Discussion 4

1. The difference operator ∇ is defined as $\nabla X_t = X_t - X_{t-1}$.

   a) Suppose $\{X_t\}$ has a linear trend, $X_t = \beta_0 + \beta_1 t + Y_t$, where β_i are constants, and $\{Y_t\}$ is stationary. Show that ∇X_t does not have a trend.

   b) Suppose $\{X_t\}$ has a quadratic trend, $X_t = \beta_0 + \beta_1 t + \beta_2 t^2 + Y_t$, where β_i are constants, and $\{Y_t\}$ is stationary. Show that $\nabla^2 X_t$ does not have a trend.

   c) Suppose $\{X_t\}$ is a random walk, $X_t = X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ is i.i.d. noise. Show that ∇X_t is stationary.

   d) Suppose $\{X_t\}$ is a random walk with drift, $X_t = \delta + X_{t-1} + \varepsilon_t$, where δ is a constant, and $\{\varepsilon_t\}$ is i.i.d. noise. Show that ∇X_t is stationary.

2. The backshift operator is defined as $B(X_t) = X_{t-1}$. Express the following using backshift notation.

   a) An $MA(3)$ process $X_t = \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3} + \varepsilon_t$, where θ_i are constants, and $\{\varepsilon_t\}$ is i.i.d. noise.

   b) An $AR(3)$ process $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \varepsilon_t$, where ϕ_i are constants, and $\{\varepsilon_t\}$ is i.i.d. noise.

   c) An $ARMA(2,3)$ process $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3} + \varepsilon_t$, where ϕ_i and θ_i are constants, and $\{\varepsilon_t\}$ is i.i.d. noise.

3. Let $\{X_t\}$ be a sequence (not necessarily $AR(p)$) with mean μ and lag 1 autocovariance $\rho(1) = \phi$.

   a) The best linear predictor of X_t using X_{t-1} is $X_t^{(f)} = \mu + \phi(X_{t-1} - \mu)$.
   Show that $E(X_t - X_t^{(f)})^2 = (1 + \phi^2)\gamma(0) - 2\phi\gamma(1)$.

   b) The best linear predictor of X_{t-2} using X_{t-1} is $X_{t-2}^{(b)} = \mu + \phi(X_{t-1} - \mu)$.
   $E(X_{t-2} - X_{t-2}^{(b)})^2 = (1 + \phi^2)\gamma(0) - 2\phi\gamma(1)$.