Research statement

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I am interested in the study of operator algebras. In particular, I am interested in noncommutative geometry, K-theory, Inverse semigroups and their C^*-algebras.

In my thesis, the noncommutative geometry of the odd dimensional quantum spheres was studied. Also the K-groups of the quantum Stiefel manifold $SU_q(n)/SU_q(n-2)$ were computed. During my postdoctoral period, I applied the theory of Inverse semigroups to study the Cuntz-Li algebras. In the next few paragraphs, after a brief introduction to noncommutative geometry, quantum groups and the local index formula, my research is described in detail.

Noncommutative geometry

The starting point of noncommutative geometry can be traced back to the Gelfand Naimark theorem which gives an anti-equivalence between the category of locally compact Hausdorff spaces and the category of commutative C^*-algebras. The correspondence is given by the map $X \mapsto C_0(X)$ where $C_0(X)$ is the algebra of continuous complex valued functions which vanish at infinity. This says that all the information about a space is actually encoded in the algebra of continuous functions on it. Thus one thinks of noncommutative C^*-algebras as noncommutative topological spaces and tries to apply topological methods to understand them. K-theory adapts well to study C^* algebras. The classification program of C^*-algebras using K-theory is a major theme of research today.

In geometry, the topological spaces that one tries to understand are smooth manifolds. Connes proposed the following notion of spectral triples as the noncommutatative counterpart of smooth manifolds. An **even spectral triple** for a $*$ algebra $\mathcal{A}$ is a triple $(\pi, \mathcal{H}, D)$ together with a $\mathbb{Z}_2$ grading γ on $\mathcal{H}$ such that

1. the map $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a $*$ representation such that $\pi(a)\gamma = \gamma\pi(a)$ for every $a \in \mathcal{A}$,
2. the operator D is an unbounded self-adjoint operator with compact resolvent such that $D\gamma = -\gamma D$, and
3. the commutator $[D, \pi(a)]$ is bounded for every $a \in \mathcal{A}$.

If no grading is present one calls it an odd spectral triple. Usually $\mathcal{A}$ will be a dense subalgebra of a C^*-algebra.

The reason for calling spectral triples as noncommutative manifolds is due to the fact that the spectral triple $(C^\infty(M), L^2(M,S), D)$ [[7]] captures all the information about the manifold M. Here

- M is a spin manifold,
- $S \to M$ is a spinor bundle and $L^2(M,S)$ denote the space of square integrable sections, and
- The operator D is the Dirac operator associated to the Levi-Civita connection.

Also Connes in [9] proved that if $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple which satisfies certain assumptions and if $\mathcal{A}$ is commutative then the spectral triple comes from a classical spectral triple $(C^\infty(M), L^2(M,S), D)$ for some smooth manifold M. Thus it makes good sense to think of spectral triples as noncommutative manifolds.

Quantum groups

The theory of quantum groups has its origin in finding a good duality theorem, analogous to Pontryagin duality theorem, for general locally compact groups. In late 1980's, Woronowicz developed a general theory of compact quantum groups and developed a Peter-Weyl theory for them in [32], [33], and in [34]. One of the main examples in Woronowicz's theory is the quantum group $SU_q(n)$. Let us recall the definition.

Let $q \in (0,1)$. The C^*-algebra $C(SU_q(n))$ is the universal unital C^*-algebra generated by n^2 elements u_{ij} satisfying the following condition

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij} \quad , \quad \sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij}$$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2 \cdots i_n} u_{j_1 i_1} \cdots u_{j_n i_n} = E_{j_1 j_2 \cdots j_n}$$

where

$$E_{i_1 i_2 \cdots i_n} \quad := \quad \begin{cases} 0 & if \quad i_1, i_2, \cdots i_n \text{ are not distinct} \\ (-q)^{n(i_1, i_2, \cdots, i_n)} & \text{otherwise} \end{cases}$$

where for a permutation σ on $\{1, 2, \cdots, n\}$, $\ell(\sigma)$ denotes its length. The C^*-algebra $C(SU_q(n))$ has a compact quantum group structure with the comultiplication Δ given by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}$$

