr{B}|$ is even, we have $F^{|\mathscr{B}|} = 1$ and so:

$$\begin{aligned}&(\mathcal{R}_{\mathscr{A}}(c) - (-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|})|D| = \\ &= \mathcal{R}_{\mathscr{B}}(c')L(a_p) + (\mathcal{R}_{\mathscr{B}}(c') - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')\cdot F^{|\mathscr{B}|})\cdot\delta(a_p)D. \\ &= \mathcal{R}_{\mathscr{B}}(c')L(a_p) + (\mathcal{R}_{\mathscr{B}}(c') - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')\cdot F^{|\mathscr{B}|})D\cdot\delta(a_p) - \\ &\quad -(\mathcal{R}_{\mathscr{B}}(c') - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')\cdot F^{|\mathscr{B}|})\cdot\partial(\delta(a_p)).\end{aligned}$$

Since $\delta(a_p)$, $\partial(\delta(a_p))$ and $L(a_p)$ are bounded, by the inductive hypothesis the above has bounded extension, completing the proof of the case where $p \in \mathscr{A}$ and $|\mathscr{B}|$ is even.

On the other hand, if $p \in \mathscr{A}$ and $|\mathscr{B}|$ is odd, then:

$$\begin{aligned}&(\mathcal{R}_{\mathscr{A}}(c) - (-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|})|D| = \\ &= \mathcal{R}_{\mathscr{B}}(c')L(a_p) + \mathcal{R}_{\mathscr{B}}(c')\delta(a_p)D - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')\delta(a_p)|D| = \\ &= \mathcal{R}_{\mathscr{B}}(c')L(a_p) + (\mathcal{R}_{\mathscr{B}}(c') - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')F^{|\mathscr{B}|})D\cdot\delta(a_p) - \\ &\quad -\mathcal{R}_{\mathscr{B}}(c')\delta(a_p)D + (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')\delta^2(a_p).\end{aligned}$$

Since $\delta(a_p)$, $\delta^2(a_p)$ and $L(a_p)$ are bounded, by the inductive hypothesis the above has bounded extension, completing the proof of the case where $p \in \mathscr{A}$ and $|\mathscr{B}|$ is odd.

Now assume that $p \notin \mathscr{A}$. Then,

$$\mathcal{R}_{\mathscr{A}}(c) = \mathcal{R}_{\mathscr{B}}(c')L(a_p)$$
$$\mathcal{P}_{\mathscr{A}}(c) = \mathcal{P}_{\mathscr{B}}(c')L(a_p).$$

Focusing on $\mathcal{P}_{\mathscr{A}}(c)$:

$$\mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|} = \mathcal{P}_{\mathscr{B}}(c')L(a_p)F^{|\mathscr{B}|}.$$

So:

$$(\mathcal{R}_{\mathscr{A}}(c)-(-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|})|D| = (\mathcal{R}_{\mathscr{B}}(c')L(a_p)-(-1)^{n_{\mathscr{B}}+|\mathscr{B}|}\mathcal{P}_{\mathscr{B}}(c')L(a_p)F^{|\mathscr{B}|})|D|.$$

Note that since F anticommutes with $[F, a_p]$, F also anticommutes with $L(a_p)$. Hence,

$$L(a_p)F^{|\mathscr{B}|} = (-1)^{|\mathscr{B}|}F^{|\mathscr{B}|}L(a_p)$$

and so

$$\begin{aligned}(\mathcal{R}_{\mathscr{A}}(c) - (-1)^{n_{\mathscr{A}}}\mathcal{P}_{\mathscr{A}}(c)F^{|\mathscr{A}|})|D| &= (\mathcal{R}_{\mathscr{B}}(c') - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')F^{|\mathscr{B}|})L(a_p)|D| \\ &= (\mathcal{R}_{\mathscr{B}}(c') - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')F^{|\mathscr{B}|})(|D|L(a_p) + L(\delta(a_p))).\end{aligned}$$

By the inductive assumption, the operator $(\mathcal{R}_{\mathscr{B}}(c') - (-1)^{n_{\mathscr{B}}}\mathcal{P}_{\mathscr{B}}(c')F^{|\mathscr{B}|})|D|$ has bounded extension. This completes the proof of the $p \notin \mathscr{A}$ case.

Hence, the statement is true for p and this completes the induction. □

LEMMA 4.1.6. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple. Suppose D has a spectral gap at 0. For all $c \in (\mathcal{A}+\mathbb{C})^{\otimes(p+1)}$ and for all $\mathscr{A} \subseteq \{1, \dots, p\}$ the operator*

$$\left(\mathcal{P}_{\mathscr{A}}(c) - \mathcal{W}_{\mathscr{A}}(c) \cdot |D|^{p-|\mathscr{A}|}\right) \cdot |D|.$$

has bounded extension.

PROOF. This proof is again similar to the proofs of Lemmas 4.1.3 and 4.1.5.

Once more, it suffices to prove the assertion for an elementary tensor $c = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in (\mathcal{A}+\mathbb{C})^{\otimes(p+1)}$. Let $c' := a_0 \otimes \cdots \otimes a_{p-1} \in (\mathcal{A}+\mathbb{C})^{\otimes p}$. The proof proceeds by induction on p.

First, if $p = 1$, then either $\mathscr{A} = \{1\}$ or $\mathscr{A} = \emptyset$. If $\mathscr{A} = \{1\}$, then

$$\mathcal{P}_{\mathscr{A}}(c) = \Gamma a_0\delta(a_1),$$
$$\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|} = \Gamma a_0\delta(a_1).$$

and if $\mathscr{A} = \emptyset$, then

$$\mathcal{P}_{\mathscr{A}}(c) = \Gamma a_0 L(a_1),$$
$$\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|} = \Gamma a_0[F, a_1]|D|.$$

Since $L(a_1) = [F, a_1]|D|$ on the dense subspace H_∞, it follows that in all cases with $p = 1$ the difference $\mathcal{P}_{\mathscr{A}}(c) - \mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|}$ is identically zero on H_∞ and therefore has trivial bounded extension to H. This establishes the $p = 1$ case.

Now suppose that $p > 1$ and the assertion has been proved for $p - 1$. Let $\mathscr{B} = \mathscr{A} \setminus \{p\}$, and we consider the two cases of $p \in \mathscr{A}$ and $p \notin \mathscr{A}$.

Suppose that $p \in \mathscr{A}$. Then,

$$\mathcal{P}_{\mathscr{A}}(c) = \mathcal{P}_{\mathscr{B}}(c')\delta(a_p),$$
$$\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|} = \mathcal{W}_{\mathscr{B}}(c')\delta(a_p)|D|^{p-1-|\mathscr{B}|}.$$

So,

$$\begin{aligned}\mathcal{P}_{\mathscr{A}}(c)|D| &= \mathcal{P}_{\mathscr{B}}(c')\delta(a_p)|D| \\ &= \mathcal{P}_{\mathscr{B}}(c')|D| - \mathcal{P}_{\mathscr{B}}(c')\delta^2(a_p). \end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|+1} &= \mathcal{W}_{\mathscr{B}}(c')\delta(a_p)|D|^{p-|\mathscr{B}|} \\ &= \mathcal{W}_{\mathscr{B}}(c')|D|^{p-|\mathscr{B}|}\delta(a_p) + \mathcal{W}_{\mathscr{B}}(c')[\delta(a_p), |D|^{p-|\mathscr{B}|}] \\ &= \mathcal{W}_{\mathscr{B}}(c')|D|^{p-|\mathscr{B}|}\delta(a_p) - \mathcal{W}_{\mathscr{B}}(c')|D|^{p-|\mathscr{B}|-1}|D|^{|\mathscr{B}|-p+1}[|D|^{p-|\mathscr{B}|}, \delta(a_p)]. \end{aligned} \tag{4.2}$$

Applying Lemma A.1.2, the operator $|D|^{|\mathscr{B}|-p+1}[|D|^{p-|\mathscr{B}|}, \delta(a_p)]$ has bounded extension. So combining (4.1) and (4.2):

$$(\mathcal{P}_{\mathscr{A}}(c) - \mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|})|D| - (\mathcal{P}_{\mathscr{B}}(c') - \mathcal{W}_{\mathscr{B}}(c')|D|^{p-1-|\mathscr{B}|})|D|$$

has bounded extension. So by the inductive hypothesis, $(\mathcal{P}_{\mathscr{A}}(c) - \mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|})|D|$ has bounded extension in the case that $p \in \mathscr{A}$.

Now suppose that $p \notin \mathscr{A}$. In this case, we have:

$$\mathcal{P}_{\mathscr{A}}(c) = \mathcal{P}_{\mathscr{B}}(c')L(a_p)$$
$$\mathcal{W}_{\mathscr{A}}(c) = \mathcal{W}_{\mathscr{B}}(c')[F, a_p].$$

Multiplying by $|D|$, we have

$$\mathcal{P}_{\mathscr{A}}(c)|D| = \mathcal{P}_{\mathscr{B}}(c')|D|L(a_p) - \mathcal{P}_{\mathscr{B}}(c')L(\delta(a_p)). \tag{4.3}$$

Note that $\mathcal{P}_{\mathscr{B}}(c')L(\delta(a_p))$ is bounded.

Also,

$$\begin{aligned}\mathcal{W}_{\mathscr{A}}(c)|D|^{p+1-|\mathscr{A}|} &= \mathcal{W}_{\mathscr{B}}(c')L(a_p)|D|^{p-|\mathscr{A}|} \\ &= \mathcal{W}_{\mathscr{B}}(c')|D|^{p-|\mathscr{A}|}L(a_p) - \mathcal{W}_{\mathscr{B}}(c')[|D|^{p-|\mathscr{A}|}, L(a_p)] \\ &= \mathcal{W}_{\mathscr{B}}(c')|D|^{p-|\mathscr{A}|}L(a_p) - \mathcal{W}_{\mathscr{B}}(c')|D|^{p-1-|\mathscr{A}|}|D|^{-p+1+|\mathscr{A}|}[|D|^{p-|\mathscr{A}|}, L(a_p)]. \end{aligned} \tag{4.4}$$

By Lemma A.1.2, the operator $|D|^{-p+1+|\mathscr{A}|}[|D|^{p-|\mathscr{A}|}, L(a_p)]$ has bounded extension. So combining (4.3) and (4.4), it follows that

$$(\mathcal{P}_{\mathscr{A}}(c) - \mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|})|D| - (\mathcal{P}_{\mathscr{B}}(c') - \mathcal{W}_{\mathscr{B}}(c')|D|^{p-1-|\mathscr{B}|})|D|L(a_p)$$

has bounded extension. So by the inductive hypothesis, it follows that $(\mathcal{P}_{\mathscr{A}}(c) - \mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|})|D|$ has bounded extension in the case $p \notin \mathscr{A}$. $\square$

The main idea used in the proof of Lemma 4.1.2 is the algebraic identity,

$$\prod_{k=1}^{p}(x_k + y_k) = \sum_{\mathscr{A} \subseteq \{1,\cdots,p\}} z_{\mathscr{A}} \tag{4.5}$$

where $z_{\mathscr{A}}$ is given by the product $z_1 z_2 \cdots z_p$, where $z_k = x_k$ for $k \in \mathscr{A}$ and $z_k = y_k$ for $k \notin \mathscr{A}$.

Now we are ready to complete the proof of Lemma 4.1.2:

Proof of Lemma 4.1.2. Since

$$[D, a] = F[|D|, a] + [F, a]|D|,$$

as an equality of operators on H_∞, it follows that we have $\partial(a) = F\delta(a) + L(a)$. Now using (4.5):

$$\begin{aligned}\Omega(c) &= \Gamma a_0 \prod_{k=1}^{p} (F\delta(a_k) + L(a_k)) \\ &= \sum_{\mathscr{A}\subseteq\{1,\cdots,p\}} \mathcal{R}_{\mathscr{A}}(c).\end{aligned}$$

So on H_∞:

$$\Omega(c)|D| = \sum_{\mathscr{A}\subseteq\{1,\cdots,p\}} \mathcal{R}_{\mathscr{A}}(c)|D|.$$

We may now apply Lemma 4.1.5 to each summand to conclude that

$$\Omega(c)|D| - \sum_{\mathscr{A}\subseteq\{1,\cdots,p\}} (-1)^{n_{\mathscr{A}}} \mathcal{P}_{\mathscr{A}}(c) F^{|\mathscr{A}|}|D|$$

has bounded extension. Now applying Lemma 4.1.6 to each summand, we have that the operator

$$\Omega(c)|D| - \sum_{\mathscr{A}\subseteq\{1,\cdots,p\}} (-1)^{n_{\mathscr{A}}} \mathcal{W}_{\mathscr{A}}(c)|D|^{p-|\mathscr{A}|+1} F^{|\mathscr{A}|}$$

has bounded extension.

Equivalently,

$$\left(\Omega(c)|D|^{2-p} - \sum_{\mathscr{A}\subseteq\{1,\ldots,p\}} (-1)^{n_{\mathscr{A}}} \mathcal{W}_{\mathscr{A}}(c) D^{2-|\mathscr{A}|}\right)|D|^{p-1}$$

has bounded extension. □

We now prove the main result of this section.

Theorem 4.1.7. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For all $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\operatorname{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) = \sum_{\mathscr{A}\subseteq\{1,2,\ldots,p\}} (-1)^{n_{\mathscr{A}}} \operatorname{Tr}(\mathcal{W}_{\mathscr{A}}(c) D^{2-|\mathscr{A}|} e^{-s^2D^2}) + O(s^{-1})$$

as $s \to 0$.

Proof. As in the preceding lemmas it suffices to prove the result for an elementary tensor $c = a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$. Let $c' = 1 \otimes a_1 \otimes \cdots \otimes a_p$. By the cyclicity of the trace and the fact that $\mathcal{A}$ commutes with Γ, we have:

$$\operatorname{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) = \operatorname{Tr}(\Omega(c')|D|^{2-p}e^{-s^2D^2}a_0)$$

and for all $\mathscr{A} \subseteq \{1, 2, \ldots, p\}$:

$$\operatorname{Tr}(\mathcal{W}_{\mathscr{A}}(c) D^{2-|\mathscr{A}|} e^{-s^2D^2}) = \operatorname{Tr}(\mathcal{W}_{\mathscr{A}}(c') D^{2-|\mathscr{A}|} e^{-s^2D^2} a_0).$$

Thus,

$$\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) - \sum_{\mathscr{A}\subseteq\{1,\dots,p\}} (-1)^{n_{\mathscr{A}}}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c)D^{2-|\mathscr{A}|}e^{-s^2D^2})$$

$$= \mathrm{Tr}\left(\left(\Omega(c')|D|^{2-p} - \sum_{\mathscr{A}\subseteq\{1,\dots,p\}} (-1)^{n_{\mathscr{A}}}\mathcal{W}_{\mathscr{A}}(c')D^{2-|\mathscr{A}|}\right)|D|^{p-1}|D|^{1-p}e^{-s^2D^2}a_0\right)$$

Hence,

$$\left|\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2} - \sum_{\mathscr{A}\subseteq\{1,\dots,p\}} (-1)^{n_{\mathscr{A}}}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c)D^{2-|\mathscr{A}|}e^{-s^2D^2})\right|$$

$$\leq \left\|\left(\Omega(c')D^{2-p} - \sum_{\mathscr{A}\subseteq\{1,\dots,p\}} (-1)^{n_{\mathscr{A}}}\mathcal{W}_{\mathscr{A}}(c')D^{2-|\mathscr{A}|}\right)|D|^{p-1}\right\|_\infty \||D|^{1-p}e^{-s^2D^2}a_0\|_1$$

The first factor is finite, by Lemma 4.1.2, and the second factor is $O(s^{-1})$, by Lemma A.1.4. This completes the proof. $\square$

4.2. Auxiliary commutator estimates

This section is a slight detour from the main task of this chapter. Here we establish bounds on the $\mathcal{L}_1$-norm of commutators of the form $[f(s|D|), x]$, where $x \in \mathcal{B}$, $s > 0$ and f is the square of a Schwartz class function on $\mathbb{R}$. These bounds are used everywhere in the subsequent sections of this chapter.

Recall that the algebra $\mathcal{B}$ is defined in Definition 2.2.7.

The following lemma serves the same purpose as [**12**, Lemma 18], but the right hand sides are different here due to the fact that we deal with non-unital spectral triples.

LEMMA 4.2.1. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple. Let h be a Schwartz class function on $\mathbb{R}$ and let $f = h^2$. Then for all $x \in \mathcal{B}$, we have*

$$\left\|[f(s|D|), x] - \frac{s}{2}\{f'(s|D|), \delta(x)\}\right\|_1 \leq \frac{1}{2}s^2\|\widehat{h''}\|_1\cdot\big(\|\delta^2(x)h(s|D|)\|_1 + \|h(s|D|)\delta^2(x)\|_1\big)$$

Here, $\{\cdot,\cdot\}$ denotes the anti-commutator.

PROOF. Since $f'(s|D|) = 2h'(s|D|)h(s|D|)$, by the Leibniz rule we have:

$$[f(s|D|), x] - \frac{s}{2}\{f'(s|D|), \delta(x)\} = [h(s|D|)^2, x] - s\{h'(s|D|)h(s|D|), \delta(x)\} =$$

$$= h(s|D|)\big([h(s|D|), x] - sh'(s|D|)\delta(x)\big) + \big([h(s|D|), x] - s\delta(x)h'(s|D|)\big)h(s|D|).$$

Applying Lemma A.2.3, we have

$$[h(s|D|), x] - sh'(s|D|)\delta(x) = -s^2\int_{-\infty}^{\infty}\int_0^1 \widehat{h''}(u)(1-v)e^{ius(1-v)|D|}\delta^2(x)e^{iusv|D|}\,dvdu$$

$$[h(s|D|), x] - s\delta(x)h'(s|D|) = -s^2\int_{-\infty}^{\infty}\int_0^1 \widehat{h''}(u)(1-v)e^{iusv|D|}\delta^2(x)e^{ius(1-v)|D|}\,dvdu.$$

Therefore,

$$
\begin{aligned}
[f(s|D|), x] - \frac{s}{2}\{f'(s|D|), \delta(x)\} = \\
& - s^2 \int_{-\infty}^{\infty} \int_0^1 \widehat{h''}(u)(1-v)e^{iu(1-v)s|D|} h(s|D|)\delta^2(x)e^{iuvs|D|}\, dvdu \\
& - s^2 \int_{-\infty}^{\infty} \int_0^1 \widehat{h''}(u)(1-v)e^{iuvs|D|}\delta^2(x)h(s|D|)e^{iu(1-v)s|D|}\, dvdu.
\end{aligned}
$$

Now applying Lemma 2.3.2 to each integral, we have

$$
\begin{aligned}
&\|[f(s|D|), x] - \frac{s}{2}\{f'(s|D|), \delta(x)\}\|_1 \\
&\qquad \le s^2 \int_{-\infty}^{\infty} \int_0^1 \left\|\widehat{h''}(u)(1-v)e^{iu(1-v)s|D|} h(s|D|)\delta^2(x)e^{iuvs|D|}\right\|_1 dvdu \\
&\qquad + s^2 \int_{-\infty}^{\infty} \int_0^1 \left\|\widehat{h''}(u)(1-v)e^{iuvs|D|}\delta^2(x)h(s|D|)e^{iu(1-v)s|D|}\right\|_1 dvdu \\
&\qquad = s^2 \|\widehat{h''}\|_1 (\|\delta^2(x)h(s|D|)\|_1 + \|h(s|D|)\delta^2(x)\|_1) \int_0^1 (1-v)dv \\
&\qquad = \frac{1}{2} s^2 \|\widehat{h''}\|_1 (\|\delta^2(x)h(s|D|)\|_1 + \|h(s|D|)\delta^2(x)\|_1).
\end{aligned}
$$

$\square$

LEMMA 4.2.2. *Let* $(\mathcal{A}, H, D)$ *be a smooth spectral triple and assume that* D *has a spectral gap at* 0. *Let* h *be a Schwartz function on* $\mathbb{R}$ *and let* $f = h^2$. *Then for every* $x \in \mathcal{B}$, *we have*

$$
\||D|^m [f(s|D|), x]\|_1 \le s\|\widehat{h'}\|_1 \left(\||D|^m h(s|D|)\delta(x)\|_1 + \||D|^m \delta(x) h(s|D|)\|_1 \right).
$$

PROOF. Since $f = h^2$, by the Leibniz rule, we have

$$
[f(s|D|), x] = h(s|D|)[h(s|D|), x] + [h(s|D|), x]h(s|D|).
$$

Using Lemma A.2.2, we have

$$
[h(s|D|), x] = s \int_{\mathbb{R}} \int_0^1 \widehat{h'}(u) e^{ius(1-v)|D|} \delta(x) e^{iusv|D|} dvdu.
$$

Thus,

$$
\begin{aligned}
|D|^m [f(s|D|), x] = s \int_{\mathbb{R}} \int_0^1 \widehat{h'}(u) e^{ius(1-v)|D|} |D|^m h(s|D|)\delta(x) e^{iusv|D|}\, dvdu \\
+ s \int_{\mathbb{R}} \int_0^1 \widehat{h'}(u) e^{ius(1-v)|D|} |D|^m \delta(x) h(s|D|) e^{iusv|D|}\, dvdu.
\end{aligned}
$$

Bounding the $\mathcal{L}_1$ norm using Lemma 2.3.2, we have

$$
\begin{aligned}
\||D|^m [f(s|D|), x]\|_1 &\le s \int_{\mathbb{R}} \int_0^1 |\widehat{h'}(u)| \||D|^m h(s|D|)\delta(x)\|_1\, dvdu \\
&\quad + s \int_{\mathbb{R}} \int_0^1 |\widehat{h'}(u)| \||D|^m \delta(x) h(s|D|)\|_1\, dvdu \\
&= s\|\widehat{h'}\|_{L_1(\mathbb{R})} (\||D|^m h(s|D|)\delta(x)\|_1 + \||D|^m \delta(x) h(s|D|)\|_1).
\end{aligned}
$$

$\square$

LEMMA 4.2.3. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For all $x \in \mathcal{B}$ and for all integers $m > -p$, we have*

$$\||D|^m [e^{-s^2D^2}, x]\|_1 = O(s^{1-p-m}), \quad s \downarrow 0.$$

PROOF. Let $h(t) = e^{-\frac{1}{2}t^2}$, $t \in \mathbb{R}$. By Lemma 4.2.2, we have

$$\||D|^m [e^{-s^2D^2}, x]\|_1 \leq s\|\widehat{h'}\|_1(\||D|^m e^{-\frac{1}{2}s^2D^2}\delta(x)\|_1 + \||D|^m \delta(x) e^{-\frac{1}{2}s^2D^2}\|_1).$$

If $m \leq 0$, then the assertion now follows from applying Lemma A.1.4 to the terms $\||D|^m e^{-\frac{1}{2}s^2D^2}\delta(x)\|_1$ and $\||D|^m \delta(x) e^{-\frac{1}{2}s^2D^2}\|_1$. Assume now $m > 0$. Using Lemma A.1.3 with the Schwartz function $t \mapsto t^m e^{-\frac{1}{2}t^2}$, we obtain

$$s \left\| |D|^m e^{-\frac{1}{2}s^2D^2}\delta(x) \right\|_1 = O(s^{1-p-m}), \quad s \to 0.$$

By Lemma A.1.1, we have

$$|D|^m \delta(x) e^{-\frac{1}{2}s^2D^2} = \sum_{k=0}^{m} \binom{m}{k} \delta^{m+1-k}(x)|D|^k e^{-\frac{1}{2}s^2D^2}.$$

Now we apply Lemma A.1.3 to each summand, using the function $t \mapsto t^k e^{-\frac{1}{2}s^2D^2}$ for the kth summand. So,

$$\begin{aligned} s\||D|^m \delta(x) e^{-\frac{1}{2}s^2D^2}\|_1 &\leq s \sum_{k=0}^{m} \binom{m}{k} O(s^{1-p-k}) \\ &= O(s^{1-p-m}), \quad s \to 0. \end{aligned}$$

□

The following lemma is used in the proof of Theorem 4.4.2.

LEMMA 4.2.4. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. Let $f(t) = e^{-t^2}$, $t \in \mathbb{R}$.*

(i) *for every $a \in \mathcal{A}$, we have*

$$\left\| [f(s|D|), a] - s\delta(a) f'(s|D|) \right\|_\infty = O(s^2), \quad s \downarrow 0.$$

(ii) *for every $a \in \mathcal{A}$, we have*

$$\left\| [f(s|D|), a] - s\delta(a) f'(s|D|) \right\|_1 = O(s^{2-p}), \quad s \downarrow 0.$$

(iii) *for every $a \in \mathcal{A}$, we have*

$$\left\| [f(s|D|), a] - s\delta(a) f'(s|D|) \right\|_{p,1} = O(s), \quad s \downarrow 0.$$

PROOF. First we prove (i): this is a simple combination of Lemma A.2.3 and the triangle inequality:

$$\|[f(s|D|), a] - sf'(s|D|)\delta(a)\|_\infty \leq s^2\|\widehat{f''}\|_1\|\delta^2(a)\|_\infty.$$

Now we prove (ii). Let $h(t) = e^{-t^2/2}$, $t \in \mathbb{R}$, so that $f = h^2$. By Lemma 4.2.1, for all $a \in \mathcal{A}$ we have

(4.6)
$$\left\| [f(s|D|), a] - \frac{s}{2}\{f'(s|D|), \delta(a)\} \right\|_1 \leq \frac{1}{2}s^2\|\hat{h''}\|_1(\|\delta^2(a)h(s|D|)\|_1 + \|h(s|D|)\delta^2(a)\|_1).$$

Using Lemma A.1.3, we have

$$\|\delta^2(a)h(s|D|)\|_1 = O(s^{-p}) \text{ and,}$$

$$\|h(s|D|)\delta^2(a)\|_1 = O(s^{-p}). \tag{4.7}$$

Combining (4.6) and (4.7), we arrive at:

$$\|[f(s|D|),a] - \frac{s}{2}\{f'(s|D|),\delta(a)\}\|_1 = O(s^{2-p}). \tag{4.8}$$

On the other hand,

$$\begin{aligned}\|[f'(s|D|),\delta(a)]\|_1 &= 2s\|[|D|e^{-s^2D^2},\delta(a)]\|_1\\ &\leq 2s\|\delta^2(a)e^{-s^2D^2}\|_1 + 2s\||D|[e^{-s^2D^2},\delta(a)]\|_1.\end{aligned}$$

Due to Lemma A.1.3, we have $2s\|\delta^2(a)e^{-s^2D^2}\|_1 = O(s^{1-p})$, and by Lemma 4.2.3 we also have $2s\||D|[e^{-s^2D^2},\delta(a)]\|_1 = O(s^{1-p})$. Therefore,

$$\|[f'(s|D|),\delta(a)]\|_1 = O(s^{1-p}), \quad s\downarrow 0. \tag{4.9}$$

By combining (4.8) and (4.9), we obtain (ii).

Finally, to prove (iii), we use the inequality

$$\|T\|_{p,1} \leq \|T\|_1^{\frac{1}{p}}\|T\|_\infty^{1-\frac{1}{p}}$$

and write

$$\begin{aligned}&\|[f(s|D|),a] - s\delta(a)f'(s|D|)\|_{p,1} \leq\\ \leq\ &\|[f(s|D|),a] - s\delta(a)f'(s|D|)\|_1^{\frac{1}{p}}\|[f(s|D|,a)] - s\delta(a)f'(s|D|)\|_\infty^{1-\frac{1}{p}} =\\ &= O(s^{2-p})^{\frac{1}{p}}\cdot O(s^2)^{1-\frac{1}{p}} = O(s).\end{aligned}$$

□

The following Lemma is used in Lemma 4.3.2, Lemma 4.3.3 and Lemma 4.5.2.

Lemma 4.2.5. *Let $(\mathcal{A},H,D)$ be a smooth $p-$dimensional spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For every $0\leq m\leq p$, and $x\in\mathcal{B}$ we have*

$$\|D^{m-p}[D^{2-m}e^{-s^2D^2},x]\|_1 = O(s^{-1}), \quad s\downarrow 0.$$

Proof. By the Leibniz rule,

$$[D^{2-m}e^{-s^2D^2},x] = [D^{2-m},a]e^{-s^2D^2} + D^{2-m}[e^{-s^2D^2},x].$$

Thus,

$$\|D^{m-p}[D^{2-m}e^{-s^2D^2},x]\|_1 \leq \|D^{2-p}[e^{-s^2D^2},x]\|_1 + \|D^{m-p}[D^{2-m},x]e^{-s^2D^2}\|_1.$$

By Lemma 4.2.3, we have $\|D^{2-p}[e^{-s^2D^2},x]\|_1 = O(s^{-1})$, so we now focus on the second summand. First, for the case when $m>2$ we apply the Leibniz rule

$$\begin{aligned}[D^{2-m},x] &= -D^{2-m}[D^{m-2},x]D^{2-m}\\ &= -\sum_{k+l=m-3} D^{k+2-m}\partial(x)D^{l+2-m}.\end{aligned}$$

Now using the triangle inequality:

$$\|D^{m-p}[D^{2-m},x]e^{-s^2D^2}\|_1 \leq \sum_{k+l=m-3}\||D|^{k+2-p}\partial(x)|D|^{l+2-m}e^{-s^2|D|^2}\|_1.$$

Applying Lemma A.1.4 to each summand, we then conclude that

$$\|D^{m-p}[D^{2-m},x]e^{-s^2D^2}\|_1 = O(s^{-1}),$$

thus proving the claim for $m > 2$.

We now deal with the remaining cases $m = 0, 1, 2$ individually. In the case $m = 2$, we have $D^{m-p}[D^{2-m},x]e^{-s^2D^2} = 0$, and so the claim follows trivially in this case.

For $m = 1$, we have

$$\|D^{m-p}[D^{2-m},x]e^{-s^2D^2}\|_1 = \||D|^{1-p}\partial(x)e^{-s^2D^2}\|_1$$

So by Lemma A.1.4, we also have in this case that the above is $O(s^{-1})$.

Finally, for $m = 0$,

$$\begin{aligned}[D^2,x] &= [|D|^2,x]\\ &= |D|\delta(x) + \delta(x)|D|\\ &= 2|D|\delta(x) - \delta^2(x).\end{aligned}$$

So by the triangle inequality:

$$\|D^{m-p}[D^{2-m},x]e^{-s^2D^2}\|_1 \le 2\||D|^{1-p}\delta(x)e^{-s^2D^2}\|_1 + \||D|^{-p}\delta^2(x)e^{-s^2D^2}\|_1$$

so an application of Lemma A.1.4 to each of the above summands yields the result. □

4.3. Exploiting Hochschild homology

Recall the multilinear mapping $\mathcal{W}_p$ from Definition 4.1.1. In this section, we prove the following:

Theorem 4.3.1. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$ we have:*

$$\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) - p\mathrm{Tr}(\mathcal{W}_p(c)De^{-s^2D^2}) = O(s^{-1}), \quad s \downarrow 0.$$

We achieve this using the commutator estimates of the preceding section.

Our strategy to prove Theorem 4.3.1 is to start from Theorem 4.1.7 and then show that:

1. all of the terms with $|\mathscr{A}| \ge 2$ are $O(s^{-1})$ (see Lemma 4.3.4)
2. the $\mathscr{A} = \emptyset$ term is $O(s^{-1})$ (see Lemma 4.3.7
3. finally we complete the proof by showing that the terms with $|\mathscr{A}| = 1$ are all equal to the $\mathscr{A} = \{p\}$ term up to terms of size $O(s^{-1})$.

The proofs in this section rely crucially on the assumption that c is a Hochschild cycle.

First, we show that terms in Theorem 4.1.7 such that there is some m with $m-1, m \in \mathscr{A}$ are $O(s^{-1})$.

Lemma 4.3.2. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. Let $m \in \{1, \ldots, p\}$ and suppose that $m-1, m \in \mathscr{A}$ (so necessarily we have $|\mathscr{A}| \ge 2$). For every Hochschild cycle $c \in \mathscr{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c)D^{2-|\mathscr{A}|}e^{-s^2D^2}) = O(s^{-1}), \quad s \downarrow 0.$$

PROOF. For $s > 0$, consider the multilinear mapping $\theta_s : \mathcal{A}^{\otimes p} \to \mathbb{C}$ defined by:

$$\theta_s(a_0\otimes\cdots\otimes a_{p-1}) = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k)\right) \delta^2(a_{m-1}) \left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right) D^{2-|\mathscr{A}|}e^{-s^2D^2}\right)$$

where b_k is as in Definition 4.1.1. Then, from the computation in Appendix A.3, we have that the Hochschild coboundary is:

$$\begin{aligned}(b\theta_s)(a_0\otimes\cdots\otimes a_p) &\\ &= (-1)^p\mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2} [b_k, a_k]\right) \delta^2(a_{m-1}) \left(\prod_{k=m}^{p-1} [b_{k+1}, a_k]\right) [D^{2-|\mathscr{A}|}e^{-s^2D^2}, a_p]\right)\\ &\quad + 2(-1)^{m-1}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(a_0\otimes\cdots\otimes a_p)D^{2-|\mathscr{A}|}e^{-s^2D^2}).\end{aligned}$$

We now claim that the first summand is $O(s^{-1})$ as $s \downarrow 0$. Indeed, dividing and multiplying by $D^{|\mathscr{A}|-p}$:

$$\begin{aligned}&\left\|\Gamma a_0 \left(\prod_{k=1}^{m-2} [b_k, a_k]\right)\delta^2(a_{m-1}) \left(\prod_{k=m}^{p-1} [b_{k+1}, a_k]\right) [D^{2-|\mathscr{A}|}e^{-s^2D^2}, a_p]\right\|_1\\ &\qquad\leq \left\|D^{|\mathscr{A}|-p}[D^{2-|\mathscr{A}|}e^{-s^2D^2}, a_p]\right\|_1\\ &\qquad\times \left\|\Gamma a_0 \left(\prod_{k=1}^{m-2} [b_k, a_k]\right) \delta^2(a_{m-1}) \left(\prod_{k=m}^{p-1} [b_{k+1}, a_k]\right) |D|^{p-|\mathscr{A}|}\right\|_\infty\end{aligned}$$

The first factor is $O(s^{-1})$ by Lemma 4.2.5, and the second factor is finite by Lemma A.4.2 and has no dependence on s.

