A GEOMETRIC APPROACH TO CATLIN'S BOUNDARY SYSTEMS

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Contents

1. Introduction

1.1. **Overview and motivation.** This is the first paper in the series aiming at understanding geometric invariants behind the tools developed by Catlin in his celebrated papers [C84a, C84b, C87] in order to obtain a priori estimates for the $\bar{\partial}$ operator, motivated by previous foundational work of Kohn [K64a, K64b, K79].

A geometric aspect of Catlin's a priori estimates proof consists of showing the existence of *weight functions* satisfying certain boundedness and positivity estimates for their complex Hessians, that are known as "Property (P)" type conditions (see e.g. [BS16] for a recent survey). A major difficulty when constructing such weight functions under geometric conditions (such as D'Angelo's finite type [D82]), is to keep the uniform nature of the estimates across points of *varying "degeneracy"* for the underlying geometry. A simple example of a degeneracy measure is the *rank of the Levi form* of the boundary $M := \partial D$ (where D is a domain in $\mathbb{C}^n$). A more refined measure is Catlin's multitype [C84b], see also §5.4. To deal with points of varying multitype, Catlin developed his machinery of boundary systems [C84b] to gain control of the multitype level sets by including them locally into certain *"containing submanifolds"* transversal to the Levi form kernel. A result of this type is the content of [C84b, Main Theorem, Part (2)], where a containing submanifold is constructed by a collection of inductively chosen *boundary system functions* that arise as certain carefully selected (vector field) derivatives of the Levi form.

In this paper we focus on geometric invariants behind the containing submanifold construction, which in some way extend and simplify the boundary system approach. To make things more concrete and explicit, we restrict ourselves to the 4th order invariants that, in particular, cover the cases of pseudoconvex hypersurfaces of the finite type at most 4, where the multitype level sets boil down to simpler *level sets of the Levi (form) rank* (see Proposition 5.13 for details). Recall that Catlin's boundary system functions [C84b] are constructed inductively with every new equation depending on chosen solutions for previous ones. In comparison, we here collect natural defining functions for the Levi rank level sets into *invariant ideal sheaves* $\mathcal{I}(q)$ on M, for each Levi rank q. The sheaf $\mathcal{I}(q)$ is generated by certain 1st order Levi form derivatives as described in Theorem 2.1 below. In particular, arbitrary defining functions from $\mathcal{I}(q)$ can be combined without any additional relations. Furthermore, additional derivatives of the Levi form along arbitrary complex vector fields L^3 (in Theorem 2.1, Part (5)), including transversal ones, are allowed for functions in $\mathcal{I}(q)$. In comparison, for a related boundary system function given by the same formula, the outside vector field L^3 would have to be in a special subbundle inside the holomorphic tangent bundle. As a result, we obtain richer classes of defining functions allowing for more control over containing submanifolds (see Example 2.2), that may potentially lead to sharper a priori estimates.

In parallel to the ideal sheaf $\mathcal{I}(q)$ construction, we introduce *invariant quartic tensors* τ^4, giving a precise control over differentials of the functions in $\mathcal{I}(q)$. This is expressed in Theorem 2.1, part (2), where the tangent space of the containing manifold S equals the real kernel of the tensor. Importantly, the full tangent space of S (rather than only the tangential part) is controlled here via the kernel of τ^4, which means that transversal vector fields must also be allowed among tensor arguments. The tensors are constructed in Lemma 4.13 as certain 2nd order Levi form derivatives taken along all possible vector fields. In comparison, only derivatives with respect to $(1,0)$ and $(0,1)$ vector fields can appear in the boundary systems.

For reader's convenience, we summarize the main results and constructions in Theorem 2.1, leaving more detailed and general statements with their proofs in the chapters following.

