Lecture 4

Stationarity
Textbook Sections: 1.3, 1.4

Stationarity

We say a sequence $\{X_t : t \in \mathbb{N}\}$ is **strictly stationary** if it satisfies

$$P(X_{t_1} \le c_1, X_{t_2} \le c_2, \cdots, X_{t_k} \le c_k) = P(X_{t_1+h} \le c_1, X_{t_2+h} \le c_2, \cdots, X_{t_k+h} \le c_k)$$

for all positive integers k, all sets of time indices $t_1, t_2, \cdots, t_k$, all sets of real values $c_1, c_2, \cdots, c_k$, and all time lags h.

This means that the joint distribution of any subset of variables X_t in the sequence remains invariant if the indices of the subset of random variables are shifted by h. Strict stationarity is very hard to verify from data, so this definition of stationarity is rarely used in practice.

A more practical concept is **weak stationarity**. A sequence $\{X_t : t \in \mathbb{N}\}$ is said to be weakly stationary if

(i) $E(X_t) = \mu_t = \mu$, and

(ii) $Cov(X_t, X_{t+h}) = \gamma(h)$ for all $h = 0, 1, 2, \cdots$.

In words, the conditions are

(i) the expected value is constant, and does not depend on time t, and

(ii) the covariance of two random variables in the sequence depends only on the lag h, and not on time t.

Keep in mind that $\gamma(0) = Cov(X_t, X_t) = Var(X_t)$, so the second condition includes the requirement that variance is constant. Sometimes the definition of weak stationarity is written using three conditions instead of two: constant mean, constant variance, and autocovariance that only depends on lag.

All strictly stationary sequences with finite variance are weakly stationary, but the converse is not true.

From now on, the term “stationarity” wlll refer to weak stationarity.

Useful Properties

A linear combination of uncorrelated stationary sequences will also be stationary.

If a sequence $\{X_t : t \in \mathbb{N}\}$ is stationary with mean μ, then the centered sequence $\{X_t^{(c)} = X_t - \mu\}$ is also stationary with mean zero. The two sequences $\{X_t^{(c)}\}$ and $\{X_t\}$ have the same autocovariance and autocorrelation functions.

Some Processes

1. A sequence $\{X_t : t \in \mathbb{N}\}$ is called **i.i.d. noise** if it is composed of i.i.d. random variables with mean 0 and variance σ^2. We will denote this by $\{X_t\} \sim \text{IID}(0, \sigma^2)$.
   This sequence is stationary.

2. A sequence $\{X_t : t \in \mathbb{N}\}$ is called **white noise** if it is composed of uncorrelated random variables with mean 0 and variance σ^2. We will denote this by $\{X_t\} \sim \text{WN}(0, \sigma^2)$.
   This sequence is stationary.

3. A sequence $\{X_t : t \in \mathbb{N}\}$ is called **a moving average of order 1,** or **MA(1)** if it has the form
   $$X_t = \theta\varepsilon_{t-1} + \varepsilon_t,$$
   where $\theta \in \mathbb{R}$ and $\{\varepsilon_t\} \sim \text{IID}(0, \sigma^2)$.
   This sequence is stationary.

4. A sequence $\{X_t : t \in \mathbb{N}\}$ is called **a moving average of order** q**,** or **MA(q)** if it has the form
   $$X_t = \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q} + \varepsilon_t,$$
   where $\theta_1, \theta_2, \cdots, \theta_q \in \mathbb{R}$ and $\{\varepsilon_t\} \sim \text{IID}(0, \sigma^2)$.
   This sequence is stationary.

5. A sequence $\{X_t : t \in \mathbb{N}\}$ is called **an autoregression of order 1,** or **AR(1)** if it has the form
   $$X_t = \phi X_{t-1} + \varepsilon_t,$$
   where $\phi \in \mathbb{R}$ and $\{\varepsilon_t\} \sim \text{IID}(0, \sigma^2)$.
   This sequence is stationary for some values of ϕ, and not stationary for others.

6. A sequence $\{X_t : t \in \mathbb{N}\}$ is called **an autoregression of order** p**,** or **AR(p)** if it has the form
   $$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \varepsilon_t,$$
   where $\phi_1, \phi_2, \cdots, \phi_p \in \mathbb{R}$ and $\{\varepsilon_t\} \sim \text{IID}(0, \sigma^2)$.
   This sequence is stationary for some values of $\phi_1, \cdots, \phi_p$, and not stationary for others.

7. A sequence $\{X_t : t \in \mathbb{N}\}$ is called **a random walk** if it has the form
   $$X_t = X_{t-1} + \varepsilon_t,$$
   where $\{\varepsilon_t\} \sim \text{IID}(0, \sigma^2)$.
   This sequence is not stationary.

Checking Stationarity

1. Suppose $\{X_t\} \sim \text{WN}(0, \sigma^2)$. Then we have
$$E(X_t) = 0, \qquad \gamma(h) = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \neq 0. \end{cases}$$
Since $E(X_t)$ and $\gamma(h)$, $h = 0, 1, 2, \cdots$ do not depend on t, the sequence is stationary.

2. Suppose $\{X_t\}$ is MA(1). Then we have $X_t = \theta\varepsilon_{t-1} + \varepsilon_t$, where $\theta \in \mathbb{R}$, and $\{\varepsilon_t\} \sim \text{IID}(0, \sigma^2)$. As shown in lecture 3,
$$\begin{aligned} E(X_t) &= E(\theta\varepsilon_{t-1} + \varepsilon_t) = \theta E(\varepsilon_{t-1}) + E(\varepsilon_t) = \theta \cdot 0 + 0 = 0, \\ \gamma(h) = Cov(X_t, X_{t+h}) &= \begin{cases} (1+\theta^2)\sigma^2 & h = 0 \\ \theta\sigma^2 & h = 1 \\ 0 & h \geq 2. \end{cases} \end{aligned}$$
Since $E(X_t)$ and $\gamma(h)$, $h = 0, 1, 2, \cdots$ do not depend on t, the sequence is stationary.

3. Suppose $\{X_t\}$ is a random walk. So $X_t = X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\} \sim \text{IID}(0, \sigma^2)$. Observe that
$$\begin{aligned} X_t = X_{t-1} + \varepsilon_t &= (X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= ((X_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1}) + \varepsilon_t \\ &= (((X_{t-4} + \varepsilon_{t-3}) + \varepsilon_{t-2}) + \varepsilon_{t-1}) + \varepsilon_t \\ &\cdots \\ &= \sum_{i=1}^{t} \varepsilon_i. \end{aligned}$$
We then have
$$E(X_t) = E\left(\sum_{i=1}^{t} \varepsilon_i\right) = \sum_{i=1}^{t} E(\varepsilon_i) = 0,$$
and
$$\begin{aligned} Cov(X_t, X_{t+h}) &= Cov\left(\sum_{i=1}^{t} \varepsilon_i, \sum_{j=1}^{t+h} \varepsilon_j\right) \\ &= Cov\left(\sum_{i=1}^{t} \varepsilon_i, \sum_{j=1}^{t} \varepsilon_j\right) \\ &= \sum_{i=1}^{t} \sum_{j=1}^{t} Cov(\varepsilon_i, \varepsilon_j) \\ &= \sum_{i=1}^{t} Cov(\varepsilon_i, \varepsilon_i) \\ &= \sum_{i=1}^{t} Var(\varepsilon_i) = t\sigma^2. \end{aligned}$$
Since $Cov(X_t, X_{t+h})$, $h = 0, 1, 2, \cdots$ depends on t, the sequence is NOT stationary.