To summarise, so far we that if $c \in \mathcal{A}^{\otimes(p+1)}$:

$$(b\theta_s)(c) = 2(-1)^{m-1}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c)D^{2-|\mathscr{A}|}e^{-s^2D^2}) + O(s^{-1}), \quad s \downarrow 0.$$

If c is a Hochschild cycle, then $(b\theta_s)(c) = \theta_s(bc) = 0$, and so

$$2(-1)^{m-1}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c)D^{2-|\mathscr{A}|}e^{-s^2D^2}) = O(s^{-1})$$

as required. □

LEMMA 4.3.3. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. Let $\mathscr{A}_1, \mathscr{A}_2 \subseteq \{1, \ldots, p\}$, with $|\mathscr{A}_1| = |\mathscr{A}_2|$ and that the symmetric difference $\mathscr{A}_1\Delta\mathscr{A}_2 = \{m-1, m\}$ for some m. Then for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\mathcal{W}_{\mathscr{A}_1}(c)D^{2-|\mathscr{A}_1|}e^{-s^2D^2}) + \mathrm{Tr}(\mathcal{W}_{\mathscr{A}_2}(c)D^{2-|\mathscr{A}_2|}e^{-s^2D^2}) = O(s^{-1}), \quad s \downarrow 0.$$

PROOF. This proof is similar to that of Lemma 4.3.2. For $s > 0$ we consider the multilinear mapping $\theta_s : \mathcal{A}^{\otimes p} \to \mathbb{C}$ given by

$$\theta_s(a_0\otimes\cdots\otimes a_{p-1}) = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k)\right) [F, \delta(a_{m-1})] \left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right) D^{2-|\mathscr{A}_1|}e^{-s^2D^2}\right).$$

Here, as in Lemma 4.3.2, the operators b_k are defined as in Definition 4.1.1, relative to the set $\mathscr{A} = \mathscr{A}_1$. From the computation in Appendix A.3,

$$\begin{aligned}
&(b\theta_s)(a_0 \otimes \cdots \otimes a_p) \\
&\qquad = (-1)^p \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2}[b_k, a_k]\right)[F, \delta(a_{m-1})]\left(\prod_{k=m}^{p-1}[b_{k+1}, a_k]\right)[D^{2-|\mathscr{A}_1|}e^{-s^2D^2}, a_p]\right) \\
&\qquad + (-1)^{m-1}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}_1}(a_0 \otimes \cdots \otimes a_p)D^{2-|\mathscr{A}_1|}e^{-s^2D^2}) \\
&\qquad + (-1)^{m-1}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}_2}(a_0 \otimes \cdots \otimes a_p)D^{2-|\mathscr{A}_2|}e^{-s^2D^2}).
\end{aligned}$$

We first show that the first summand above is $O(s^{-1})$. Indeed,

$$\begin{aligned}
&\left\|\Gamma a_0 \left(\prod_{k=1}^{m-2}[b_k, a_k]\right)[F, \delta(a_{m-1})]\left(\prod_{k=m}^{p-1}[b_{k+1}, a_k]\right)[D^{2-|\mathscr{A}_1|}e^{-s^2D^2}, a_p]\right\|_1 \\
&\qquad\qquad \leq \left\|D^{|\mathscr{A}_1|-p}[D^{2-|\mathscr{A}_1|}e^{-s^2D^2}, a_p]\right\|_1 \\
&\qquad\qquad \times \left\|\Gamma a_0 \prod_{k=1}^{m-2}[b_k, a_k][F, \delta(a_{m-1})]\prod_{k=m}^{p-1}[b_{k+1}, a_k] \cdot |D|^{p-|\mathscr{A}_1|}\right\|_\infty.
\end{aligned}$$

The first factor above is $O(s^{-1})$ due to Lemma 4.2.5, and the second factor is finite by Lemma A.4.3 and has no dependence on s.

Summarising the above, if $c \in \mathcal{A}^{\otimes(p+1)}$ we have

$$\begin{aligned}
(b\theta_s)(c) &= (-1)^{m-1}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}_1}(c)D^{2-|\mathscr{A}_1|}e^{-s^2D^2}) \\
&\quad + (-1)^{m-1}\mathrm{Tr}(\mathcal{W}_{\mathscr{A}_2}(c)D^{2-|\mathscr{A}_2|}e^{-s^2D^2}) + O(s^{-1})
\end{aligned}$$

as $s \downarrow 0$. Hence, if c is a Hochschild cycle then $(b\theta_s)(c) = \theta_s(bc) = 0$, and this completes the proof. $\square$

Lemma 4.3.4. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$ and for every $\mathscr{A} \subset \{1, \cdots, p\}$ with $|\mathscr{A}| \geq 2$, we have*

$$\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c)D^{2-|\mathscr{A}|}e^{-s^2D^2}) = O(s^{-1}), \quad s \downarrow 0.$$

Proof. Let m be the maximum element in $\mathscr{A}$ and let n be the maximal element in $\mathscr{A}\setminus\{m\}$. If $n = m-1$, then the assertion is already proved in Lemma 4.3.2. If $n < m-1$, then $m - n > 1$, and hence $n + j \notin \mathscr{A}$ for all $1 \leq j < m - n$. Now for each $0 \leq j < m - n$ we define $\mathscr{A}_j$ to be $\mathscr{A}$ with n replaced with $n + j$. That is:

$$\mathscr{A}_j := (\mathscr{A}\setminus\{n\}) \cup \{j + n\}, \quad 0 \leq j < m - n.$$

Then by construction:

(1) $|\mathscr{A}_j| = |\mathscr{A}|$ and $\mathscr{A}_j \Delta \mathscr{A}_{j-1} = \{n + j, n + j - 1\}$ for all $1 \leq j < m - n$.
(2) $\mathscr{A}_0 = \mathscr{A}$ and $m - 1, m \in \mathscr{A}_{m-n-1}$.

Hence if $1 \leq j < m - n$ the subsets $\mathscr{A}_j$ and $\mathscr{A}_{j-1}$ satisfy the conditions of Lemma 4.3.3. So for all Hochschild cycles $c \in \mathcal{A}^{\otimes(p+1)}$:

$$\mathrm{Tr}(\mathcal{W}_{\mathscr{A}_{j-1}}(c)D^{2-|\mathscr{A}_{i-1}|}e^{-s^2D^2}) = -\mathrm{Tr}(\mathcal{W}_{\mathscr{A}_j}(c)D^{2-|\mathscr{A}_j|}e^{-s^2D^2}) + O(s^{-1}), \quad s \downarrow 0.$$

So by induction, we have:

$$\begin{aligned}\operatorname{Tr}(\mathcal{W}_{\mathscr{A}_0}(c)D^{2-|\mathscr{A}_0|}e^{-s^2D^2}) &= (-1)^{m-n-1}\operatorname{Tr}(\mathcal{W}_{\mathscr{A}_{m-n-1}}(c)D^{2-|\mathscr{A}_{m-n-1}|}e^{-s^2D^2}) \\ &\quad + O(s^{-1}), \quad s\downarrow 0.\end{aligned} \tag{4.10}$$

On the other hand, since $m-1, m \in \mathscr{A}_{m-n-1}$, we may apply Lemma 4.3.2 to $\mathscr{A}_{m-n-1}$ to obtain:

$$\operatorname{Tr}(\mathcal{W}_{\mathscr{A}_{m-n-1}}(c)D^{2-|\mathscr{A}_{m-n-1}|}e^{-s^2D^2}) = O(s^{-1}), \quad s\downarrow 0.$$

Combining (4.10) and (4.3), we get

$$\operatorname{Tr}(\mathcal{W}_{\mathscr{A}_0}(c)D^{2-|\mathscr{A}_0|}e^{-s^2D^2}) = O(s^{-1}).$$

since $\mathscr{A}_0 = \mathscr{A}$, the proof is complete. □

Recall the mapping ch from Definition 2.2.15.

LEMMA 4.3.5. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For every $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\|\operatorname{ch}(c)D^2e^{-s^2D^2}\|_1 = O(s^{-1}), \quad s\downarrow 0.$$

PROOF. Recall that on H_∞ we have $[F, a_p]|D| = L(a_p)$. So on H_∞:

$$\begin{aligned}[F,a_{p-1}][F,a_p]|D|^2 &= [F,a_{p-1}]\cdot L(a_p)\cdot|D| \\ &= [F,a_{p-1}]\cdot|D|L(a_p) - [F,a_{p-1}]\cdot[|D|, L(a_p)] \\ &= [F,a_{p-1}]|D|\cdot L(a_p) - [F,a_{p-1}]\cdot\delta(L(a_p)) \\ &= L(a_{p-1})\cdot L(a_p) - [F,a_{p-1}]\cdot L(\delta(a_p)).\end{aligned}$$

So for $c = a_0\otimes\cdots\otimes a_p \in \mathcal{A}^{\otimes(p+1)}$ we have

$$\begin{aligned}\operatorname{ch}(c)\cdot D^2e^{-s^2D^2} &= \Gamma\left(\prod_{k=0}^{p-2}[F,a_k]\right)|D|^{p-1}\cdot|D|^{1-p}L(a_{p-1})L(a_p)e^{-s^2D^2} \\ &\quad - \Gamma\left(\prod_{k=0}^{p-1}[F,a_k]\right)|D|^{p-1}\cdot|D|^{1-p}L(\delta(a_p))e^{-s^2D^2}.\end{aligned}$$

Using Lemma A.4.4, the operators $\left(\prod_{k=0}^{p-2}[F,a_k]\right)|D|^{p-1}$ and $\left(\prod_{k=0}^{p-1}[F,a_k]\right)|D|^{p-1}$ both have bounded extension and no dependence on s. From Lemma A.1.4, we have that:

$$\begin{aligned}\||D|^{1-p}L(a_{p-1})L(a_p)e^{-s^2D^2}\|_1 &= O(s^{-1}) \\ \||D|^{1-p}L(\delta(a_p))e^{-s^2D^2}\|_1 &= O(s^{-1}).\end{aligned}$$

So by the triangle inequality: $\|\operatorname{ch}(c)D^2e^{-s^2D^2}\|_1 = O(s^{-1})$ as $s\downarrow 0$. □

LEMMA 4.3.6. *Let $(\mathcal{A}, H, D)$ be a spectral triple of dimension p, where p has the same parity as $(\mathcal{A}, H, D)$. If $c \in \mathcal{A}^{\otimes(p+1)}$ then*

$$\operatorname{ch}(c) = \mathcal{W}_\emptyset(c) + F\mathcal{W}_\emptyset(c)F.$$

Proof. Let $c = a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$. Recall that

$$\mathcal{W}_\emptyset(c) = \Gamma a_0 \prod_{k=1}^{p} [F, a_k].$$

Using the fact that F anticommutes with $[F, a_k]$ for all k, we have:

$$\begin{aligned}\mathrm{ch}(c) &= \Gamma F[F, a_0] \prod_{k=1}^{p} [F, a_k] \\ &= \Gamma a_0 \prod_{k=1}^{p} [F, a_k] - \Gamma F a_0 F \prod_{k=1}^{p} [F, a_k]. \\ &= \mathcal{W}_\emptyset(c) + (-1)^{p+1} \Gamma F a_0 \left(\prod_{k=1}^{p} [F, a_k] \right) F.\end{aligned}$$

Since $\Gamma^2 = 1$,

$$\mathrm{ch}(c) = \mathcal{W}_\emptyset(c) + (-1)^{p+1} \Gamma F \Gamma \mathcal{W}_\emptyset(c) F.$$

Since the parity of p matches the parity of Γ, we have $\Gamma F = (-1)^{p+1} F \Gamma$. This completes the proof. $\square$

Lemma 4.3.7. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For every $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\mathcal{W}_\emptyset(c) D^2 e^{-s^2 D^2}) = O(s^{-1}), \quad s \downarrow 0.$$

Proof. By Lemma 4.3.6, we have:

$$\mathrm{Tr}(\mathrm{ch}(c) D^2 e^{-s^2 D^2}) = \mathrm{Tr}(\mathcal{W}_\emptyset(c) D^2 e^{-s^2 D^2}) + \mathrm{Tr}(F \mathcal{W}_\emptyset(c) F D^2 e^{-s^2 D^2}).$$

However since F commutes with $D^2 e^{-s^2 D^2}$ and $F^2 = 1$ we have:

$$2\mathrm{Tr}(\mathcal{W}_\emptyset(c) D^2 e^{-s^2 D^2}) = \mathrm{Tr}(\mathrm{ch}(c) D^2 e^{-s^2 D^2}).$$

However by Lemma 4.3.5,

$$|\mathrm{Tr}(\mathrm{ch}(c) D^2 e^{-s^2 D^2})| = O(s^{-1}).$$

Hence $\mathrm{Tr}(\mathcal{W}_\emptyset(c) D^2 e^{-s^2 D^2}) = O(s^{-1})$. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 4.3.1. Let $c \in \mathcal{A}^{\otimes(p+1)}$. Then using Theorem 4.1.7:

$$\mathrm{Tr}(\Omega(c)|D|^{2-p} e^{-s^2 D^2}) = \sum_{\mathscr{A} \subseteq \{1,\ldots,p\}} (-1)^{n_{\mathscr{A}}} \mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c) D^{2-|\mathscr{A}|} e^{-s^2 D^2}) + O(s^{-1}), \quad s \downarrow 0.$$

Applying Lemma 4.3.4 to every summand with $|\mathscr{A}| \geq 2$, and Lemma 4.3.7 to the summand $\mathscr{A} = \emptyset$, it follows that:

$$\mathrm{Tr}(\Omega(c)|D|^{2-p} e^{-s^2 D^2}) = \sum_{k=1}^{p} (-1)^{n_{\{k\}}} \mathrm{Tr}(\mathcal{W}_k(c) D e^{-s^2 D^2}) + O(s^{-1}), \quad s \downarrow 0.$$

Recall that $n_{\mathscr{A}} = |\{(j,k) \in \{1,\ldots,p\}^2 \ : \ j \in \mathscr{A}, k \notin \mathscr{A}\}|$. So in particular, $n_{\{k\}} = p - k$. Hence:

$$\mathrm{Tr}(\Omega(c)|D|^{2-p} e^{-s^2 D^2}) = \sum_{k=1}^{p} (-1)^{p-k} \mathrm{Tr}(\mathcal{W}_k(c) D e^{-s^2 D^2}) + O(s^{-1}), \quad s \downarrow 0. \tag{4.11}$$

For any $1 \leq k \leq p-1$, the sets $\mathscr{A}_1 = \{k\}$ and $\mathscr{A}_2 = \{k+1\}$ satisfy the conditions of Lemma 4.3.3 with $m = k+1$. So we have

$$\mathrm{Tr}(\mathcal{W}_k(c)De^{-s^2D^2}) = -\mathrm{Tr}(\mathcal{W}_{k+1}(c)De^{-s^2D^2}) + O(s^{-1}), \quad s \downarrow 0,$$

Hence by induction, for any $1 \leq k \leq p$:

$$\mathrm{Tr}(\mathcal{W}_k(c)De^{-s^2D^2}) = (-1)^{p-k}\mathrm{Tr}(\mathcal{W}_pDe^{-s^2D^2}) + O(s^{-1}), \quad s \downarrow 0. \tag{4.12}$$

Substituting (4.12) into each summand of (4.11) we finally get:

$$\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) = p\mathrm{Tr}(\mathcal{W}_p(c)De^{-s^2D^2}) + O(s^{-1}).$$

□

4.4. Preliminary heat semigroup asymptotic

In this section, we move closer to proving Theorem 1.2.2. We will show that if $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1 and Hypothesis 4.0.1 (in particular, D has a spectral gap at 0), then for a Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$ we have

$$\mathrm{Tr}(\mathcal{W}_p(c)|D|^{2-p}e^{-s^2D^2}) = \frac{1}{4}\mathrm{Tr}(\mathrm{ch}(c))s^{-2} + O(s^{-1}), s \downarrow 0. \tag{4.13}$$

By the Theorem 4.3.1, this is a formula very close to Theorem 1.2.2: the only difference is the assumption of Hypothesis 4.0.1 and that the occurance of $|D|$ should be replaced with $(1+D^2)^{1/2}$.

We start with the following asymptotic result.

Lemma 4.4.1. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For all $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\mathrm{ch}(c)e^{-s^2D^2}) = \mathrm{Tr}(\mathrm{ch}(c)) + O(s), \quad s \downarrow 0.$$

Proof. We wish to show that

$$\mathrm{Tr}(\mathrm{ch}(c)(1 - e^{-s^2D^2})) = O(s)$$

so it suffices to prove

$$\|\mathrm{ch}(c)(1 - e^{-s^2D^2})\|_1 = O(s).$$

Let $c = a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$. Then using $[F, a_p] = |D|^{-1}\partial(a_p) - |D|^{-1}\delta(a_p)F$, we have (on H_∞)

$$\begin{aligned}\mathrm{ch}(c)(1 - e^{-s^2D^2}) &= \Gamma F\left(\prod_{k=0}^{p-1}[F, a_k]\right)(|D|^p|D|^{-p})[F, a_p](1 - e^{-s^2D^2})\\ &= \Gamma F\left(\prod_{k=0}^{p-1}[F, a_k]\right)|D|^p \cdot |D|^{-p-1}\partial(a_p)(1 - e^{-s^2D^2})\\ &\quad - \Gamma F\left(\prod_{k=0}^{p-1}[F, a_k]\right)|D|^p \cdot |D|^{-p-1}\delta(a_p)(1 - e^{-s^2D^2})F.\end{aligned}$$

Thus,

$$\|\mathrm{ch}(c)(1-e^{-s^2D^2})\|_1 \leq \left\| \left(\prod_{k=0}^{p-1}[F,a_k]\right)|D|^p \right\|_\infty \cdot \left\| |D|^{-p-1}\partial(a_p)(1-e^{-s^2D^2}) \right\|_1 + \left\| \left(\prod_{k=0}^{p-1}[F,a_k]\right)|D|^p \right\|_\infty \cdot \left\| |D|^{-p-1}\delta(a_p)(1-e^{-s^2D^2}) \right\|_1 .$$

In both summands, the first factor is finite by Lemma A.4.4, the second factor is $O(s)$ by Lemma A.1.6. □

THEOREM 4.4.2. *Let $(\mathcal{A},H,D)$ be a spectral triple satisfying Hypothesis 1.2.1 and Hypothesis 4.0.1. For every Hochschild cycle $c\in\mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\mathcal{W}_p(c)De^{-s^2D^2}) = \frac{\mathrm{Ch}(c)}{2}s^{-2} + O(s^{-1}), \quad s\downarrow 0. \tag{4.14}$$

PROOF. Let $c\in\mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. By Lemma 4.4.1, we have

$$\mathrm{Tr}(\mathrm{ch}(c)e^{-s^2D^2}) = \mathrm{Tr}(\mathrm{ch}(c)) + O(s), \quad s\downarrow 0.$$

Since the spectral triple and p both have the same parity, we may apply Lemma 4.3.6 to get:

$$2\mathrm{Tr}(\mathcal{W}_\emptyset(c)e^{-s^2D^2}) = \mathrm{Tr}(\mathrm{ch}(c)) + O(s), \quad s\downarrow 0, \tag{4.15}$$

for all $c\in\mathcal{A}^{\otimes(p+1)}$

Define the multilinear mappings $\mathcal{K}_s$, $\mathcal{H}_s : \mathcal{A}^{\otimes(p+1)}\to\mathbb{C}$ by setting

$$\mathcal{K}_s(a_0\otimes\cdots\otimes a_p) = \mathrm{Tr}(\Gamma a_0\left(\prod_{k=1}^{p-1}[F,a_k]\right)[Fe^{-s^2D^2},a_p]),$$

$$\mathcal{H}_s(a_0\otimes\cdots\otimes a_p) = \mathrm{Tr}(\Gamma a_0\left(\prod_{k=1}^{p-1}[F,a_k]\right)F[e^{-s^2D^2},a_p]).$$

By the Leibniz rule, we have

$$[F,a_p]e^{-s^2D^2} = [Fe^{-s^2D^2},a_p] - F[e^{-s^2D^2},a_p].$$

Therefore:

$$\mathrm{Tr}(\mathcal{W}_\emptyset(c)e^{-s^2D^2}) = \mathcal{K}_s(c) - \mathcal{H}_s(c). \tag{4.16}$$

Now combining (4.15) and (4.16), we arrive at

$$2\mathcal{K}_s(c) - 2\mathcal{H}_s(c) = \mathrm{Tr}(\mathrm{ch}(c)) + O(s), \quad s\downarrow 0, \tag{4.17}$$

However it is shown in Appendix A.3 that $\mathcal{K}_s$ is a Hochschild coboundary, and thus since c is a Hochschild cycle we have $\mathcal{K}_s(c)=0$.

Using (4.17), we obtain

$$-2\mathcal{H}_s(c) = \mathrm{Tr}(\mathrm{ch}(c)) + O(s), \quad s\downarrow 0, \tag{4.18}$$

Define the multilinear mapping $\mathcal{V}_s : \mathcal{A}^{\otimes(p+1)}\to\mathbb{C}$ by setting

$$\mathcal{V}_s(a_0\otimes\cdots\otimes a_p) = \mathrm{Tr}(\Gamma a_0\left(\prod_{k=1}^{p-1}[F,a_k]\right)F\delta(a_p)|D|e^{-s^2D^2}).$$

Let $\frac{1}{q} = 1 - \frac{1}{p}$. By the Hölder inequality in the form (2.7):

$$
\begin{aligned}
&|(\mathcal{H}_s + 2s^2\mathcal{V}_s)(a_0 \otimes \cdots \otimes a_p)| \\
&\qquad = \Big|\mathrm{Tr}\Big(\Gamma a_0\Big(\prod_{k=1}^{p-1}[F,a_k]\Big)F\cdot\Big([e^{-s^2D^2},a_p] + 2s^2\delta(a_p)|D|e^{-s^2D^2}\Big)\Big)\Big| \\
&\qquad \leq \Big\|\Gamma a_0\prod_{k=1}^{p-1}[F,a_k]F\Big\|_{q,\infty}\Big\|[e^{-s^2D^2},a_p] + 2s^2\delta(a_p)|D|e^{-s^2D^2}\Big\|_{p,1}.
\end{aligned}
$$

The first factor on the above right hand side is finite by Proposition 3.1.5, and the second factor above is $O(s)$ by Lemma 4.2.4.(iii). Therefore, we have

$$
(\mathcal{H}_s + 2s^2\mathcal{V}_s)(c) = O(s), \quad s \downarrow 0, \tag{4.19}
$$

Combining (4.19) and (4.18), we arrive at

$$
4s^2\mathcal{V}_s(c) = \mathrm{Tr}(\mathrm{ch}(c)) + O(s), \quad s \downarrow 0, \tag{4.20}
$$

for all Hochschild cycles $c \in \mathcal{A}^{\otimes(p+1)}$.

From the definition of $\mathcal{W}_p$, if $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$:

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{W}_p(a_0\otimes\cdots\otimes a_p)De^{-s^2D^2}) &= \mathrm{Tr}(\Gamma a_0\left(\prod_{k=1}^{p-1}[F,a_k]\right)\delta(a_p)De^{-s^2D^2}) \\
&= \mathrm{Tr}(\Gamma a_0\left(\prod_{k=1}^{p-1}[F,a_k]\right)\delta(a_p)F|D|e^{-s^2D^2}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big|\mathcal{V}_s(a_0\otimes\cdots\otimes a_p) - \mathrm{Tr}(\mathcal{W}_p(a_0\otimes\cdots\otimes a_p)De^{-s^2D^2})\Big| \\
&\qquad = \Big|\mathrm{Tr}(\Gamma a_0\left(\prod_{k=1}^{p-1}[F,a_k]\right)[F,\delta(a_p)]|D|e^{-s^2D^2})\Big| \\
&\qquad = \Big|\mathrm{Tr}(\Gamma\left(\prod_{k=1}^{p-1}[F,a_k]\right)[F,\delta(a_p)]|D|e^{-s^2D^2}a_0)\Big| \\
&\qquad \leq \Big\|\Gamma\left(\prod_{k=1}^{p-1}[F,a_k]\right)[F,\delta(a_p)]|D|^p\Big\|_\infty\cdot\Big\||D|^{1-p}e^{-s^2D^2}a_0\Big\|_1.
\end{aligned}
$$

By Lemma A.1.4, we have that this is $O(s^{-1})$ and therefore, we have

$$
\mathcal{V}_s(c) = \mathrm{Tr}(\mathcal{W}_p(c)De^{-s^2D^2}) + O(s^{-1}), \quad s \downarrow 0, \tag{4.21}
$$

for all $c \in \mathcal{A}^{\otimes(p+1)}$.

Combining (4.20) and (4.21), we arrive at

$$
4s^2\mathrm{Tr}(\mathcal{W}_p(c)De^{-s^2D^2}) = \mathrm{Tr}(\mathrm{ch}(c)) + O(s), \quad s \downarrow 0
$$

for all Hochschild cycles $c \in \mathcal{A}^{\otimes(p+1)}$. Dividing by $4s^2$,

$$
\mathrm{Tr}(\mathcal{W}_p(c)De^{-s^2D^2}) = \frac{1}{4}s^{-2}\mathrm{Tr}(\mathrm{ch}(c)) + O(s^{-1}).
$$

Since $\mathrm{Ch}(c) = \frac{1}{2}\mathrm{Tr}(\mathrm{ch}(c))$, this formula coincides with (4.14). $\square$

We remark that (4.13) follows as a simple combination of Theorems 4.3.1 and 4.4.2.

4.5. Heat semigroup asymptotic: the proof of the first main result

In this section, we finally complete the proof of Theorem 1.2.2. We start by removing the assumption of Hypothesis 4.0.1.

The following two lemmas show that if the parity of p does not match $(\mathcal{A}, H, D)$, then the statement of (4.13) becomes trivial.

LEMMA 4.5.1. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. Suppose that D has a spectral gap at 0. Suppose that the dimension p is odd but $(\mathcal{A}, H, D)$ is even. Then:*

(i) *for every $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) = 0, \quad s > 0.$$

(ii) *for every $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\mathrm{ch}(c)) = 0.$$

PROOF. Let us prove (i). Since $\Gamma D = -D\Gamma$ and Γ commutes with $a \in \mathcal{A}$, we have $\Gamma[D,a] = -[D,a]\Gamma$ on H_∞. Hence $\Gamma\partial(a) = -\partial(a)\Gamma$ for all $a \in \mathcal{A}$. Thus for $c = a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$ since p is odd we have:

$$\Omega(c) = \Gamma a_0 \prod_{k=1}^{p} \partial(a_k) = -a_0 \left(\prod_{k=1}^{p} \partial(a_k)\right)\Gamma.$$

However $\Gamma D^2 = D^2\Gamma$, so by the spectral theorem we have $\Gamma e^{-s^2D^2} = e^{-s^2D^2}$. So multiplying by $e^{-s^2D^2}$ on the right and taking the trace,

$$\begin{aligned}\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) &= -\mathrm{Tr}(a_0\left(\prod_{k=1}^{p} \partial(a_k)\right)|D|^{2-p}e^{-s^2D^2}\Gamma)\\ &= -\mathrm{Tr}(\Gamma a_0\left(\prod_{k=1}^{p} \partial(a_k)\right)|D|^{2-p}e^{-s^2D^2}).\end{aligned}$$

This proves (i)

The argument for (ii) is similar. Note that we have $\Gamma[F,a] = -[F,a]\Gamma$ for every $a \in \mathcal{A}$. Thus since $p+1$ is even:

$$\begin{aligned}\mathrm{ch}(c) &= \Gamma F \prod_{k=0}^{p} [F, a_k]\\ &= -F\Gamma \prod_{k=0}^{p} [F, a_k]\\ &= -F \cdot \prod_{k=0}^{p} [F, a_k]\Gamma.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Tr}(\mathrm{ch}(c)) &= -\mathrm{Tr}(\Gamma F \prod_{k=0}^{p}[F, a_k]) \\ &= -\mathrm{Tr}(\mathrm{ch}(c)).\end{aligned}$$

This proves the second assertion. $\square$

Now, we deal with the other case where the parity of $(\mathcal{A}, H, D)$ does not match p.

LEMMA 4.5.2. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. Suppose D has a spectral gap at 0 and $(\mathcal{A}, H, D)$ is odd but p is even.*

(i) *for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2}) = O(s^{-1}), \quad s \downarrow 0.$$

(ii) *for every $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\mathrm{Tr}(\mathrm{ch}(c)) = 0.$$

PROOF. First, we prove (i). Consider the multilinear mapping $\theta_s : \mathcal{A}^{\otimes p} \to \mathbb{C}$ defined by the formula

$$\theta_s(a_0 \otimes \cdots \otimes a_{p-1}) = \mathrm{Tr}(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) |D|^{2-p}e^{-s^2D^2}).$$

The Hochschild coboundary $b\theta_s$ is computed in Section A.3 by the formula:

$$\begin{aligned}(b\theta_s)(a_0 \otimes \cdots \otimes a_p) = 2\mathrm{Tr}(a_0 &\left(\prod_{k=1}^{p} \partial(a_k)\right) |D|^{2-p}e^{-s^2D^2}) \\ &+ \mathrm{Tr}(a_0 \left(\prod_{k=1}^{p-1} \partial(a_k)\right) [|D|^{2-p}e^{-s^2D^2}, \partial(a_p)]) \\ &+ \mathrm{Tr}(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) [|D|^{2-p}e^{-s^2D^2}, a_p]).\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}&\left|\mathrm{Tr}(a_0 \left(\prod_{k=1}^{p-1} \partial(a_k)\right) [|D|^{2-p}e^{-s^2D^2}, \partial(a_p)])\right| \\ &\qquad \le \|a_0\|_\infty \prod_{k=1}^{p-1} \|\partial(a_k)\|_\infty \cdot \left\|[|D|^{2-p}e^{-s^2D^2}, \partial a_p])\right\|_1.\end{aligned}$$

By Lemma 4.2.5, we have

$$\mathrm{Tr}(a_0 \left(\prod_{k=1}^{p-1} \partial(a_k)\right) [|D|^{2-p}e^{-s^2D^2}, \partial(a_p)]) = O(s^{-1}), \quad s \downarrow 0. \tag{4.22}$$

Similarly, we have

$$\left|\mathrm{Tr}(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) [|D|^{2-p}e^{-s^2D^2}, a_p])\right| \le \prod_{k=0}^{p-1} \|\partial(a_k)\|_\infty \cdot \left\|[|D|^{2-p}e^{-s^2D^2}, a_p])\right\|_1.$$

By Lemma 4.2.5, we have

$$\mathrm{Tr}(\left(\prod_{k=0}^{p-1}\partial(a_k)\right)[|D|^{2-p}e^{-s^2D^2},a_p])=O(s^{-1}),\quad s\downarrow 0. \tag{4.23}$$

For every $c\in\mathcal{A}^{\otimes(p+1)}$, it follows from (4.22) and (4.23) that

$$(b\theta_s)(c)=2\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2})+O(s^{-1}),\quad s\downarrow 0.$$

If c is a Hochschild cycle, then $(b\theta_s)(c)=0$. Thus,

$$\mathrm{Tr}(\Omega(c)|D|^{2-p}e^{-s^2D^2})=O(s^{-1}),\quad s\downarrow 0.$$

for every Hochschild cycle c. This completes the proof of (i).

The proof of (ii) is similar to Lemma 4.5.1.(ii). For all $a\in\mathcal{A}$, we have $F[F,a]=-[F,a]F$. Since $p+1$ is odd,

$$F\cdot\prod_{k=0}^{p}[F,a_k]=-\left(\prod_{k=0}^{p}[F,a_k]\right)F.$$

Hence,

$$\mathrm{Tr}(F\prod_{k=0}^{p}[F,a_k])=-\mathrm{Tr}(F\prod_{k=0}^{p}[F,a_k]).$$

This proves (ii). □

The preceding two lemmas show how to remove the assumption that the parities of p and $(\mathcal{A},H,D)$ match. It remains to remove the assumption that D has a spectral gap at 0. For this purpose, we use the doubling trick. The "doubling trick" in this form follows [**12**, Definition 6].

Let $\mu>0$. We define another spectral triple $(\pi(\mathcal{A}),H_0,D_\mu)$, where

$$H_0=\mathbb{C}^2\otimes H,$$
$$D_\mu=\begin{pmatrix}D&\mu\\ \mu&-D\end{pmatrix}.$$

and π is the same representation of $\mathcal{A}$ as in Definition 2.2.17. That is:

$$\pi(a):=\begin{pmatrix}a&0\\0&0\end{pmatrix}.$$

For a tensor $c\in\mathcal{A}^{\otimes(p+1)}$, we denote $\pi(c)$ for the corresponding element of $(\pi(\mathcal{A}))^{\otimes(p+1)}$ obtained by applying the map $\pi^{\otimes(p+1)}$ to c. The spectral triple $(\pi(\mathcal{A}),H_0,D_\mu)$ is equipped with grading operator

$$\Gamma_0=\begin{pmatrix}\Gamma&0\\0&(-1)^{\deg}\Gamma\end{pmatrix}$$

where Γ is the (possibly trivial) grading of $(\mathcal{A},H,D)$ (see Definition 2.2.17).

Let Ω_μ and ch_μ be the multilinear mappings Ω and ch (just as in Definition 2.2.15) as applied to the spectral triple $(\pi(\mathcal{A}),H_0,D_\mu)$.

Lemma 4.5.3. *Let $(\mathcal{A},H,D)$ be a spectral triple satisfying Hypothesis 1.2.1. Let F_0 be as in Definition 2.2.17. If $a\in\mathcal{A}$, then as $\mu\downarrow 0$ we have:*

$$[\mathrm{sgn}(D_\mu),\pi(a)]\to[F_0,\pi(a)]$$

in $\mathcal{L}_{p+1}$.

