STA13: Elementary Statistics

Lecture 6
Book Sections 3.1-3.4

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Probability as a Tool

Probability is a **tool** that helps us deal with situations that involve

- randomness
- uncertainty
- chance variation

Such situations arise all over the place:

- Quality control
- Genetics
- Finance and Economics
- Weather Forecasting
- Health Insurance
- Casinos
- Sports Predictions
- Speech and Facial Recognition
- many, many, many more

The Basics

Now we'll look at simpler examples in more detail.

Some basic questions we may want to answer:

- In simple terms, what is being done?
- What are the possible outcomes?
- Are some outcomes more likely than others?
- Might it be useful to group the outcomes?

The Basics

- In simple terms, what is being done?
  Die: A fair six-sided die is rolled once.
  Coin: A fair coin is flipped once.
- What are the possible outcomes?
  Die: 1, 2, 3, 4, 5, 6
  Coin: Heads(H), Tails(T)
- Are some outcomes more likely than others?
  Die: All are equally likely since die is fair.
  Coin: All are equally likely since coin is fair.
- Might it be useful to group the outcomes?
  Die: Odds/Evens? Multiples of three?
  Coin: Can't really group

Experiment

Probability is used in situations where outcomes occur randomly. Such situations are called experiments.

Examples:

- Toss two fair coins once
- Draw a 5-card hand from a shuffled deck

Sample Space

The set of all possible outcomes is called the sample space.
It is usually denoted as S={outcome 1, outcome 2, ... }

Examples:

- Toss a coin
  S={HH, HT, TH, TT}
- Draw a 5-card hand from a shuffled deck
  S is the set of all possible five-card hands.

Simple Events

A simple event is the outcome resulting when an experiment is performed just once.

Examples:

- Toss a coin
  Simple events: HH, HT, TH, TT
- Draw a five-card hand from a shuffled deck
  Simple events: all possible five-card hands

Simple Events

One and only one simple event occurs each time an experiment is performed once.

We can assign probabilities to individual simple events.

We can also group simple events into more complex events.

Event

A collection (grouping) of simple events is called an event.

An event is just a subset of the sample space.

Events are usually denoted by capital English letters.

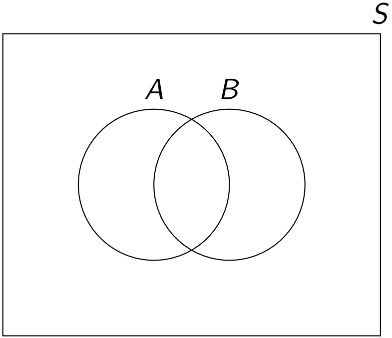
Examples:

- Toss a coin
  both coins come up the same (HH or TT)
- Draw a 5-card hand from a shuffled deck
  Event A: all cards are of the same suit
  Event B: there is at least one king

Event

It's useful to use Venn diagrams to visualize events

Suppose a fair die is tossed. Where do the simple events belong on the diagram?



A: The number facing up is even.

B: The number facing up is less than five.

Mutually Exclusive

Two events are called mutually exclusive, or disjoint, if they cannot occur at the same time. (If one occurs, the other cannot occur)

Examples:

- Toss a coin
  Heads / Tails
- Draw a 5-card hand from a shuffled deck
  Getting 4 kings / Getting 3 aces
  Getting a straight / Getting a 2-of-a-kind

Probability

We can think of the probability of an event as the likelihood that the event will occur, or as the frequency with which the event occurs.

As sample size n becomes "large:"

Relative frequency $\rightarrow$ Probability

Properties of Probability

- Probability of an event is a number between 0 and 1.
- The probabilities of all the simple events add up to 1.
- The probability of an event A is denoted by P(A)
- P(A) is the sum of the probabilities of all the simple events that make up A.
- P(A)=1 means event A always occurs.
- P(A)=0 means event A never occurs.

Equal Probabilities

The simplest case: all simple events are equally likely

$N =$ number of simple events

Probability of any simple event: $\frac{1}{N}$

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

We will cover rules that make it easier to calculate N and n_A.

Equal Probabilities

Example 1: Roll a fair die once

- $S = \{1, 2, 3, 4, 5, 6\}$ and $N = 6$
- A $=$ an odd number comes up
  A$=\{1,\ 3,\ 5\}$
  $n_A = 3$
  $P(A) = \frac{3}{6} = \frac{1}{2}$
- B $=$ a multiple of three comes up
  B$=\{3,\ 6\}$
  $n_B = 2$
  $P(B) = \frac{2}{6} = \frac{1}{3}$

Equal Probabilities

Example 2: Toss two fair coins once

- $S = \{HH, HT, TH, TT\}$ and $N = 4$
- A = both coins come up the same
  A={HH, TT}
  $n_A = 2$
  $P(A) = \frac{2}{4} = \frac{1}{2}$
- B = at least one coin comes up tails
  B={TH, HT or TT}
  $n_B = 3$
  $P(B) = \frac{3}{4}$

Set Relations

Next time we'll cover counting rules to calculate n_A and N.

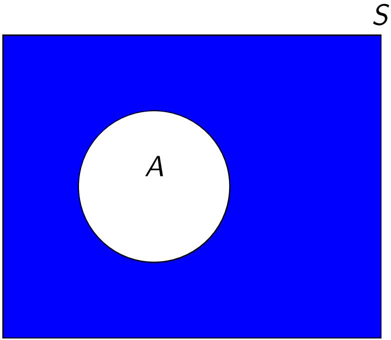
Now we need methods to answer questions like:

- What is the probability that A does NOT occur?
- What is the probability that A and B both occur?
- What is the probability that either A or B occurs?

