PROJECT TITLE AIMS AND BACKGROUND FUTURE FELLOWSHIP CANDIDATE PROPOSED PROJECT QUALITY AND INNOVATION FEASIBILITY AND STRATEGIC ALIGNMENT BENEFIT AND COLLABORATION COMMUNICATION OF RESULTS MANAGEMENT OF DATA REFERENCES ACKNOWLEDGEMENTS (IF REQUIRED)

Project Title Actions of quantum groups and noncommutative manifolds.

Aims and Background

Broad Aim This proposal covers a novel viewpoint on noncommutative Riemannian manifolds and aims bringing together an international team of experts with a new approach to fundamental and long-standing problems in geometry and harmonic analysis.

There exist two theories of Riemannian manifolds: the classical one (which treats manifold as a locally Euclidean space) and the modern one introduced by Connes (which treats manifold as a spectral triple). A triple consists of $*-$algebra $\mathcal{A}$ represented on a Hilbert space H and an unbounded self-adjoint operator D on H. It is easy to compare both notions of a manifold: in the classical case, one lets $\mathcal{A}=C^{\infty}(X)$ (algebra of smooth functions on X), $H=L_2\Omega(X)$ (square-integrable forms on X) and lets D to be a Hodge-Dirac operator. Famous Connes Reconstruction Theorem tells us that so-defined spectral triple captures all the geometric information about a manifold.

Manifolds are often equipped with a Lie group action. For instance, an orthogonal group acts on the real sphere. Moreover, rotations are exactly those automorphisms of the sphere which commute with Laplace-Beltrami operator. To take group action into account, one requires that Dirac-type operator commutes with the action of the group.

In the noncommutative realm, we have an action of a given Lie group (or even quantum group) on a $*-$algebra. Natural noncommutative analogue of "commutes with a group action" is an equivariance property.

The CI will investigate actions of groups (more broadly, quantum groups) on noncommutative manifolds. Those actions compel an equivariant Dirac operator which captures all the geometric information about the manifold (famous Connes Reconstruction Theorem).

Our proposal gives a firm and unified foundation to many such attempts.

General Background Operator algebras provide a natural edifice to many areas of classical and modern Mathematics. Among them, a particular role is played by C^*-algebras (uniformly closed $*-$subalgebras in $B(H)$) and von Neumann algebras (weakly closed subalgebras in $B(H)$). These additional structures make the algebras suitable for studying noncommutative analysis.

Gelfand-Naimark theorem delivers a duality between category of compact topological spaces and that of commutative C^*-algebras. So, topology is incorporated into a theory of C^*-algebras. A similar result due to von Neumann theorem provides a duality between category of measure spaces and that of commutative von Neumann algebras. In other words, measure theory is a part of von Neumann algebras theory.

A conventional wisdom suggests now a dictionary between classical and quantum worlds. While topology and measure theory are supplied with natural quantum counter-parts, geometry was left behind for a long time. Starting from von Neumann himslef, people tried to find a quantum analogue of geometry (that is, to construct a functor from useful geometric categories to treatable quantum categories). So far, the most successful attempt is due to Connes who introduced the notion of spectral triple and promoted it as an analogue of Riemannian manifold.

Notion of a Riemannian manifold is a pillar of geometry. Usually, a manifold is defined as a topological space in which every point admits a neighborhood which is homeomorphic to a linear space. A manifold equipped with a smooth metric tensor is called Riemannian.

With every Riemannian manifold, one can associate (i) algebra $\mathcal{A}=C^{\infty}(X)$ of all smooth functions on X; (ii) Hilbert space $H=L_2\Omega(X)$ of all square-integrable forms on X; (iii) Hodge-Dirac operator D on $L_2\Omega(X)$. The following properties of the triple $(\mathcal{A},H,D)$ are crucial:

1. if $\pi:\mathcal{A}\to B(H)$ is the representation of $\mathcal{A}$ by multiplication operators, then the commutator $[D,\pi(a)]$ is bounded for all $a\in\mathcal{A}$;
2. the operator $\pi(a)(D+i)^{-1}$ is compact and its singular values decrease as $k^{-\frac{1}{d}}$, where $d=\dim(X)$.