PROOF. We have

$$\operatorname{sgn}(D_\mu) = \begin{pmatrix} \frac{D}{(D^2+\mu^2)^{1/2}} & \frac{\mu}{(D^2+\mu^2)^{1/2}} \\ \frac{\mu}{(D^2+\mu^2)^{1/2}} & -\frac{D}{(D^2+\mu^2)^{1/2}} \end{pmatrix}.$$

Hence,

$$[\operatorname{sgn}(D_\mu), \pi(a)] = \begin{pmatrix} \left[\frac{D}{(D^2+\mu^2)^{1/2}}, a\right] & -a\frac{\mu}{(D^2+\mu^2)^{1/2}} \\ \frac{\mu}{(D^2+\mu^2)^{1/2}}a & 0. \end{pmatrix}$$

On the other hand, we have

$$[F_0, \pi(a)] = \begin{pmatrix} [\operatorname{sgn}(D), a] & -aP \\ Pa & 0 \end{pmatrix}.$$

Therefore:

$$\begin{aligned}
&\Big\|[\operatorname{sgn}(D_\mu), \pi(a)] - [F_0, \pi(a)]\Big\|_{p+1} \\
&\qquad \le \Big\|\Big(\operatorname{sgn}(D) - \frac{D}{(D^2+\mu^2)^{\frac12}}\Big)a\Big\|_{p+1} + \Big\|a\Big(\operatorname{sgn}(D) - \frac{D}{(D^2+\mu^2)^{\frac12}}\Big)\Big\|_{p+1} \\
&\qquad\quad + \Big\|\Big(P - \frac{\mu}{(D^2+\mu^2)^{\frac12}}\Big)a\Big\|_{p+1} + \Big\|a\Big(P - \frac{\mu}{(D^2+\mu^2)^{\frac12}}\Big)\Big\|_{p+1} \\
&\qquad = \Big\|a^*\Big(\operatorname{sgn}(D) - \frac{D}{(D^2+\mu^2)^{\frac12}}\Big)^2 a\Big\|_{\frac{p+1}{2}}^{\frac12} + \Big\|a\Big(\operatorname{sgn}(D) - \frac{D}{(D^2+\mu^2)^{\frac12}}\Big)^2 a^*\Big\|_{\frac{p+1}{2}}^{\frac12} \\
&\qquad\quad + \Big\|a^*\Big(P - \frac{\mu}{(D^2+\mu^2)^{\frac12}}\Big)^2 a\Big\|_{\frac{p+1}{2}}^{\frac12} + \Big\|a\Big(P - \frac{\mu}{(D^2+\mu^2)^{\frac12}}\Big)^2 a^*\Big\|_{\frac{p+1}{2}}^{\frac12}.
\end{aligned}$$

By functional calculus, as $\mu \downarrow 0$ we have:

$$\Big(\operatorname{sgn}(D) - \frac{D}{(D^2+\mu^2)^{\frac12}}\Big)^2 \downarrow 0,$$
$$\Big(P - \frac{\mu}{(D^2+\mu^2)^{\frac12}}\Big)^2 \downarrow 0$$

in the weak operator topology. Hence,

$$a^*\Big(\operatorname{sgn}(D) - \frac{D}{(D^2+\mu^2)^{\frac12}}\Big)^2 a \downarrow 0, \quad a\Big(\operatorname{sgn}(D) - \frac{D}{(D^2+\mu^2)^{\frac12}}\Big)^2 a^* \downarrow 0, \quad \mu \downarrow 0,$$

$$a^*\Big(P - \frac{\mu}{(D^2+\mu^2)^{\frac12}}\Big)^2 a \downarrow 0, \quad a\Big(P - \frac{\mu}{(D^2+\mu^2)^{\frac12}}\Big)^2 a^* \downarrow 0, \quad \mu \downarrow 0.$$

also in the weak operator topology. The assertion follows now from the order continuity of the $\mathcal{L}_{\frac{p+1}{2}}$ norm. □

LEMMA 4.5.4. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathcal{A}^{\otimes(p+1)}$, then*

$$\lim_{\mu\to 0} \operatorname{ch}_\mu(\pi(c)) = \operatorname{ch}_0(c)$$

in the $\mathcal{L}_1$-norm.

PROOF. It suffices to prove the assertion for $c = a_0 \otimes \cdots \otimes a_p$. Then we have

$$\begin{aligned}\mathrm{ch}_0(c) - \mathrm{ch}_\mu(\pi(c)) &= \Gamma_0(F_0 - \mathrm{sgn}(D_\mu)) \prod_{k=0}^{p} [F_0, \pi(a_k)] \\ &\quad - \Gamma_0 \mathrm{sgn}(D_\mu)\Big(\prod_{k=0}^{p} [\mathrm{sgn}(D_\mu), \pi(a_k)] - \prod_{k=0}^{p} [F_0, \pi(a_k)]\Big).\end{aligned}$$

Next we use the fact that if $q \geq 1$ and if $A_\mu \to A$ in the strong operator topology, $B_\mu \to B \in \mathcal{L}_q$ in the $\mathcal{L}_q$ norm, and $\sup_{\mu \downarrow 0} \|A_\mu\|_\infty < \infty$ then $A_\mu B_\mu \to AB$ in $\mathcal{L}_1$ (see [**60**, Chapter 2, Example 3]).

We have that $\mathrm{sgn}(D_\mu) - F_0 \to 0$ in the strong operator topology, and since $\prod_{k=0}^{p} [F_0, \pi(a_k)] \in \mathcal{L}_1$, it follows that the first summand converges to 0 in the $\mathcal{L}_1$ norm.

For the second summand, we apply Lemma 4.5.3. Since for each k we have that $[\mathrm{sgn}(D_\mu), \pi(a_k)] \to [F_0, \pi(a_k)]$ in $\mathcal{L}_{p+1}$, it follows that the second summand converges to 0 in $\mathcal{L}_1$. □

LEMMA 4.5.5. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. For every $c \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\Big\| \Omega_\mu(\pi(c))(1 \otimes (1+D^2)^{-\frac{p}{2}} e^{-s^2 D^2}) \Big\|_1 = O(s^{-1}), \quad s \downarrow 0.$$

PROOF. Once more it suffices to prove the assertion for an elementary tensor $c = a_0 \otimes \cdots \otimes a_p$. We have

$$\begin{aligned}&\Big\| \Omega_\mu(\pi(c))(1 \otimes (1+D^2)^{-\frac{p}{2}} e^{-s^2 D^2}) \Big\|_1 \\ &\quad \leq \Big\| \Gamma_0 \pi(a_0) \prod_{k=1}^{p-1} [D_\mu, \pi(a_k)] \Big\|_\infty \Big\| [D_\mu, \pi(a_p)](1 \otimes (1+D^2)^{-\frac{p}{2}} e^{-s^2 D^2}) \Big\|_1 \\ &\quad \leq \|a_0\|_\infty \prod_{k=1}^{p-1} \|[D_\mu, \pi(a_k)]\|_\infty \Big\| [D, a_p](1+D^2)^{-\frac{p}{2}} e^{-s^2 D^2} \Big\|_1 \\ &\qquad + 2\mu \|a_0\|_\infty \prod_{k=1}^{p-1} \|[D_\mu, \pi(a_k)]\|_\infty \Big\| a_p (1+D^2)^{-\frac{p}{2}} e^{-s^2 D^2} \Big\|_1.\end{aligned}$$

The assertion follows by applying Lemma A.1.4 (with $m_1 = 0$ and $m_2 = p-1$) to the odd spectral triple $(\mathcal{A}, H, F(1+D^2)^{\frac{1}{2}})$. □

We note that since π is an algebra homomorphism, if $c \in \mathcal{A}^{\otimes(p+1)}$ is a Hochschild cycle then so is $\pi(c)$.

LEMMA 4.5.6. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. Let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. We have*

$$(\mathrm{Tr}_2 \otimes \mathrm{Tr})(\Omega_\mu(\pi(c))(1 \otimes (1+D^2)^{1-\frac{p}{2}} e^{-s^2 D^2})) = \frac{p}{2} \mathrm{Ch}_\mu(\pi(c)) s^{-2} + O(s^{-1}), \quad s \downarrow 0.$$

PROOF. For $\mu > 0$, the spectral triple $(\pi(\mathcal{A}), H_0, D_\mu)$ satisfies Hypothesis 1.2.1 and the spectral gap assumption. This allows us to apply Theorems 4.3.1 and 4.4.2 (or Lemmas 4.5.1 and 4.5.2) to the Hochschild cycle $\pi(c) \in (\pi(\mathcal{A}))^{\otimes(p+1)}$.

A combination of Theorems 4.3.1 and 4.4.2 (if the parities of p and $(\pi(\mathcal{A}), H_0, D_\mu)$ match) or one of Lemmas 4.5.1 and 4.5.2 (if parities of p and $(\pi(A), H_0, D_\mu)$ do not match) yields

$$(\mathrm{Tr}_2 \otimes \mathrm{Tr})(\Omega_\mu(\pi(c))|D_\mu|^{2-p}e^{-s^2 D_\mu^2}) = \frac{p}{2}\mathrm{Ch}_\mu(\pi(c))s^{-2} + O(s^{-1}), \quad s \downarrow 0.$$

Noting that $D_\mu^2 = D_0^2 + \mu^2$, and $e^{-s^2\mu^2} = O(1)$ we obtain

$$(\mathrm{Tr}_2 \otimes \mathrm{Tr})(\Omega_\mu(\pi(c))|D_\mu|^{2-p}e^{-s^2 D_0^2}) = \frac{p}{2}\mathrm{Ch}_\mu(\pi(c))s^{-2} + O(s^{-1}), \quad s \downarrow 0.$$

By Lemma 4.5.5, we have

$$\begin{aligned}
&\left\|\Omega_\mu(\pi(c))|D_\mu|^{2-p}e^{-s^2 D_0^2} - \Omega_\mu(\pi(c))|D_1|^{2-p}e^{-s^2 D_0^2}\right\|_1 \\
&\qquad \le \left\|\Omega_\mu(\pi(c))|D_1|^{-p}e^{-s^2 D_0^2}\right\|_1 \left\|(|D_\mu|^{2-p} - |D_1|^{2-p}) \cdot |D_1|^{p-2}\right\|_\infty \\
&\qquad = O(s^{-1}).
\end{aligned}$$

□

We are now ready to prove the main result of this chapter.

Proof of Theorem 1.2.2. For $\mathscr{A} \subseteq \{1, \cdots, p\}$, we define the multilinear functional on $a_0 \otimes \cdots a_p \in \mathcal{A}^{\otimes(p+1)}$ by:

$$\mathcal{T}_{\mathscr{A}}(a_0 \otimes \cdots \otimes a_p) = \Gamma_0 \pi(a_0) \prod_{k=1}^{p} y_k(a_k).$$

Here,

$$y_k(a) := \begin{cases} \begin{pmatrix} \partial(a) & 0 \\ 0 & 0 \end{pmatrix}, k \notin \mathscr{A} \\ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, k \in \mathscr{A}. \end{cases}$$

In particular,

$$\mathcal{T}_{\emptyset}(a_0 \otimes \cdots a_p) = \begin{pmatrix} \Omega(a_0 \otimes \cdots \otimes a_p) & 0 \\ 0 & 0 \end{pmatrix}.$$

For $c \in \mathcal{A}^{\otimes(p+1)}$, we apply (4.5) to get

$$\Omega_\mu(\pi(c)) = \sum_{\mathscr{A} \subseteq \{1,\cdots,p\}} \mu^{|\mathscr{A}|}\mathcal{T}_{\mathscr{A}}(c).$$

For each $0 \le k \le p$, we set

$$f_k(s) = \sum_{|\mathscr{A}|=k} (\mathrm{Tr}_2 \otimes \mathrm{Tr})(\mathcal{T}_{\mathscr{A}}(c) \cdot (1 \otimes (1 + D^2)^{1-\frac{p}{2}} e^{-s^2 D^2})).$$

That is, $f_k(s)$ is the coefficient of μ^k in $(\mathrm{Tr}_2 \otimes \mathrm{Tr})(\Omega_\mu(\pi(c))(1 \otimes (1+D^2)^{1-\frac{p}{2}} e^{-s^2 D^2}))$.

Now if $c \in \mathcal{A}^{\otimes(p+1)}$ is a Hochschild cycle, then Lemma 4.5.6 yields

$$\sum_{k=0}^{p} \mu^k f_k(s) = \frac{p}{2}\mathrm{Ch}_\mu(\pi(c))s^{-2} + O(s^{-1}), \quad s \downarrow 0. \tag{4.24}$$

Select a set $\{\mu_0, \dots, \mu_p\}$ of distinct positive numbers, and for each $0 \le l \le p$ we may take $\mu = \mu_l$ in (4.24) to arrive at:

$$\sum_{k=0}^{p} \mu_l^k f_k(s) = \frac{p}{2} \mathrm{Ch}_{\mu_l}(\pi(c)) s^{-2} + O(s^{-1}), \quad s \downarrow 0, \quad 0 \le l \le p.$$

Since the Vandermonde matrix $\{\mu_l^k\}_{0 \le l, k \le p}$ is invertible, it follows that there exist $\{\alpha_0, \dots, \alpha_k\}$ such that:

$$f_k(s) = \frac{\alpha_k}{s^2} + O(s^{-1}), \quad s \downarrow 0, \quad 0 \le k \le p.$$

Substituting this back to (4.24), we obtain

$$\sum_{k=0}^{p} \mu^k \alpha_k = \frac{p}{2} \mathrm{Ch}_\mu(\pi(c)).$$

In particular

$$\alpha_0 = \frac{p}{2} \lim_{\mu \to 0} \mathrm{Ch}_\mu(\pi(c))$$

So by Lemma 4.5.4,

$$\alpha_0 = \frac{p}{2} \mathrm{Ch}(c).$$

Hence,

$$f_0(s) = \frac{p}{2} \mathrm{Ch}(c) s^{-2} + O(s^{-1}), \quad s \downarrow 0.$$

The assertion follows now from the definition of f_0. □

CHAPTER 5

Residue of the $\zeta-$function and the Connes character formula

In this chapter we complete the proofs of Theorem 1.2.3 and Theorem 1.2.5.

For a spectral triple $(\mathcal{A}, H, D)$ satisfying Hypothesis 1.2.1, we define the zeta function of a Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$ by the formula

$$\zeta_{c,D}(z) := \mathrm{Tr}(\Omega(c)(1+D^2)^{-z/2}), \quad \Re(z) > p+1.$$

Indeed, by Hypothesis 1.2.1.(iii) if $\Re(z) \geq p+1$ then the operator $\Omega(c)(1+D^2)^{-z/2}$ is trace class, and so $\zeta_{c,D}$ is well defined when $\Re(z) > p+1$. In Section 5.1 we prove that $\zeta_{c,D}$ is holomorphic and has analytic continuation to the set

$$\{z \in \mathbb{C} \ : \ \Re(z) > p-1\} \setminus \{p\}.$$

We also show that the point p is a simple pole for $\zeta_{c,D}$, and that the corresponding residue of $\zeta_{c,D}$ at p is equal to $p\mathrm{Ch}(c)$, thus completing the proof of Theorem 1.2.3.

Then we undertake the more difficult task of proving Theorem 1.2.5. We achieve this by a new characterisation of universal measurability in Section 5.5, which allows us to deduce Theorem 1.2.5 as a corollary of Theorem 1.2.3.

The most novel feature of this chapter, and of this manuscript as a whole, is a certain integral representation of the difference $B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z$ for two positive bounded operators A and B and $z \in \mathbb{C}$ with $\Re(z) > 0$ (see Lemma 5.2.1). This result first appeared in [**25**, Lemma 5.2] for the special case where z is real and positive. With this integral representation we are able to prove the analyticity of the function

$$z \mapsto \mathrm{Tr}(XB^zA^z) - \mathrm{Tr}(X(A^{\frac{1}{2}}BA^{\frac{1}{2}})^z)$$

for a bounded operator X and for z in a certain domain in the complex plane, under certain assumptions on A and B. This result is stated in full in Section 5.4.

In sections 5.6 and 5.7 we complete the proof of Theorem 1.2.5.

5.1. Analyticity of the ζ-function for $\Re(z) > p-1$, $z \neq p$

This section contains the proof of Theorem 1.2.3. The proof is relatively short, since we are able to use Theorem 1.2.2.

LEMMA 5.1.1. *Let $h \in L_\infty(0,1)$ and $u \in L_\infty(1,\infty)$. Then,*

(i) *the function*

$$F(z) := \int_0^1 s^{z-1}h(s)ds, \quad \Re(z) > 0,$$

is analytic.

(ii) *the function*

$$G(z) := \int_1^\infty s^{z-1} u(s) e^{-s} ds, \quad z \in \mathbb{C},$$

is analytic.

PROOF. Let us prove (i). Define

$$F_n(z) = \int_{\frac{1}{n}}^1 s^{z-1} h(s) ds, \quad \Re(z) > 0.$$

Then for $\Re(z) > 0$ we have:

$$\begin{aligned} |F(z) - F_n(z)| &= \left| \int_0^{\frac{1}{n}} s^{z-1} h(s) ds \right| \\ &\leq \int_0^{\frac{1}{n}} s^{\Re(z)-1} |h(s)| ds \\ &\leq \|h\|_\infty \int_0^{\frac{1}{n}} s^{\Re(z)-1} ds \\ &= \frac{\|h\|_\infty}{\Re(z)} n^{-\Re(z)}. \end{aligned}$$

So for every $\varepsilon > 0$, we have that F_n converges uniformly to F on the set $\{z : \Re(z) > \varepsilon\}$.

We now show that for each n the function F_n is entire. Indeed, we have the power series expansion

$$\begin{aligned} s^{z-1} &= e^{(z-1)\log(s)} \\ &= \sum_{k \geq 0} \frac{(\log(s))^k}{k!} (z-1)^k. \end{aligned}$$

which converges uniformly on compact subsets of $\mathbb{C}$. Therefore, interchanging the integral and summation, we have that for all $z \in \mathbb{C}$

$$F_n(z) = \sum_{k \geq 0} \frac{1}{k!} \left(\int_{\frac{1}{n}}^1 (\log(s))^k h(s) ds \right) (z-1)^k, \quad z \in \mathbb{C}. \tag{5.1}$$

This power series has infinite radius of convergence, since

$$\left| \int_{\frac{1}{n}}^1 (\log(s))^k h(s) ds \right| \leq \|h\|_\infty (\log(n))^k.$$

So each F_n is entire.

In summary, the sequence $\{F_n\}_{n \geq 1}$ of entire functions converges to F uniformly on the half-plane $\{z : \Re(z) > \varepsilon\}$. Since ε is arbitrary, the sequence $\{F_n\}_{n \geq 0}$ converges uniformly to F on compact subsets of the half plane $\{z : \Re(z) > 0\}$. Thus, F is holomorphic on this half-plane.

To prove (ii), we consider the functions

$$G_n(z) := \int_1^n s^{z-1} u(s) e^{-s} ds, \quad z \in \mathbb{C}$$

Exactly the same argument as above shows that each G_n is entire. For all $n \geq 1$, we have:

$$\begin{aligned}|G(z) - G_n(z)| &\leq \int_n^\infty s^{\Re(z)-1}|u(s)|e^{-s}\,ds \\ &\leq \|u\|_\infty \int_n^\infty s^{\Re(z)-1}e^{-s}\,ds\end{aligned}$$

So for any $N > 0$, we have that G_n converges uniformly to G in the set $\{z \in \mathbb{C} : \Re(z) < N\}$, and therefore on compact subsets of the plane. Hence, G is entire. ☐

We are now able to prove Theorem 1.2.3.

Theorem. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1, and let $c \in \mathcal{A}^{\otimes(p+1)}$ be a Hochschild cycle. Then the function*

$$\zeta_{c,D}(z) := \mathrm{Tr}(\Omega(c)(1+D^2)^{-z/2}), \quad \Re(z) > p+1$$

is analytic, and has analytic continuation to the set $\{z \in \mathbb{C} : \Re(z) > p-1\} \setminus \{p\}$ and a simple pole at p with residue $p\mathrm{Ch}(c)$.

Proof of Theorem 1.2.3. Let $z \in \mathbb{C}$ with $\Re(z) > 1$. Then for all $x > 0$, we have

$$\begin{aligned}\int_0^\infty s^{z-1}e^{-s^2x^2}\,ds &= x^{-z}\int_0^\infty t^{z-1}e^{-t^2}\,dt \\ &= x^{-z}\int_0^\infty u^{\frac{z-1}{2}}e^{-u}\frac{u^{-\frac{1}{2}}}{2}\,du \\ &= \frac{x^{-z}}{2}\Gamma\left(\frac{z}{2}\right).\end{aligned}$$

Thus,

$$x^{2-z} = \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^\infty s^{z-1}x^2e^{-x^2s^2}\,ds$$

So by the functional calculus, for $\Re(z) > 2$ we have an integral in the weak operator topology:

$$(1+D^2)^{1-\frac{z}{2}} = \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^\infty s^{z-1}(1+D^2)e^{-s^2(1+D^2)}\,ds.$$

We now multiply on the left by the bounded operator $\Omega(c)(1+D^2)^{-p/2}$ to arrive at:

$$\Omega(c)(1+D^2)^{1-\frac{z+p}{2}} = \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^\infty s^{z-1}\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}\,ds.$$

We claim that this integral converges in $\mathcal{L}_1$. First, if $p = 1$ then consider the function $h_1(t) = te^{-t^2}$. Applying Lemma A.1.3 with the function h_1, we have:

$$\begin{aligned}\|s^{\Re(z)-1}\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}\|_1 &= s^{\Re(z)-2}\|\Omega(c)h_1(s(1+D^2)^{1/2})\|_1 \\ &= O(s^{\Re(z)-3}), \quad s \downarrow 0.\end{aligned}$$

On the other hand, if $p > 1$ then define $h_p(t) = (1+t^2)^{1-p/2}e^{-t^2}$. Now applying Lemma A.1.3 with the function h_p:

$$\begin{aligned}\|s^{z-1}\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}\|_1 &\leq \left\|\left(\frac{1+D^2}{1+s^2D^2}\right)^{1-\frac{p}{2}}\right\|_\infty \|s^{z-1}\Omega(c)h_p(s|D|)\|_1 \\ &= s^{\Re(z)+p-3}\|\Omega(c)h_p(s|D|)\|_1 \\ &= O(s^{\Re(z)-3}), \quad s \downarrow 0.\end{aligned}$$

So in both cases, since $\Re(z) > 2$, the function $s \mapsto \|s^{z-1}\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}\|_1$ is integrable on $[0,1]$.

For $s > 1$, we have

$$e^{-s^2D^2} \leq e^{-D^2} \leq (1+D^2)^{-3/2}.$$

so we have that

$$\|s^{z-1}\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}\|_1 \leq s^{\Re(z)-1}\|\Omega(c)(1+D^2)^{-\frac{p+1}{2}}\|_1.$$

Hence, $s \mapsto \|s^{z-1}\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}\|_1$ is integrable on the interval $(1,\infty)$.

By Lemma 2.3.2, for $\Re(z) > 2$ we therefore have:

$$\begin{aligned}\|\Omega(c)(1+D^2)^{1-\frac{z+p}{2}}\|_1 &\leq \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^\infty \|s^{z-1}\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}\|_1\,ds \\ &< \infty\end{aligned}$$

and

$$\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{z+p}{2}}) = \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^\infty s^{z-1}\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)})\,ds. \tag{5.2}$$

We will now apply the result of Theorem 1.2.2 to the integrand. First we define a function h on $(0,\infty)$ by:

$$h(s) := \begin{cases} e^s\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}), & s \geq 1 \\ s\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2(1+D^2)}) - \frac{p}{2}\mathrm{Ch}(c)s^{-1}, & 0 < s < 1.\end{cases}$$

By Theorem 1.2.2, the function h is bounded on the interval $(0,1)$. For $s > 1$, we have a constant c such that.

$$|h(s)| \leq Ce^{s-s^2}$$

Hence, h is bounded on $[0,\infty)$. Substituting h in (5.2):

$$\begin{aligned}\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{z+p}{2}}) &= \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^1 s^{z-1}(s^{-1}h(s) + \frac{p}{2}\mathrm{Ch}(c)s^{-2})\,ds \\ &+ \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_1^\infty s^{z-1}e^{-s}h(s)\,ds.\end{aligned}$$

By Lemma 5.1.1.(ii), the second term in the above sum has extension to an entire function, and so we focus on the first term. We have,

$$\begin{aligned}\frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^1 s^{z-1}(s^{-1}h(s) + \frac{p}{2}\mathrm{Ch}(c)s^{-2})\,ds &= \frac{2}{\Gamma\left(\frac{z}{2}\right)}\int_0^1 s^{z-2}h(s)\,ds \\ &+ \frac{p}{\Gamma\left(\frac{z}{2}\right)}\mathrm{Ch}(c)\int_0^1 s^{z-3}\,ds.\end{aligned}$$

Due to Lemma 5.1.1.(i), the first term in the above sum has extension to an analytic function for $\Re(z-1) > 0$. That is, for $\Re(z) > 1$.

As for the second term, since we are still working with $\Re(z) > 2$ we may compute:

$$\int_0^1 s^{z-3}\,ds = (z-2)^{-1}.$$

So in summary, the function initially defined for $\Re(z) > 2$ given by:

$$z \mapsto \mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{p+z}{2}}) - \frac{p}{\Gamma\left(\frac{z}{2}\right)}(z-2)^{-1}$$

is analytic on the set $\Re(z) > 2$, and has analytic continuation to the set $\Re(z) > 1$. Since the function $\frac{1}{\Gamma\left(\frac{z}{2}\right)}$ is entire, and $\Gamma(1) = 1$, we may equivalently say that the function

$$z \mapsto \zeta_{c,D}(z+p-2) - p\mathrm{Ch}(c)(z-2)^{-1}, \quad \Re(z) > 2$$

has analytic continuation to the set $\Re(z) > 1$. In other words, for $\Re(z) > p$

$$\zeta_{c,D}(z) - p\mathrm{Ch}(c)(z-p)^{-1}$$

has analytic continuation to the set $\Re(z) > p-1$. This is exactly the statement of the theorem. □

5.2. An Integral Representation for $B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z$

In this section, we follow the convention that for all $s \in \mathbb{R}$ we have $0^{is} = 0$, so in particular we have the unusual convention that $0^{i0} = 0$. This section is devoted to the proof of Theorem 5.2.1 (stated below). Theorem 5.2.1 is a strengthening of [**25**, Lemma 5.2] (which corresponds to the special case where z is real and B is compact). Theorem 5.2.1 also substantially strengthens [**10**, Proposition 4.4].

Here we work with abstract operators on a separable Hilbert space H. Given a positive bounded operator A on H, and a complex number z with $\Re(z) > 0$, the operator A^z may be defined by continuous functional calculus.

Theorem 5.2.1. *Let A and B be bounded, positive operators on H, and let $z \in \mathbb{C}$ with $\Re(z) > 1$. Let $Y := A^{1/2}BA^{1/2}$. We define the mapping $T_z : \mathbb{R} \to \mathcal{L}_\infty$ by,*

$$T_z(0) := B^{z-1}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}}] + [BA^{\frac{1}{2}}, A^{\frac{1}{2}}]Y^{z-1},$$
$$T_z(s) := B^{z-1+is}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}+is}]Y^{-is} + B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1-is}, \quad s \neq 0.$$

We also define the function $g_z : \mathbb{R} \to \mathbb{C}$ by:

$$g_z(0) := 1 - \frac{z}{2},$$
$$g_z(t) := 1 - \frac{e^{\frac{z}{2}t} - e^{-\frac{z}{2}t}}{(e^{\frac{t}{2}} - e^{-\frac{t}{2}})(e^{\left(\frac{z-1}{2}\right)t} + e^{-\left(\frac{z-1}{2}\right)t})}.$$

Then:

(i) *The mapping $T_z : \mathbb{R} \to \mathcal{L}_\infty$ is continuous in the weak operator topology.*
(ii) *We have:*

$$B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z = T_z(0) - \int_{\mathbb{R}} T_z(s)\widehat{g}_z(s)\,ds.$$

REMARK 5.2.2. *For $\Re(z) > 1$, the function g_z is Schwartz, and hence so is the (rescaled) Fourier transform $\widehat{g}_z$.*

PROOF. For $t \neq 0$, we may rewrite $g_z(t)$ as:

$$g_z(t) = \frac{1}{2}\left(\frac{\tanh((z-1)t/2)}{\tanh(t/2)} - 1\right).$$

Letting $s = t/2$ and $w = z - 1$, it then suffices to show that for $\Re(w) > 0$ the function

$$f_w(s) = \frac{\tanh(ws)}{\tanh(s)} - 1, \quad s \neq 0$$

with $f_w(0) = w - 1$ is Schwartz. Since $\lim_{s\to 0} \frac{\tanh(s)}{s} = 1$, it is evident that f_w is continuous at 0 and that f_w is smooth in $[-1, 1]$.

It suffices now to show that the function $\tanh(ws) - \tanh(s)$ is Schwartz, since for $|s| > 1$ the function $\frac{1}{\tanh(s)}$ is smooth and bounded with all derivatives bounded. For $s > 1$, we can write,

$$\tanh(ws) - \tanh(s) = \tanh(ws) - 1 + (1 - \tanh(s))$$

and then note that since $\Re(w) > 0$, the functions $\tanh(ws) - 1$ and $1 - \tanh(s)$ have rapid decay as $s \to \infty$, with all derivatives to all orders also of rapid decay as $s \to \infty$. Similarly, for $s < -1$, we can write $\tanh(ws) - \tanh(s) = \tanh(ws) + 1 - (\tanh(s) + 1)$ and then use the fact that $\tanh(s) + 1$ and $\tanh(ws) + 1$ have rapid decay, with all derivatives of rapid decay, as $s \to -\infty$. □

LEMMA 5.2.3. *Let $A_k, X_k \in \mathcal{L}_\infty$, $1 \leq k \leq n$, and let $X_k \geq 0$, $1 \leq k \leq n$. The function from $\mathbb{R}$ to $\mathcal{L}_\infty$ given by:*

$$s \mapsto \prod_{k=1}^{n} A_k X_k^{is}, s \in \mathbb{R}$$

is continuous in the strong operator topology (and in particular in the weak operator topology).

PROOF. If uniformly bounded nets $\{A_i\}_{i\in\mathbb{I}}$ and $\{B_i\}_{i\in\mathbb{I}}$ converge in the strong operator topology to A and B respectively, then the net $\{A_iB_i\}_{i\in\mathbb{I}}$ converges to AB in strong operator topology. This fact is standard and can be found e.g. in [**8**, Proposition 2.4.1]. Therefore it suffices to show that for each $k = 1, \ldots, n$ that the function $s \mapsto X_k^{is}$ is continuous in the strong operator topology.

We note if X_k has a spectral gap at 0, then $\log(X_k)$ is well defined by continuous functional calculus and $X_k^{is} = \exp(is\log(X_k))$ is strongly continuous by the Stone-von Neumann theorem.

If X_k does not necessarily have a spectral gap, then instead we use the Borel function:

$$\log_0(t) := \begin{cases} \log(t), & t > 0 \\ 0, & t = 0. \end{cases}$$

Hence, for $s \in \mathbb{R}$ and $t \geq 0$,

$$\exp(is\log_0(t)) = \begin{cases} t^{is}, & t > 0, \\ 1, & t = 0. \end{cases}$$

Recalling our convention stated at the start of this section, that $0^{is} = 0$ for all $t \geq 0$, we have:

$$t^{is} = \exp(is\log_0(t))(1 - \chi_{\{0\}}(t)).$$

Let P_k be the support projection of X_k (i.e., the projection onto the orthogonal complement of the kernel of X_k). Then since $P_k = 1 - \chi_{\{0\}}(X_k)$ by Borel functional calculus we have:

$$X_k^{is} = P_k \exp(is\log_0(X_k)).$$

Since the operator $\log_0(X_k)$ is self-adjoint, by the Stone-von Neumann theorem it follows that $s \mapsto \exp(is\log_0(X_k))$ is strongly continuous. Hence, $s \mapsto X_k^{is}$ is strongly continuous and the proof is complete. □

LEMMA 5.2.4. *Let X and Y be positive bounded operators and $z \in \mathbb{C}$ with $\Re(z) > 1$. Set $V_z := X^{z-1}(X - Y) + (X - Y)Y^{z-1}$. Then,*

$$X^z - Y^z = V_z - \int_{\mathbb{R}} X^{is} V_z Y^{-is} \widehat{g}_z(s) ds.$$

The integral is understood in the weak operator topology sense. The function g_z is the same as in the statement of Theorem 5.2.1.

PROOF. We define the function $\phi_{1,z}$ on $[0, \infty) \times [0, \infty)$ by:

$$\begin{aligned} \phi_{1,z}(\lambda, \mu) &:= g_z(\log(\frac{\lambda}{\mu})) \quad \lambda, \mu > 0, \\ \phi_{1,z}(0, \mu) &:= 0, \quad \mu \geq 0, \\ \phi_{1,z}(\lambda, 0) &:= 0, \quad \lambda \geq 0. \end{aligned}$$

We caution the reader that $\phi_{1,z}$ is not continuous at $(0, 0)$ unless $z = 2$ (indeed, $\phi_{1,z}(\lambda, \lambda) = g_z(0) = 1 - \frac{z}{2}$ for all $\lambda > 0$). If we rewrite the definition of g_z in terms of exponentials, then for $t \neq 0$ we get

$$g_z(t) = 1 - \frac{e^{\frac{z}{2}t} - e^{-\frac{z}{2}t}}{(e^{\frac{t}{2}} - e^{-\frac{t}{2}})(e^{(\frac{z-1}{2})t} + e^{-(\frac{z-1}{2})t})},$$

and therefore:

$$\phi_{1,z}(\lambda, \mu) = 1 - \frac{\lambda^z - \mu^z}{(\lambda - \mu)(\lambda^{z-1} + \mu^{z-1})}, \quad \lambda, \mu > 0, \lambda \neq \mu. \tag{5.3}$$

We claim that

$$\phi_{1z}(\lambda, \mu) = \int_{\mathbb{R}} \widehat{g}_z(s) \lambda^{is} \mu^{-is} ds, \quad \lambda, \mu \geq 0. \tag{5.4}$$

Indeed, since g_z is Schwartz we can use the Fourier inversion theorem:

$$g_z(t) = \int_{\mathbb{R}} \widehat{g}_z(s) e^{ist} ds, \quad t \in \mathbb{R}.$$

If $\lambda, \mu > 0$, we simply substitute $t = \log(\lambda/\mu)$. For $\lambda = 0$ or $\mu = 0$, then the right hand side of (5.4) vanishes, as does $\phi_{1,z}$ by definition. Hence (5.4) is valid for all $\lambda, \mu \geq 0$.

Thus, by the definition of the double operator integral (2.14), we have:

$$T_{\phi_{1,z}}^{X,Y}(A) = \int_{\mathbb{R}} \widehat{g}_z(s) X^{is} A Y^{-is} ds. \tag{5.5}$$

Indeed, since g_z is a Schwartz function, it follows that $\widehat{g}_z \in L_1(\mathbb{R})$ and so the condition (2.13) holds. Therefore, (5.5) follows as a consequence of (2.14). Here, the integral on the right hand side of (5.5) is understood in the weak operator topology sense.

Measurability of the function $s \mapsto X^{is}AY^{-is}$ in the weak operator topology is guaranteed by Lemma 5.2.3 and condition (2.10) follows from the inequality

$$\|\widehat{g}_z(s)X^{is}AY^{-is}\|_\infty \le |\widehat{g}_z(s)| \cdot \|A\|_\infty, \quad s \in \mathbb{R},$$

and from the fact that $\widehat{g}_z$ is a Schwartz (and in particular integrable) function. So it follows that $T^{X,Y}_{\phi_{1,z}}$ is bounded in the operator norm from $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$.