1.2. **More details on invariant tensors and sheaves.** Our first step in defining invariant tensors is a byproduct result giving a *complete set of cubic invariants* for a general real hypersurface M, without pseudoconvexity assumption. This is achieved by constructing an invariant cubic tensor τ^3 obtained by differentiating the Levi form along vector fields with values in the Levi kernel, see Lemma 3.3. Remarkably, to obtain tensoriality of the Levi form derivatives, it is of crucial importance to require *both vector fields inside the Levi form to take values in the Levi kernel* as explained in Example 3.1. This stands further, in remarkable contrast with the cubic tensor ψ_3 defined by Ebenfelt [E] (by means of the Lie derivatives of the contact form), where only one of the arguments needs to be in the Levi kernel. On the other hand, Ebenfelt's tensor ψ_3 does not allow for transversal directions as τ^3 does. It turns out that the pair (ψ_3, τ^3) yields a complete set of cubic invariants, as demonstrated by a normal form (of order 3) in Proposition 3.4 eliminating all other terms that are not part of either of the tensors.

We also investigate the construction based on double Lie brackets (also considered by Webster [W95]). This approach, however, forces all vector fields to be in the complexified holomorphic tangent bundle, leading only to a restriction of the tensor τ^3. Again, the double Lie bracket construction is only tensorial when both vector fields inside the inner bracket take their values in the Levi kernel (see Example 3.1).

As mentioned earlier, the cubic tensor τ^3 is constructed without any pseudoconvexity assumption. On the other hand, in presence of pseudoconvexity, the whole tensor τ^3 must vanish identically (Lemma 3.11). The only cubic terms that may survive are of the form (3.12) which can never appear in the lowest weight terms, and hence never play a role in Catlin's multitype and boundary system theory.

Motivated by the above, we next look for quartic tensors. It turns out (Example 4.1) that this time, neither second order Levi form derivatives nor quartic Lie brackets provide tensorial invariants even when all vector field arguments take their values in the Levi kernel. To overcome this problem, we *restrict the choice of the vector fields involved* by requiring a certain kind of condition of "Levi kernel inclusion up to higher order" (Definition 4.2). In Lemma 4.6 we show that this additional condition always holds for any vector field that is Levi-orthogonal to a maximal Levi-nondegenerate subbundle, which, in particular, are some of the vector fields in a Catlin's boundary system. However, the mentioned Levi-orthogonality lacks some invariance as it depends on the choice of the subbundle. In contrast, the Levi kernel inclusion up to order 1 is invariant and only depends on the 1-jet of the vector field at the reference point.

With that restriction in place, an invariant quartic tensor τ^4 can now be defined in a similar fashion. Then it's restriction τ^{40} enters the lowest weight normal form with weights $\geq 1/4$, see Proposition 4.15. It turns out, the restriction τ^{40} provides exactly the missing information at the lowest weight level for hypersurfaces of finite type 4 (where finite type 3 cannot occur for pseudoconvex points in view of Corollary 3.12). For example, both D'Angelo finite type 4 and Catlin's multitype up to entry 4 can be completely characterized in terms of τ^{40}. In fact, having the finite type 4 is equivalent to the nonvanishing of τ^{40} on complex lines (Proposition 5.1), whereas having a multitype up to entry 4 is equivalent to τ^{40} having trivial kernel (Propositions 5.13).

In §6 we use the quartic tensor τ^4 to characterize the differentials of functions in the ideal sheaf $\mathcal{I}(q)$ as well as the minimal tangent spaces of containing manifolds defined by a transversal set of functions in $\mathcal{I}(q)$.

Finally, in §7 we obtain a characterization for a Catlin's boundary system, where the most difficult part of obtaining vector field directions of nonvanishing Levi form derivatives at the lowest weight is replaced by the nonvanishing of the tensor τ^{40} on the vector fields' values at the reference point, a purely algebraic condition.

In a forthcoming paper will shall extend the present geometric approach towards its *approximate versions* with necessary control to perform the induction step in the subelliptic estimate proof.

