STA13: Elementary Statistics

Lecture 16
Book Sections 6.1 - 6.6

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Hypotheses Tests

Sometimes we already have an idea about the parameter.

We may wish to see...

- How well does the data in the sample match our existing guess?
- How valid does the guess seem now that we have seen sample data?

Hypothesis tests are a tool that allows us to do that.

Elements of Hypotheses Tests

These are the main elements involved in hypothesis tests.

It's a good idea to approach problems with this list in mind.

- Significance level
- Parameter of interest
- Two Hypotheses
- Test Statistic
- P-value
- Conclusion

Significance Level

The significance level of a test is a number between 0 and 1 that serves as a statement about the reliability of the test.

- Denoted by α. This will match our CI notation.
- α is chosen by the person designing the test.
- Common choices are $\alpha = 0.1$, $\alpha = 0.05$, or $\alpha = 0.01$.
- Smaller α leads to a more reliable test.
- α is a statement about the reliability of the testing procedure, not about the reliability of results from any one sample.

Parameter

The first thing to identify is the parameter of interest.

We'll deal with hypothesis tests for

- the population mean μ
- the population proportion p
- difference of two means $\mu_1 - \mu_2$
- difference of two proportions $p_1 - p_2$

We'll first focus on μ and p, then extend the methods to the differences after Midterm 2.

Hypotheses

There are always two competing hypotheses.

- The Null hypothesis (H_0) is the existing belief or statement about the value of the parameter. This is the "status quo."

- The Alternative hypothesis (H_A) is a competing belief about the value of the parameter. This is sometimes called the research hypothesis.

H_0 is pronounced "H not", and H_A is pronounced "H a".

Null Hypothesis H_0

This is the original statement about the parameter.

- We write
$$H_0 : \mu = \mu_0$$
$$H_0 : p = p_0$$
- $H_0 : \mu = 25$
(The existing belief is that the true mean is 25)
- $H_0 : p = 0.5$
(The existing belief is that the true proportion is 0.5)

Alternative Hypothesis H_A

The alternative hypothesis proposes a way the parameter is different from the value in H_0 (μ_0 or p_0).

There are three options we'll consider:

Parameter	Mean	Proportion
H_A	$\mu < \mu_0$	$p < p_0$
H_A	$\mu > \mu_0$	$p > p_0$
H_A	$\mu \neq \mu_0$	$p \neq p_0$

Two Hypotheses

Any hypothesis test requires two hypotheses.

Possibilities when testing form the mean. Here $\mu_0 = 25$.

- $H_0 : \mu = 25$ and $H_A : \mu < 25$
- $H_0 : \mu = 25$ and $H_A : \mu > 25$
- $H_0 : \mu = 25$ and $H_A : \mu \neq 25$

Possibilities when testing for the proportion. Here $p_0 = 0.5$.

- $H_0 : p = 0.5$ and $H_A : p < 0.5$
- $H_0 : p = 0.5$ and $H_A : p > 0.5$
- $H_0 : p = 0.5$ and $H_A : p \neq 0.5$

The choice will depend on the setting of the problem.

Two Hypotheses

Suppose we choose $H_0 : \mu = 25$ and $H_A : \mu > 25$.

- This would test the claim that the true mean is larger than 25 against the status quo belief that the true mean is 25.
- Check if the observed statistic is significantly larger than what we'd expect if the true mean were 25.
- This is an example of a one-sided test. (Or a one-sided alternative hypothesis).

Two Hypotheses

Suppose we choose $H_0 : \mu = 25$ and $H_A : \mu < 25$.

- This would test the claim that the true mean is smaller than 25 against the status quo belief that the true mean is 25.
- Check if the observed statistic is significantly smaller than what we'd expect if the true mean were 25.
- This is an example of a one-sided test. (Or a one-sided alternative hypothesis).

Two Hypotheses

Suppose we choose $H_0 : \mu = 25$ and $H_A : \mu \neq 25$.

- This would test the claim that the true mean is different from 25 against the status quo belief that the true mean is 25.
- Check if the observed statistic is significantly different from (either larger or smaller) what we'd expect if the true mean were 25.
- This is an example of a two-sided test. (Or a two-sided alternative hypothesis).

Test Statistic

- The test statistic is a summary of the observed data.
- It helps us make a decision about the parameter.
- We check the test statistic for evidence that H_0 is wrong.

Test Statistic

The general form of the test statistic is:

$$\frac{\text{(point est. of parameter) - (mean of point est. according to } H_0\text{)}}{\text{(standard error of point estimate)}}$$

A bit of review:

Parameter	μ	p
Point Estimate	$\bar{x}$	$\hat{p}$
Mean of Pt. Est. by H_0	μ_0	p_0
Standard Error of Pt. Est.	$\frac{s}{\sqrt{n}}$	$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Null Distribution

- Notice that the value of the statistic depends on the null hypothesis and the data.

- The null distribution is the sampling distribution of the statistic assuming the null hypothesis is true.