We introduce two more functions on $[0,\infty) \times [0,\infty)$. First,

$$\phi_{2,z}(\lambda,\mu) = (\lambda^{z-1} + \mu^{z-1})(\lambda - \mu), \quad \lambda,\mu \ge 0$$

and secondly,

$$\phi_{3,z}(\lambda,\mu) = (\lambda^{z-1} + \mu^{z-1})(\lambda - \mu) - (\lambda^z - \mu^z), \quad \lambda,\mu \ge 0.$$

Both functions are bounded on compact subsets of $[0,\infty)^2$, and so in particular on $\mathrm{Spec}(X) \times \mathrm{Spec}(Y)$, since by assumption both X and Y are bounded.

The equality $\phi_{3,z} = \phi_{1,z}\phi_{2,z}$ holds on $[0,\infty) \times [0,\infty)$. Indeed this follows from (5.3) for $\lambda,\mu > 0$, $\lambda \ne \mu$. For $\lambda = \mu > 0$ one has $\phi_{1,z}(\lambda,\lambda) = 1 - \frac{z}{2}$, $\phi_{2,z}(\lambda,\lambda) = 0$ and $\phi_{3,z}(\lambda,\lambda) = 0$. If $\lambda = 0$ or $\mu = 0$ one has $\phi_{1,z}(\lambda,\mu) = 0$ and $\phi_{3,z}(\lambda,\mu) = 0$.

Using formulae (2.16) and (2.14), we obtain that $T^{X,Y}_{\phi_{2,z}} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ and

$$T^{X,Y}_{\phi_{2,z}}(A) = X^zA - X^{z-1}AY + XAY^{z-1} - AY^z.$$

Since $\phi_{3,z}$ bounded on $\mathrm{Spec}(X) \times \mathrm{Spec}(Y)$, we also get that $T^{X,Y}_{\phi_{3,z}}$ is bounded in the operator norm from $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$, and

$$T^{X,Y}_{\phi_{3,z}}(A) = (X^zA - X^{z-1}AY + XAY^{z-1} - AY^z) - (X^zA - AY^z).$$

We note at this point that $T^{X,Y}_{\phi_{3,z}}(1) = V_z$.

We have $\phi_{3,z} = \phi_{1,z}\phi_{2,z}$ on $\mathrm{Spec}(X) \times \mathrm{Spec}(Y)$, and thus by (2.15):

$$\begin{aligned} T^{X,Y}_{\phi_{1,z}}(V_z) &= T^{X,Y}_{\phi_{1,z}}(T^{X,Y}_{\phi_{2,z}}(1)) \\ &= T^{X,Y}_{\phi_{3,z}}(1) \\ &= V_z - (X^z - Y^z). \end{aligned}$$

The assertion follows now from (5.5). □

We are now able to prove Theorem 5.2.1 in the special case where the spectrum of B is a finite set.

Lemma 5.2.5. *Theorem 5.2.1 holds under the additional assumption that the spectrum of B consists of a finite set of points.*

Proof. Suppose that $\mathrm{Spec}(B) = \{\lambda_1,\dots,\lambda_n\}$, where each $\lambda_k \ge 0$ is distinct. By the spectral theorem there exists n mutually orthogonal projections $\{P_k\}_{k=1}^n$ such that

$$B = \sum_{k=1}^n \lambda_k P_k$$

and $\sum_{k=1}^n P_k = 1$. We have,

$$B^z = \sum_{k=1}^n \lambda_k^z P_k.$$

Therefore,

$$\begin{aligned} B^z A^z - Y^z &= \sum_{k=1}^{n} (P_k \lambda_k^z A^z - P_k Y^z) \\ &= \sum_{k=1}^{n} P_k((\lambda_k A)^z - Y^z). \end{aligned} \tag{5.6}$$

Applying Lemma 5.2.4 to each term in the above sum, with $X = \lambda_k A$, if

$$V_{k,z} = (\lambda_k A)^{z-1}(\lambda_k A - Y) + (\lambda_k A - Y)Y^{z-1}$$

then

$$(\lambda_k A)^z - Y^z = V_{k,z} - \int_{\mathbb{R}} (\lambda_k A)^{is} V_{k,z} Y^{-is} \widehat{g}_z(s)\, ds.$$

Now substituting into (5.6), we have:

$$\begin{aligned} B^z A^z - Y^z &= \sum_{k=1}^{n} P_k \left(V_{k,z} - \int_{\mathbb{R}} (\lambda_k A)^{is} V_{k,z} Y^{-is} \widehat{g}_z(s)\, ds \right) \\ &= \sum_{k=1}^{n} P_k V_{k,z} - \int_{\mathbb{R}} \left(\sum_{k=1}^{n} P_k (\lambda_k A)^{is} V_{k,z} \right) Y^{-is}\, \widehat{g}_z(s)\, ds. \end{aligned}$$

By the definition of $V_{k,z}$:

$$V_{k,z} = (\lambda_k A)^z - (\lambda_k A)^{z-1} Y + \lambda_k A Y^{z-1} - Y^z$$

and so

$$\begin{aligned} \sum_{k=1}^{n} P_k V_{k,z} &= B^z A^z - B^{z-1} A^{z-1} Y + BAY^{z-1} - Y^z \\ &= B^{z-1}(BA^z - A^{z-1}Y) + (BA - Y)Y^{z-1} \\ &= B^{z-1}(BA^z - A^{z-1}A^{1/2}BA^{1/2}) + (BA - A^{1/2}BA^{1/2})Y^{z-1} \\ &= [BA^{1/2}, A^{z-1/2}] + [BA^{1/2}, A^{1/2}]Y^{z-1} \\ &= T_z(0). \end{aligned}$$

We may also compute the sum in the integrand:

$$\begin{aligned} \sum_{k=1}^{n} P_k (\lambda_k A)^{is} V_{k,z} &= \sum_{k=1}^{n} P_k \left((\lambda_k A)^{z+is} - (\lambda_k A)^{z-1+is} Y + (\lambda_k A)^{1+is} Y^{z-1} - (\lambda_k A)^{is} Y^z \right) \\ &= \sum_{k=1}^{n} j P_k \lambda_k^{z+is} \cdot A^{p+is} - \sum_{k=1}^{n} P_k \lambda_k^{z-1+is} \cdot A^{z-1+is} Y \\ &\quad + \sum_{k=1}^{n} P_k \lambda_k^{1+is} A^{1+is} Y^{z-1} - \sum_{k=1}^{n} P_k \lambda_k^{is} \cdot A^{is} Y^z. \end{aligned}$$

By functional calculus, we have

$$\begin{aligned} \sum_{k=1}^{n} P_k (\lambda_k A)^{is} V_{k,z} &= B^{z+is} A^{z+is} - B^{z-1+is} A^{z-1+is} Y + B^{1+is} A^{1+is} Y^{z-1} - B^{is} A^{is} Y^z \\ &= B^{z-1+is}(BA^{z+is} - A^{z-1+is}Y) + B^{is}(BA^{1+is} - A^{is}Y)Y^{z-1} \\ &= B^{z-1+is}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}+is}] + B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1}. \end{aligned}$$

So multiplying on the right by Y^{-is},

$$\left(\sum_{k=1}^{n} P_k(\lambda_k A)^{is} V_{k,z}\right) Y^{-is} = B^{z-1+is}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}+is}]Y^{-is} + B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1-is}.$$

We recognise this right hand side as exactly $T_z(s)$, and so

$$B^z A^z - Y^z = T_z(0) - \int_{\mathbb{R}} T_z(s)\widehat{g}_z(s)\,ds$$

as required. □

We now explain how to deduce the general version of Lemma 5.2.1 from the special case of Lemma 5.2.5 (i.e, when $\mathrm{Spec}(B)$ is a finite set). To do this we will select a sequence $\{B_n\}_{n=1}^{\infty}$ with $B_n \to B$ in the uniform norm and such that the spectrum of each B_n is finite.

The following lemma shows that under certain conditions, if $B_n \to B$ in the uniform norm, then $B_n^{is} \to B^{is}$ in the weak operator topology for each fixed $s \in \mathbb{R}$.

For a bounded operator T we denote $\mathrm{supp}(T)$ for the projection onto the orthogonal complement of $\ker(T)$ (this is the support projection of T).

LEMMA 5.2.6. *Let C be a positive bounded operator, and let $\{C_n\}_{n=1}^{\infty}$ be a sequence of positive bounded operators such that $C_n \to C$ in the operator norm, and for each n we have $\mathrm{supp}(C_n) = \mathrm{supp}(C)$. Then for all $s \in \mathbb{R}$, we have that $C_n^{is} \to C^{is}$ in the weak operator topology.*

PROOF. By definition, we need to show that for all $s \in \mathbb{R}$ and $\xi, \eta \in H$ we have

$$\lim_{n\to\infty} \langle C_n^{is}\xi, \eta\rangle = \langle C^{is}\xi, \eta\rangle.$$

By assumption, $\mathrm{supp}(C_n) = \mathrm{supp}(C)$ for every $n \geq 0$. By taking a quotient by the closed subspace $\ker(C)$ if necessary, we may assume without loss of generality that $\mathrm{supp}(C_n) = \mathrm{supp}(C) = 1$ for every $n \geq 0$. Also without loss of generality, we assume that $\|C\|_\infty \leq 1$ and $\sup_{n\geq 0}\|C_n\|_\infty \leq 1$. Let $\xi, \eta \in H$ be such that $\|\xi\| = \|\eta\| = 1$.

Fix $\varepsilon > 0$. Since $\ker(C) = \{0\}$, we may select $m > 1$ such that

$$\|\chi_{[0,\frac{1}{m}]}(C))\xi\| < \varepsilon.$$

Let $0 \leq \phi \leq 1$ be a smooth function supported on the interval $[\frac{1}{m+1}, 2]$ such that $\phi = 1$ on the interval $[\frac{1}{m}, 1]$. We note that therefore

$$\begin{aligned}\|(1-\phi(C))\xi\| &= \|(1-\phi(C))\chi_{[0,\frac{1}{m})}\xi\|\\ &\leq \|\chi_{[0,\frac{1}{m})}\xi\|\\ &< \varepsilon.\end{aligned}$$

Let $\psi(t) := t^{is}\phi(t)$. Since ϕ and ψ are smooth and compactly supported, it follows that their first and second derivatives are in $L_2(\mathbb{R})$. These conditions are sufficient for ϕ and ψ to be operator Lipschitz (see [**51**, Lemma 6, Lemma 7]): i.e., there are constants C_ϕ and C_ψ such that

$$\begin{aligned}\|\phi(C_n) - \phi(C)\|_\infty &\leq C_\phi\|C_n - C\|_\infty,\\ \|\psi(C_n) - \psi(C)\|_\infty &\leq C_\psi\|C_n - C\|_\infty.\end{aligned}$$

Select $N > 0$ such that for all $n > N$ we thus have,

$$\|\phi(C_n) - \phi(C)\|_\infty \leq \epsilon \text{ and,}$$
$$\|\phi(C_n)C_n^{is} - \phi(C)C^{is}\|_\infty \leq \epsilon.$$

For $n > N$ we have:

$$\begin{aligned}\langle C_n^{is}\xi, \eta\rangle - \langle C^{is}\xi, \eta\rangle &= \langle C_n^{is}(1 - \phi(C_n))\xi, \eta\rangle + \langle C^{is}(\phi(C) - 1)\xi, \eta\rangle \\ &\quad + \langle (C_n^{is}\phi(C_n) - C^{is}\phi(C))\xi, \eta\rangle. \end{aligned} \tag{5.7}$$

For the first term in (5.7) above, we have:

$$\begin{aligned}|\langle C_n^{is}(1 - \phi(C_n))\xi, \eta\rangle| &\leq \|(1 - \phi(C_n))\xi\| \\ &\leq \|(1 - \phi(C))\xi\| + \|\phi(C_n) - \phi(C)\|_\infty \\ &< 2\varepsilon.\end{aligned}$$

Next, for the second term in (5.7), we have

$$\begin{aligned}|\langle C^{is}(\phi(C) - 1)\xi, \eta\rangle| &\leq \|(1 - \phi(C))\xi\|_\infty \\ &< \varepsilon.\end{aligned}$$

Finally, for the third term in (5.7),

$$|\langle (C_n^{is}\phi(C_n) - C^{is}\phi(C))\xi, \eta\rangle| \leq \|\psi(C_n) - \psi(C)\|_\infty \leq \varepsilon.$$

So in summary, for $n \geq N$ we have:

$$|\langle C_n^{is}\xi, \eta\rangle - \langle C^{is}\xi, \eta\rangle| \leq 4\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, the assertion follows. □

We are now ready to prove Theorem 5.2.1.

PROOF OF THEOREM 5.2.1. Without loss of generality, $\|B\|_\infty = 1$ (if not, then we replace the couple (A, B) with a couple $(cA, c^{-1}B)$ with a suitable constant $c > 0$). In this case we have $\mathrm{Spec}(B) \subseteq [0, 1]$ and $1 \in \mathrm{Spec}(B)$. For every $n \geq 1$, set

$$B_n = \sum_{m=1}^{n} \frac{m}{n}\chi_{(\frac{m-1}{n}, \frac{m}{n}]}(B).$$

Recall that $Y := A^{1/2}BA^{1/2}$, and let $Y_n := A^{1/2}B_nA^{1/2}$, and let $T_{n,z}(s)$ be defined as $T_z(s)$ with the occurances of B replaced with B_n and Y replaced with Y_n.

By construction, the spectrum of B consists of at most n points, indeed by the spectral mapping theorem:

$$\mathrm{Spec}(B_n) \subseteq \left\{\frac{m}{n}\right\}_{m=1}^{n}.$$

Since $1 \in \mathrm{Spec}(B)$, we always have that $\chi_{(\frac{n-1}{n}, 1]}(B) \neq 0$, so $1 \in \mathrm{Spec}(B_n)$ and $\|B_n\|_\infty = 1$. We also have that $\mathrm{supp}(B_n) = \mathrm{supp}(B)$, and $\mathrm{supp}(Y_n) = \mathrm{supp}(Y)$.

Moreover, B_n converges in norm to B, since $\|B - B_n\|_\infty \leq \frac{1}{n}$. Thus by Lemma 5.2.6, for any $s \in \mathbb{R}$ we also have that $B_n^{is} \to B^{is}$ in the weak operator topology. Similarly, $Y_n \to Y$ in the norm topology and $Y_n^{is} \to Y^{is}$ in the weak operator topology.

It follows now that for each $s \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\Re(z) > 1$, we have that $T_{n,z}(s) \to T_z(s)$ in the weak operator topology. One can also see that $\sup_{s\in\mathbb{R}} \sup_{n\geq 1} \|T_{n,z}(s)\|_\infty < \infty$.

In other words, for every $\xi, \eta \in H$, we have

$$\langle T_{n,z}(s)\xi, \eta\rangle \to \langle T_z(s)\xi, \eta\rangle.$$

Since $|\langle T_{n,z}(s)\xi, \eta\rangle| \leq \sup_{n\geq 1} \|T_{n,z}(s)\|_\infty$, and this is bounded in s, we may use the Dominated Convergence theorem to obtain

$$\int_{\mathbb{R}} \langle T_n(s)\xi, \eta\rangle \widehat{g}_z(s) ds \to \int_{\mathbb{R}} \langle T(s)\xi, \eta\rangle \widehat{g}_z(s) ds$$

By Lemma 5.2.5, for all $n \geq 1$ and $\xi, \eta \in H$ we have:

$$\langle (B_n^z A^z - Y_n^z)\xi, \eta\rangle = \langle T_{n,z}(0)\xi, \eta\rangle - \int_{\mathbb{R}} \langle T_{n,z}(s)\xi, \eta\rangle \widehat{g}_z(s)\, ds. \tag{5.8}$$

As already discussed, $B_n^z \to B^z$ in the weak operator topology, and similarly $Y_n^z \to Y^z$ in the weak operator topology. Hence both sides of (5.8) converge, and:

$$\langle (B^z A^z - Y^z)\xi, \eta\rangle = \langle T_z(0)\xi, \eta\rangle - \int_{\mathbb{R}} \langle T_z(s)\xi, \eta\rangle \widehat{g}_z(s)\, ds.$$

Since ξ and η are arbitrary, this completes the proof. $\square$

5.3. Analyticity of the mapping $z \mapsto g_z$

So far we have considered the function,

$$g_z(t) := 1 - \frac{e^{\frac{z}{2}t} - e^{-\frac{z}{2}t}}{(e^{\frac{t}{2}} - e^{-\frac{t}{2}})(e^{\left(\frac{z-1}{2}\right)t} + e^{-\left(\frac{z-1}{2}\right)t})}, t \neq 0$$

with $g_z(0) := 1 - \frac{z}{2}$ as a Schwartz function of t with a fixed parameter $z \in \mathbb{C}$, with $\Re(z) > 1$.

We may equally well consider g as a function of z. That is, the mapping $z \mapsto g_z$ defines a function:

$$\{z \in \mathbb{C} \ : \ \Re(z) > 1\} \to \mathcal{S}(\mathbb{R}).$$

As a matter of fact, the function $z \mapsto g_z$ is holomorphic with values in the Hilbert-Sobolev space:

$$H^2(\mathbb{R}) := \{f \in L_2(\mathbb{R}) \ : \ f', f'' \in L_2(\mathbb{R})\},$$

equipped with the Sobolev norm:

$$\|f\|^2_{H^2(\mathbb{R})} := \|f\|^2_{L_2(\mathbb{R})} + \|f'\|^2_{L_2(\mathbb{R})} + \|f''\|^2_{L_2(\mathbb{R})}.$$

We remind the reader of the meaning of Banach space valued holomorphy. If $D \subseteq \mathbb{C}$ is a domain, X is a Banach space and $f : D \to X$ then the following two conditions are equivalent:

(a) For any continuous linear functional $\varpi \in X^*$, the function $\varpi \circ f : D \to \mathbb{C}$ is holomorphic.
(b) For any $z \in \mathbb{C}$, the limit in the norm topology of X

$$f'(z) = \lim_{\zeta \to z} \frac{f(z) - f(\zeta)}{z - \zeta}$$

exists.

The equivalence of these conditions is well known, see e.g. [**57**, Theorem 3.31].

We work with the first condition. Since $H^2(\mathbb{R})$ is a Hilbert space, for any continuous linear functional ϖ on $H^2(\mathbb{R})$ there exists $h \in H^2(\mathbb{R})$ such that:

$$\varpi(g_z) = \int_{\mathbb{R}} g_z(t)h(t)\,dt + \int_{\mathbb{R}} g_z'(t)h'(t)\,dt + \int_{\mathbb{R}} g_z''(t)h''(t)\,dt.$$

So we focus on proving that for each $h \in H^2(\mathbb{R})$, the mapping $z \mapsto \varpi(g_z)$ is holomorphic.

LEMMA 5.3.1. *Let $G : \{z \in \mathbb{C} \,:\, \Re(z) > 1\} \to H^2(\mathbb{R})\}$ be the function given by $G(z) = g_z$. Then G is continuous on its domain.*

PROOF. It suffices to prove that the mappings $G : \{z \in \mathbb{C} \,:\, \Re(z) > 1\} \to H^2(-1,1)\}$, $G : \{z \in \mathbb{C} \,:\, \Re(z) > 1\} \to H^2(1,\infty)\}$ and $G : \{z \in \mathbb{C} \,:\, \Re(z) > 1\} \to H^2(-\infty,-1)\}$ are continuous on their domains.

We write the first function as follows:

$$g_z = 1 - a_z b c_z, \quad \Re(z) > 1,$$

where

$$a_z(t) = \frac{\sinh(\frac{zt}{2})}{t}, \quad b(t) = \frac{t}{2\sinh(\frac{t}{2})}, \quad c_z(t) = \frac{1}{\cosh((\frac{z-1}{2})\,t)}.$$

Since $z \to a_z$ and $z \to c_z$ are continuous $C^2[-1,1]-$valued mappings, the first assertion follows.

We rewrite our function as follows.

$$g_z = -\frac{\sinh(\frac{(z-2)t}{2})}{2\sinh(\frac{t}{2})\cosh(\frac{(z-1)t}{2})}, \quad t \in \mathbb{R}.$$

Equivalently,

$$g_z = -a_z b c_z,$$

where

$$a_z = e^{-\frac{(z+1)t}{2}} - e^{-\frac{(3z-3)t}{2}}, \quad b = \frac{e^{-t}}{1-e^{-t}}, \quad c_z = \frac{1}{1+e^{-(z-1)t}}.$$

Since $z \to a_z$ and $z \to c_z$ are continuous $C^2(1,\infty)-$valued mappings and $b \in H^2(1,\infty)$, the second assertion follows. □

THEOREM 5.3.2. *Let $G : \{z \in \mathbb{C} \,:\, \Re(z) > 1\} \to H^2(\mathbb{R})\}$ be the function given by $G(z) = g_z$. Then G is holomorphic on its domain.*

PROOF. To show that G is holomorphic with values in $H^2(\mathbb{R})$, it suffices to show for all continuous linear functionals ϖ on $H^2(\mathbb{R})$ that $z \mapsto \varpi(G(z))$ is holomorphic.

Since $H^2(\mathbb{R})$ is a Hilbert space, it suffices to show for all $h \in H^2(\mathbb{R})$ that:

$$z \mapsto \int_{\mathbb{R}} g_z(t)h(t) + g_z'(t)h'(t) + g_z''(t)h''(t)\,dt, \quad \Re(z) > 1 \tag{5.9}$$

is holomorphic.

Let γ be a simple closed curve contained in $\{z \,:\, \Re(z) > 1\}$. By Lemma 5.3.1, the function

$$(z,t) \mapsto g_z(t)h(t) + g_z'(t)h'(t) + g_z''(t)h''(t), \quad t \in \mathbb{R}, z \in \gamma$$

is integrable on $\gamma \times \mathbb{R}$. Indeed,

$$\int_\gamma \Big(\int_{\mathbb{R}} |g_z(t)h(t) + g_z'(t)h'(t) + g_z''(t)h''(t)|dt\Big)|dz| \leq$$

$$\leq \int_\gamma \|g_z\|_{H^2(\mathbb{R})}|dz| \leq \text{length}(\gamma)\cdot \sup_{z\in\gamma}\|g_z\|_{H^2(\mathbb{R})} < \infty.$$

We may apply Fubini's theorem to conclude that:

$$\int_\gamma \varpi(G_z)\,dz = \int_{\mathbb{R}}\Big(\int_\gamma (g_z(t)h(t)+g_z'(t)h'(t)+g_z''(t)h''(t))dz\Big)dt.$$

For each fixed t it follows from the definition that the functions $g_z(t)$, $g_z'(t)$ and $g_z''(t)$ are holomorphic in z. Hence $\int_\gamma \varpi(G(z))\,dz = 0$ for all simple closed curves γ contained in $\{z\ :\ \Re(z) > 1\}$. Since $z\mapsto \varpi(G(z))$ is continuous, by Morera's theorem $z\mapsto\varpi(G(z))$ is holomorphic in the domain $\{z\ :\ \Re(z)>1\}$. Since ϖ is arbitrary, G is $H^2(\mathbb{R})$-valued holomorphic. □

5.4. The function $z\to \mathrm{Tr}(XB^zA^z)-\mathrm{Tr}(X(A^{\frac12}BA^{\frac12})^z)$ admits analytic continuation to $\{\Re(z)>p-1\}$

As indicated in the title, in this section we prove (under certain assumptions on A and B) that for all $X\in\mathcal{L}_\infty$ the mapping

$$z\mapsto \mathrm{Tr}(XB^zA^z)-\mathrm{Tr}(X(A^{\frac12}BA^{\frac12})^z)$$

defined initially on $\{z\in\mathbb{C}\ :\ \Re(z)>p\}$ is holomorphic, and admits analytic continuation to the set $\{z\ :\ \Re(z)>p-1\}$. The precise assumptions on A and B are as follows:

CONDITION 5.4.1. *Let $p>2$ and let $0\leq A,B\in\mathcal{L}_\infty$ satisfy the conditions*

(i) $B^pA\in\mathcal{L}_{1,\infty}$
(ii) $B^{q-2}[B,A]\in\mathcal{L}_1$ *for every* $q>p$
(iii) $A^{\frac12}BA^{\frac12}\in\mathcal{L}_{p,\infty}$
(iv) $[B,A^{\frac12}]\in\mathcal{L}_{\frac{p}{2},\infty}$.

The main result of this section is the following:

THEOREM 5.4.2. *Let $p>2$ and let A and B satisfy Condition 5.4.1. If $X\in\mathcal{L}_\infty$, then the mapping*

$$z\to \mathrm{Tr}(XB^zA^z)-\mathrm{Tr}(X(A^{\frac12}BA^{\frac12})^z),\quad \Re(z)>p,$$

admits an analytic continuation to the half-plane $\{\Re(z)>p-1\}$.

LEMMA 5.4.3. *Assume that $p\geq1$ and that A,B satisfy Condition 5.4.1. Then for all $r\geq1$ we have $B^{\frac{p}{r}}A\in\mathcal{L}_{r,\infty}$. More precisely, we have the following norm bound:*

$$\|B^{\frac{p}{r}}A\|_{r,\infty}\leq e\|A\|_\infty^{1-\frac1r}\|B^pA\|_{1,\infty}^{\frac1r}.$$

PROOF. We show this is a consequence of the Araki-Lieb-Thirring inequality (2.1) and (2.4).

Fix $r\geq1$. Then by the Araki-Lieb-Thirring inequality:

$$|B^{p/r}A|^r \prec\prec_{\log} B^pA^r.$$

Now using (2.4):

$$\begin{aligned}\|B^{p/r}A\|_{r,\infty} &= \||B^{p/r}A|^r\|_{1,\infty}\\ &\leq e\|B^pA^r\|_{1,\infty}\\ &\leq e\|A\|_\infty^{r-1}\|B^pA\|_{1,\infty}.\end{aligned}$$

$\square$

The next lemma provides a sufficient condition for a function to be holomorphic with values in a Banach ideal of $\mathcal{L}_\infty$.

LEMMA 5.4.4. *Assume that $D \subseteq \mathbb{C}$ is a domain (i.e., a connected open set) and that $F : D \to \mathcal{L}_\infty$ is a holomorphic function. If $\mathcal{I}$ is a Banach-normed ideal of $\mathcal{L}_\infty$ such that F takes values in $\mathcal{I}$ and $F : D \to \mathcal{I}$ is continuous, then F is also an $\mathcal{I}$-valued holomorphic function.*

PROOF. Fix a contour $\gamma \subset D$. such that the interior of γ is contained in D. Then for all z in the interior of γ, we have

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{F(w)}{w - z}\, dw$$

A priori, the integral is a weak integral. However, $F : D \to \mathcal{I}$ is continuous and, therefore, the integral is Bochner in $\mathcal{I}$.

In order to show that F is holomorphic, it suffices to show that it is differentiable. Let

$$G(z) = \frac{1}{2\pi i} \int_\gamma \frac{F(w)}{(w - z)^2}\, dw$$

Once more, since $F : D \to \mathcal{I}$ is continuous, then this integral is defined as an $\mathcal{I}$-valued Bochner integral. If F were holomorphic, then G would be the derivative of F. The proof will be completed upon showing that G is indeed the derivative of F considered as an $\mathcal{I}$-valued mapping.

For every z and z_0 in the interior of γ, we have:

$$\frac{F(z) - F(z_0)}{z - z_0} - G(z_0) = \frac{z - z_0}{2\pi i} \int_\gamma \frac{F(w)}{(w - z)(w - z_0)^2}\, dw$$

Again, this integral is an $\mathcal{I}$-valued Bochner integral.

Thus,

$$\begin{aligned} \Big\| \frac{F(z) - F(z_0)}{z - z_0} - G(z_0) \Big\|_{\mathcal{I}} &\le \frac{|z - z_0|}{2\pi} \int_\gamma \|F(w)\|_{\mathcal{I}} \frac{1}{|w - z| \cdot |w - z_0|^2}\, |dw| \\ &\le \sup_{w \in \gamma} \|F(w)\|_{\mathcal{I}} \cdot \frac{|z - z_0|}{2\pi} \int_\gamma \frac{1}{|w - z| \cdot |w - z_0|^2}\, |dw|. \end{aligned}$$

Since f is continuous, the right hand side tends to 0 as $z \to z_0$. Hence, $G = F'$ and so F is holomorphic. $\square$

LEMMA 5.4.5. *Let $0 \le A \in \mathcal{L}_\infty$. The function $z \to A^z$ is $\mathcal{L}_\infty-$valued holomorphic on the half-plane $\{z \, : \, \Re(z) > 0\}$.*

PROOF. For $z_0, z \in \mathbb{C}$ with $\Re(z_0) > 0$ and $\Re(z) > 0$, we define the operator $A^{z_0}\log(A)$ by means of functional calculus (the convention $0^{z_0}\log(0) = 0$ is used).

Hence,

$$\begin{aligned} \Big\| \frac{A^z - A^{z_0}}{z - z_0} - A^{z_0}\log(A) \Big\|_\infty &\le \sup_{0 \le \lambda \le \|A\|_\infty} \Big| \frac{\lambda^z - \lambda^{z_0}}{z - z_0} - \lambda^{z_0}\log(\lambda) \Big| \\ &= O(z - z_0), \quad z_0 \to z. \end{aligned}$$

Hence, for all z with $\Re(z) > 0$

$$A^z - A^{z_0} = A^{z_0}\log(A)(z - z_0) + o(1), \quad z_0 \to z$$

and so A^z is $\mathcal{L}_\infty$-valued holomorphic with derivative $A^z\log(A)$. $\square$

LEMMA 5.4.6. *Let $p>2$ and assume A and B satisfy condition 5.4.1. The mapping*

$$z\to[B,A^z],\quad \Re(z)>1,$$

is a holomorphic $\mathcal{L}_{\frac{p}{2},\infty}-$valued function.

PROOF. We take care to note that since $p>2$, the ideal $\mathcal{L}_{p/2,\infty}$ can be equipped with a norm generating the same topology as that of the canonical quasi-norm (see [**3**, Chapter 4, Lemma 4.5]). Denote such a norm as $\|\cdot\|'_{\mathcal{L}_{p/2,\infty}}$.

Denote

$$F(z)=[B,A^z].$$

Since $F(z)=BA^z-A^zB$ it now follows from Lemma 5.4.5 that F is $\mathcal{L}_\infty$-valued holomorphic. Due to Lemma 5.4.4, it now suffices to show that F is $\mathcal{L}_{p/2,\infty}$-valued and $\mathcal{L}_{p/2,\infty}$-continuous.

Let ϕ be a compactly supported smooth function on $\mathbb{R}$, such that $0\le\phi\le 1$ and $\phi=1$ on the interval $[0,\|A\|_\infty]$. Define

$$\phi_z(t)=|t|^z\phi(t),\quad t\in\mathbb{R},\Re(z)>1.$$

Since $\phi=1$ on the spectrum of A, we have that $\phi_z(A)=A^z$ and so for all z,z_1,z_2 with $\Re(z),\Re(z_1),\Re(z_2)>1$,

$$F(z)=[B,\phi_z(A)]$$
$$F(z_1)-F(z_2)=[B,(\phi_{z_1}-\phi_{z_2})(A)].$$

Now we refer to [**50**], where it is proved that if A and B are self-adjoint operators and $r>1$ is such that $[A,B]\in\mathcal{L}_{r,\infty}$, then for all Lipschitz functions f, there is a constant c_r such that:

$$\|[f(A),B]\|_{\mathcal{L}_{r,\infty}}\le c_r\|f'\|_{L_\infty(\mathbb{R})}\|[A,B]\|_{\mathcal{L}_{r,\infty}}.$$

Since $p>2$, we may apply this result with $r=p/2$ and since ϕ is smooth and compactly supported, we may take $f=\phi_z$. Thus,

$$\|F(z)\|'_{\mathcal{L}_{p/2,\infty}}\le c_{p/2}\|\phi_z'\|_{L_\infty(\mathbb{R})}\|[B,A]\|'_{\mathcal{L}_{p/2,\infty}}$$

and similarly taking $f=\phi_{z_1}-\phi_{z_2}$,

$$\|F(z_1)-F(z_2)\|'_{\mathcal{L}_{p/2,\infty}}\le c_{p/2}\|\phi'_{z_1}-\phi'_{z_2}\|_{L_\infty(\mathbb{R})}\|[B,A]\|'_{\mathcal{L}_{p/2,\infty}}.$$

Note that for $z_1\to z_2$ we have:

$$\|\phi'_{z_1}-\phi'_{z_2}\|_{L_\infty(\mathbb{R})}\to 0$$

and hence F is $\mathcal{L}_{p/2,\infty}$-valued continuous. The result now follows. $\square$

LEMMA 5.4.7. *Let $p>2$ and let A and B satisfy Condition 5.4.1. Then:*

(i) *The mapping $F_0(z):=B^{z-1}[B,A^{z-\frac12}]A^{\frac12}+[B,A^{\frac12}]A^{\frac12}Y^{z-1}$ is an $\mathcal{L}_1-$valued holomorphic function for the domain $\Re(z)>p-1$.*
(ii) *The mapping $F_1(z):=B^{z-1}A^{z-1}$ is an $\mathcal{L}_{\frac{p}{p-2},1}-$valued holomorphic function for the domain $\Re(z)>p-1$.*
(iii) *The mapping $F_2(z):=B^{z-1}[BA,A^{z-1}]$ is an $\mathcal{L}_1-$valued holomorphic function for the domain $\Re(z)>p-1$.*
(iv) *The mapping $F_3(z):=Y^{z-1}$ is an $\mathcal{L}_{\frac{p}{p-2},1}-$valued holomorphic function for the domain $\Re(z)>p-1$. (Recall that $Y=A^{1/2}BA^{1/2}$).*

PROOF. We first prove (ii). Fix $q \in (p, p+2)$. If $\Re(z) > q-1$, then

$$B^{z-1}A^{z-1} = B^{z-q+1}B^{q-2}A \cdot A^{z-2}.$$

By Lemma 5.4.3, we have that

$$B^{q-2}A \in \mathcal{L}_{\frac{p}{q-2},\infty} \subset \mathcal{L}_{\frac{p}{p-2},1}$$

and correspondingly by Lemma 5.4.5 the mapping $z \mapsto B^{z-1}A^{z-1}$ is continuous in the $\mathcal{L}_{p/(p-2),1}$ norm.

Moreover Lemma 5.4.5 implies that the mappings $z \to B^{z-q+1}$ and $z \to A^{z-2}$ are $\mathcal{L}_\infty$−valued holomorphic for $\Re(z) > q-1$. Thus by Lemma 5.4.4 the mapping $z \to B^{z-1}A^{z-1}$ is $\mathcal{L}_{\frac{p}{p-2},1}$−valued holomorphic for $\Re(z) > q-1$. Since $q > p$ is arbitrary, (ii) follows.