Complement

The complement of an event A is the event which occurs when A does not occur. It is denoted by A^c, and it consists of all the simple events that are not in A.

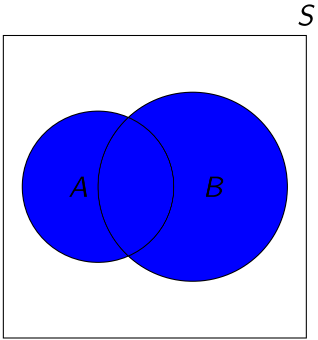
A^c is the blue part:



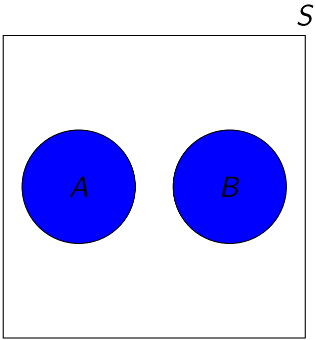
Union

The union of events A and B is the event that occurs when at least one of A and B occurs. It is denoted by $A \cup B$, and it consists of all of the simple events that are either in A or in B or in both A and B.

$A \cup B$ is the blue part:



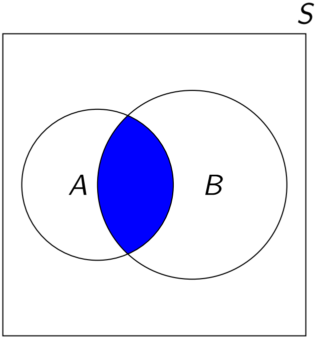
$A \cup B$ is the blue part:



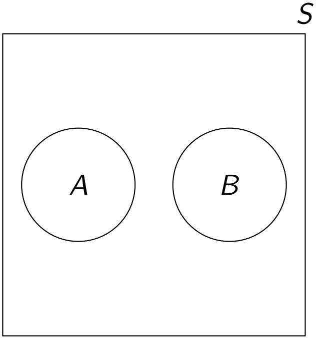
Intersection

The intersection of events A and B is the event that occurs when both A and B occur. It is denoted by $A \cap B$, and it consists of all of the simple events that are in both A and B.

$A \cap B$ is the blue part:



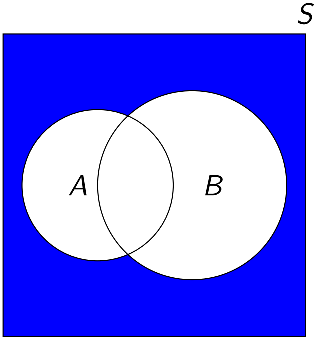
$A \cap B$ is the blue part:



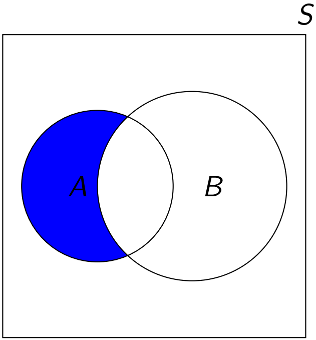
Properties

Complements, unions and intersections can be combined.

$(A \cup B)^c$ is the blue part:



$A \cap B^c$ is the blue part:



Properties

Here are a couple useful properties:

$$A = (A \cap B) \cup (A \cap B^c)$$

$$A \cap B = (A^c \cup B^c)^c$$

$$A^c \cap B^c = (A \cup B)^c$$

Complement Property

The probabilites of all the simple events always add up to 1.

$P(A) =$ sum of probabilities of all simple events in A

$P(A^c) =$ sum of probabilities of all simple events not in A

Putting these facts together we get:

$$P(A^c) = 1 - P(A)$$

Union and Intersection

The formula for $P(A \cup B)$ is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

By rearranging the terms, we can also get:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Special case:

If A and B are disjoint, then:

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

Back to Probability

Keep in mind that A^c, $A \cup B$ and $A \cap B$ are also events.

They are subsets of the sample space and they have probabilities.

We can compute their probabilities in two ways:

- Add up the probabilities of the simple events that fall into each event.
- Use a formula

Example

In a group of 30 students, 12 belong to the Spanish club, 7 belong to the Ski club, and 2 belong to both.

One of the 30 students is selected at random.

What is the probability that the student belongs to at least one of the two clubs?

The given information can be expressed as:

A = the student is in Spanish club
B = the student is in Ski Club

$P(A) = \frac{12}{30} = 0.4$
$P(B) = \frac{7}{30} = 0.233$
$P(A \cap B) = \frac{2}{30} = 0.067$

The answer to the question is:
$P(A \cup B) = P(A) + P(B) - (A \cap B) = 0.566$

Example

Same setup:

In a group of 30 students, 12 belong to the Spanish club, 7 belong to the Ski club, and 2 belong to both.

One of the 30 students is selected at random.

What is the probability that the student belongs to at least one of the two clubs?

Why do we need to subtract $P(A \cap B)$?

When is it wrong to write $P(A \cup B) = P(A) + P(B)$?

When is it correct to write $P(A \cup B) = P(A) + P(B)$?