STA13: Elementary Statistics

Lecture 7
Book Sections 3.5-3.7

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Probability Framework

Last time we covered the basic probability framework:

- Experiment
- Sample space
- Simple event
- Event
- Probability of an event

Equal Probabilities

When all the simple events have equal probabilities:

- N = number of simple events
- Probability of any simple event: $\frac{1}{N}$
- Probability of event A:

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

The Multiplicative Rule

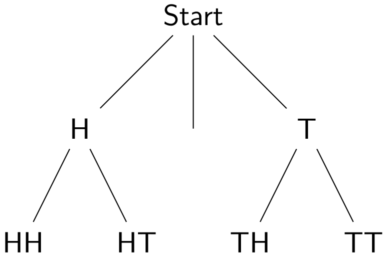
Sometimes an experiment has two stages.

- m possible outcomes for the first stage
- n possible outcomes for the second stage
- mn ways the experiment can proceed

The rule is sometimes called "the mn rule."

The Multiplicative Rule

Example: Toss a two fair coins once



Stage 1: Toss 1st coin
2 outcomes

Stage 2: Toss 2nd coin
2 outcomes

Total:
$2 \times 2 = 4$ outcomes

The Multiplicative Rule

The *mn* rule can easily be extended to more stages:

- k stages
- n_1 outcomes for 1^{st} stage,
  n_2 outcomes for 2^{nd} stage,...
  n_k outcomes for k^{th} stage
- Total number of ways the experiment can proceed:
  $n_1 \times n_2 \times \cdots \times n_k$

If 4 fair dice are tossed, how many outcomes are there?

Factorial

We will make use of the factorial notation:

$n! = n \times (n-1) \times (n-2) \times \ldots \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Special case: $0! = 1$

Combinations

Some experiments consist of choosing k out of n possibilities, where only the selection matters, and not the order.

The number of ways to choose k out of n objects in any order:

$$_nC_k = \frac{n!}{(n-k)!k!}$$

Pronounced "n choose k"
Written $_nC_k$ or $\binom{n}{k}$

n = total number of objects/possibilities
k = how many of those are chosen

Combinations

When does order notmatter?

- Choosing 3 out of 10 candidates to fill three secretary positions
- Drawing five cards from a deck

Counting Rules

To find N and n_A we need to count simple events.

If the sample space is large, counting them may be difficult.

There are some rules that help us count events:

- The multiplicative rule
- Number of ways to choose k out of n options
  - if order matters:
    $$ {}_nP_k = \frac{n!}{(n-k)!} $$
  - if order does not matter:
    $$ {}_nC_k = \frac{n!}{(n-k)!k!} $$

Example 1

How many ways are there to choose a five-card hand from a shuffled deck of cards?

- How many options are there? $n = 52$
- How many of them are we choosing? $k = 5$
- Does order matter? No.

Answer:

$$_{52}C_5 = \frac{52!}{(52-5)!5!} = \frac{52!}{47!5!} = 2598960$$

Example 2

You draw a five-card hand from a shuffled deck of cards. What is the probability that the hand will contain at least one king?

- What is the sample space?
- Are the simple events equally likely?
- What is the event of interest? (A)
- How many simple events are there? (N)
- How many simple events are in A? (n_A)

Example 2

You draw a five-card hand from a shuffled deck of cards. What is the probability that the hand will contain at least one king?

- What is the sample space? All five-card hands.
- Are the simple events equally likely? Yes.
- What is the event of interest? (A)
  A is the event that the hand contains at least one king.
- How many simple events are there? (N)
  $N =_{52} C_5 = 2598960$
- How many simple events are in A? (n_A) On next slide.

Example 2

Number of hands with *exactly one* king - use the *mn* rule:

$$\underbrace{\text{number of ways to choose one king}}_{m} \times \underbrace{\text{number of ways to choose four non-kings}}_{n}$$

We can then find the number of hands with exactly

one king: ${}_4C_1 \times {}_{48}C_4 = 778320$
two kings: ${}_4C_2 \times {}_{48}C_3 = 103776$
three kings: ${}_4C_3 \times {}_{48}C_2 = 4512$
four kings: ${}_4C_4 \times {}_{48}C_1 = 48$

$n_A = 778320 + 103776 + 4512 + 48 = 886656$

Answer:

$$P(A) = \frac{n_A}{N} = \frac{886656}{2598960} \approx 0.34$$

Another Look at Example 2

A five-card hand is drawn from a shuffled deck of cards. A is the event that the hand contains at least one king. Find $P(A)$.

Finding the number of simple events in A was tedious.

Let's find the number of simple events in A^c instead.

Another Look at Example 2

What is A^c?

- the event that the hand contains no kings
- the set of all hands that contain no kings

How many simple events are in A^c?

- number of hands with no kings
- number of ways to choose 5 cards out of the 48 cards that are not kings
- $\binom{48}{5} = 1712304$

$P(A^c) = \frac{1712304}{2598960} \approx 0.66$

Answer: $P(A) = 1 - P(A^c) \approx 0.34$

Conditional Probability

Consider the following:

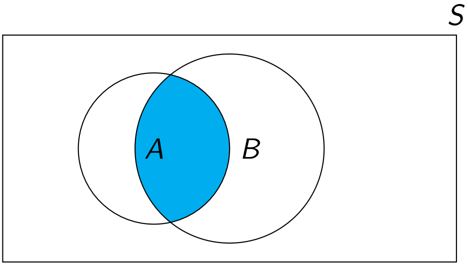
- P(Dmitriy gets 100 on midterm II)
- P(Dmitriy gets 100 on midterm II, given that he studied really really hard)
- P(Dmitriy gets 100 on midterm II, given that he ate a sandwich today)

Conditional Probability

The conditional probability of A given B is the probability that A occurs given that B has already occurred.

It is written as $P(A|B)$, and computed as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$



Bayes Rule

The formula on the previous slide is called Bayes' Rule. Here are some useful properties:

- denominator is probability of the "given" event: $P(B|A) = \frac{P(A \cap B)}{P(A)}$ if $P(A) \neq 0$
- rearrange: $P(A \cap B) = P(A|B) \times P(B)$
- similarly: $P(A \cap B) = P(B|A) \times P(A)$
- rearrange some more: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Example

The table below contains information for 500 randomly selected US males aged 50-60.

	high cholesterol (LDL)	low cholesterol (LDL)
heart attack	130	50
no heart attack	40	280

A single male is randomly chosen from the sample of 500.

What is the probability that he has had a heart attack?

What is the probability that he has had a heart attack given that he has high LDL levels?

Independence

Events A and B are independent if the probability of A occurring is not influenced by whether or not B occurs.

Otherwise A and B are called dependent.

A and B are independent if and only if $P(A) = P(A|B)$ (if $P(B) \neq 0$).

Independence

Properties of independence:

- If A and B are independent, then the following are independent:
  - B and A
  - A and B^c
  - A^c and B
  - A^c and B^c

- S is independent with any event A. Why?

- If A, B are independent, then $P(A \cap B) = P(A)P(B)$.

Recap

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  Special case: $P(A \cup B) = P(A) + P(B)$ if A, B disjoint
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$ if $P(B) \neq 0$
- A and B are independent if and only if $P(A) = P(A|B)$
- $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$
  Special case: $P(A \cap B) = P(A)P(B)$ if A, B indep.