Now we prove (iii). Again fix $q \in (p, p+2)$ and let z satisfy $\Re(z) > q-1$. We rewrite F_2 as:

$$\begin{aligned} F_2(z) &= B^zA^z - B^{z-1}A^{z-1}BA \\ &= B^{z-1}A^{z-1} \cdot [A,B] + [B, B^{z-1}A^z] \\ &= B^{z-1}A^{z-1} \cdot [A,B] + B^{z-1} \cdot [B,A] \cdot A^{z-1} + B^{z-1}A \cdot [B, A^{z-1}]. \end{aligned} \tag{5.10}$$

The first summand in (5.10) can be written as

$$B^{z-q+1} \cdot B^{q-2}A \cdot A^{z-2} \cdot [A,B].$$

Due to Lemma 5.4.5, the mappings $z \to B^{z-q+1}$ and $z \to A^{z-2}$ are $\mathcal{L}_\infty$−valued holomorphic for $\Re(z) > q-1$. By Lemma 5.4.3, we have that

$$B^{q-2}A \in \mathcal{L}_{\frac{p}{q-2},\infty} \subset \mathcal{L}_{\frac{p}{p-2},1}.$$

The element $[A,B] = A^{1/2}[A^{1/2}, B] + [A^{1/2}, B]A^{1/2}$ belongs to $\mathcal{L}_{\frac{p}{2},\infty}$. Hence, the first summand is $\mathcal{L}_1$-valued holomorphic for $\Re(z) > q-1$.

The second summand in (5.10) can be written as

$$z \to B^{z-q+1} \cdot B^{q-2}[B,A] \cdot A^{z-1}.$$

By our assumption of Condition 5.4.1, the operator $B^{q-2}[B,A]$ belongs to $\mathcal{L}_1$ and accordingly the map $z \to B^{z-1} \cdot [B,A] \cdot A^{z-1}$ is $\mathcal{L}_1$-continuous. Once again due to Lemma 5.4.5, the mappings $z \to B^{z-q+1}$ and $z \to A^{z-1}$ are $\mathcal{L}_\infty$−valued holomorphic for $\Re(z) > q-1$. Hence, the second summand of (5.10) is $\mathcal{L}_1$-valued holomorphic for $\Re(z) > q-1$.

We now treat the third summand of (5.10). By Lemma 5.4.6, the mapping

$$z \to [B, A^{z-1}], \quad \Re(z) > q-1,$$

is a holomorphic $\mathcal{L}_{\frac{p}{2},\infty}$−valued function. Due to Condition 5.4.1, we have that

$$B^{q-2}A \in \mathcal{L}_{\frac{p}{q-2},\infty} \subset \mathcal{L}_{\frac{p}{p-2},1}.$$

Hence

$$z \to B^{z-q+1} \cdot B^{q-2}A \cdot [B, A^{z-1}], \quad \Re(z) > q-1,$$

is an $\mathcal{L}_1$-valued holomorphic function.

So, all three summands in (5.10) are holomorphic for $\Re(z) > q-1$. Thus, $z \to F_2(z)$ is holomorphic for $\Re(z) > q-1$. Since $q > p$ is arbitrary, it follows that $z \to F_2(z)$ is holomorphic for $\Re(z) > p-1$. This proves (iii).

For (iv), we note that this is an immediate consequence of the assumption of Condition 5.4.1.(iii) and Lemma 5.4.5.

Finally we prove (i). We write $F_0(z)$ as:

$$\begin{aligned} F_0(z) &= B^{z-1}[B, A^{z-1}A^{1/2}]A^{1/2} + [B, A^{1/2}]A^{1/2}Y^{z-1} \\ &= F_2(z) + B^{z-1}A^{z-1}[B, A^{1/2}]A^{1/2} + [B, A^{1/2}]A^{1/2}F_3(z) \\ &= F_2(z) + F_1(z)[B, A^{1/2}]A^{1/2} + [B, A^{1/2}]F_3(z). \end{aligned}$$

Hence, by (ii), (iii) and (iv) and Condition 5.4.1.(iv),

$$F_0(z) \in \mathcal{L}_1 + \mathcal{L}_{p/(p-2),1} \cdot \mathcal{L}_{p/2,\infty} + \mathcal{L}_{p/2,\infty}\mathcal{L}_{p/(p-2),\infty}.$$

so by the Hölder-type inequality (2.7), $F_0(z) \in \mathcal{L}_1$, and it continuous in the $\mathcal{L}_1$-norm. So by Lemma 5.4.4, F_0 is $\mathcal{L}_1$-valued holomorphic. □

LEMMA 5.4.8. *Let $s \to H(s)$ be a bounded $\mathcal{L}_\infty$−valued function measurable in the weak operator topology (see Definition 2.3.1). Let g_z be as in Theorem 5.2.1. Define*

$$G(z) := \int_{\mathbb{R}} H(s)\widehat{g}_z(s)ds, \quad \Re(z) > 1$$

as a weak operator topology integral.

(i) *G is an $\mathcal{L}_\infty$−valued holomorphic function for the domain $\Re(z) > 1$.*
(ii) *if there is $r > 1$ such that for all $s \in \mathbb{R}$ we have $\|H(s)\|_{r,\infty} \leq 1 + |s|$, then G is an $\mathcal{L}_{r,\infty}$−valued holomorphic function for the domain $\Re(z) > 1$.*

PROOF. Define:

$$g_{1,z} = \frac{\partial}{\partial z} g_z, \quad z \in \mathbb{C}, \quad \Re(z) > 1.$$

Define the $\mathcal{L}_\infty$-valued function G_1 by:

$$G_1(z) = \int_{\mathbb{R}} H(s)\hat{g}_{1,z}(s)ds.$$

We will show that G is $\mathcal{L}_\infty$-valued holomorphic by showing that G_1 is the derivative of G.

Let $z, z_0 \in \mathbb{C}$ have real part greater than 1. Then we have

$$\frac{G(z) - G(z_0)}{z - z_0} - G_1(z_0) = \int_{\mathbb{R}} H(s)\Big(\frac{\hat{g}_z(s) - \hat{g}_{z_0}(s)}{z - z_0} - \hat{g}_{1,z_0}(s)\Big)ds.$$

So by the triangle inequality,

$$\Big\|\frac{G(z) - G(z_0)}{z - z_0} - G'(z_0)\Big\|_\infty \leq \operatorname*{ess\,sup}_{s\in\mathbb{R}} \|H(s)\|_{\mathcal{L}_\infty} \int_{\mathbb{R}} \Big|\frac{\hat{g}_z(s) - \hat{g}_{z_0}(s)}{z - z_0} - \hat{g}_{1,z_0}(s)\Big|ds.$$

By [**51**, Lemma 7], we have an absolute constant c_{abs} such that:

$$\int_{\mathbb{R}} \Big|\frac{\hat{g}_z(s) - \hat{g}_{z_0}(s)}{z - z_0} - \hat{g}_{1,z_0}(s)\Big|ds \leq c_{abs}\Big(\Big\|\frac{g_z - g_{z_0}}{z - z_0} - g_{1,z_0}\Big\|_2 + \Big\|\frac{g'_z - g'_{z_0}}{z - z_0} - g'_{1,z_0}\Big\|_2\Big).$$

The assertion (i) follows now from Theorem 5.3.2, and hence moreover we have that $G_1 = G'$.

Let us now establish (ii). Indeed, by Lemma 2.3.3, we have

$$\|G(z)\|_{r,\infty} \leq \frac{r}{r-1}\int_{\mathbb{R}}(1+|s|)|\hat{g}_z(s)|ds,$$

$$\|G'(z)\|_{r,\infty} \leq \frac{r}{r-1}\int_{\mathbb{R}}(1+|s|)|\hat{g}_{1,z}(s)|ds,$$

$$\Big\|\frac{G(z)-G(z_0)}{z-z_0}-G'(z_0)\Big\|_{r,\infty} \leq \frac{r}{r-1}\int_{\mathbb{R}}(1+|s|)\Big|\frac{\hat{g}_z(s)-\hat{g}_{z_0}(s)}{z-z_0}-\hat{g}_{1,z_0}(s)\Big|ds.$$

Once more using [**51**, Lemma 7], we have

$$\int_{\mathbb{R}}(1+|s|)|\hat{g}_z(s)|ds = \|\hat{g}_z\|_1 + \|\hat{g'}_z\|_1 \leq c_{\mathrm{abs}}\|g_z\|_{W^{2,2}}.$$

Similarly,

$$\int_{\mathbb{R}}(1+|s|)|\hat{g}_{1,z}(s)|ds = \|\hat{g}_{1,z}\|_1 + \|\hat{g'}_{1,z}\|_1 \leq c_{abs}\|g_{1,z}\|_{W^{2,2}}$$

and

$$\int_{\mathbb{R}}(1+|s|)\Big|\frac{\hat{g}_z(s)-\hat{g}_{z_0}(s)}{z-z_0}-\hat{g}_{1,z_0}(s)\Big|ds =$$

$$= \Big\|\frac{\hat{g}_z-\hat{g}_{z_0}}{z-z_0}-\hat{g}_{1,z_0}\Big\|_1 + \Big\|\frac{\hat{g'}_z-\hat{g'}_{z_0}}{z-z_0}-\hat{g'}_{1,z_0}\Big\|_1 \leq$$

$$\leq c_{abs}\Big\|\frac{g_z-g_{z_0}}{z-z_0}-g_{1,z_0}\Big\|_{W^{2,2}}.$$

We now deduce (ii) from Theorem 5.3.2. □

LEMMA 5.4.9. *Let $p>2$ and let A and B satisfy Condition 5.4.1. If $X\in\mathcal{L}_\infty$, then*

(i) *the mapping*

$$G_1(z) := \int_{\mathbb{R}}[BA^{\frac12}, A^{\frac12+is}]Y^{-is}XB^{is}\hat{g}_z(s)ds,$$

is an $\mathcal{L}_{\frac{p}{2},\infty}$−valued holomorphic function for $\Re(z)>1$.

(ii) *the mapping*

$$G_2(z) := \int_{\mathbb{R}}A^{is}Y^{-is}XB^{is}\hat{g}_z(s)ds,$$

is $\mathcal{L}_\infty$−valued holomorphic for $\Re(z)>1$.

(iii) *the mapping*

$$G_3(z) := \int_{\mathbb{R}}Y^{-is}XB^{is}[BA^{\frac12}, A^{\frac12+is}]\hat{g}_z(s)ds,$$

is $\mathcal{L}_{\frac{p}{2},\infty}$−valued holomorphic for $\Re(z)>1$.

PROOF. By Lemma 5.4.8 (i), the functions G_1, G_2 and G_3 are $\mathcal{L}_\infty$−valued holomorphic. In particular, this proves (ii). We now prove the first and third assertions.

Set

$$H_1(s) = [BA^{\frac12}, A^{\frac12+is}]Y^{-is}XB^{is}, \quad s\in\mathbb{R},$$

and

$$H_2(s) = Y^{-is}XB^{is}[BA^{\frac12}, A^{\frac12+is}], \quad s\in\mathbb{R}.$$

For every $s \in \mathbb{R}$, we have for $j = 1, 2$,

$$\Big\|H_j(s)\Big\|_{\frac{p}{2},\infty} \le \|X\|_\infty \cdot \Big\|[BA^{\frac12}, A^{\frac12+is}]\Big\|_{\frac{p}{2},\infty},$$

Let $\phi_s(t) := |t|^{1+2is}$, $t \in \mathbb{R}$. We now write

$$[BA^{\frac12}, A^{\frac12+is}] = [BA^{\frac12}, \phi_s(A^{\frac12})].$$

We again refer to [**50**]. Since $p > 2$ and ϕ_s is a Lipschitz function, we have

$$\begin{aligned}\|[BA^{\frac12}, A^{\frac12+is}]\|_{\frac{p}{2},\infty} &\le c_p\|\phi_s'\|_\infty\|[BA, A^{\frac12}]\|_{\frac{p}{2},\infty}\\ &\le c_p(1+|s|)\|[BA, A^{\frac12}]\|_{\frac{p}{2},\infty}.\end{aligned}$$

Hence for $j = 1, 2$ we have:

$$\|H_j(s)\|_{\frac{p}{2},\infty} \le c_p(1+|s|)\|[BA, A^{\frac12}]\|_{\frac{p}{2},\infty}\|X\|_\infty.$$

The assertions (i) and (iii) now follow from Lemma 5.4.8.(ii) upon taking $j = 1$ for (i) and $j = 2$ for (iii). ☐

LEMMA 5.4.10. *Let $p > 2$ and let A and B satisfy Condition 5.4.1. Let $T_z(s)$, $s \in \mathbb{R}$ be defined as in Theorem 5.2.1. Then if $\Re(z) > p - 1$ we have:*

$$\int_{\mathbb{R}} \|T_z(s)\|_1 \cdot |\hat{g}_z(s)| ds < \infty.$$

PROOF. We recall the definition of $T_z(s)$:

$$T_z(s) = B^{z-1+is}[BA^{\frac12}, A^{z-\frac12+is}]Y^{-is} + B^{is}[BA^{\frac12}, A^{\frac12+is}]Y^{z-1-is}, \quad s \in \mathbb{R}.$$

Consider the first summand in the definition of $T_z(s)$. Using the Leibniz rule:

$$\begin{aligned}B^{z-1+is}[BA^{\frac12}, A^{z-\frac12+is}]Y^{-is} &= B^{z-1+is}[BA^{\frac12}, A^{z-1}A^{\frac12+is}]Y^{-is}\\ &= B^{is}B^{z-1}A^{z-1}[BA^{\frac12}, A^{\frac12+is}]Y^{-is}\\ &\quad + B^{is}B^{z-1}[BA, A^{z-1}]A^{is}Y^{-is}.\end{aligned}$$

By the $\mathcal{L}_1$-triangle inequality, we have

$$\begin{aligned}&\|B^{z-1+is}[BA^{\frac12}, A^{z-\frac12+is}]Y^{-is}\|_1\\ &\qquad \le \|B^{z-1}A^{z-1}[BA^{\frac12}, A^{\frac12+is}]\|_1 + \|B^{z-1}[BA, A^{z-1}]\|_1.\end{aligned}$$

We now apply the Hölder-type inequality (2.7),

$$\begin{aligned}&\|B^{z-1+is}[BA^{\frac12}, A^{z-\frac12+is}]Y^{-is}\|_1\\ &\qquad \le \|B^{z-1}A^{z-1}\|_{\frac{p}{p-2},1}\|[BA^{\frac12}, A^{\frac12+is}]\|_{\frac{p}{2},\infty} + \|B^{z-1}[BA, A^{z-1}]\|_1.\end{aligned} \tag{5.11}$$

Consider the function $\phi_s(t) = |t|^{1+2is}$, $t \in \mathbb{R}$. Immediately, ϕ_s is Lipschitz and $\|\phi_s'\|_{L_\infty} \le 2(1+|s|)$. Since $p > 2$, we may apply the result of [**50**] to obtain:

$$\|[BA^{1/2}, \phi(A^{1/2})]\|_{p/2,\infty} \le C_p(1+|s|)\|[BA^{1/2}, A^{1/2}]\|_{p/2,\infty}.$$

Therefore,

$$\|[BA^{\frac12}, A^{\frac12+is}]\|_{\frac{p}{2},\infty} \le C_p(1+|s|)\|[BA^{\frac12}, A^{\frac12}]\|_{\frac{p}{2},\infty}. \tag{5.12}$$

Combining (5.11) and (5.12), we have

$$\|B^{z-1+is}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}+is}]Y^{-is}\|_1$$
$$\leq C_p(1+|s|)\|[BA^{\frac{1}{2}}, A^{\frac{1}{2}}]\|_{\frac{p}{2},\infty}\|B^{z-1}A^{z-1}\|_{\frac{p}{p-2},1} + \|B^{z-1}[BA, A^{z-1}]\|_1. \tag{5.13}$$

Let us now consider the second summand in $T_z(s)$. Using the Hölder inequality in the form of (2.7), we obtain

$$\begin{aligned}\|B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1-is}\|_1 &\leq \|[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1}\|_1 \\ &\leq \|[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]\|_{\frac{p}{2},\infty}\|Y^{z-1}\|_{\frac{p}{p-2},1}.\end{aligned} \tag{5.14}$$

Now combining (5.14) with (5.12), we arrive at:

$$\begin{aligned}&\|B^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1-is}\|_1 \\ &\qquad \leq C_p(1+|s|)\|[BA^{\frac{1}{2}}, A^{\frac{1}{2}}]\|_{\frac{p}{2},\infty}\|Y^{z-1}\|_{\frac{p}{p-2},1}.\end{aligned} \tag{5.15}$$

Now we may combine (5.13) and (5.15):

$$\begin{aligned}\|T_z(s)\|_1 &\leq \|B^{z-1}[BA, A^{z-1}]\|_1 \\ &\quad + c_{abs}(1+|s|)\|[BA^{\frac{1}{2}}, A^{\frac{1}{2}}]\|_{\frac{p}{2},\infty} \cdot \Big(\|B^{z-1}A^{z-1}\|_{\frac{p}{p-2},1} + \|Y^{z-1}\|_{\frac{p}{p-2},1}\Big).\end{aligned}$$

By Lemma 5.4.7,

$$\|B^{z-1}[BA, A^{z-1}]\|_1, \|B^{z-1}A^{z-1}\|_{\frac{p}{p-2},1}, \|Y^{z-1}\|_{\frac{p}{p-2},1} < \infty, \quad \Re(z) > p-1.$$

Hence,

$$\|T_z(s)\|_1 \leq C_{A,B,z} \cdot (1+|s|).$$

We now have

$$\int_{\mathbb{R}} \|T_z(s)\|_1 \cdot |\hat{g}_z(s)| ds \leq C_{A,B,z} \int_{\mathbb{R}} (1+|s|)|\hat{g}_z(s)| ds.$$

By [**51**, Lemma 7], we have

$$\int_{\mathbb{R}} (1+|s|)|\hat{g}_z(s)| ds = \|g_z\|_2 + \|g_z'\|_2 \leq c_{abs}\|g_z\|_{W^{2,2}}.$$

Thus,

$$\int_{\mathbb{R}} \|T_z(s)\|_1 |\hat{g}_z(s)| ds \leq C_{A,B,z}\|g_z\|_{W^{2,2}}.$$

The assertion follows now from Theorem 5.3.2. □

Corollary 5.4.11. *If $p > 2$ and A and B satisfy condition 5.4.1, then for all $\Re(z) > p-1$ we have:*

$$B^zA^z - (A^{1/2}BA^{1/2})^z \in \mathcal{L}_1.$$

Proof. The integral formula from Lemma 5.2.1 is valid for $\Re(z) > 1$. Since $p > 2$, we therefore have a weak operator topology integral:

$$B^zA^z - (A^{1/2}BA^{1/2})^z = T_z(0) - \int_{\mathbb{R}} T_z(s)\widehat{g}_z(s)\, ds.$$

From Lemma 5.4.10, we have that

$$\int_{\mathbb{R}} \|T_z(s)\|_1 \cdot |\hat{g}_z(s)| ds < \infty.$$

Thus by Lemma 2.3.2, we have that $\int_{\mathbb{R}} T_z(s)\widehat{g}_z(s) \in \mathcal{L}_1$.

Recalling the definition of $T_z(0)$:

$$T_z(0) := B^{z-1}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}}] + [BA^{\frac{1}{2}}, A^{\frac{1}{2}}]Y^{z-1}$$

So by (5.11) and (5.14), $T_z(0) \in \mathcal{L}_1$. Therefore, $B^zA^z - (A^{1/2}BA^{1/2})^z \in \mathcal{L}_1$. □

LEMMA 5.4.12. *Assume that $p > 2$ and let A and B satisfy Condition 5.4.1. If $X \in \mathcal{L}_\infty$, then for $\Re(z) > p$:*

$$\operatorname{Tr}\Big(X\Big(B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z\Big)\Big) = \operatorname{Tr}(XF_0(z)) - \sum_{k=1}^{3}\operatorname{Tr}(G_k(z)F_k(z))$$

Here, the functions F_k are as in Lemma 5.4.7 and the functions G_k are as in Lemma 5.4.9.

PROOF. By Lemma 5.4.10,

$$\int_{\mathbb{R}} \|T_z(s)\|_1 |\widehat{g}_z(s)|\, ds < \infty.$$

So by Lemma 2.3.2, we have:

$$\int_{\mathbb{R}} T_z(s)\widehat{g}_z(s)ds \in \mathcal{L}_1$$

and

$$\operatorname{Tr}\Big(X\int_{\mathbb{R}} T_z(s)\widehat{g}_z(s)ds\Big) = \int_{\mathbb{R}} \operatorname{Tr}(XT_z(s))\widehat{g}_z(s)ds.$$

By Theorem 5.2.1, we have:

$$\operatorname{Tr}\Big(X\Big(B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z\Big)\Big) = \operatorname{Tr}(XT_z(0)) - \int_{\mathbb{R}} \operatorname{Tr}(XT_z(s))\hat{g}_z(s)ds \tag{5.16}$$

for $\Re(z) > p$.

Observing that $F_0(z) = T_z(0)$, we have:

$$\operatorname{Tr}\Big(X\Big(B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z\Big)\Big) = \operatorname{Tr}(X \cdot F_0(z)) - \int_{\mathbb{R}} \operatorname{Tr}(XT_z(s))\hat{g}_z(s)ds. \tag{5.17}$$

By (5.11) and (5.14), we have

$$B^{z-1}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}+is}] \in \mathcal{L}_1,$$
$$[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1} \in \mathcal{L}_1.$$

Now by the definition of $T_z(s)$, we have

$$\begin{aligned}\operatorname{Tr}(XT_z(s)) &= \operatorname{Tr}(Y^{-is}XB^{is}B^{z-1}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}+is}]) \\ &\quad + \operatorname{Tr}(Y^{-is}XB^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1}).\end{aligned}$$

By an application of the Leibniz rule, we have

$$B^{z-1}[BA^{\frac{1}{2}}, A^{z-\frac{1}{2}+is}] = B^{z-1}A^{z-1}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}] + B^{z-1}[BA, A^{z-1}]A^{is}.$$

Each of the above terms is $\mathcal{L}_1$, by (5.11) and Lemma 5.4.7.(iii) respectively. Therefore,

$$\begin{aligned}\operatorname{Tr}(X \cdot T_z(s)) &= \operatorname{Tr}([BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{-is}XB^{is}B^{z-1}A^{z-1}) \\ &\quad + \operatorname{Tr}(A^{is}Y^{-is}XB^{is}B^{z-1}[BA, A^{z-1}]) + \operatorname{Tr}(Y^{-is}XB^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{z-1}).\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} &\mathrm{Tr}(XT_z(s))\hat{g}_z(s)ds \\ &= \mathrm{Tr}\Big(\int_{\mathbb{R}}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]Y^{-is}XB^{is}\hat{g}_z(s)ds\cdot B^{z-1}A^{z-1}\Big) \\ &\quad + \mathrm{Tr}\Big(\int_{\mathbb{R}} A^{is}Y^{-is}XB^{is}\hat{g}_z(s)ds\cdot B^{z-1}[BA, A^{z-1}]\Big) \\ &\quad + \mathrm{Tr}\Big(\int_{\mathbb{R}} Y^{-is}XB^{is}[BA^{\frac{1}{2}}, A^{\frac{1}{2}+is}]\hat{g}_z(s)ds\cdot Y^{z-1}\Big).\end{aligned}$$

Using the notations from Lemma 5.4.7 and Lemma 5.4.9, we may summarise the above equality as:

$$\int_{\mathbb{R}} \mathrm{Tr}(XT_z(s))\hat{g}_z(s)ds = \sum_{k=1}^{3}\mathrm{Tr}(G_k(z)F_k(z)). \tag{5.18}$$

Combining (5.17) and (5.18) completes the proof. □

We are now ready to complete the proof of Theorem 5.4.2.

Proof of Theorem 5.4.2. We will show that the function

$$A(z) := \mathrm{Tr}(X\cdot F_0(z)) - \sum_{k=1}^{3}\mathrm{Tr}(G_k(z)F_k(z))$$

is analytic for $\Re(z) > p-1$.

For the F_0 term, we use Lemma 5.4.7.(i): the mapping $z\to XF_0(z)$ is $\mathcal{L}_1-$valued analytic for $\Re(z) > p-1$. For the G_1F_1 term, we use Lemma 5.4.7.(ii) and Lemma 5.4.9.(i) to see that the mapping $z\to G_1(z)F_1(z)$ is $\mathcal{L}_1-$valued analytic for $\Re(z) > p-1$. For the G_2F_2 term, we use Lemma 5.4.7.(iii) and Lemma 5.4.9.(ii) to see that the mapping $z\to G_2(z)F_2(z)$ is $\mathcal{L}_1-$valued analytic for $\Re(z) > p-1$. Finally, for the G_3F_3 term, we use Lemma 5.4.7. (iv) and Lemma 5.4.9.(iii) to see that mapping $z\to G_3(z)F_3(z)$ is $\mathcal{L}_1-$valued analytic for $\Re(z) > p-1$.

Hence, A is holomorphic in the set $\Re(z) > p-1$. By Lemma 5.4.12,

$$A(z) = \mathrm{Tr}\Big(X\Big(B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z\Big)\Big)$$

and so the proof is complete. □

5.5. Criterion for universal measurability in terms of a $\zeta-$function

In this section we provide a sufficient condition for universal measurability of operators in $\mathcal{L}_{1,\infty}$.

We recall that a linear functional φ on the weak Schatten ideal $\mathcal{L}_{1,\infty}$ is called a trace if for all unitary operators U and $T\in\mathcal{L}_{1,\infty}$, we have $\varphi(U^*TU) = \varphi(T)$. Equivalently, for all bounded operators A we have $\varphi(AT) = \varphi(TA)$. We say that φ is normalised if

$$\varphi\left(\mathrm{diag}\left\{\frac{1}{n+1}\right\}_{n\geq 0}\right) = 1.$$

An operator $T\in\mathcal{L}_{1,\infty}$ is called *universally measurable* if all normalised traces take the same value on T.

In this section we prove Theorem 1.2.7, which provides a sufficient condition for operators of the form AV, $A \in \mathcal{L}_\infty$, $V \in \mathcal{L}_{1,\infty}$ to be universally measurable. This result is new, and is sufficiently powerful to allow us to prove Theorem 1.2.5. A similar characterisation is provided in [**63**, Theorem 4.13], but the result provided here is stronger. A previously known characterisation of universal measurability in terms of a heat trace can be found in [**15**, Proposition 6].

Let b be a signed Borel measure on $[0,\infty)$. Recall that b can be written as a difference of two positive measures, $b = b_+ - b_-$ such that b_+ and b_- are mutually singular to each other. Given a Borel set S, the total variation of b on S is defined to be $\mathrm{Var}_S(b) := b_+(S) + b_-(S)$.

We will consider measures b which satisfy,

$$\sup_{x\geq 0} \mathrm{Var}_{[x,x+1]}(b) = c_b < \infty. \tag{5.19}$$

Clearly any measure of finite total variation will satisfy this condition, as will some measures with infinite total variation such as Lebesgue measure and the measure $d\nu(t) = \sin(t)dt$.

Lemma 5.5.1. *Let b be a signed Borel measure satisfying the condition* (5.19) *and let f be the Laplace transform of b, that is,*

$$f(z) := \int_0^\infty e^{-tz}\,db(t), \quad \Re(z) > 0.$$

The function f is analytic on the half-plane $\{\Re(z) > 0\}$.

Proof. For every $n \geq 0$, let

$$f_n(z) = \int_0^n e^{-tz} db(t), \quad z \in \mathbb{C}.$$

Each function f_n, $n \geq 0$, is entire. Let $\varepsilon > 0$. For $\Re(z) > \varepsilon$, we have

$$|f(z) - f_n(z)| = |\int_n^\infty e^{-tz} db(t)| \leq \sum_{k\geq n} e^{-k\varepsilon}\mathrm{Var}_{[k,k+1]}(b) \leq c_b \frac{e^{-n\varepsilon}}{1-e^{-\varepsilon}}.$$

Therefore, $f_n \to f$ uniformly on the half-plane $\{\Re(z) > \varepsilon\}$. Since $\varepsilon > 0$ is arbitrary, it follows that $f_n \to f$ uniformly on the compact subsets of the half-plane $\{\Re(z) > 0\}$. Thus, f is analytic on the half-plane $\{\Re(z) > 0\}$. □

For $x \geq 0$, we denote

$$b(x) := b([0,x]).$$

We caution the reader that $b(x)$ does not denote $b(\{x\})$ nor the Radon-Nikodym derivative of b at x.

The following lemma is very similar to [**61**, Lemma 2.1.3]. However we require a slightly different formulation of that result and we were unable to find an English-language version of its proof, and so we include a self-contained proof here.

Lemma 5.5.2. *Let b be a signed Borel measure on $[0,\infty)$ satisfying* (5.19), *and and let f be the Laplace transform of b. For $x \geq 1$, $x \to \infty$, we have*

$$b(x) = \frac{1}{2\pi}\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{1}{x}+it} f(\frac{1}{x}+it)e^{(\frac{1}{x}+it)x}dt + O(1), \quad x \geq 1.$$

PROOF. Let $x \geq 1$. By definition we have:

$$\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{1}{x}+it} f(\frac{1}{x}+it) e^{(\frac{1}{x}+it)x} dt = \int_{-1}^{1}\int_{0}^{\infty} \frac{(1-t^2)^2}{\frac{1}{x}+it} e^{(\frac{1}{x}+it)(x-s)} db(s) dt.$$

Examining the integrand, we see that the function

$$(s,t) \mapsto \frac{(1-t^2)^2}{x^{-1}+t} e^{(\frac{1}{x}+it)(x-s)}$$

is bounded above in absolute value by

$$(s,t) \mapsto exe^{-s/x}.$$

Since $x \geq 0$ the function $s \to e^{-s/x}$ is in $L_1([0,\infty), b)$ we may interchange the integrals to get:

$$\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{1}{x}+it} f(\frac{1}{x}+it) e^{(\frac{1}{x}+it)x}\, dt = \int_{0}^{\infty} \left(\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{1}{x}+it} e^{(\frac{1}{x}+it)(x-s)}\, dt \right) db(s)$$

Now we refer to Proposition A.5.1, where it is proved that:

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{1}{x}+it} e^{(\frac{1}{x}+it)(x-s)} dt = (1+x^{-2})^2 \chi_{[0,x]}(s) + \min\{1, (x-s)^{-2}\} \cdot O(1).$$

Thus,

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{1}{x}+it} f(\frac{1}{x}+it) e^{(\frac{1}{x}+it)x} dt = \int_{0}^{x} (1+x^{-2})^2 db(s)$$
$$+ \int_{0}^{\infty} \min\{1, (x-s)^{-2}\} \cdot O(1)\, db(s).$$

We let $h(s)$ be the total variation of b on $[0,s]$. That is, $h(s) := \mathrm{Var}_{[0,s]}(b)$. Then by the triangle inequality, there is a positive constant c_{abs} such that

$$\begin{aligned}
\Big| \int_{0}^{\infty} \min\{1, (x-s)^{-2}\} \cdot O(1) \cdot db(s) \Big| &\leq c_{abs} \cdot \int_{0}^{\infty} \min\{1, (x-s)^{-2}\} dh(s) \\
&= c_{abs} \cdot \sum_{k \geq 0} \int_{k}^{k+1} \min\{1, (x-s)^{-2}\} dh(s) \\
&\leq c_{abs} \cdot \sum_{k \geq 0} \sup_{s \in [k,k+1]} \min\{1, (x-s)^{-2}\} \cdot \int_{k}^{k+1} dh(s) \\
&\leq c_{abs} \cdot c_b \cdot \sum_{k \geq 0} \sup_{s \in [k,k+1]} \min\{1, (x-s)^{-2}\} \\
&= O(1)
\end{aligned}$$

Thus,

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{(1-t^2)^2}{\frac{1}{x}+it} f(\frac{1}{x}+it) e^{(\frac{1}{x}+it)x} dt = \int_{0}^{x} (1+x^{-2})^2 db(s) + O(1). \tag{5.20}$$

By the definition of $b(x)$,

$$|b(x) - b(0)| \leq \mathrm{Var}_{[0,x]}(b) \leq c_b(1+x), \quad x > 0.$$

Therefore,

$$\int_{0}^{x} (1+x^{-2})^2 db(s) = (1+x^{-2})^2 \cdot (b(x) - b(0)) = b(x) + O(1), \quad x \geq 1.$$

A combination of the latter inequality with (5.20) completes the proof. □

The following Lemma is similar in spirit to (but much stronger than) the well-known Wiener-Ikehara Tauberian theorem [**59**, Theorem 14.1].

LEMMA 5.5.3. *Let b be a signed Borel measure on $[0,\infty)$ satisfying (5.19), and let f be the Laplace transform of b. If there exists $\epsilon>0$ such that f has analytic continuation to a half-plane $\{z\,:\,\Re(z)>-\epsilon\}$, then for $x\geq 1$*

$$b(x)=O(1),\quad x\to\infty.$$

PROOF. We write $f(z)=f(0)+zf_0(z)$, where f_0 is an analytic function on the half-plane $\{z\,:\,\Re(z)>-\epsilon\}$. In particular, since the (closure of the) set

$$\Big\{\frac{1}{x}+it:\ x\geq 1,\ t\in[-1,1]\Big\},$$

is a compact subset in $\{\Re(z)>-\epsilon\}$, it follows that

$$\sup_{x\geq 1}\sup_{t\in[-1,1]}|f_0(\frac{1}{x}+it)|<\infty. \tag{5.21}$$

The assertion of Lemma 5.5.2 is now written as follows.

$$\begin{aligned}\frac{1}{2\pi}\int_{-1}^{1}(1-t^2)^2f_0(\frac{1}{x}+it)e^{(\frac{1}{x}+it)x}dt+\frac{1}{2\pi}\int_{-1}^{1}\frac{(1-t^2)^2}{\frac{1}{x}+it}f(0)e^{(\frac{1}{x}+it)x}dt&\\ =b(x)+O(1),\quad x\to\infty.&\end{aligned} \tag{5.22}$$

The first summand in the left hand side is bounded for $x\geq 1$. due to (5.21). By Proposition A.5.1, we have

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{(1-t^2)^2}{\frac{1}{x}+it}e^{(\frac{1}{x}+it)x}dt=1+O(x^{-2}),\quad x\to\infty. \tag{5.23}$$

Combining (5.22) and (5.23), we get:

$$b(x)+O(1)=f(0)+O(1),\quad x\to\infty.$$

So $b(x)=O(1)$ as $x\to\infty$. □

In order to continue our discussion of measurability we refer to the concept of a modulated operator. This theory was introduced in [**37**] and is developed extensively in [**45**, Section 11.2]. If $V\in\mathcal{L}_{1,\infty}$ is positive, and $T\in\mathcal{L}_\infty$, we say that T is V-modulated if

$$\sup_{t>0}t^{1/2}\|T(1+tV)^{-1}\|_{\mathcal{L}_2}<\infty.$$

It can be easily seen that if T is V-modulated, and $A\in\mathcal{L}_\infty$, then AT is V-modulated (this is also [**45**, Proposition 11.2.2]). It is proved in [**45**, Lemma 11.2.8] that V is V-modulated, and therefore that AV is V-modulated.