Recall that $-D^2$ is a Hodge-Laplace operator and its component acting on 0 order forms is Laplace-Beltrami operator.

A triple $(\mathcal{A},H,D)$ satisfying the conditions above is called spectral. Celebrated Reconstruction Theorem due to Connes dictates that every spectral triple with commutative $\mathcal{A}$ (satisfying few natural conditions) comes from a $d-$dimensional Riemannian manifold X.

Hodge-Dirac operator allows an easy description of isometries of the manifold X. If $\gamma:X\to X$ is an isometry and if $U:L_2\Omega(X)\to L_2\Omega(X)$ is a composition (with γ) operator, then U is unitary, U preserves $\mathcal{A}$ and commutes with D. Conversely, every operator U as above comes from an isometry $\gamma:X\to X$.

As is typical in applications, manifolds are equipped with isometric action of Lie groups.

After the seminal work of Weyl, it become a tradition to measure geometric (and often topological) quantities in terms of heat semi-group expansion. If (X,g) is a compact Riemannian manifold, then resolvent of Δ_g (Laplace-Beltrami operator) is compact. Moreover, heat semi-group $e^{t\Delta_g}$ belongs to the trace class for every $t>0$. For every $f\in C^\infty(X)$, Minakshisundaram-Plejel theorem asserts an existence of an asymptotic expansion

$$\mathrm{Tr}(M_f e^{t\Delta_g})\approx(4\pi t)^{-\frac{d}{2}}\cdot\sum_{n\geq0}a_n(f)t^n,\quad t\downarrow0. \tag{1}$$

Here, d is the dimension of X. Here,

$$a_k(f)=\int_X A_k\cdot f d\mathrm{vol}_g, \tag{2}$$

where vol_g is the standard volume element on X given in local coordinates by the formula

$$d\mathrm{vol}_g=(\det(g))^{\frac12}(x)dx.$$

As established by Weyl, $A_0=1$. Further computations show that

$$A_1=\frac16R,$$

where R is the scalar curvature.

For non-commutative manifolds, we no longer have coordinates. Suppose that a non-commutative manifold is equipped with a Laplacian Δ. The task now is to

1. find a normal state h on $\mathcal{A}''$ such that
$$\mathrm{Tr}(\pi(x)e^{t\Delta})\approx(4\pi t)^{-\frac{d}{2}}h(x),\quad t\downarrow0,$$
for every $x\in\mathcal{A}''$;
2. prove a non-commutative version of Minakshisundaram-Plejel theorem, i.e. generalise the formula (1) as follows
$$\mathrm{Tr}(\pi(x)e^{t\Delta})\approx(4\pi t)^{-\frac{d}{2}}\sum_{n\geq0}a_n(x)t^n,\quad t\downarrow0,$$
for every $x\in\mathcal{A}''$;
3. prove normality of coefficients as in (2) with respect to the volume state, i.e., show that
$$a_k(x)=h(xA_k),\quad x\in\mathcal{A}'',$$
for every $k\geq0$ and for some $A_k\in\mathcal{A}$;
4. compute respective A_k;

When this mission is accomplished, one can *define* a scalar curvature of a non-commutative manifold by setting $R=6A_1$.

One of the fundamental tools in noncommutative geometry is the Chern character. The Connes Character Formula (also known as the Hochschild character theorem) provides an expression for the class of the Chern character in Hochschild cohomology, and it is an important tool in the computation of the Chern character. The formula has been applied to many areas of noncommutative geometry and its applications: such as the local index formula [**?**], the spectral characterisation of manifolds [**?**] and recent work in mathematical physics [**?**].

