Lecture 10

Autoregressive Processes as Models
Textbook Sections: 2.1-2.3

Definition and Notation

A zero-mean sequence $\{X_t\}$ is said to be an autoregressive process of order p if it is of the form

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t, \tag{1}$$

where $\{\varepsilon_t\}$ is white noise with variance σ^2. Similarly, a sequence $\{X_t\}$ with mean μ is said to be an autoregressive model of order p if it is of the form

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \cdots + \phi_p(X_{t-p} - \mu) + \varepsilon_t, \tag{2}$$

where $\{\varepsilon_t\}$ is mean zero while noise with variance σ^2.

Note that model (2) can be simply rewritten as

$$X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t,$$

where $\phi_0 = \mu(1 - \phi_1 - \cdots - \phi_p)$. This is simply a linear regression model with the dependent variable X_t and the independent variables $X_{t-1}, ..., X_{t-p}$. **This model is particularly simple for forecasting.**

Note that the autocovariances and autocorrelations of the two sequences given in (1) and (2) are the same.

To keep the notation simple, we will focus on model (1). If the data appears to come from a population with non-zero mean, we can always center the data by subtracting $\bar{X}$, and fit a zero-mean model to the centered data.

Choosing the Order p

Typically, the order p is not known in advance, and must be chosen based on the data. We can select the order by fitting several AR models with different orders p, and choosing among them using a model selection criterion such as AIC, BIC, or $AICc$.

Using $AICc$ for this purpose is common in practice, and is recommended by the authors of our textbook. However, it is not a bad idea to use several criteria, and check if they agree.

Keep in mind that both BIC and $AICc$ have a stronger penalty for the number of parameters in the model than AIC does. This means that AIC may lead to the selection of a more complicated model.

Finally, when selecting a model, you should actually look at the criterion values of at least the top few model choices (if not all the models under consideration). If two top models have very similar criterion values, inspect both fits. If there isn't much difference, then it may be better to use the simpler model even if it's criterion value is slightly worse.

The AR Polynomial and Stationarity

Conditions on the autoregressive coefficients $\phi_1, ..., \phi_p$ are needed to guarantee that the sequence is stationary. Define the autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ be a polynomial in z. This polynomial has p roots which can be real or complex valued.

An autoregressive process is stationary if and only if the absolute values of all the roots of the AR polynomial are larger than 1.

When $p = 1$, the root of the polynomial is $1/\phi_1$. The condition that the absolute value of $1/\phi_1$ is larger than 1 is equivalent to the condition $-1 < \phi_1 < 1$.

Be careful when writing out the AR polynomial of a given process. It is possible for an $AR(p)$ process to have $\phi_j = 0$ for some $j < p$, and this will be reflected in the AR polynomial missing the term z^j. For example, if $X_t = 0.1X_{t-1} - 0.2X_{t-3} + \varepsilon_t$, then $\phi(z) = 1 - 0.1z + 0.2z^3$. Notice that there is a zero coefficient for X_{t-2} in the definition of X_t, and a zero coefficient for z^2 in the polynomial.