The relevance of the notion of a V-modulated operator to measurability comes from [**45**, Theorem 11.2.3], which states that if $V\geq 0$ is in $\mathcal{L}_{1,\infty}$, $\ker(V)=0$, T is V-modulated and $\{e_n\}_{n\geq 0}$ is an eigenbasis for V ordered so that $Ve_n=\mu(n,V)e_n$ for $n\geq 0$, then

(a) $T\in\mathcal{L}_{1,\infty}$ and $\mathrm{diag}\{\langle Te_n,e_n\rangle\}_{n=0}^{\infty}\in\mathcal{L}_{1,\infty}$
(b) We have:

$$\sum_{k=0}^{n}\lambda(k,T)-\sum_{k=0}^{n}\langle Te_k,e_k\rangle=O(1),\quad n\to\infty.$$

Recall that $\{\lambda(k,T)\}_{k=0}^{\infty}$ denotes an eigenvalue sequence for T, ordered with non-increasing absolute value.

LEMMA 5.5.4. *Let* $0 \le V \in \mathcal{L}_{1,\infty}$ *satisfy* $\ker(V)=0$ *and let* $A \in \mathcal{L}_{\infty}$. *Let* $\{e_k\}_{k=0}^{\infty}$ *be an eigenbasis for* V *ordered such that* $Ve_k = \mu(k,V)e_k$. *We have*

$$\sum_{k=0}^{n} \lambda(k,AV) = \sum_{\mu(k,V) > \frac{\|V\|_{1,\infty}}{n}} \langle Ae_k, e_k\rangle \mu(k,V) + O(1).$$

Here, e_k *is the eigenvector of* V *corresponding to the eigenvalue* $\mu(k,V)$.

PROOF. We have that AV is $V-$modulated. [**45**, Theorem 11.2.3] now states that as $n \to \infty$

$$\begin{aligned}\sum_{k=0}^{n} \lambda(k,AV) &= \sum_{k=0}^{n} \langle AVe_k, e_k\rangle + O(1)\\ &= \sum_{k=0}^{n} \langle Ae_k, e_k\rangle \mu(k,V) + O(1).\end{aligned}$$

For $n \ge 1$, let

$$m(n) = \max\{k \in \mathbb{N} :\ \mu(k,V) > \frac{\|V\|_{1,\infty}}{n}\}.$$

Using this notation, we write

$$\sum_{\mu(k,V) > \frac{\|V\|_{1,\infty}}{n}} \langle Ae_k, e_k\rangle \mu(k,V) = \sum_{k=0}^{m(n)} \langle Ae_k, e_k\rangle \mu(k,V).$$

We have $\mu(k,V) \le \frac{\|V\|_{1,\infty}}{k+1}$ for every $k \ge 0$ and, therefore,

$$m(n) \le \max\left\{k \in \mathbb{Z}_+ :\ \frac{\|V\|_{1,\infty}}{k+1} > \frac{\|V\|_{1,\infty}}{n}\right\} = n-2 < n.$$

On the other hand, we have $\mu(k,V) \le \frac{\|V\|_{1,\infty}}{n}$ for all $k > m(n)$. Thus,

$$\begin{aligned}\Big|\sum_{k=m(n)+1}^{n} \langle Ae_k, e_k\rangle \mu(k,V)\Big| &\le \|A\|_{\infty} \sum_{k=m(n)+1}^{n} \mu(k,V)\\ &\le \frac{\|A\|_{\infty}\|V\|_{1,\infty}}{n} \sum_{k=m(n)+1}^{n} 1\\ &= O(1).\end{aligned}$$

Finally, we have

$$\begin{aligned}\sum_{k=0}^{n} \lambda(k,AV) &= \sum_{k=0}^{n} \langle Ae_k, e_k\rangle \mu(k,V) + O(1)\\ &= \sum_{k=0}^{m(n)} \langle Ae_k, e_k\rangle \mu(k,V) + O(1)\\ &= \sum_{\mu(k,V) > \frac{\|V\|_{1,\infty}}{n}} \langle Ae_k, e_k\rangle \mu(k,V) + O(1).\end{aligned}$$

$\square$

We now conclude with the proof of Theorem 1.2.7, a sufficient condition for universal measurability in terms of a ζ-function. Recall that if f is a meromorphic function of a complex variable z with a simple pole at $z=0$, then $\mathrm{Res}_{z=0}f(z)$ denotes the coefficient of z^{-1} in the Laurent expansion of f, or equivalently the value of $zf(z)$ at $z=0$.

THEOREM. *Let $0 \le V \in \mathcal{L}_{1,\infty}$ and let $A \in \mathcal{L}_\infty$. Define the ζ-function:*

$$\zeta_{A,V}(z) := \mathrm{Tr}(AV^{1+z}), \quad \Re(z) > 0.$$

If there exists $\varepsilon > 0$ such that $\zeta_{A,V}$ admits an analytic continuation to the set $\{z : \Re(z) > -\varepsilon\} \setminus \{0\}$ with a simple pole at 0, then for every normalised trace φ on $\mathcal{L}_{1,\infty}$ we have:

$$\varphi(AV) = \mathrm{Res}_{z=0}\zeta_{A,V}(z).$$

In particular, AV is universally measurable.

PROOF. Assume without loss of generality that $\ker(V) = 0$. Select an orthonormal basis $\{e_k\}_{k=0}^\infty$ such that $Ve_k = \mu(k,V)e_k$.

We define:

$$b(t) := -t\mathrm{Res}_{w=0}\zeta_{A,V}(w) + \sum_{\mu(k,V)>e^{-t}} \langle Ae_k, e_k\rangle \mu(k,V), \quad t \ge 0.$$

Since b is a linear combination of monotone functions, it is of locally bounded variation. Hence there is a signed Borel measure b such that $b([0,x]) = b(x)$, $x \ge 0$.

We first prove that b satisfies (5.19). If $x \ge 0$, then:

$$\begin{aligned}\mathrm{Var}_{[x,x+1]}b &\le |\mathrm{Res}_{w=0}\zeta_{A,V}(w)| + \sum_{\mu(k,V)\in(e^{-1-x},e^{-x}]} |\langle Ae_k, e_k\rangle|\mu(k,V)\\ &\le |\mathrm{Res}_{w=0}\zeta_{A,V}(w)| + 2\|A\|_\infty e^{-x} \sum_{\mu(k,V)\in(e^{-1-x},e^{-x}]} 1\\ &\le |\mathrm{Res}_{w=0}\zeta_{A,V}(w)| + 2\|A\|_\infty e^{-x} \sum_{\mu(k,V)\in(e^{-1-x},\infty]} 1.\end{aligned}$$

Since $V \in \mathcal{L}_{1,\infty}$, we have that

$$\sum_{\mu(k,V)\in(e^{-1-x},\infty]} 1 \le e^{1+x}\|V\|_{1,\infty}, \quad x \in \mathbb{R}.$$

Therefore,

$$\mathrm{Var}_{[x,x+1]}b \le |\mathrm{Res}_{w=0}\zeta_{A,V}(w)| + 2e\|A\|_\infty\|V\|_{1,\infty}.$$

So b indeed satisfies (5.19).

Let $\alpha_k := \log(\frac{1}{\mu(k,V)})$ and $b_k := \langle Ae_k, e_k\rangle\mu(k,V)$, then the function b has a jump discontinuity at the point α_k of magnitude b_k. Let $\Re(z) > 0$. Using the identity $e^{-\alpha_k z} = \mu(k,V)^z$, we have:

$$\begin{aligned}\int_0^\infty e^{-zt}db(t) &= \sum_{k\geq 0} e^{-\alpha_k z}\cdot b_k - \mathrm{Res}_{w=0}\zeta_{A,V}(w)\int_0^\infty e^{-zt}dt\\ &= \sum_{k\geq 0}\mu(k,V)^z\cdot\langle Ae_k,e_k\rangle\mu(k,V) - \mathrm{Res}_{w=0}\zeta_{A,V}(w)\int_0^\infty e^{-zt}dt\\ &= \sum_{k\geq 0}\langle AV^{1+z}e_k,e_k\rangle - \mathrm{Res}_{w=0}\zeta_{A,V}(w)\int_0^\infty e^{-zt}dt\\ &= \mathrm{Tr}(AV^{1+z}) - \frac{1}{z}\mathrm{Res}_{w=0}\zeta_{A,V}(w).\end{aligned}$$

By assumption, the above right hand side has analytic continuation to the set $\{z\ :\ \Re(z) > -\varepsilon\}$.

Thus, since b satisfies (5.19) the assumptions of Lemma 5.5.3 are satisfied, and we may then conclude that $b(t) = O(1)$, for $t\to\infty$. Thus by the definition of b:,

$$\sum_{\mu(k,V)>e^{-t}}\langle Ae_k,e_k\rangle\mu(k,V) = t\cdot\mathrm{Res}_{w=0}\zeta_{A,V}(w) + O(1),\quad t\to\infty.$$

Setting $e^{-t} = \frac{\|V\|_{1,\infty}}{n}$, we obtain

$$\sum_{\mu(k,V)>\frac{\|V\|_{1,\infty}}{n}}\langle Ae_k,e_k\rangle\mu(k,V) = \log(n)\cdot\mathrm{Res}_{w=0}\zeta_{A,V}(w) + O(1),\quad t\to\infty.$$

By Lemma 5.5.4, we have

$$\sum_{k=0}^n\lambda(k,AV) = \log(n)\cdot\mathrm{Res}_{w=0}\zeta_{A,V}(w) + O(1),\quad t\to\infty.$$

The assertion follows now from Theorem 2.1.5. □

5.6. Proof of Theorem 1.2.5, $p > 2$

In this section we complete the proof of Theorem 1.2.5 under the restriction that $p > 2$. We require this restriction in order to directly apply the results of Section 5.4. We will handle the $p = 1$ and $p = 2$ cases separately in the next section.

LEMMA 5.6.1. *Let $(\mathcal{A},H,D)$ be a spectral triple satisfying Hypothesis 1.2.1. Let $0\leq a\in\mathcal{A}$. If $p>2$, then the operators $A = a^2$ and $B = (1+D^2)^{-\frac{1}{2}}$ satisfy Condition 5.4.1.*

PROOF. Let $D_0 = F(1+D^2)^{\frac{1}{2}}$. By Lemma 3.1.4, the spectral triple $(\mathcal{A},H,D_0)$ satisfies Hypothesis 1.2.1. Since $|D_0|\geq 1$, we have that $\||D_0|^{-1}\|_\infty\leq 1$.

Let us establish Condition 5.4.1.(i). We have

$$B^pA = |D_0|^{-p}a^2.$$

Since $(\mathcal{A},H,D)$ is p-dimensional, we have that $|D_0|^{-p}a\in\mathcal{L}_{1,\infty}$, and so $B^pA\in\mathcal{L}_{1,\infty}$.

Next let us prove Condition 5.4.1.(ii). Let $q > p$. We have

$$\begin{aligned}B^{q-2}[B,A] &= |D_0|^{2-q}[|D_0|^{-1}, a^2]\\ &= -|D_0|^{1-q}\delta_0(a^2)|D_0|^{-1}\\ &= -|D_0|^{1-q}\delta_0(a)a|D_0|^{-1} - |D_0|^{1-q}a\delta_0(a)|D_0|^{-1}.\end{aligned}$$

Referring to Lemma 3.1.3, we have

$$|D_0|^{1-q}\delta_0(a) \in \mathcal{L}_{p/(q-1),\infty}, \quad |D_0|^{1-q}a \in \mathcal{L}_{p/(q-1),\infty},$$
$$\delta_0(a)|D_0|^{-1} \in \mathcal{L}_{p,\infty}, \quad a|D_0|^{-1} \in \mathcal{L}_{p,\infty}.$$

Therefore,

$$B^{q-2}[B,A] \in \mathcal{L}_{p/(q-1),\infty} \cdot \mathcal{L}_{p,\infty} + \mathcal{L}_{p/(q-1),\infty} \cdot \mathcal{L}_{p,\infty}$$

So by the Hölder inequality, $B^{q-2}[B,A] \in \mathcal{L}_{p/q,\infty}$. Since $q > p$, it then follows that $B^{q-2}[B,A] \in \mathcal{L}_1$.

Now we establish Condition 5.4.1.(iii). We have

$$A^{\frac{1}{2}}BA^{\frac{1}{2}} = a|D_0|^{-1}a.$$

Thus,

$$\|A^{\frac{1}{2}}BA^{\frac{1}{2}}\|_{p,\infty} \leq \|a\|_\infty \|a|D_0|^{-1}\|_{p,\infty}.$$

The above right hand side is finite by Lemma 3.1.3.

Finally we prove Condition 5.4.1.(iv). We recall the notation that $\delta_0(a)$ denotes the bounded extension of $[|D_0|, a]$. Using (2.2), we have:

$$[B, A^{\frac{1}{2}}] = [|D_0|^{-1}, a] = -|D_0|^{-1}\delta_0(a)|D_0|^{-1}.$$

By Theorem 9 in [**62**], we have

$$|D_0|^{-1}\delta_0(a)|D_0|^{-1} \prec\prec \delta_0(a)|D_0|^{-2}.$$

By Lemma 3.1.3, we have

$$\delta_0(a)|D_0|^{-2} \in \mathcal{L}_{\frac{p}{2},\infty}.$$

Since the norm in the space $\mathcal{L}_{\frac{p}{2},\infty}$ is monotone with respect to the Hardy-Littlewood submajorisation (recall that $p > 2$), it follows that also

$$[B, A^{\frac{1}{2}}] = -|D_0|^{-1}\delta_0(a)|D_0|^{-1} \in \mathcal{L}_{\frac{p}{2},\infty}.$$

∎

Now we may prove Theorem 1.2.5 for the case $p > 2$.

Theorem (1.2.5, $p > 2$ case). *Assume $p > 2$ and let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a local Hochschild cycle, then for every normalised trace φ on $\mathcal{L}_{1,\infty}$ we have:*

$$\varphi(\Omega(c)(1+D^2)^{-\frac{p}{2}}) = \mathrm{Ch}(c).$$

Proof. By Theorem 1.2.3, the function

$$\zeta_{c,D}(z) := \mathrm{Tr}(\Omega(c)(1+D^2)^{-z/2}), \quad \Re(z) > p$$

admits an analytic continuation to the set $\{z : \Re(z) > p-1\} \setminus \{p\}$, and p is a simple pole for $\zeta_{c,D}$ with residue $p\mathrm{Ch}(c)$.

Let $c = \sum_{j=1}^m a_0^j \otimes a_1^j \otimes \cdots \otimes a_p^j$. We assume that c is local, i.e. that there exists $0 \leq a \in \mathcal{A}$ such that for all j we have $aa_0^j = a_0^j$. Equivalently, $(1-a)a_0^j = 0$.

So $\operatorname{im}(a_0^j) \subseteq \ker(1-a)$. Since the support projection $\operatorname{supp}(1-a)$ is the projection onto the orthogonal complement of the kernel, we have:

$$\operatorname{supp}(1-a)a_0^j = 0.$$

By functional calculus, $\operatorname{supp}(1-a) = 1 - \chi_{\{1\}}(a)$, so moreover we have that $\chi_{\{1\}}(a)a_0^j = a_0^j$. Therefore, for all $z \in \mathbb{C}$ with $\Re(z) > 0$:

$$a^{2z}a_0^j = a^{2z}\chi_{\{1\}}(a)a_0^j = 1^{2z}a_0^j = a_0^j.$$

Recall that $\Omega(c) = \sum_{j=0}^m \Gamma a_0^j \partial(a_1^j)\cdots\partial(a_p^j)$. Since a commutes with Γ, we have for all $\Re(z) > 0$,

$$a^{2z}\Omega(c) = \Omega(c). \tag{5.24}$$

Let $A = a^2$ and $B = (1+D^2)^{-\frac{1}{2}}$, as in Lemma 5.6.1. Then $B^zA^z = (1+D^2)^{-z/2}a^{2z}$, and hence

$$\begin{aligned}\operatorname{Tr}(\Omega(c)B^zA^z) &= \operatorname{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}a^{2z})\\ &= \operatorname{Tr}(a^{2z}\Omega(c)(1+D^2)^{-\frac{z}{2}})\\ &= \operatorname{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}).\end{aligned}$$

By Theorem 1.2.3, it then follows that $z \mapsto \operatorname{Tr}(\Omega(c)B^zA^z)$ admits an analytic continuation to the set $\{\Re(z) > p-1\}\setminus\{p\}$ with a simple pole at $z = p$ and residue $p\operatorname{Ch}(c)$.

Condition 5.4.1 holds for A and B by Lemma 5.6.1. Hence we may apply Theorem 5.4.2 to conclude that

$$z \to \operatorname{Tr}\Big(\Omega(c)\Big(B^zA^z - (A^{\frac{1}{2}}BA^{\frac{1}{2}})^z\Big)\Big)$$

admits an analytic continuation to the set $\{\Re(z) > p-1\}$.

By Lemma 5.6.1, $A^{\frac{1}{2}}BA^{\frac{1}{2}} \in \mathcal{L}_{p,\infty}$. Hence, the function (defined *a priori* for $\Re(z) > p$)

$$z \to \operatorname{Tr}(\Omega(c)(A^{\frac{1}{2}}BA^{\frac{1}{2}})^z)$$

admits an analytic continuation to the set $\{\Re(z) > p-1\}\setminus\{p\}$, with $z = p$ being a simple pole with residue $p\operatorname{Ch}(c)$. Consider $V = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^p \in \mathcal{L}_{1,\infty}$. It has just been demonstrated that

$$z \to \operatorname{Tr}(\Omega(c)V^z)$$

admits an analytic continuation to the set $\{\Re(z) > 1-\frac{1}{p}\}\setminus\{1\}$, with a simple pole at $z = 1$ and the corresponding residue being $\operatorname{Ch}(c)$.

By Theorem 1.2.7, we therefore have

$$\varphi(\Omega(c)V) = \operatorname{Ch}(c)$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

By Corollary 5.4.11 we have:

$$V - B^pA^p = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^p - B^pA^p \in \mathcal{L}_1.$$

Since φ vanishes on $\mathcal{L}_1$, it follows that

$$\varphi(\Omega(c)B^pA^p) = \varphi(\Omega(c)V) - \varphi(\Omega(c)(V - B^pA^p)) = \varphi(\Omega(c)V)$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$. Now using (5.24) with $z = p$:

$$\varphi(\Omega(c)(1+D^2)^{-\frac{p}{2}}) = \varphi(a^{2p}\Omega(c)(1+D^2)^{-\frac{p}{2}}) = \varphi(\Omega(c)B^pA^p) = \operatorname{Ch}(c)$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$. □

5.7. Proof of Theorem 1.2.5, $p = 1, 2$

In this final section we complete the proof of Theorem 1.2.5 by dealing with the remaining cases of $p = 1$ and $p = 2$. We require adjustment for these cases since Theorem 5.4.2 is inapplicable for $p \le 2$.

LEMMA 5.7.1. *Let $(\mathcal{A}, H, D)$ satisfy Hypothesis 1.2.1. Suppose that spectrum of the operator D does not intersect the interval $(-1, 1)$. Then for all $x \in \mathrm{dom}(\delta)$ we have an absolute constant c_{abs} such that*

$$\|[|D|^{\frac{1}{3}}, x]\|_1 \le c_{\mathrm{abs}} \|[|D|, x]\|_1,$$

and for all $r \in (1, \infty)$ a constant $c_r > 0$ such that

$$\|[|D|^{\frac{1}{3}}, x]\|_{r,\infty} \le c_r \|[|D|, x]\|_{r,\infty}.$$

These inequalities are understood to be trivially true if the right hand side is infinite.

PROOF. We only prove the first assertion. One can prove the second inequality by an identical argument, with the $\mathcal{L}_{r,\infty}$ quasi-norm in place of the $\mathcal{L}_1$ norm.

Let f be a smooth function on $\mathbb{R}$ such that $f(t) = |t|^{\frac{1}{3}}$ for $|t| > 1$. For $\varepsilon > 0$, set $f_\varepsilon(t) = f(t)e^{-\varepsilon^2 t^2}$, $t \in \mathbb{R}$. Then,

$$\begin{aligned} f_\varepsilon'(t) &= (f'(t) - 2\varepsilon^2 t f(t))e^{-\varepsilon^2 t^2}, \\ f_\varepsilon''(t) &= (f''(t) - 4\varepsilon^2 t f'(t) + (4t\varepsilon^4 - 2\varepsilon^2) f(t))e^{-\varepsilon^2 t^2} \end{aligned}$$

Since for $|t| > 1$ we have $f'(t) = \frac{1}{3}|t|^{-2/3}$, we have that as $\varepsilon \to 0$ the L_2-norm $\|f_\varepsilon'\|_{L_2(\mathbb{R})}$ is uniformly bounded. Similarly, since for $|t| > 1$, $f''(t) = -\frac{2}{9}|t|^{-5/3}$, we also have that $\|f_\varepsilon''(t)\|_{L_2(\mathbb{R})}$ is uniformly bounded.

We have that (see e.g. [**51**, Lemma 7]):

$$\|\widehat{f_\epsilon'}\|_1 \le c_{abs}\Big(\|f_\epsilon'\|_2 + \|f_\epsilon''\|_2\Big).$$

and so if $\varepsilon \in (0, 1)$, $\|\widehat{f_\varepsilon'}\|_1$ is uniformly bounded.

By Lemma A.2.2 (taken with $s = 1$), we have the identity:

$$[f_\epsilon(|D|), x] = \int_{-\infty}^{\infty} \Big(\int_0^1 \hat{f}_\epsilon'(u) e^{iu(1-v)|D|} \delta(x) e^{iuv|D|} dv \Big) du.$$

So taking the $\mathcal{L}_1$-norm, we conclude from Lemma 2.3.3 that

$$\begin{aligned} \Big\|[f_\epsilon(|D|), x]\Big\|_1 &\le \|\hat{f}_\epsilon'\|_1 \|\delta(x)\|_1 \\ &\le c_{\mathrm{abs}} \|\delta(x)\|_1. \end{aligned}$$

Fix $N > 0$. We have

$$\Big\|\chi_{[0,N]}(|D|)[f_\epsilon(|D|), x]\chi_{[0,N]}(|D|)\Big\|_{r,\infty} \le \frac{c_{abs} r}{r-1} \|\delta(x)\|_{r,\infty}.$$

Since as $\varepsilon \to 0$, we have that f_ε converges uniformly to f on the set $[0, N]$, we have that: As $\epsilon \to 0$, we have

$$\chi_{[0,N]}(|D|)[f_\epsilon(|D|), x]\chi_{[0,N]}(|D|) \to \chi_{[0,N]}(|D|)[|D|^{\frac{1}{3}}, x]\chi_{[0,N]}(|D|)$$

in the operator norm. By the Fatou property of the $\mathcal{L}_1$-norm:

$$\left\| \chi_{[0,N]}(|D|) \cdot [|D|^{\frac{1}{3}}, x] \cdot \chi_{[0,N]}(|D|) \right\|_1 \leq c_{\text{abs}} \|\delta(x)\|_1.$$

Since the above inequality is true for arbitrary $N > 0$, we may take the limit $N \to \infty$ and again using the Fatou property of the $\mathcal{L}_1$ norm, we arrive at

$$\left\| [|D|^{\frac{1}{3}}, x] \right\|_1 \leq c_{\text{abs}} \|\delta(x)\|_1.$$

□

As a replacement for Lemma 5.6.1 in the $p = 1$ case we use the following:

LEMMA 5.7.2. *Let $(\mathcal{A}, H, D)$ be a $1-$dimensional spectral triple satisfying Hypothesis 1.2.1. If $0 \leq a \in \mathcal{A}$, then the operators $A = a^4$ and $B = (1+D^2)^{-\frac{1}{6}}$ satisfy Condition 5.4.1 with $p = 3$.*

PROOF. This proof is similar to that of Lemma 5.6.1.

Let $D_0 = F(1 + D^2)^{\frac{1}{2}}$. By Lemma 3.1.4, the $1-$dimensional spectral triple $(\mathcal{A}, H, D_0)$ satisfies Hypothesis 1.2.1. Since $|D_0| \geq 1$, we have that $\||D_0|^{-1}\|_\infty \leq 1$.

Let us establish Condition 5.4.1.(i). We have

$$B^p A = |D_0|^{-1} a^4 \in \mathcal{L}_{1,\infty}$$

since by assumption $(\mathcal{A}, H, D)$ is 1-dimensional.

Next we establish Condition 5.4.1.(ii). Let $q \in (3, 4)$. Using (2.2), we have on H_∞:

$$\begin{aligned} B^{q-2}[B, A] &= |D_0|^{\frac{2-q}{3}} [|D_0|^{-\frac{1}{3}}, a^4] \\ &= -|D_0|^{\frac{1-q}{3}} [|D_0|^{\frac{1}{3}}, a^4] |D_0|^{-\frac{1}{3}} \\ &= -[|D_0|^{\frac{1}{3}}, |D_0|^{\frac{1-q}{3}} a^4 |D_0|^{-\frac{1}{3}}]. \end{aligned}$$

By Lemma 5.7.1, we have

$$\|B^{q-2}[B, A]\|_1 \leq c_{abs} \|[|D_0|, |D_0|^{\frac{1-q}{3}} a^4 |D_0|^{-\frac{1}{3}}]\|_1.$$

Still working on H_∞, we also have:

$$\begin{aligned} [|D_0|, |D_0|^{\frac{1-q}{3}} a^4 |D_0|^{-\frac{1}{3}}] &= |D_0|^{\frac{1-q}{3}} \delta_0(a^4) |D_0|^{-\frac{1}{3}} \\ &= |D_0|^{\frac{1-q}{3}} \delta_0(a^2) a^2 |D_0|^{-\frac{1}{3}} + |D_0|^{\frac{1-q}{3}} a^2 \delta_0(a^2) |D_0|^{-\frac{1}{3}}. \end{aligned}$$

Applying Lemma 3.1.3, we have

$$[|D_0|, |D_0|^{\frac{1-q}{3}} a^4 |D_0|^{-\frac{1}{3}}] \in \mathcal{L}_{\frac{3}{q-1},\infty} \cdot \mathcal{L}_{3,\infty} \subset \mathcal{L}_{\frac{3}{q},\infty}$$

by the Holder inequality, since $q > 3$, $\mathcal{L}_{3/q,\infty} \subset \mathcal{L}_1$, and so $B^{q-2}[B, A] \in \mathcal{L}_1$.

Now we establish Condition 5.4.1.(iii). We may compute:

$$A^{\frac{1}{2}} B A^{\frac{1}{2}} = a^2 |D_0|^{-\frac{1}{3}} a^2.$$

Thus,

$$\|A^{\frac{1}{2}} B A^{\frac{1}{2}}\|_{3,\infty} \leq \|a\|_\infty^3 \|a|D_0|^{-\frac{1}{3}}\|_{3,\infty}.$$

The right hand side is finite by Lemma 3.1.3.

Finally we verify Condition 5.4.1.(iv). We have:

$$\begin{aligned}[B,A^{\frac{1}{2}}] &= [|D_0|^{-\frac{1}{3}},a^2]\\ &= -|D_0|^{-\frac{1}{3}}[|D_0|^{\frac{1}{3}},a^2]|D_0|^{-\frac{1}{3}}\\ &= -[|D_0|^{\frac{1}{3}},|D_0|^{-\frac{1}{3}}a^2|D_0|^{-\frac{1}{3}}].\end{aligned}$$

By Lemma 5.7.1, we have

$$\|[B,A^{\frac{1}{2}}]\|_{\frac{3}{2},\infty} \le c_{\mathrm{abs}}\|[|D_0|,|D_0|^{-\frac{1}{3}}a^2|D_0|^{-\frac{1}{3}}]\|_{\frac{3}{2},\infty}.$$

Moreover, by the Leibniz rule

$$\begin{aligned}[|D_0|,|D_0|^{-\frac{1}{3}}a^2|D|^{-\frac{1}{3}}] &= |D_0|^{-\frac{1}{3}}\delta_0(a^2)|D_0|^{-\frac{1}{3}}\\ &= |D_0|^{-\frac{1}{3}}\delta_0(a)a|D_0|^{-\frac{1}{3}} + |D_0|^{-\frac{1}{3}}a\cdot\delta_0(a)|D_0|^{-\frac{1}{3}}.\end{aligned}$$

By Lemma 3.1.3 and the Hölder inequality, we have

$$[|D_0|,|D_0|^{-\frac{1}{3}}a^2|D_0|^{-\frac{1}{3}}] \in \mathcal{L}_{3,\infty}\cdot\mathcal{L}_{3,\infty} \subset \mathcal{L}_{\frac{3}{2},\infty}.$$

□

For the case $p=2$, we instead use:

LEMMA 5.7.3. *Let $(\mathcal{A},H,D)$ be a $2-$dimensional spectral triple satisfying Hypothesis 1.2.1. If $0\le a\in\mathcal{A}$, then the operators $A=a^4$ and $B=(1+D^2)^{-\frac{1}{6}}$ satisfy Condition 5.4.1 with $p=6$.*

PROOF. This proof is similar to those of Lemmas 5.6.1 and 5.7.2.

Let $D_0 = F(1+D^2)^{\frac{1}{2}}$. By Lemma 3.1.4, the $2-$dimensional spectral triple $(\mathcal{A},H,D_0)$ satisfies Hypothesis 1.2.1. By rescaling D if necessary, we may assume without loss of generality that the spectrum of D_0 does not intersect the interval $(-1,1)$.

Let us establish Condition 5.4.1.(i). We have

$$B^pA = |D_0|^{-2}a^4 \in \mathcal{L}_{1,\infty}$$

by Hypothesis 1.2.1.

Next we establish Condition 5.4.1.(ii). Let $q\in(6,7)$. We have

$$\begin{aligned}B^{q-2}[B,A] &= |D_0|^{\frac{2-q}{3}}[|D_0|^{-\frac{1}{3}},a^4]\\ &= -|D_0|^{\frac{1-q}{3}}[|D_0|^{\frac{1}{3}},a^4]|D_0|^{-\frac{1}{3}}\\ &= -[|D_0|^{\frac{1}{3}},|D_0|^{\frac{1-q}{3}}a^4|D_0|^{-\frac{1}{3}}].\end{aligned}$$

By Lemma 5.7.1, we have

$$\|B^{q-2}[B,A]\|_1 \le c_{abs}\|[|D_0|,|D_0|^{\frac{1-q}{3}}a^4|D_0|^{-\frac{1}{3}}]\|_1.$$

However,

$$\begin{aligned}[|D_0|,|D_0|^{\frac{1-q}{3}}a^4|D_0|^{-\frac{1}{3}}] &= |D_0|^{\frac{1-q}{3}}\delta_0(a^4)|D_0|^{-\frac{1}{3}}\\ &= |D_0|^{\frac{1-q}{3}}\delta_0(a^2)a^2|D_0|^{-\frac{1}{3}} + |D_0|^{\frac{1-q}{3}}a^2\delta_0(a^2)|D_0|^{-\frac{1}{3}}.\end{aligned}$$

By Lemma 3.1.3, we have by the Hölder inequality:

$$[|D_0|,|D_0|^{\frac{1-q}{3}}a^4|D|^{-\frac{1}{3}}] \in \mathcal{L}_{\frac{6}{q-1},\infty}\cdot\mathcal{L}_{6,\infty} \subset \mathcal{L}_{\frac{6}{q},\infty}$$

Since $q>6$, we have that $\mathcal{L}_{6/q,\infty}\subset\mathcal{L}_1$, and so $B^{q-2}[B,A]\in\mathcal{L}_1$.

Now we prove Condition 5.4.1.(iii). We have

$$A^{\frac{1}{2}} B A^{\frac{1}{2}} = a^2 |D_0|^{-\frac{1}{3}} a^2.$$

Thus,

$$\|A^{\frac{1}{2}} B A^{\frac{1}{2}}\|_{6,\infty} \leq \|a\|_\infty^3 \|a|D_0|^{-\frac{1}{3}}\|_{6,\infty}.$$

The above right hand side is finite by Lemma 3.1.3.

Finally, let us establish Condition 5.4.1.(iv). We may compute on H_∞:

$$\begin{aligned}[B, A^{\frac{1}{2}}] &= [|D_0|^{-\frac{1}{3}}, a^2] \\ &= -|D_0|^{-\frac{1}{3}} [|D_0|^{\frac{1}{3}}, a^2] |D_0|^{-\frac{1}{3}} \\ &= -[|D_0|^{\frac{1}{3}}, |D_0|^{-\frac{1}{3}} a^2 |D_0|^{-\frac{1}{3}}].\end{aligned}$$

Therefore using Lemma 5.7.1, we have

$$\|[B, A^{\frac{1}{2}}]\|_{3,\infty} \leq c_{abs} \|[|D_0|, |D_0|^{-\frac{1}{3}} a^2 |D_0|^{-\frac{1}{3}}]\|_{3,\infty}.$$

Applying the Leibniz rule,

$$\begin{aligned}[|D_0|, |D_0|^{-\frac{1}{3}} a^2 |D_0|^{-\frac{1}{3}}] &= |D_0|^{-\frac{1}{3}} \delta_0(a^2) |D_0|^{-\frac{1}{3}} \\ &= |D_0|^{-\frac{1}{3}} \delta_0(a) a |D_0|^{-\frac{1}{3}} + |D_0|^{-\frac{1}{3}} a \delta_0(a) |D_0|^{-\frac{1}{3}}.\end{aligned}$$

By Lemma 3.1.3, we then have from the Hölder inequality:

$$[D_0, |D_0|^{-\frac{1}{3}} a^2 |D_0|^{-\frac{1}{3}}] \in \mathcal{L}_{6,\infty} \cdot \mathcal{L}_{6,\infty} \subset \mathcal{L}_{3,\infty}.$$

□

We may now at last complete the proof of Theorem 1.2.5.