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2. Notation and main results

We shall work in the smooth (C^∞) category unless stated otherwise. Let $M \subset \mathbb{C}^n$ be a real hypersurface. We write $T := TM$ for its tangent bundle, $H = HM \subset T$ for the complex (or holomorphic) tangent bundle, $Q := T/H$ for the normal bundle, as well as

$$\mathbb{C}T := \mathbb{C} \otimes T, \quad \mathbb{C}H := \mathbb{C} \otimes H, \quad \mathbb{C}Q := \mathbb{C} \otimes Q,$$

for their respective complexifications. Further, H^{10} and $H^{01} = \overline{H^{10}}$ denote $(1,0)$ and $(0,1)$ bundles respectively, such that $\mathbb{C}H = H^{10} \oplus \overline{H^{10}}$. By a slight abuse of notation, we write $L \in V$ when a vector field L is a local section in V, which can be a bundle or a sheaf.

On the dual side, Ω stands for the bundle of all real 1-forms, $\mathbb{C}\Omega$ for all complex 1-forms, Ω^0 for all *contact forms*, i.e. forms form Ω that are vanishing on H and real-valued on T, and $\mathbb{C}\Omega^0$ for the corresponding complexification.

Recall that a *(local) defining function* of M is any real-valued function ρ with $d\rho \neq 0$ such that M given by $\rho = 0$. For any defining function ρ, the one-form $\theta := i\partial\rho$ spans (over $\mathbb{R}$) the bundle Ω^0 of all contact forms.

We shall consider the standard pairing $\langle \theta, L \rangle := \theta(L)$ for $\theta \in \mathbb{C}\Omega$, $L \in \mathbb{C}T$. By a slight abuse, we keep the same notation also for the pairing

$$\langle \cdot, \cdot \rangle \colon \mathbb{C}\Omega^0 \times \mathbb{C}Q \to \mathbb{C}$$

between the (complex) contact forms and the normal bundle. With this notation, we regard the *Levi form* at a point $p \in M$ as $\mathbb{C}$-bilinear map

$$\tau^2_p \colon H^{10}_p \times \overline{H^{10}_p} \to \mathbb{C}Q_p,$$

which is uniquely determined by the identity

$$\text{(2.1)} \qquad (\textbf{levi}-\textbf{id})\ \langle \theta_p, \tau^2_p(L^2_p, L^1_p)\rangle = -i\langle \theta, [L^2, L^1]\rangle_p, \quad L^2 \in H^{10},\ L^1 \in \overline{H^{10}}.$$

The normalization of τ^2 used here is chosen such that for the quadric

$$\rho = -2\mathsf{Re}\, w + q(z, \bar{z}) = 0, \quad (w, z) \in \mathbb{C} \times \mathbb{C}^{n-1},$$

and the contact form $\theta = i\partial\rho$, we have

$$\langle \theta_0, \tau^2_0(\partial_{z_j}, \partial_{\bar{z}_k})\rangle = \partial_{z_j}\partial_{\bar{z}_k} q,$$

or more generally

$$\text{(2.2)} \qquad (\textbf{levi}-\textbf{calc})\ \langle \theta_0, \tau^2_0(v^2, v^1)\rangle = \partial_{v^2}\partial_{v^1} q,$$

where $v^2 \in H^{10}_0$, $v^1 \in \overline{H^{10}_0}$. Here ∂_v denotes the directional derivative along the vector v, which is thought to be applied at every point.

We say that a *point $p \in M$ is of Levi rank q*, if the Levi form τ^2_p at p has rank q. A subbundle $V \subset H^{10}$ is called *Levi-nondegenerate*, if the Levi form is nondegenerate on $V \times \overline{V}$. For every such subbundle V, we write

$$V^\perp \subset H^{10}, \quad V^\perp = \cup_x V^\perp_x, \quad V^\perp_x = \{v \in H^{10}_x : \tau^2_x(v, \bar{v}^1) = 0 \text{ for all } v^1 \in V\},$$

for the orthogonal complement with respect to the Levi form, which is necessarily a subbundle.