Let $1 \le m \le n-1$. Call the generators of $SU_q(n-m)$ as v_{ij}. The map $\phi : C(SU_q(n)) \to C(SU_q(n-m))$ defined by

$$\phi(u_{ij}) \quad := \quad \begin{cases} v_{i-m, j-m} & \text{if } m+1 \ \le i, j \ \le n \\ \delta_{ij} & \text{otherwise} \end{cases}$$

is a surjective unital C^*-algebra homomorphism such that $\Delta \circ \phi = (\phi \otimes \phi)\Delta$. In this way the quantum group $SU_q(n-m)$ is a subgroup of the quantum group $SU_q(n)$. The C^*-algebra of the quotient $SU_q(n)/SU_q(n-m)$ is denoted by $C(S_q^{n,m})$ and are called quantum Stiefel manifolds. The C^*-algebra $C(S_q^{n,1})$ is denoted by $C(S_q^{2n-1})$ and are called the odd dimensional spheres.

The rich interplay between Lie groups and differential geometry naturally raises the question of understanding the interaction between quantum groups and noncommutative geometry. Papers [2],[31],[21] attempt to put quantum groups within the framework of Connes' noncommutativie geometry.

Local index formula

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Denote the the sign of the operator D by F. One can show that $[F, a]$ is compact for every $a \in \mathcal{A}$. In other words $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module. It is called an even Fredholm module or an odd one depending on whether one has a grading or not.

Let $(\mathcal{A}, \mathcal{H}, F)$ be an odd Fredholm module i.e $[F, a]$ is compact for every $a \in \mathcal{A}$ and F is a selfadjoint unitary. Let $P := \frac{1+F}{2}$. For a unitary $u \in \mathcal{A}$, the operator $PuP : P\mathcal{H} \to P\mathcal{H}$ is Fredholm and hence has an index. In this way one obtains a map $Ind_F : K_1(\mathcal{A}) \to \mathbb{Z}$ defined by the equation

$$Ind_F([u]) := Ind(P \otimes 1)u(P \otimes 1).$$

In geometric examples, it is difficult to compute the sign of the Dirac operator and one needs a formula for the index completely in terms of D. Under certain assumptions, Connes and Moscovici in [6] showed that the index map can be computed in terms of residues at the poles of certain meromorphic functions associated with D. The formula they obtained is called the local index formula. But for a while, except for the classical spectral triple coming from the Dirac operator on spin manifolds, examples of spectral triples which satisfy the assumptions of the local index formula were not known.

Chakraborty and Pal in [2] produced a satisfactory spectral triple on the quantum group $SU_q(2)$ which is also equivariant. Connes made a detailed study of this spectral triple in [8] from the local index formula point of view. Similar computation was done in [31].

Moreover in the case of manifolds one does not see the contribution of the residues at the poles lower than the dimension. But in noncommutative examples, the residues at the lower poles also contribute. This is a purely noncommutative phenomenon. Thus one would like to understand the local index formula for other examples also.

Summary of the work done in my thesis

- In [22], we showed that the $SU_q(n)$ equivariant spectral triple on the odd dimensional quantum sphere S_q^{2n-1}, constructed by Chakraborty and Pal in [3], satisfies the hypothesis of the local index formula. Also the dimension spectrum was calculated. First the case $q = 0$ was considered. Then, it was shown that the general case follows from the case $q = 0$.

- The computations in [8], [31] and [22] are case specific. The general principle underlying these computations were explained in [5]. This is achieved through a construction called quantum double suspension. We considered a property of spectral triples called the weak heat kernel asymptotic expansion property. We showed that a spectral triple having the weak heat kernel expansion property satisfies the hypothesis of the local index formula. We also showed that this property is stable under quantum double suspension. This construction produces new examples for which local index formula holds.

- To construct non-trivial spectral triples on a specific algebra, one needs to have an idea of its K-groups. In that regard, the K-groups of the quantum homogeneous space $SU_q(n)/SU_q(n-2)$ were computed in [4]. Using the irreducible representations of the algebra $C(SU_q(n)/SU_q(n-2))$ obtained by Vainerman and Podkolzin in [23], certain exact sequences connecting $C(SU_q(n)/SU_q(n-2))$ and its quotients were derived as in [24]. Then applying the six term sequence of K-theory, it was shown that the K-groups of $C(SU_q(n)/SU_q(n-2))$ are isomorphic to $\mathbb{Z}^2$.