THEOREM. *Assume $p = 1$ or $p = 2$ and let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a local Hochschild cycle, then for every normalised trace φ on $\mathcal{L}_{1,\infty}$ we have:*

$$\varphi(\Omega(c)(1 + D^2)^{-\frac{p}{2}}) = \mathrm{Ch}(c).$$

PROOF. By Theorem 1.2.3, the function

$$\zeta_{c,D}(z) = \mathrm{Tr}(\Omega(c)(1 + D^2)^{-z/2}), \quad \Re(z) > p$$

admits an analytic continuation to the set $\{z \,:\, \Re(z) > p - 1\} \setminus \{p\}$, and the point p is a simple pole with corresponding residue $p\mathrm{Ch}(c)$.

Let $c = \sum_{j=1}^m a_0^j \otimes \cdots \otimes a_p^j$. Since c is local, we may choose $0 \leq a \in \mathcal{A}$ such that $aa_0^j = a_0^j$ for all j.

By exactly the same argument as in the $p > 2$ case, we can show that for all $\Re(z) > 0$:

$$a^{4z} \Omega(c) = \Omega(c). \tag{5.25}$$

We let $A = a^4$ and $B = (1 + D^2)^{-\frac{1}{6}}$ as in Lemmas 5.7.2 and 5.7.3.

We have that:

$$\begin{aligned}\mathrm{Tr}(\Omega(c) B^z A^z) &= \mathrm{Tr}(a^{4z} \Omega(c)(1 + D^2)^{-\frac{z}{6}}) \\ &= \mathrm{Tr}(\Omega(c)(1 + D^2)^{-\frac{z}{6}}), \quad \Re(z) > 3p.\end{aligned}$$

We recognise the above function as being precisely $z \mapsto \zeta_{c,D}(z/3)$. Hence by Theorem 1.2.3, the function $z \mapsto \mathrm{Tr}(\Omega(c) B^z A^z)$ admits an analytic continuation to the set $\{z : \Re(z) > 3(p-1)\} \setminus \{3p\}$, with a simple pole at $3p$ with corresponding residue $3p\mathrm{Ch}(c)$.

Assume now that $p = 1$. By Lemma 5.7.2, Condition 5.4.1 holds for A and B (with $p = 3$). By Theorem 5.4.2 the function

$$z \to \mathrm{Tr}\Big(\Omega(c)\Big(B^z A^z - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^z\Big)\Big)$$

admits an analytic continuation to the set $\{\Re(z) > 2\}$.

By Lemma 5.7.2, $A^{\frac{1}{2}} B A^{\frac{1}{2}} \in \mathcal{L}_{3,\infty}$. Hence, the function (defined *a priori* for $\Re(z) > 3$)

$$z \to \mathrm{Tr}(\Omega(c)(A^{\frac{1}{2}} B A^{\frac{1}{2}})^z)$$

admits an analytic continuation to the set $\{\Re(z) > 2\} \setminus \{3\}$. with a pole at $z = 3$ and corresponding residue $3\mathrm{Ch}(c)$. Define $V_1 := (A^{\frac{1}{2}} B A^{\frac{1}{2}})^3 \in \mathcal{L}_{1,\infty}$. Then,

$$\mathrm{Tr}(\Omega(c)(A^{1/2} B A^{1/2})^z) = \mathrm{Tr}(\Omega(c) V_1^{z/3}).$$

We now know that function

$$z \mapsto \mathrm{Tr}(\Omega(c) V_1^{z/3})$$

admits an analytic continuation to the set $\{\Re(z) > 2\} \setminus \{3\}$ with a simple pole at 3 and corresponding residue $3\mathrm{Ch}(c)$. So by rescaling the argument, we can equivalently say that the function

$$z \mapsto \mathrm{Tr}(\Omega(c) V_1^z)$$

has analytic continuation to the set $\{z : \Re(z) > 2/3\} \setminus \{1\}$ with a simple pole at 1 with corresponding residue $\mathrm{Ch}(c)$.

Thus by Theorem 1.2.7, for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$ we have

$$\varphi(\Omega(c) V_1) = \mathrm{Ch}(c).$$

Due to Lemma 5.7.2, we have that $V_1 - B^3 A^3 \in \mathcal{L}_1$, and since φ vanishes on $\mathcal{L}_1$ it follows that

$$\varphi(\Omega(c) B^3 A^3) = \mathrm{Ch}(c).$$

So

$$\varphi(a^{12}\Omega(c)(1 + D^2)^{-1/2}) = \mathrm{Ch}(c).$$

By taking $z = 3$ in (5.25), we have that $a^{12}\Omega(c) = \Omega(c)$, this completes the proof in the case $p = 1$.

Now assume that $p = 2$. By Lemma 5.7.3, Condition 5.4.1 holds for A and B (with $p = 6$). By Theorem 5.4.2 the function

$$z \to \mathrm{Tr}\Big(\Omega(c)\Big(B^z A^z - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^z\Big)\Big)$$

admits an analytic extension to the set $\{\Re(z) > 5\}$.

By Lemma 5.7.3, $A^{\frac{1}{2}} B A^{\frac{1}{2}} \in \mathcal{L}_{6,\infty}$. Hence, the function (defined *a priori* for $\Re(z) > 6$)

$$z \to \mathrm{Tr}(\Omega(c)(A^{\frac{1}{2}} B A^{\frac{1}{2}})^z)$$

admits an analytic extension to the set $\{\Re(z) > 5\} \setminus \{6\}$. The point $z = 6$ is a simple pole with corresponding residue $6\mathrm{Ch}(c)$. Consider $V_2 = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^6 \in \mathcal{L}_{1,\infty}$. We have so far shown that the function

$$z \to \mathrm{Tr}(\Omega(c) V_2^z)$$

admits an analytic extension to the set $\{z : \Re(z) > \frac{5}{6}\} \setminus \{1\}$. The point $z = 1$ is a simple pole with corresponding residue $\mathrm{Ch}(c)$.

Hence, by Theorem 1.2.7, for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, we have

$$\varphi(\Omega(c)V_2) = \mathrm{Ch}(c).$$

By Lemma 5.7.3, the operator $V_2 - B^6A^6$ is trace class. Thus,

$$\varphi(\Omega(c)B^6A^6) = \mathrm{Ch}(c).$$

So $\varphi(a^{12}\Omega(c)(1 + D^2)^{-1}) = \mathrm{Ch}(c)$. Since $a^{12}\Omega(c) = \Omega(c)$, this completes the proof for the case $p = 2$. $\square$

APPENDIX A

Appendix

A.1. Properties of the algebra $\mathcal{B}$

For this section, $(\mathcal{A}, H, D)$ is a smooth spectral triple. Recall from Definition 2.2.7 that $\mathcal{B}$ is the $*-$algebra generated by all elements of the form $\delta^k(a)$ or $\partial(\delta^k(a))$, $k \geq 0$, $a \in \mathcal{A}$. Recall that we define $H_\infty := \bigcap_{k\geq 1} \mathrm{dom}(D^k)$, and that for all $T \in \mathcal{B}$, we have $T : H_\infty \to H_\infty$, and for all $k \geq 0$ we have $D^k, |D|^k : H_\infty \to H_\infty$.

The following should be compared with [**13**, Lemma 6.2]. See also the discussion following [**33**, Lemma 10.22].

Lemma A.1.1. *For every $x \in \mathcal{B}$ and for every $m \geq 0$, we have the following equalities of linear (potentially unbounded) operators on H_∞:*

$$|D|^m x = \sum_{k=0}^{m} \binom{m}{k} \delta^{m-k}(x)|D|^k \text{ and,}$$

$$x|D|^m = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} |D|^k \delta^{m-k}(x).$$

Proof. We prove only the first equality as the proof of the second one follows by an identical argument.

This formula can be seen by induction on m. Indeed, for $m = 1$, this is simply the claim that since $\mathcal{B} \subseteq \mathrm{dom}_\infty(\delta)$ we have an equality of operators on H_∞:

$$|D|x = \delta(x) + x|D|.$$

Now suppose that the claim is true for $m - 1$. Then on H_∞ we have

$$\begin{aligned}
|D|^m x &= |D| \cdot |D|^{m-1} x \\
&= |D| \cdot \sum_{k=0}^{m-1} \binom{m-1}{k} \delta^{m-1-k}(x)|D|^k \\
&= \sum_{k=0}^{m-1} \binom{m-1}{k} |D|\delta^{m-1-k}(x) \cdot |D|^k \\
&= \sum_{k=0}^{m-1} \binom{m-1}{k} \delta^{m-1-k}(x)|D|^{k+1} + \sum_{k=0}^{m-1} \binom{m-1}{k} \delta^{m-k}(x)|D|^k \\
&= \sum_{k=1}^{m} \binom{m-1}{k-1} \delta^{m-k}(x)|D|^k + \sum_{k=0}^{m-1} \binom{m-1}{k} \delta^{m-k}(x)|D|^k \\
&= \sum_{k=0}^{m} \binom{m}{k} \delta^{m-k}(x)|D|^k.
\end{aligned}$$

and so the statement follows for m. $\square$

LEMMA A.1.2. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple, and assume that D has a spectral gap at 0. Then for all $x \in \mathcal{B}$ and $m \geq 0$ we have*

(i) *The operators $|D|^{-m}x|D|^{m}, |D|^{m}x|D|^{-m} : H_\infty \to H_\infty$ have bounded extension*

(ii) *$|D|^{1-m}[|D|^m, x] : H_\infty \to H_\infty$ has bounded extension.*

PROOF. By Lemma A.1.1, on H_∞ we have:

$$|D|^m x|D|^{-m} = \sum_{k=0}^{m} \binom{m}{k} \delta^{m-k}(x)|D|^{k-m},$$

$$|D|^{-m} x|D|^{m} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} |D|^{k-m}\delta^{m-k}(x).$$

Clearly, the expressions on the right hand side have bounded extension. This proves the first assertion.

By Lemma A.1.1, we have

$$\begin{aligned}[|D|^m, x] &= |D|^m x - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} |D|^{k}\delta^{m-k}(x)\\ &= \sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m}{k} |D|^{k}\delta^{m-k}(x).\end{aligned}$$

Therefore,

$$|D|^{1-m}[|D|^m, x] = \sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m}{k} |D|^{k+1-m}\delta^{m-k}(x).$$

Since $x \in \mathrm{dom}_\infty(\delta)$, for each $0 \leq k \leq m-1$ the operator $|D|^{k+1-m}\delta^{m-k}(x)$ has bounded extension. Hence, $|D|^{1-m}[|D|^m, x]$ has bounded extension. $\square$

LEMMA A.1.3. *Assume that $(\mathcal{A}, H, D)$ satisfies Hypothesis 1.2.1. Let h be a Borel function on $\mathbb{R}$ such that*

$$t \mapsto (1+t^2)^{\frac{p+1}{2}} h(t), \quad t \in \mathbb{R}.$$

is bounded. Then for all $x \in \mathcal{B}$ and $s > 0$ the operator $xh(sD)$ is in $\mathcal{L}_1$, and:

$$\|xh(sD)\|_1 = O(s^{-p}), \quad s \downarrow 0.$$

PROOF. Let $s > 0$. Clearly,

$$(1+s^2D^2)^{-\frac{p+1}{2}} = |(1-isD)^{-p-1}|.$$

Setting $\lambda = \frac{1}{s}$, we obtain from Hypothesis 1.2.1 that:

$$\begin{aligned}\|x(1+s^2D^2)^{-\frac{p+1}{2}}\|_1 &= s^{-p-1}\|x(D+\frac{i}{s})^{-p-1}\|_1\\ &= s^{-p-1} \cdot O(s)\\ &= O(s^{-p}), \quad s \downarrow 0.\end{aligned}$$

Since the operator $(1+s^2D^2)^{\frac{p+1}{2}} h(sD)$ is bounded, with

$$\|(1+s^2D^2)^{\frac{p+1}{2}} h(sD)\|_\infty \leq \sup_{t\in\mathbb{R}} (1+t^2)^{\frac{p+1}{2}} |h(t)|$$

We can conclude that:

$$\begin{aligned}\|xh(sD)\|_1 &\le \|x(1+s^2D^2)^{-\frac{p+1}{2}}\|_1\|(1+s^2D^2)^{\frac{p+1}{2}}h(sD)\|_\infty\\ &= O(s^{-p}), \quad s\downarrow 0.\end{aligned}$$

□

We note in particular that the assumption on h in Lemma A.1.3 is satisfied if h is a Schwartz function.

LEMMA A.1.4. *Let $(\mathcal{A},H,D)$ satisfy Hypothesis 1.2.1, assume that D has a spectral gap at 0 and let $x\in\mathcal{B}$. For all non-negative integers $m_1,m_2\ge 0$ with $m_1+m_2<p$, we have*

$$\||D|^{-m_1}x|D|^{-m_2}e^{-s^2D^2}\|_1 = O(s^{m_1+m_2-p}), \quad s\downarrow 0.$$

PROOF. Suppose first that $m_1>0$. Using the triangle inequality:

$$\begin{aligned}\||D|^{-m_1}x|D|^{-m_2}e^{-s^2D^2}\|_1 &\le \sum_{k,l=0}^\infty \|\chi_{[2^k,2^{k+1}]}(|D|)|D|^{-m_1}x|D|^{-m_2}e^{-s^2D^2}\chi_{[2^l,2^{l+1}]}(|D|)\|_1\\ &\le \sum_{k,l=0}^\infty 2^{-km_1-lm_2}\cdot e^{-2^{2l}s^2}\cdot\|\chi_{[2^k,2^{k+1}]}(|D|)x\chi_{[2^l,2^{l+1}]}(|D|)\|_1\\ &\le \sum_{k,l=0}^\infty 2^{-km_1-lm_2}\cdot e^{-2^{2l}s^2}\cdot\|\chi_{[0,2^{k+1}]}(|D|)x\chi_{[0,2^{l+1}]}(|D|)\|_1.\end{aligned}$$

If $m=\min\{k,l\}$, then

$$\begin{aligned}\left\|\chi_{[0,2^{k+1}]}(|D|)x\chi_{[0,2^{l+1}]}(|D|)\right\|_1 &\le \min\left\{\left\|x\chi_{[0,2^{m+1}]}(|D|)\right\|_1, \left\|\chi_{[0,2^{m+1}]}(|D|)x\right\|_1\right\}\\ &\le e\min\left\{\left\|xe^{-2^{-2(m+1)}|D|^2}\right\|_1, \left\|x^*e^{-2^{-2(m+1)}|D|^2}\right\|_1\right\}\\ &= O(2^{mp}).\end{aligned}$$

Thus,

$$\||D|^{-m_1}x|D|^{-m_2}e^{-s^2D^2}\|_1 \le \Big(\sum_{k,l=0}^\infty 2^{-km_1-lm_2}\cdot e^{-2^{2l}s^2}\cdot 2^{p\cdot\min\{k,l\}}\Big)\cdot O(1).$$

Since $p-m_1>0$, it follows that

$$\begin{aligned}\sum_{\substack{k,l\ge 0\\ k\le l}} 2^{-km_1-lm_2}\cdot e^{-2^{2l}s^2}\cdot 2^{p\cdot\min\{k,l\}} &= \sum_{\substack{k,l\ge 0\\ k\le l}} 2^{(p-m_1)k}\cdot 2^{-lm_2}\cdot e^{-2^{2l}s^2}\\ &\le 2\cdot\sum_{l=0}^\infty 2^{(p-m_1)l}\cdot 2^{-lm_2}\cdot e^{-2^{2l}s^2}.\end{aligned}$$

Now due to our assumption that $m_1>0$, it follows that

$$\begin{aligned}\sum_{\substack{k,l\ge 0\\ k\ge l}} 2^{-km_1-lm_2}\cdot e^{-2^{2l}s^2}\cdot 2^{p\cdot\min\{k,l\}} &= \sum_{\substack{k,l\ge 0\\ k\ge l}} 2^{-km_1}\cdot 2^{(p-m_2)l}\cdot e^{-2^{2l}s^2}\\ &\le 2\cdot\sum_{l=0}^\infty 2^{-lm_1}\cdot 2^{(p-m_2)l}\cdot e^{-2^{2l}s^2}.\end{aligned}$$

Note that for $m > 0$, we have

$$\sum_{l=0}^{\infty} 2^{lm} e^{-2^{2l} s^2} = O(s^{-m}), \quad s \downarrow 0.$$

Therefore,

$$\begin{aligned}\||D|^{-m_1} x |D|^{-m_2} e^{-s^2 D^2}\|_1 &\le \Big(\sum_{l=0}^{\infty} 2^{l(p-m_1-m_2)} \cdot e^{-2^{2l} s^2}\Big) \cdot O(1) \\ &= O(s^{m_1+m_2-p}).\end{aligned}$$

This completes the proof for the case $m_1 > 0$.

To complete the proof we now deal with the case where $m_1 = 0$. We have

$$\begin{aligned}\||D|^{-m_1} x |D|^{-m_2} e^{-s^2 D^2}\|_1 &\le \sum_{l=0}^{\infty} \Big\| x |D|^{-m_2} e^{-s^2 D^2} \chi_{[2^l, 2^{l+1}]}(|D|)\Big\|_1 \\ &\le \sum_{l=0}^{\infty} 2^{-lm_2} \cdot e^{-2^{2l} s^2} \cdot \Big\| x \chi_{[2^l, 2^{l+1}]}(|D|)\Big\|_1 \\ &\le \sum_{l=0}^{\infty} 2^{-lm_2} \cdot e^{-2^{2l} s^2} \cdot \Big\| x \chi_{[0, 2^{l+1}]}(|D|)\Big\|_1.\end{aligned}$$

Thus,

$$\begin{aligned}\||D|^{-m_1} x |D|^{m_2} e^{-s^2 D^2}\|_1 &\le \Big(\sum_{l=0}^{\infty} 2^{l(p-m_2)} \cdot e^{-2^{2l} s^2}\Big) \cdot O(1) \\ &= O(s^{m_2-p}).\end{aligned}$$

This completes the proof for the case $m_1 = 0$. □

Lemma A.1.5. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and assume that D has spectral gap at 0. For all $x \in \mathcal{B}$ we have*

$$\|x|D|^{-p-1}(1 - e^{-s^2 D^2})\|_1 = O(s), \quad s \downarrow 0.$$

Proof. By the triangle inequality, we have

$$\begin{aligned}\|x|D|^{-p-1}(1 - e^{-s^2 D^2})\|_1 &\le \|x|D|^{-p-1}(1 - e^{-s^2 D^2})\chi_{(\frac{1}{s}, \infty)}(|D|)\|_1 \\ &\quad + \|x|D|^{-p-1}(1 - e^{-s^2 D^2})\chi_{[0, \frac{1}{s}]}(|D|)\|_1. \end{aligned} \tag{A.1}$$

Let us estimate the first summand of (A.1). Since for $t > 1$ we have:

$$t^{-p-1}(1 - e^{-t^2}) \le t^{-p-1} \le 2^{\frac{p+1}{2}} \cdot (t^2 + 1)^{-\frac{p+1}{2}},$$

it follows that

$$|D|^{-p-1}(1 - e^{-s^2 D^2})\chi_{(\frac{1}{s}, \infty)}(|D|) \le s^{p+1} \cdot 2^{\frac{p+1}{2}} \cdot (1 + s^2 D^2)^{-\frac{p+1}{2}}.$$

So we may estimate the first summand by:

$$\begin{aligned}\|x|D|^{p-1}(1 - e^{-s^2 D^2})\chi_{(\frac{1}{s}, \infty)}(|D|)\|_1 &\le 2^{\frac{p+1}{2}} \Big\| x(D^2 + s^{-2})^{-\frac{p+1}{2}} \Big\|_1 \\ &= 2^{\frac{p+1}{2}} \Big\| x(D + \frac{i}{s})^{-p-1} \Big\|_1 \\ &= O(s)\end{aligned}$$

Where the final step follows from Hypothesis 1.2.1.(iii).

Let us estimate the second summand of (A.1). We have

$$t^{-p-1}(1-e^{-t^2}) \leq t^{1-p} \leq et^{1-p} \cdot e^{-t^2}, \quad t \in [0,1],$$

so it follows that

$$|D|^{-p-1}(1-e^{-s^2D^2})\chi_{[0,\frac{1}{s}]}(|D|) \leq es^{p+1} \cdot (s|D|)^{1-p} \cdot e^{-s^2D^2}.$$

So, the second summand in (A.1) can be estimated by:

$$\|x|D|^{-p-1}(1-e^{-s^2D^2})\chi_{[0,\frac{1}{s})]}(|D|)\|_1 \leq es^2 \Big\| x|D|^{1-p}e^{-s^2D^2} \Big\|_1.$$

From Lemma A.1.4, this is $O(s)$ as $s \downarrow 0$. □

Lemma A.1.6. *Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying Hypothesis 1.2.1 and assume that D has a spectral gap at 0. For every $x \in \mathcal{B}$, we have*

$$\||D|^{-p-1}x(1-e^{-s^2D^2})\|_1 = O(s), \quad s \downarrow 0.$$

Proof. On the subspace H_∞, we have:

$$\begin{aligned} |D|^{-p-1}x &= x|D|^{-p-1} + [|D|^{-p-1}, x] \\ &= x|D|^{-p-1} - |D|^{-p-1}[|D|^{p+1}, x]|D|^{-p-1}. \end{aligned}$$

By Lemma A.1.1, we have (again on H_∞):

$$\begin{aligned} [|D|^{p+1}, x] &= |D|^{p+1}x - x|D|^{p+1} \\ &= \left(\sum_{k=0}^{p+1} \binom{p+1}{k} \delta^{p+1-k}(x)|D|^k\right) - x|D|^{p+1} \\ &= \sum_{k=0}^{p} \binom{p+1}{k} \delta^{p+1-k}(x)|D|^k. \end{aligned}$$

Thus on H_∞:

$$|D|^{-p-1}x = x|D|^{-p-1} - \sum_{k=0}^{p} \binom{p+1}{k} |D|^{-p-1}\delta^{p+1-k}(x)|D|^{k-p-1}.$$

Multiplying on the right by $(1-e^{-s^2D^2})$, on H_∞ we have:

(A.2)

$$|D|^{-p-1}x(1-e^{-s^2D^2}) = x|D|^{-p-1}(1-e^{-s^2D^2}) - \sum_{k=0}^{p} \binom{p+1}{k} |D|^{-p-1}\delta^{p+1-k}(x)|D|^{k-p-1}(1-e^{-s^2D^2})$$

From Lemma A.1.5, we have that $x|D|^{-p-1}(1-e^{-s^2D^2})$ has bounded extension to an operator in $\mathcal{L}_1$ with norm bounded by $O(s)$.

For $0 \leq k \leq p$, we have

$$\begin{aligned} &\Big\| |D|^{-p-1}\delta^{p+1-k}(x)|D|^{k-p-1}(1-e^{-s^2D^2}) \Big\|_1 \\ &\qquad \leq \Big\| |D|^{-p-1}\delta^{p+1-k}(x) \Big\|_1 \cdot \Big\| |D|^{k-p} \Big\|_\infty \\ &\qquad\quad \cdot s \Big\| (s|D|)^{-1}(1-e^{-s^2D^2}) \Big\|_\infty. \end{aligned}$$

The first factor here is finite by Remark 3.1.1. The second factor is finite because $k \leq p$. The third factor is $O(s)$ by the functional calculus. Thus,

$$\||D|^{-p-1}\delta^{p+1-k}(x)|D|^{k-p-1}(1-e^{-s^2D^2})\|_1 = O(s), \quad s \downarrow 0.$$

Hence, each summand in (A.2) extends to a trace class operator with norm bounded by $O(s)$, $s \downarrow 0$. By the triangle inequality, this completes the proof. □

A.2. Integral formulae for commutators

In this section of the appendix, we collect results concerning formulae for commutators with functions of D. Many of the results of this section will be known to the expert reader, but since they are scattered around various sources we provide them here with short and self-contained proofs.

In this section, $(\mathcal{A}, H, D)$ is a smooth p-dimensional spectral triple.

The following is essentially a consequence of the classical Duhamel formula.

LEMMA A.2.1. *Let $(\mathcal{A}, H, D)$ be a smooth spectral triple. If $x \in \mathcal{B}$ then for all $t \in \mathbb{R}$:*

$$[e^{it|D|}, x] = it\int_0^1 e^{it(1-v)|D|}\delta(x)e^{itv|D|}\,dv.$$

Here, the integral is understood in the weak operator topology sense.

PROOF. For $n \geq 0$, we define the projection

$$p_n := \chi_{[0,n]}(|D|).$$

Since $|D|$ is non-negative, as $n \to \infty$ the sequence of projections $\{p_n\}_{n\geq 0}$ converges in the strong operator topology to the identity. We define the functions ξ and η by:

$$\xi(v) := p_n \exp(it(1-v)|D|),$$
$$\eta(v) := x\exp(itv|D|)p_n, \quad v \in \mathbb{R}.$$

Since $p_n|D| \leq n$, the operator valued functions ξ and η are continuous and differentiable in the uniform norm.

Since ξ and η are continuous and differentiable, we have:

$$\xi(1)\eta(1) - \xi(0)\eta(0) = \int_0^1 \xi'(v)\eta(v) + \xi(v)\eta'(v)\,dv.$$

where since ξ, ξ', η and η' are continuous, this may be considered as a Bochner integral. Therefore in particular, this is an integral in the weak operator topology.

We can compute the terms in the integrand as:

$$\xi'(v)\eta(v) = -itp_n\exp(it(1-v)|D|)|D|x\exp(itv|D|)p_n,$$
$$\xi(v)\eta'(v) = itp_n\exp(it(1-v)|D|)x|D|\exp(itv|D|)p_n.$$

Thus,

$$\xi(1)\eta(1) - \xi(0)\eta(0) = -it\int_0^1 p_n\exp(it(1-v)|D|)\delta(x)\exp(itv|D|)p_n\,dv.$$

Since the operator $\exp(it(1-v)|D|)\delta(x)\exp(itv|D|)$ is a continuous function of v (in weak operator topology), the weak operator topology integral

$$\int_0^1 \exp(it(1-v)|D|)\delta(x)\exp(itv|D|)\,dv$$

exists, and we have:

$$p_n\int_0^1 \exp(it(1-v)|D|)\delta(x)\exp(itv|D|)\,dv\cdot p_n = \int_0^1 p_n\exp(it(1-v)|D|)\delta(x)\exp(itv|D|)p_n\,dv.$$

Therefore:

$$\xi(1)\eta(1) - \xi(0)\eta(0) = p_n \int_0^1 \exp(it(1-v)|D|)\delta(x)\exp(itv|D|)\,dv \cdot p_n.$$

On the other hand, we can compute $\xi(1), \eta(1), \xi(0)$ and $\eta(0)$ directly:

$$\begin{aligned}\xi(1)\eta(1) - \xi(0)\eta(0) &= p_n x \exp(it|D|)p_n - p_n \exp(it|D|)x p_n \\ &= p_n[x, \exp(it|D|)]p_n.\end{aligned}$$

Thus,

$$p_n[\exp(it|D|), x]p_n = itp_n \int_0^1 \exp(it(1-v)|D|)\delta(x)\exp(itv|D|)\,dv \cdot p_n.$$

Since $n \geq 0$ is arbitrary, we may take $n \to \infty$ and since p_n converges in the strong operator topology to the identity we get the desired equality. □

Combining Lemma A.2.1 with the Fourier inversion theorem yields a formula for $[f(s|D|), x]$ for quite general functions f. The following formula is well known and appears in many places, for example [**8**, Theorem 3.2.32].

Lemma A.2.2. *If $\widehat{f}, \widehat{f'} \in L_1(\mathbb{R})$, then for all $x \in \mathcal{B}$, and $s > 0$,*

$$[f(s|D|), x] = s\int_{-\infty}^{\infty} \left(\int_0^1 \widehat{f'}(u)e^{ius(1-v)|D|}\delta(x)e^{iusv|D|}dv \right) du.$$

Here, the integral is understood in a weak operator topology sense.

Proof. Indeed, by the Fourier inversion formula, by functional calculus we have a weak operator topology integral:

$$f(s|D|) = \int_{\mathbb{R}} \widehat{f}(u)e^{ius|D|}du.$$

Therefore,

$$[f(s|D|), x] = \int_{-\infty}^{\infty} \widehat{f}(u)[e^{ius|D|}, x]du.$$

By Lemma A.2.1, we have a weak operator topology integral:

$$[e^{ius|D|}, x] = ius \int_0^1 e^{ius(1-v)|D|}\delta(x)e^{iusv|D|}dv.$$

Thus,

$$[f(s|D|), x] = s\int_{-\infty}^{\infty} iu\widehat{f}(u)\Big(\int_0^1 e^{ius(1-v)|D|}\delta(x)e^{iusv|D|}dv \Big) du.$$

Since $iu\widehat{f}(u) = \widehat{f'}(u)$, the assertion follows. □

Lemma A.2.3. *If $\widehat{f}, \widehat{f'}, \widehat{f''} \in L_1(\mathbb{R})$, then for all $x \in \mathcal{B}$ and $s > 0$ we have:*

$$[f(s|D|), x] - sf'(s|D|)\delta(x) = -s^2 \int_{-\infty}^{\infty} \left(\int_0^1 \widehat{f''}(u)(1-v)e^{ius(1-v)|D|}\delta^2(x)e^{iusv|D|}dv \right) du.$$

$$[f(s|D|), x] - s\delta(x)f'(s|D|) = -s^2 \int_{-\infty}^{\infty} \left(\int_0^1 \widehat{f''}(u)(1-v)e^{iusv|D|}\delta^2(x)e^{ius(1-v)|D|}dv \right) du.$$

Here once again the integrals are understood in the weak operator topology.

Proof. We only prove the first equality as the proof of the second one is similar.

By the Fourier inversion theorem and functional calculus we have a weak operator topology integral representation:

$$f'(s|D|) = \int_{\mathbb{R}} \widehat{f'}(u) e^{ius|D|} du.$$

So multiplying on the left by the bounded operator $\delta(x)$,

$$\begin{aligned} sf'(s|D|)\delta(x) &= s \int_{\mathbb{R}} \widehat{f'}(u) e^{ius|D|} \delta(x) du \\ &= s \int_{-\infty}^{\infty} \left(\int_0^1 \widehat{f'}(u) e^{ius|D|} \delta(x) dv \right) du. \end{aligned}$$

Now representing $[f(s|D|), x]$ by the integral representation given by Lemma A.2.2, we infer that

$$\begin{aligned} [f(s|D|), x] - sf'(s|D|)\delta(x) &= s \int_{-\infty}^{\infty} \Big(\int_0^1 \widehat{f'}(u) \Big(e^{ius(1-v)|D|} \delta(x) e^{iusv|D|} - e^{ius|D|} \delta(x) \Big) dv \Big) du \\ &= s \int_{-\infty}^{\infty} \Big(\int_0^1 \widehat{f'}(u) \Big(e^{ius(1-v)|D|} [\delta(x), e^{iusv|D|}] \Big) dv \Big) du. \qquad \text{(A.3)} \end{aligned}$$

Applying Lemma A.2.1 to $\delta(x) \in \mathcal{B}$, we have:

$$[\delta(x), e^{iusv|D|}] = -iusv \int_0^1 e^{iusv(1-w)|D|} \delta^2(x) e^{iusvw|D|} dw. \qquad \text{(A.4)}$$

Combining (A.3) and (A.4), we obtain

(A.5)

$$[f(s|D|), x] - sf'(s|D|)\delta(x) = -s^2 \int_{-\infty}^{\infty} \Big(\int_0^1 \Big(\int_0^1 \widehat{f''}(u) v e^{ius(1-vw)|D|} \delta^2(x) e^{iusvw|D|} dw \Big) dv \Big) du.$$

We focus on the inner integral. Performing a linear change of variables, $w_0 = vw$, we get:

$$\int_0^1 v e^{ius(1-vw)|D|} \delta^2(x) e^{iusvw|D|} \, dw = \int_0^v e^{ius(1-w_0)|D|} \delta^2(x) e^{iusw_0|D|} \, dw_0,$$

and therefore:

$$\int_0^1 \int_0^1 v e^{ius(1-vw)|D|} \delta^2(x) e^{iusvw|D|} \, dw dv = \int_0^1 \int_0^v e^{ius(1-w)|D|} \delta^2(x) e^{iusw|D|} \, dw dv.$$

Since the integrand is continuous, we may apply Fubini's theorem to interchange the integrals:

$$\begin{aligned} \int_0^1 \int_0^1 v e^{ius(1-vw)|D|} \delta^2(x) e^{iusvw|D|} \, dw dv &= \int_0^1 \int_w^1 e^{ius(1-w)|D|} \delta^2(x) e^{iusw|D|} \, dv dw \\ &= \int_0^1 (1-w) e^{ius(1-w)|D|} \delta^2(x) e^{iusw|D|} \, dw. \end{aligned}$$

So from (A.5):

$$[f(s|D|), x] - sf'(s|D|)\delta(x) = -s^2 \int_{-\infty}^{\infty} \int_0^1 (1-w) e^{ius(1-w)|D|} \delta^2(x) e^{iusw|D|} \, dw du.$$

□

A.3. Hochschild coboundary computations

In this part of the appendix we include some of the lengthy algebraic computations required for Sections 4.3 and 4.4. Recall that for a multilinear functional $\theta : \mathcal{A}^{\otimes p} \to \mathbb{C}$ the Hochschild coboundary $b\theta : \mathcal{A}^{\otimes(p+1)} \to \mathbb{C}$ is defined in terms of the Hochschild boundary b by $b\theta(c) = \theta(bc)$.