Test Statistics for the Population Mean

1. Test for mean μ, large sample case
   - setting: $n \geq 30$
   - test statistic: $z = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$
   - null distribution: Standard normal, $N(0,1)$
2. Test for mean μ, small sample case
   - setting: $n < 30$, population is normal (or we assume so)
   - test statistic: $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$
   - null distribution: t distribution with $n-1$ degrees of freedom

Test Statistic for the Population Proportion

Test for population proportion p, large sample

- setting: $n\hat{p} > 15$ and $n(1-\hat{p}) > 15$
- test statistic: $z = \frac{\hat{p}-p_0}{\sqrt{\hat{p}(1-\hat{p})/n}}$
- null distribution: Standard normal, $N(0,1)$

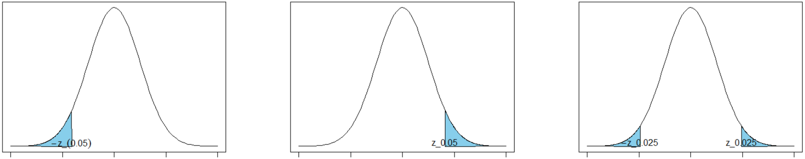
Rejection Region

- We use the test statistic to determine if there is significant evidence against H_0.
- The rejection region is the set of values of the test statistic for which we will reject H_0.
- The rejection region depends on
  - the significance level α
  - the alternative hypothesis H_A

Rejection Region

The rejection region is determined as follows.

Alternative	$H_A : \mu < \mu_0$	$H_A : \mu > \mu_0$	$H_A : \mu \neq \mu_0$
Rejection region	$z < -z_\alpha$	$z > z_\alpha$	$\|z\| > z_{\alpha/2}$



The area under the curve in the rejection region is always α.

P-value

The p-value is the probability of observing a test statistic "as extreme as ours or more" if the null hypothesis H_0 is true.

If our statistic is very improbable (small p-value) under H_0, then we have reason to reject H_0.

The p-value depends on

- the test statistic
- the null distribution of the test statistic
- the alternative hypothesis H_A

P-value

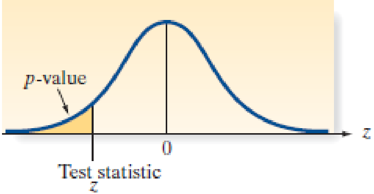
The p-value is the probability of observing a test statistic "as extreme as ours or more" if the null hypothesis H_0 is true.

- Always between 0 and 1.
- The smaller the p-value, the more evidence against H_0.
- In particular, we reject H_0 if the p-value is less than α.

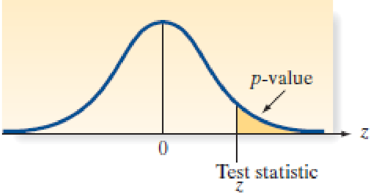
P-value of a Z-statistic

1. Calculate test statistic z.
2. Use the z-table to find the p-value.
   - H_A has $<$: p-value is $P(Z < z)$
   - H_A has $>$: p-value is $P(Z > z)$
   - H_A has $\neq$: p-value is $2P(Z > |z|)$

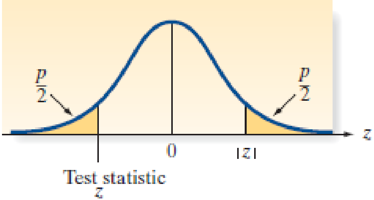
P-value of a Z-statistic



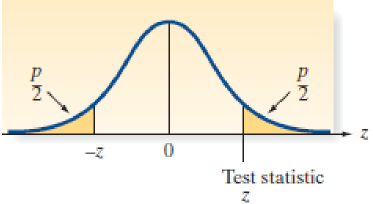
a. Lower–tailed test, H_a: $\mu < \mu_0$



b. Upper–tailed test, H_a: $\mu > \mu_0$



a. Test statistic z negative



b. Test statistic z positive

P-value of a T-statistic

1. Calculate test statistic t.
2. Compute degrees of freedom $n - 1$.
3. Use the t-table to approximate the p-value.
   - H_A has $<$: p-value is $P(T < t)$
   - H_A has $>$: p-value is $P(T > t)$
   - H_A has $\neq$: p-value is $2P(T > |t|)$

The P-value and Making Conclusions

We have two things:

- a guess about the parameter (H_0)
- a summary of observed data (test statistic)

The p-value helps us see how likely it is to observe data like ours if the initial guess about the parameter is correct.

If the data is unlikely (low p-value), then we reject the guess.

Conclusion

Finally we make a conclusion about the parameter. There are two possibilities:

- "Reject H_0 at significance level α."
- "Fail to reject H_0 at significance level α."

Making Conclusions

Reject the null hypothesis H_0 at significance level α if

- p-value $< \alpha$
- test statistic is in the rejection region

Fail to reject the null hypothesis H_0 at significance level α if

- p-value $> \alpha$
- test statistic is not in the rejection region

Conclusion and α

Keep in mind that...

- The conclusion depends on the significance level (α), so it is important to include the value of α.

- For the same sample and H_0, it is possible to reject H_0 at a larger α, but fail to reject H_0 at a smaller α.

- The significance level α describes the confidence associated with the testing procedure, not with the results of any one specific sample.