In its original formulation, [**?**], the Character Formula is stated as follows: Let $(\mathcal{A},H,D)$ be a p-dimensional compact spectral triple with (possibly trivial) grading Γ. By the definition of a spectral triple, for all $a\in\mathcal{A}$ the commutator $[D,a]$ has an extension to a bounded operator $\partial(a)$ on H. Assume for simplicity that $\ker(D)=\{0\}$ and set $F=\mathrm{sgn}(D)$. For all $a\in\mathcal{A}$ the commutator $[F,a]$ is a compact operator in the weak Schatten ideal $\mathcal{L}_{p,\infty}$.

Consider the following two linear maps on the algebraic tensor power $\mathcal{A}^{\otimes(p+1)}$, defined on an elementary tensor $c=a_0\otimes a_1\otimes\cdots\otimes a_p\in\mathcal{A}^{\otimes(p+1)}$ by setting

$$\mathrm{Ch}(c):=\frac12\mathrm{Tr}(\Gamma F[F,a_0][F,a_1]\cdots[F,a_p]),\quad \Omega(c):=\Gamma a_0\partial a_1\partial a_2\cdots\partial a_p.$$

Then the Connes Character Formula states that if c is a Hochschild cycle (as defined in Section **??**) then

$$\mathrm{Tr}_\omega(\Omega(c)(1+D^2)^{-p/2})=\mathrm{Ch}(c)$$

for every Dixmier trace Tr_ω. In other words, the multilinear maps Ch and $c \mapsto \mathrm{Tr}_\omega(\Omega(c)(1+D^2)^{-p/2})$ define the same class in Hochschild cohomology.

There has been great interest in generalising the tools and results of noncommutative geometry to the "non-compact" (i.e., non-unital) setting. The definition of a spectral triple associated to a non-unital algebra originates with Connes [?], was furthered by the work of Rennie [?, ?] and Gayral, Gracia-Bondía, Iochum, Schücker and Varilly [?]. Earlier, similar ideas appeared in the work of Baaj and Julg [?]. Additional contributions to this area were made by Carey, Gayral, Rennie and Sukochev [?, ?]. The conventional definition of a non-compact spectral triple is to replace the condition that $(1+D^2)^{-1/2}$ be compact with the assumption that for all $a \in \mathcal{A}$ the operator $a(1+D^2)^{-1/2}$ is compact.

This raises an important question: is the Connes Character Formula true for locally compact spectral triples?

In this project we aim to answer this question, using recently developed techniques of operator integration.

Specific aims There are **put number** specific aims.

Aim 1: Investigate when Chern character provides an asymptotic expansion for the heat semi-group. More precisely, when

$$\mathrm{Tr}(\Omega(c)e^{-s^2D^2}) = \mathrm{Ch}(c)s^{-p} + O(s^{1-p}), \quad s \downarrow 0, \tag{3}$$

for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$? This question is closely related (albeit, not equivalent) to the question about analyticity of the $\zeta-$function. Show that the function (defined a priori for $\Re(z) > p$)

$$z \to \mathrm{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}})$$

admits an analytic extension to the half-plane $\Re(z) > p-1$ so that

$$\lim_{z\to p}(z-p)\mathrm{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}) = p\mathrm{Ch}(c)?$$

Aim 2: The purpose of the Connes Character Formula is to compute the Hochschild class of the Chern character by a "local" formula, here stated in terms of singular traces. Show that

$$\varphi(\Omega(c)(1+D^2)^{-\frac{p}{2}}) = \mathrm{Ch}(c). \tag{4}$$

for every (normalised) trace φ on $\mathcal{L}_{1,\infty}$ and for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$. Equivalently, show that

$$\sum_{k=0}^{n} \lambda(k, \Omega(c)(1+D^2)^{-p/2}) = \mathrm{Ch}(c)\log(n) + O(1), \quad n \to \infty.$$

Here, $\lambda(k,T)$ means the $k-$th eigenvalue (counted with algebraic multiplicity) of a compact operator T.

Future Fellowship Candidate