Explicitly, for an elementary tensor $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$ we have:

$$(b\theta)(a_0 \otimes \cdots a_p) = \theta(a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_p) + \sum_{k=1}^{p-1} (-1)^k \theta(a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_p) + (-1)^p \theta(a_p a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1}).$$

A.3.1. Coboundaries in Section 4.3. Let $\mathscr{A} \subseteq \{1, \dots, p\}$. Let $T := D^{2-|\mathscr{A}|}|D|^{p+1} e^{-s^2 D^2}$. Following the notation of Definition 4.1.1, we define for $a \in \mathcal{A}$,

$$b_k(a) := \begin{cases} \delta(a), & k \in \mathscr{A}, \\ [F, a], & k \notin \mathscr{A}. \end{cases}$$

Fix $1 \leq m \leq p-1$. We introduce a pair of multilinear mappings, θ_s^1 and θ_s^2, defined on $a_0 \otimes \cdots \otimes a_{p-1} \in \mathcal{A}^{\otimes p}$ by:

$$\theta_s^1(a_0 \otimes \cdots \otimes a_{p-1}) := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k) \right) \delta^2(a_{m-1}) \left(\prod_{k=m}^{p-1} b_{k+1}(a_k) \right) T \right).$$

and

$$\theta_s^2(a_0 \otimes \cdots \otimes a_{p-1}) := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k) \right) [F, \delta(a_{m-1})] \left(\prod_{k=m}^{p-1} b_{k+1}(a_k) \right) T \right).$$

The mapping that here is denoted θ_s^1 is exactly the multilinear mapping θ_s introduced in Lemma 4.3.2, and similarly the multilinear mapping θ_s^2 is the multilinear mapping θ_s introduced in Lemma 4.3.3. For $1 \leq k < m$ we also introduce X_k^1 and X_k^2 defined by:

$$X_k^1 := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{k-1} b_l(a_l) \right) a_k \left(\prod_{l=k}^{m-2} b_l(a_{l+1}) \right) \delta^2(a_m) \left(\prod_{l=m+1}^{p} b_l(a_l) \right) \cdot T \right),$$

$$X_k^2 := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{k-1} b_l(a_l) \right) a_k \left(\prod_{l=k}^{m-2} b_l(a_{l+1}) \right) [F, \delta(a_m)] \left(\prod_{l=m+1}^{p} b_l(a_l) \right) \cdot T \right).$$

Now if $m \leq k \leq p$, we define Y_k^1 and Y_k^2 by:

$$Y_k^1 := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l) \right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{k-1} b_{l+1}(a_l) \right) a_k \left(\prod_{l=k+1}^{p} b_l(a_l) \right) \cdot T \right),$$

$$Y_k^2 := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l) \right) [F, \delta(a_{m-1})] \left(\prod_{l=m}^{k-1} b_{l+1}(a_l) \right) a_k \left(\prod_{l=k+1}^{p} b_l(a_l) \right) \cdot T \right).$$

Lemma A.3.1. *For $j = 1, 2$ and $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$. we have:*

$$\theta_s^j(a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_p) = X_1^j$$

Proof. This follows immediately from the definition. □

LEMMA A.3.2. *For $j = 1, 2$, $1 \leq k < m-1$ and $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$ we have*

$$\theta_s^j(a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_p) = X_k^j + X_{k+1}^j.$$

PROOF. We will only describe the $j = 1$ case, since the $j = 2$ case is identical. By the definition of θ_s^j, we have

$$\begin{aligned}
&\theta_s^1(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_p) \\
&\qquad = \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{k-1} b_l(a_l)\right) b_k(a_k a_{k+1}) \left(\prod_{l=k+1}^{m-2} b_l(a_{l+1})\right) \delta^2(a_m) \left(\prod_{l=m}^{p-1} b_{l+1}(a_{l+1})\right) \cdot T\right).
\end{aligned}$$

Now applying the Leibniz rule to $b_k(a_k a_{k+1})$,

$$\begin{aligned}
&\theta_s^1(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_p) \\
&\quad = \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{k-1} b_l(a_l)\right) a_k b_k(a_{k+1}) \left(\prod_{l=k+1}^{m-2} b_l(a_{l+1})\right) \delta^2(a_m) \left(\prod_{l=m}^{p-1} b_{l+1}(a_{l+1})\right) \cdot T\right) \\
&\qquad + \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{k-1} b_l(a_l)\right) b_k(a_k) a_{k+1} \left(\prod_{l=k+1}^{m-2} b_l(a_{l+1})\right) \delta^2(a_m) \left(\prod_{l=m}^{p-1} b_{l+1}(a_{l+1})\right) \cdot T\right) \\
&\quad = \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{k-1} b_l(a_l)\right) a_k \left(\prod_{l=k}^{m-2} b_l(a_{l+1})\right) \delta^2(a_m) \left(\prod_{l=m+1}^{p} b_l(a_l)\right) T\right) \\
&\qquad + \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{k} b_l(a_l)\right) a_{k+1} \left(\prod_{l=k+1}^{m-2} b_l(a_{l+1})\right) \delta^2(a_m) \left(\prod_{l=m+1}^{p} b_l(a_l)\right) \cdot T\right) \\
&\quad = X_k^1 + X_{k+1}^1
\end{aligned}$$

as required. ∎

LEMMA A.3.3. *For $m \leq k < p$, $j = 1, 2$ and $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$ we have*

$$\theta_s^j(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_p) = Y_k^j + Y_{k+1}^j.$$

PROOF. Again we demonstrate only the $j = 1$ case since the $j = 2$ case is identical. By definition we have:

$$\begin{aligned}
&\theta_s^1(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_p) \\
&\quad = \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{k-1} b_{l+1}(a_l)\right) b_{k+1}(a_k a_{k+1}) \left(\prod_{l=m+1}^{p-1} b_{l+1}(a_{l+1})\right) \cdot T\right).
\end{aligned}$$

Applying the Leibniz rule to $b_{k+1}(a_k a_{k+1})$ we have:

$$\begin{aligned}
&\theta_s^1(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_p) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{k-1} b_{l+1}(a_l)\right) a_k b_{k+1}(a_{k+1}) \left(\prod_{l=k+1}^{p-1} b_{l+1}(a_{l+1})\right) \cdot T\right) \\
&\qquad + \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{k-1} b_{l+1}(a_l)\right) b_{k+1}(a_k) a_{k+1} \left(\prod_{l=k+1}^{p-1} b_{l+1}(a_{l+1})\right) \cdot T\right) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{k-1} b_{l+1}(a_l\right) a_k \left(\prod_{l=k+1}^{p} b_l(a_l)\right) \cdot T\right) \\
&\qquad + \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{k} b_{l+1}(a_l)\right) a_{k+1} \left(\prod_{l=k+2}^{p} b_l(a_l)\right) \cdot T\right) \\
&\quad = Y_k^1 + Y_{k+1}^1.
\end{aligned}$$

□

We recall also the multilinear maps $\mathcal{W}_{\mathscr{A}}$ from Definition 4.1.1.

Lemma A.3.4. *Let $c = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$. If we assume that $m-1, m \in \mathscr{A}$, then we have:*

$$\begin{aligned}
&\theta_s^1(a_0 \otimes a_1 \otimes \cdots \otimes a_{m-2} \otimes a_{m-1}a_m \otimes a_{m+1} \otimes \cdots \otimes a_p) \\
&\quad = X_{m-1}^1 + Y_m^1 + 2\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c) \cdot T).
\end{aligned}$$

Now if $\mathscr{B} \subseteq \{1, \ldots, p\}$ is such that $|\mathscr{B}| = |\mathscr{A}|$ and $\mathscr{A} \Delta \mathscr{B} = \{m-1, m\}$ (where Δ denotes the symmetric difference) then

$$\begin{aligned}
&\theta_s^2(a_0 \otimes a_1 \otimes \cdots \otimes a_{m-2} \otimes a_{m-1}a_m \otimes a_{m+1} \otimes \cdots \otimes a_p) \\
&\quad = X_{m-1}^2 + Y_m^2 + \mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c) \cdot T) + \mathrm{Tr}(\mathcal{W}_{\mathscr{B}}(c) \cdot T).
\end{aligned}$$

Proof. Using the repeated Leibniz rule, we obtain

$$\begin{aligned}
\delta^2(a_{m-1}a_m) &= \delta^2(a_{m-1})a_m + 2\delta(a_{m-1})\delta(a_m) + a_{m-1}\delta^2(a_m), \\
[F, \delta(a_{m-1}a_m)] &= [F, \delta(a_{m-1})]a_m + [F, a_{m-1}]\delta(a_m) + \delta(a_{m-1})[F, a_m] + a_{m-1}[F, \delta(a_m)].
\end{aligned}$$

Let us focus on proving the assertion relating to θ_s^1, since the other assertion is identical.

By the definition of θ_s^1, we have

$$\begin{aligned}
&\theta_s^1(a_0 \otimes a_1 \otimes \cdots \otimes a_{m-2} \otimes a_{m-1}a_m \otimes a_{m+1} \otimes \cdots \otimes a_p) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}a_m) \left(\prod_{l=m}^{p-1} b_{l+1}(a_{l+1})\right) \cdot T\right) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}a_m) \left(\prod_{l=m+1}^{p} b_l(a_l)\right) \cdot T\right).
\end{aligned}$$

Applying the Leibniz rule to the $\delta^2(a_{m-1}a_m)$ term:

$$\begin{aligned}
&\theta_s^1(a_0 \otimes a_1 \otimes \cdots \otimes a_{m-2} \otimes a_{m-1}a_m \otimes a_{m+1} \otimes \cdots \otimes a_p) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) a_{m-1}\delta^2(a_m) \left(\prod_{l=m+1}^{p} b_l(a_l)\right) \cdot T\right) \\
&\qquad + \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) 2\delta(a_{m-1})\delta(a_m) \left(\prod_{l=m+1}^{p} b_l(a_l)\right) \cdot T\right) \\
&\qquad + \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1})a_m \left(\prod_{l=m+1}^{p} b_l(a_l)\right) \cdot T\right) \\
&\quad = X_{m-1} + 2\mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c) \cdot T) + Y_m.
\end{aligned}$$

□

Lemma A.3.5. *For $a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$, we have*

$$\begin{aligned}
&\theta_s^1(a_p a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{p-1}) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) [T, a_p]\right) + Y_p^1.
\end{aligned}$$

We also have:

$$\begin{aligned}
&\theta_s^2(a_p a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{p-1}) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) [F, \delta(a_{m-1})] \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) [T, a_p]\right) + Y_p^2.
\end{aligned}$$

Proof. We prove only the assertion involving θ_s^1, since one can prove the other result by an identical argument. Since Γ commutes with a_p, we have:

$$\begin{aligned}
&\theta_s^1(a_p a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{p-1}) \\
&\quad = \mathrm{Tr}\left(\Gamma a_p a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) \cdot T\right) \\
&\quad = \mathrm{Tr}\left(a_p \Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) \cdot T\right).
\end{aligned}$$

Using the cyclicity of the trace, we have

$$\begin{aligned}
&\theta_s^1(a_p a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{p-1}) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) \cdot T a_p\right) \\
&\quad = \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) \cdot [T, a_p]\right) + Y_p^1.
\end{aligned}$$

□

Note that for $j = 1, 2$, the telescopic sum

$$X_1^j + \sum_{k=1}^{m-1}(-1)^k(X_k^j + X_{k+1}^j) + (-1)^{m-1}(X_{m-1}^j + Y_m^j) + \sum_{k=m}^{p-1}(-1)^k(Y_k^j + Y_{k+1}^j) + (-1)^p Y_p^j.$$

vanishes.

Therefore, combining Lemmas A.3.1, A.3.2, A.3.3, A.3.4 and A.3.5 we have:

$$(b\theta_s^1)(a_0 \otimes \cdots \otimes a_p) = 2 \cdot (-1)^{m-1} \cdot \mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c) \cdot T) + (-1)^p \cdot \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) \delta^2(a_{m-1}) \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) \cdot [T, a_p]\right).$$

Similarly, if $\mathscr{B}$ is such that $|\mathscr{A}| = |\mathscr{B}|$ and $\mathscr{A} \Delta \mathscr{B} = \{m-1, m\}$ then:

$$(b\theta_s^2)(a_0 \otimes \cdots \otimes a_p) = (-1)^{m-1} \cdot \mathrm{Tr}(\mathcal{W}_{\mathscr{A}}(c) \cdot T) + (-1)^{m-1} \cdot \mathrm{Tr}(\mathcal{W}_{\mathscr{B}}(c) \cdot T) + (-1)^p \cdot \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{l=1}^{m-2} b_l(a_l)\right) [F, \delta(a_{m-1})] \left(\prod_{l=m}^{p-1} b_{l+1}(a_l)\right) \cdot [T, a_p]\right).$$

This completes the computation of the coboundaries for θ_s^1 and θ_s^2.

A.3.2. Coboundaries in Section 4.4. Again in this subsection, $(\mathcal{A}, H, D)$ is a smooth spectral triple where D has a spectral gap at 0. Let $T = Fe^{-s^2D^2}$. Note that this is different to T in the preceding section.

We define the multilinear mapping $\mathcal{L}_s : \mathcal{A}^{\otimes p} \to \mathbb{C}$ on $a_0 \otimes \cdots \otimes a_{p-1} \in \mathcal{A}^{\otimes p}$ by:

$$\mathcal{L}_s(a_0 \otimes \cdots \otimes a_{p-1}) := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{p-1}[F, a_k]\right) \cdot T\right).$$

We also define the multilinear mapping $\mathcal{K}_s : \mathcal{A}^{\otimes(p+1)} \to \mathbb{C}$ by:

$$\mathcal{K}_s(a_0 \otimes \cdots \otimes a_p) := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{p-1}[F, a_k]\right) \cdot [T, a_p]\right).$$

By definition, $\mathcal{K}_s$ is exactly the mapping from Theorem 4.4.2.

For $1 \leq m \leq p$, we define X_m by:

$$X_m := \mathrm{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-1}[F, a_k]\right) a_m \left(\prod_{k=m+1}^{p}[F, a_k]\right) T\right).$$

We have

$$\begin{aligned}\mathcal{L}_s(a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_p) &= \mathrm{Tr}(\Gamma a_0 a_1 \prod_{k=2}^{p}[F, a_k] \cdot T) \\ &= X_1.\end{aligned}$$

Applying the Leibniz rule to $[F, a_{m-1}a_m]$:

$$\begin{aligned}
&\mathcal{L}_s(a_0 \otimes a_1 \otimes \cdots \otimes a_{m-2} \otimes a_{m-1}a_m \otimes a_{m+1} \otimes \cdots \otimes a_p) \\
&\quad = \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2}[F, a_k]\right)[F, a_{m-1}a_m]\left(\prod_{k=m+1}^{p}[F, a_k]\right) \cdot T\right) \\
&\quad = \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-2}[F, a_k]\right) a_{m-1}\left(\prod_{k=m}^{p}[F, a_k]\right) \cdot T\right) \\
&\qquad + \operatorname{Tr}\left(\Gamma a_0 \left(\prod_{k=1}^{m-1}[F, a_k]\right) a_m\left(\prod_{k=m+1}^{p}[F, a_k]\right) \cdot T\right) \\
&\quad = X_m + X_{m+1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{L}_s(a_pa_0 \otimes a_1 \otimes \cdots \otimes a_{p-1}) &= \operatorname{Tr}(\Gamma a_pa_0 \prod_{k=1}^{p-1}[F, a_k] \cdot T) \\
&= \operatorname{Tr}(\Gamma a_0 \prod_{k=1}^{p-1}[F, a_k] \cdot Ta_p) \\
&= \operatorname{Tr}(\Gamma a_0 \prod_{k=1}^{p-1}[F, a_k] \cdot [T, a_p]) + \operatorname{Tr}(\Gamma a_0 \prod_{k=1}^{p-1}[F, a_k] \cdot a_pT) \\
&= \mathcal{K}_s(a_0 \otimes \cdots \otimes a_p) + X_p.
\end{aligned}$$

Thus,

$$\begin{aligned}
(b\mathcal{L}_s)(a_0 \otimes \cdots \otimes a_p) &= \mathcal{K}_s(a_0 \otimes \cdots \otimes a_p) \\
&\qquad + X_1 + \Big(\sum_{m=1}^{p-1}(-1)^m(X_m + X_{m+1})\Big) + (-1)^pX_p.
\end{aligned}$$

The latter sum telescopes and indeed vanishes, so it follows that $b\mathcal{L}_s = \mathcal{K}_s$.

A.3.3. Coboundaries in Section 4.5. In this section, $(\mathcal{A}, H, D)$ is a smooth spectral triple where D has a spectral gap at 0, and $T := |D|^{2-p}e^{-s^2D^2}$. We define the multilinear mapping $\theta_s : \mathcal{A}^{\otimes p} \to \mathbb{C}$ on $a_0 \otimes \cdots a_{p-1} \in \mathcal{A}^{\otimes p}$ by:

$$\theta_s(a_0 \otimes \cdots \otimes a_{p-1}) = \operatorname{Tr}\left(\left(\prod_{k=0}^{p-1}\partial(a_k)\right)T\right).$$

For $0 \leq k \leq p$ we also define:

$$X_k = \operatorname{Tr}\left(\left(\prod_{l=0}^{k-1}\partial(a_l)\right)a_k\left(\prod_{l=k+1}^{p}\partial(a_l)\right)T\right).$$

So in particular,

$$X_0 = \operatorname{Tr}\left(a_0\left(\prod_{l=1}^{p}\partial(a_l)\right)T\right).$$

Applying the Leibniz rule to $\partial(a_k a_{a+1})$ we get:

$$\begin{aligned}
&\theta_s(a_0 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_p) \\
&\quad = \mathrm{Tr}\left(\left(\prod_{l=0}^{k-1} \partial(a_l)\right) \partial(a_k a_{k+1}) \left(\prod_{l=k+2}^{p} \partial(a_l)\right) \cdot T\right) \\
&\quad = \mathrm{Tr}\left(\left(\prod_{l=0}^{k-1} \partial(a_l)\right) a_k \left(\prod_{l=k+1}^{p} \partial(a_l)\right) \cdot T\right) \\
&\qquad + \mathrm{Tr}\left(\left(\prod_{l=0}^{k} \partial(a_l)\right) a_{k+1} \left(\prod_{l=k+2}^{p} \partial(a_l)\right) \cdot T\right) \\
&\quad = X_k + X_{k+1}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\theta_s(a_p a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1}) \\
&\quad = \mathrm{Tr}\left(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) \cdot T a_p\right) \\
&\qquad + \mathrm{Tr}\left(a_0 \left(\prod_{l=1}^{p-1} \partial(a_k)\right) \cdot T\partial(a_p)\right) \\
&\quad = X_p + \mathrm{Tr}\left(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) \cdot [T, a_p]\right) \\
&\qquad + X_0 + \mathrm{Tr}\left(a_0 \left(\prod_{k=1}^{p-1} \partial(a_k)\right) \cdot [T, \partial(a_p)]\right).
\end{aligned}$$

If we assume that p is even, then:

$$\begin{aligned}
&(b\theta_s)(a_0 \otimes \cdots \otimes a_p) \\
&\quad = \left(\sum_{k=0}^{p-1} (-1)^k (X_k + X_{k+1})\right) + (X_p + X_0) \\
&\qquad + \mathrm{Tr}\left(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) \cdot [T, a_p]\right) + \mathrm{Tr}\left(a_0 \left(\prod_{k=1}^{p-1} \partial(a_k)\right) \cdot [T, \partial(a_p)]\right) \\
&\quad = 2X_0 + \mathrm{Tr}\left(a_0 \left(\prod_{k=1}^{p-1} \partial(a_k)\right) [T, \partial a_p]\right) + \mathrm{Tr}\left(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) [T, a_p]\right) \\
&\quad = 2\mathrm{Tr}\left(a_0 \left(\prod_{k=1}^{p} \partial(a_k)\right) \cdot T\right) + \mathrm{Tr}\left(a_0 \left(\prod_{k=1}^{p-1} \partial(a_k)\right) [T, \partial a_p]\right) \\
&\qquad + \mathrm{Tr}\left(\left(\prod_{k=0}^{p-1} \partial(a_k)\right) [T, a_p]\right).
\end{aligned}$$

This completes the computation of the coboundaries.

A.4. Technical estimates for Section 4.4.1

For this section, $(\mathcal{A}, H, D)$ is assumed to be a spectral triple satisfying Hypothesis 1.2.1 and we assume that D has a spectral gap at 0.

The results of this section are very similar to that of Lemma 4.1.3. However additional technicalities make the proofs more involved and therefore are included here in the appendix. We make use of the mappings b_k from Definition 4.1.1, defined in terms of $m \geq 1$ and a subset $\mathscr{B} \subseteq \{1, \ldots, m\}$ on $a \in \mathcal{A}$

$$b_k(a) := \begin{cases} \delta(a), & k \in \mathscr{B}, \\ [F, a], & k \notin \mathscr{B}. \end{cases}$$

LEMMA A.4.1. *Let $m \geq 1$ and Let $n \in \mathbb{Z}$. Then for all $\mathscr{B} \subseteq \{1, \ldots, m\}$ the operator on H_∞ given by:*

$$|D|^{-n} \left(\prod_{k=1}^{m} b_k(a_k) \right) |D|^{n+m-|\mathscr{B}|}$$

has bounded extension, where b_k is defined in terms of $\mathscr{B}$.

PROOF. We prove the assertion by induction on m. If $m = 1$ then there are two possible cases, $\mathscr{B} = \emptyset$ and $\mathscr{B} = \{1\}$.

If $\mathscr{B} = \emptyset$, then on H_∞ we have:

$$\begin{aligned} |D|^{-n} \left(\prod_{k=1}^{m} b_k(a_k) \right) |D|^{n+m-|\mathscr{B}|} &= |D|^{-n}[F, a_1]|D|^{n+1} \\ &= |D|^{-n} L(a_1)|D|^{n} \\ &= |D|^{-n}\partial(a_1)|D|^{n} - F|D|^{-n}\delta(a_1)|D|^{n}. \end{aligned}$$

By Lemma A.1.2, the operators $|D|^{-n}\partial(a_1)|D|^{n}$ and $|D|^{-n}\delta(a_1)|D|^{n}$ have bounded extension, so this proves the $\mathscr{B} = \emptyset$ case. On the other hand, if $\mathscr{B} = \{1\}$, then on H_∞:

$$|D|^{-n} \left(\prod_{k=1}^{m} b_k(a_k) \right) |D|^{n+m-|\mathscr{B}|} = |D|^{-n}\delta(a_1)|D|^{n}.$$

Again by Lemma A.1.2, the operator $|D|^{-n}\delta(a_1)|D|^{n}$ has bounded extension. This completes the proof of the $\mathcal{B} = \{1\}$ case.

Now assume that $m > 1$ and the assertion is true for $m - 1$. Define $\mathscr{C} := \mathscr{B} \setminus \{m\}$. and let $n_1 = n + (m-1) - |\mathscr{C}|$. Then on H_∞:

$$\begin{aligned} &|D|^{-n} \left(\prod_{k=1}^{m} b_k(a_k) \right) |D|^{n+m-|\mathscr{B}|} \\ &= \left(|D|^{-n} \left(\prod_{k=1}^{m-1} b_k(a_k) \right) |D|^{n+(m-1)-|\mathscr{C}|} \right) \left(|D|^{-n_1} b_m(a_m)|D|^{n_1+1-|\mathscr{B}\cap\{m\}|} \right). \end{aligned} \tag{A.6}$$

By the inductive assumption, the first factor has bounded extension.

If $m \in \mathscr{B}$, then the second factor in (A.6) is:

$$|D|^{-n_1}\delta(a_m)|D|^{n_1}$$

which has bounded extension by Lemma A.1.2.(i). On the other hand, if $m \notin \mathscr{B}$, then the second factor in (A.6) is

$$|D|^{-n_1} F(a_m)|D|^{n_1+1} = |D|^{-n_1}(\partial(a_m) - F\delta(a_m))|D|^{n_1}$$

which also has bounded extension by Lemma A.1.2.(i). In either case, the second factor of (A.6) has bounded extension. So the assertion is proved for m, completing the proof by induction. $\square$

LEMMA A.4.2. *Let $\mathscr{A} \subseteq \{1, \cdots, p\}$ Assume that there is $m > 1$ be such that $m-1, m \in \mathscr{A}$. The operator on H_∞ given by:*

$$\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k)\right) \delta^2(a_{m-1}) \left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right) |D|^{p-|\mathscr{A}|}$$

has bounded extension.

PROOF. Let $n_1 = |\{1, \cdots, m-2\} \setminus \mathscr{A}|$ and $n_2 = |\{m+1, \cdots, p\} \setminus \mathscr{A}|$ so that immediately $n_1 \leq m-2-|\mathscr{A}|$ and $n_2 \leq p-m-|\mathscr{A}|$. On H_∞ we have

$$\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k)\right) \delta^2(a_{m-1}) \left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right) |D|^{p-|\mathscr{A}|} = \mathrm{I} \cdot \mathrm{II} \cdot \mathrm{III}.$$

Here,

$$\begin{aligned} \mathrm{I} &= \Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k)\right) |D|^{n_1}, \\ \mathrm{II} &= |D|^{-n_1} \delta^2(a_{m-1})|D|^{n_1}, \\ \mathrm{III} &= |D|^{-n_1} \left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right) |D|^{n_1+n_2}. \end{aligned}$$

The operators I and III have bounded extension by Lemma A.4.1. On the other hand, II has bounded extension due to Lemma A.1.2. $\square$

LEMMA A.4.3. *Let $\mathscr{A} \subset \{1, \cdots, p\}$ and assume that there is $m > 1$ be such that $m-1 \in \mathscr{A}$ and $m \notin \mathscr{A}$. The operator on H_∞ given by*

$$\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k)\right) [F, \delta(a_{m-1})] \left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right) |D|^{p-|\mathscr{A}|}$$

has bounded extension.

PROOF. Let $n_1 = |\{1, \cdots, m-2\} \setminus \mathscr{A}|$ and $n_2 = |\{m+1, \cdots, p\} \setminus \mathscr{A}|$, so that we immediately have $n_1 \leq m-2-|\mathscr{A}|$ and $n_2 \leq p-m-|\mathscr{A}|$ as in Lemma A.4.2. We have

$$\Gamma a_0 \left(\prod_{k=1}^{m-2} b_k(a_k)\right) [F, \delta(a_{m-1})] \left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right) |D|^{p-|\mathscr{A}|} = \Gamma a_0 \cdot \mathrm{I} \cdot \mathrm{II} \cdot \mathrm{III}.$$

Here,

$$\begin{aligned}
\mathrm{I} &= \left(\prod_{k=1}^{m-2} b_k(a_k)\right)|D|^{n_1},\\
\mathrm{II} &= |D|^{-n_1}[F,\delta(a_{m-1})]|D|^{1+n_1},\\
\mathrm{III} &= |D|^{-n_1-1}\left(\prod_{k=m}^{p-1} b_{k+1}(a_k)\right)|D|^{1+n_1+n_2}.
\end{aligned}$$

The operators I and III have bounded extension by Lemma A.4.1. On the other hand,

$$\mathrm{II} = |D|^{-n_1}(\partial(\delta(a_{m-1})) - F\delta^2(a_{m-1}))|D|^{n_1}$$

which has bounded extension by Lemma A.1.2.(i). □

LEMMA A.4.4. *For every $m \geq 1$, the operator on H_∞ given by*

$$\Big(\prod_{k=0}^{m}[F,a_k]\Big)|D|^{m+1}$$

has bounded extension.

PROOF. We prove the assertion by induction on m. For $m = 0$, on H_∞ we have

$$[F,a_0]|D| = \partial(a_0) - F\delta(a_0)$$

which has has bounded extension.

Now let $m > 1$ and assume that the assertion is true for $m-1$. On H_∞ we write

$$\Big(\prod_{k=0}^{m}[F,a_k]\Big)|D|^{m+1} = \Big(\Big(\prod_{k=0}^{m-1}[F,a_k]\Big)|D|^{m}\Big)\cdot\Big(|D|^{-m}[F,a_m]|D|^{m+1}\Big).$$

The first factor has bounded extension by the induction assumption. The second factor is

$$|D|^{-m}L(a_m)|D|^m = |D|^{-m}\partial(a_m)|D|^m - F\cdot|D|^{-m}\delta(a_m)|D|^m$$

which has bounded extension by Lemma A.1.2.(i). Hence, the assertion holds for m, completing the inductive proof. □

A.5. Subkhankulov's computation

The following assertion is identical to [**61**, Lemma 2.1.1]. However to the best of our knowledge there is no published proof in English, and [**61**] is not easily accessible. For the convenience of the reader we include a proof here.

PROPOSITION A.5.1. *For all $u \in (0,1)$ and $v \in \mathbb{R}$, we have*

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{(1-t^2)^2}{u+it}e^{(u+it)v}dt = (1+u^2)^2\chi_{[0,\infty]}(v) + e^{uv}\cdot\min\{1,v^{-2}\}\cdot O(1).$$

PROOF. We will deal separately with the $v \geq 0$ and $v < 0$ cases. First, assume that $v \geq 0$.

Let γ_0 be a smooth curve in $\mathbb{C}$ without self-intersections such that

(1) γ_0 starts at i and ends at $-i$.
(2) γ_0 lies in the half-plane $\{\Re(z) \leq 0\}$,

(3) The distance between γ_0 and the interval $[-1, 0]$ is greater than or equal to 1,
(4) the length of γ_0 is at most 10.
(5) γ_0 is contained in the disc $\{z \,:\, |z| \leq 10\}$.

Let γ_1 be the interval starting at $-i$ and ending at i and let the contour γ be the concatenation of γ_0 and γ_1.

Define

$$f(z) := \frac{(1+z^2)^2}{u+z}, \quad z \in \mathbb{C} \setminus \{-u\}.$$

So that by definition:

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{(1-t^2)^2}{u+it} e^{(u+it)v}\, dt = \frac{1}{2\pi i} e^{uv} \int_{\gamma_1} f(z) e^{zv}\, dz. \tag{A.7}$$

Since $z \mapsto e^{zv}$ is entire, the function $z \to f(z)e^{zv}$ is holomorphic in the set $\mathbb{C} \setminus \{-u\}$ and has a simple pole at at $z = -u$ with corresponding residue $(1+u^2)^2 e^{-uv}$. By construction, the point $-u$ is in the interior of the curve γ and so by the Cauchy integral formula we have:

$$\frac{1}{2\pi i} \int_{\gamma} f(z) e^{zv} dz = (1+u^2)^2 e^{-uv}.$$

Since $\gamma = \gamma_0 \cup \gamma_1$:

$$\frac{1}{2\pi i} \int_{\gamma_0} f(z) e^{zv} dz + \frac{1}{2\pi i} \int_{\gamma_1} f(z) e^{zv} dz = (1+u^2)^2 e^{-uv}. \tag{A.8}$$

By definition γ_0 has length at most 10, so by the triangle inequality we have the bound:

$$\left| \int_{\gamma_0} f(z) e^{zv} dz \right| \leq 10 \sup_{z \in \gamma_0} |f(z)| |e^{zv}|.$$

Since γ_0 is contained in the half-plane $\{z \;:\; \Re(z) \leq 0\}$ we also have that $\sup_{z \in \gamma_0} |e^{zv}| \leq 1$ and therefore:

$$\left| \int_{\gamma_0} f(z) e^{zv} dz \right| \leq 10 \sup_{z \in \gamma_0} |f(z)|.$$

For $z \in \gamma$ we have that $|z| \leq 10$ and $|u + z| \geq 1$ so it follows that

$$\sup_{z \in \gamma_0} |f(z)| \leq (1 + 10^2)^2 \leq 10^5.$$

Therefore, we have

$$\int_{\gamma_0} f(z) e^{zv} dz = O(1). \tag{A.9}$$

Using integration by parts twice and taking into account that $f(\pm i) = f'(\pm i) = 0$, we obtain

$$\int_{\gamma_0} f(z) e^{zv} dz = v^{-2} \int_{\gamma_0} f''(z) e^{vz} dz.$$

Thus,

$$v^2 \left| \int_{\gamma_0} f(z) e^{zv} dz \right| \leq 10 \sup_{z \in \gamma_0} |f''(z)|.$$

We may compute $f''(z)$ directly as:

$$f''(z) = (4+12z)\frac{1}{u+z} - 8z(1+z^2)\frac{1}{(u+z)^2} + (1+z^2)^2\frac{2}{(u+z)^3}.$$

Since $|z| \leq 10$ and $|u+z| \geq 1$ for every $z \in \gamma_0$, it follows that

$$\begin{aligned}\sup_{z\in\gamma_0} |f''(z)| &\leq (4+12\cdot 10) + 8\cdot 10\cdot(1+10^2) + 2\cdot(1+10^2)^2 \\ &\leq 10^5.\end{aligned}$$

Therefore, we have:

$$\int_{\gamma_0} f(z)e^{zv}dz = O(v^{-2}), \tag{A.10}$$

Hence,

$$\int_{\gamma_0} f(z)e^{zv}dz = \min\{1, v^{-2}\}\cdot O(1). \tag{A.11}$$

Combining (A.8) and (A.11), we obtain

$$\frac{1}{2\pi i}\int_{\gamma_1} f(z)e^{zv}dz = (1+u^2)^2 e^{-uv} + \min\{1, v^{-2}\}\cdot O(1).$$

So using (A.7) and (A.8):

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{(1-t^2)^2}{u+it} e^{(u+it)v}dt = (1+u^2)^2 + e^{uv}\cdot \min\{1, v^{-2}\}\cdot O(1).$$

This completes the proof of the $v \geq 0$.

Now assume that $v < 0$. This proof is similar, but instead we consider a contour in the half plane $\{z : \Re(z) \geq 0\}$. Let γ_2 be a smooth curve without self-intersections such that

(1) γ_2 starts at $-i$ and ends at i.
(2) γ_2 lies in the half-plane $\{\Re(z) \geq 0\}$.
(3) the distance between γ_2 and $[-1, 0]$ is greater than or equal to 1.
(4) the length of γ_2 is at most 10.
(5) γ_2 is contained in the disc $\{z : |z| \leq 10\}$.

As in the $v \geq 0$ case, γ_1 denotes the interval joining $-i$ and i, and now write γ' for the concatenation of γ_1 and γ_2.

Since f is holomorphic in the half-plane $\Re(z) \geq 0$, we have:

$$\int_{\gamma'} f(z)e^{zv}dz = 0.$$

Since $\gamma' = \gamma_1 \cup \gamma_2$ we have:

$$\frac{1}{2\pi i}\int_{\gamma_1} f(z)e^{zv}dz + \frac{1}{2\pi i}\int_{\gamma_2} f(z)e^{zv}dz = 0. \tag{A.12}$$

Since by definition γ_2 has length at most 10, and we are assuming $v < 0$ we have:,

$$\begin{aligned}\left|\int_{\gamma_2} f(z)e^{zv}dz\right| &\leq 10 \sup_{z\in\gamma_2} |f(z)e^{zv}| \\ &\leq 10 \sup_{z\in\gamma_2} |f(z)| \\ &\leq 10(1+10^2)^2. \\ &= O(1).\end{aligned}$$

Using integration by parts and taking into account once again that $f(\pm i) = f'(\pm i) = 0$, we obtain in also in the $v < 0$ case that:

$$\int_{\gamma_2} f(z)e^{zv}dz = v^{-2}\int_{\gamma_2} f''(z)e^{vz}dz.$$

Thus,

$$\begin{aligned}v^2\left|\int_{\gamma_2} f(z)e^{zv}dz\right| &\leq 10 \sup_{z\in\gamma_2} |f''(z)||e^{zv}| \\ &\leq 10^6 \\ &= O(1).\end{aligned}$$

Therefore,

$$\int_{\gamma_2} f(z)e^{zv}dz = \min\{1, v^{-2}\} \cdot O(1). \tag{A.13}$$

Combining (A.12) and (A.13), we obtain

$$\frac{1}{2\pi i}\int_{\gamma_1} f(z)e^{zv}dz = \min\{1, v^{-2}\} \cdot O(1).$$

Hence, by (A.7) we conclude the proof for the $v < 0$ case. □

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