Outline of the work done during my postdoctoral period

During my postdoc, I got interested in the theory of inverse semigroups and its C^*-algebras.

A semigroup S is called an inverse semigroup if for every $s \in S$ there exists a unique $t \in S$ such that $sts = s$ and $tst = t$. The unique t is denoted by s^*. The canonical example of an inverse semigroup is the inverse semigroup of partial bijections. Wagner-Preston theorem tells that an abstract inverse semigroup can be embedded inside an inverse semigroup of partial isometries.

In [30], I applied the theory of inverse semigroups to reprove Sheu's result ([25]) that odd dimensional quantum spheres are groupoid C^*-algebras. It was shown that the C^*-algebra of the odd dimensional sphere is generated by an inverse semigroup. It was shown that the tight groupoid of the attached inverse semigroup is isomorphic to the groupoid obtained by Sheu. The proof I believe is constructive. Sheu in [25] raised the question of whether the C^*-algebra of the quantum group $SU_q(n)$ is a groupoid C^*-algebra. The answer to it is still not known. I believe the inverse semigroup methods should help in answering Sheu's question.

Cuntz-Li algebras

Cuntz initiated the study of ring C^*-algebras in [10]. It was further studied by Cuntz and Li in more detail in [13], [14], [15] and in [19].

Let R be an integral domain with only finite quotients. Examples of such rings are $\mathbb{Z}$ and the ring of integers in a number field. The ring C^*-algebra associated to R, denoted $\mathfrak{A}[R]$, is the universal C^*-algebra generated by isometries $\{s_m : m \in R\backslash\{0\}\}$ and the set of unitaries $\{u(k) : k \in R\}$ which satisfy the relations reflecting the semigroup multiplication in $R \rtimes R^\times$ and one important relation among the range projections of $\{s_m : m \in R^\times\}$. Cuntz and Li proved that $\mathfrak{A}[R]$ is purely infinite, simple and is Morita equivalent to a crossed product C^*-algebra. In fact, $\mathfrak{A}[R]$ is a groupoid C^*-algebra. For instance, when $R = \mathbb{Z}$, $\mathfrak{A}[\mathbb{Z}]$ is Morita-equivalent to $C_0(\mathbb{A}_f) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}^\times)$. Here $\mathbb{A}_f$ denotes the ring of finite adeles. It is a remarkable result due to Cuntz and Li that $\mathfrak{A}[\mathbb{Z}]$ is Morita-equivalent to $C_0(\mathbb{R}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}^\times)$. This Morita-equivalence is now beginning to be referred as Cuntz-Li duality theorem. Using this duality theorem, Cuntz, Li and their collaborators computed the K-groups of the ring C^*-algebra associated to the ring of integers in a number field. See [11], [12] [15], and [20].

Alternative approaches to Cuntz-Li algebras were considered in [1] and [18]. A generalisation due to Quigg, Landstad and Kaliszweski is the following. Consider a semidirect product $N \rtimes H$ and a normal subgroup M of N. Let $P := \{a \in H : aMa^{-1} \subset M\}$. Then P is a semigroup. Under certain hypothesis regarding the triple $(N \rtimes H, M, P)$, let $\mathfrak{A}[N \rtimes H, M]$ be the universal C^*-algebra generated by isometries $\{s_a : a \in P\}$ and unitaries $\{u(m) : m \in M\}$ which satisfy the relation reflecting the semigroup multiplication of $M \rtimes P$ and one more important relation among the range projections $e_a := s_a s_a^*$. In [18], it was shown that when H is abelian, $\mathfrak{A}[N \rtimes H, M]$ is Morita-equivalent to a crossed product algebra of the form $C_0(\overline{N}) \rtimes (N \rtimes H)$.

My contribution

My contribution to the Cuntz-Li algebras is to apply inverse semigroups to study them. In [27], the case of an integral domain was considered. In [29], the case of a semi-direct product $N \rtimes H$ was considered together with a normal subgroup M. But in [29], the assumption that H is abelian was relaxed . An inverse semigroup T generated by $\{s_a, u(m)\}$ were attached and it was shown that $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to the C^*-algebra of the tight groupoid of T. Exel's theory of tight representations of an inverse semigroup was used to prove this. (See [16], [17].)

The advantage of using inverse semigroups is that it completely removes the need to guess the correct groupoid. The main example considered in [29] is the pair $(\mathbb{Q}^n \rtimes \Gamma, \mathbb{Z}^n)$ where Γ is a subgroup of $GL_n(\mathbb{Q})$ acting on $\mathbb{Q}^n$ by matrix multiplication. A duality theorem analogous to that of the Cuntz-Li duality theorem was proved. For instance, it was shown that the Cuntz-Li algebra associated to the pair $(\mathbb{Q}^n \rtimes GL_n(\mathbb{Q}), \mathbb{Z}^n)$ is Morita-equivalent to the crossed product

$C_0(\mathbb{R}^n) \rtimes (\mathbb{Q}^n \rtimes GL_n(\mathbb{Q}))$. In [26], structural results about the inverse semigroup T, considered in [29], was proved. It was shown that T is strongly $0 - E$ unitary and is an F^*-inverse semigroup. Also T was described in terms of a McAlister triple.

A K-theory computation: Let $A \in M_n(\mathbb{R})$ be an invertible matrix. Consider the semi-direct product $\mathbb{R}^n \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}^n$ by matrix multiplication. Consider a strongly continuous action (α, τ) of $\mathbb{R}^n \rtimes \mathbb{Z}$ on a C^*-algebra B where α is a strongly continuous action of $\mathbb{R}^n$ and τ is an automorphism. Moreover one has the commutation rule $\tau\alpha_\xi = \alpha_{A\xi}\tau$ for $\xi \in \mathbb{R}^n$.

The map τ induces a map $\widetilde{\tau}$ on $B \rtimes_\alpha \mathbb{R}^n$. In [28], it was shown that, at the K-theory level, τ commutes with the Connes-Thom map if $\det(A) > 0$ and anticommutes if $\det(A) < 0$. As an application, the K-groups of the Cuntz-Li algebra associated to an integer dilation matrix were recomputed.

Future Plans

Some of the questions that arise out of my work are the following and I would like to investigate them in the future.

1. The local index cocycles in the local index formula are made up of the multilinear functionals $\phi_{n,k}$ where $\phi_{n,k}$ is given by
$$\phi_{n,k}(a_0, a_1, \cdots, a_n) = \int a_0[D, a_1]^{(k_1)}[D, a_2]^{(k_2)} \cdots [D, a_n]^{k_n}|D|^{-(n+2|k|)}$$
where $\int b = Res_{z=0}Tr(b|D|^{-z})$. Then the local index cocycle is given by the functionals $\phi_n = \sum_k c_{n,k}\phi_{n,k}$ where the constants $c_{n,k}$ are universal. It is interesting to know whether the constants $c_{n,k}$ are unique. To understand the uniqueness one needs to have a family of spectral triples which satisfies the assumptions of the local index formula. So one can compute the index pairing and understand what relations $c_{n,k}$ satisfy. Constructing spectral triples on the quantum homogeneous spaces associated to the quantum group $SU_q(n)$ might shed some light on this question.

2. As already mentioned, the question of whether the C^*-algebra $C(SU_q(n))$ is a groupoid C^*-algebra was considered by Albert Sheu in [24]. I believe that the theory of Inverse semigroups can be used to prove such a result. The problem is to understand whether one can set $q = 0$. For $q = 0$, there is a natural inverse semigroup associated to $C(SU_0(n))$.

3. I would like to understand the K-theory computation of ring C^*-algebras by Cuntz, Li and their collaborators. Computing the K-theory of Cuntz-Li algebras associated to subgroups of $GL_n(\mathbb{Q})$ would be a challenging and an interesting question. The following question arises naturally in this context.

   Question : If Γ is a torsion free subgroup of $GL_n^+(\mathbb{Q})$, is it true that $B \rtimes (\mathbb{R}^n \rtimes \Gamma)$ and $B \rtimes \Gamma$ are KK-equivalent ?

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