Finally, we write $K^{10}_p \subset H^{10}_p$ and $K^{01}_p = \overline{K^{10}_p} \subset H^{01}_p$ for the Levi kernel components at p, $\mathbb{C}K_p = K^{10}_p \otimes \overline{K^{10}_p}$ for the complexification and $K_p = \mathbb{C}K_p \cap T_p$ for the corresponding real part.

The following is an overview of some of the main results:

Theorem 2.1. (main) *Let $M \subset \mathbb{C}^n$ be a pseudoconvex real hypersurface. Then for every $q \in \{0, \dots, n-1\}$, there exist an invariant submodule sheaf $\mathcal{S}^{10}(q)$ of $(1,0)$ vector fields, an invariant ideal sheaf $\mathcal{I}(q)$ of complex functions, and for every $p \in M$ of Levi rank q, an invariant quartic tensor*

$$\tau^4_p \colon \mathbb{C}T_p \times \mathbb{C}T_p \times K^{10}_p \times \overline{K^{10}_p} \to \mathbb{C}Q_p,$$

and a real submanifold $S \subset M$ through p, such that the following hold:

1. *S contains the set of all points $x \in M$ of Levi rank q in a neighborhood of p.*
2. *The tangent space of S at p equals the real part of the kernel of τ^4_p:*

$$T_p S = \mathsf{Re}\, \ker \tau^4_p = \{v \in T_p : \tau^4_p(v, v^3, v^2, v^1) = 0 \text{ for all } v^3, v^2, v^1\}.$$

3. *In suitable holomorphic coordinates vanishing at p, M admits the normal form*

$$2\mathsf{Re}\, w = \sum_{j=1}^{q} |z_{2j}|^2 + \varphi_4(z_4, \bar{z}_4) + o_w(1), \quad (w, z_2, z_4) \in \mathbb{C} \times \mathbb{C}^q \times \mathbb{C}^{n-q-1},$$

where $o_w(1)$ has weight greater than 1, with weights 1, 1/2 and 1/4 being assigned to the components of w, z_2 and z_4 respectively, and where φ_4 is a homogenous polynomial of degree 4 representing a restriction of the quartic tensor τ^4_p in the sense that

$$\tau^4_p(v^4, v^3, v^2, v^1) = \partial_{v^4}\partial_{v^3}\partial_{v^2}\partial_{v^1}\varphi_4$$

holds for $v^4, v^3 \in \mathbb{C}K_0$ and $v^2, \overline{v}^1 \in K^{10}_0$.

(4) *S is given by*

$$S = \{f^1 = \ldots = f^m = 0\}, \quad df^1 \wedge \ldots \wedge df^m \neq 0, \quad f^j \in \mathsf{Re}\,\mathcal{I}(q).$$

In fact, any $f \in \mathsf{Re}\,\mathcal{I}(q)$ vanishes on the set of point of Levi rank q.

(5) *The ideal sheaf $\mathcal{I}(q)$ is generated by all functions f of the form*

$$f = L^3\langle\theta, [L^2, L^1]\rangle,$$

where $\theta \in \Omega^0$ is a contact form, $L^3 \in \mathbb{C}T$ arbitrary complex vector field, and $L^2, \overline{L}^1 \in \mathcal{S}^{10}(q)$ arbitrary sections of the submodule sheaf.

(6) *The submodule sheaf $\mathcal{S}^{10}(q)$ is generated by all $(1,0)$ vector fields L satisfying $L \in V_L^{\perp}$, with $V_L \subset H^{10}$ a Levi-nondegenerate subbundle of rank q in a neighborhood of p.*

In addition, when M is of finite type at most 4 at p, the following also holds:

(i) *The intersection $T_pS \cap K_p$ with the Levi kernel K_p is totally real.*

(ii) *For every $v \in K^{10}_p$, the tensor τ^4_p does not identically vanish on*

$$(\mathbb{C}v + \mathbb{C}\overline{v}) \times (\mathbb{C}v + \mathbb{C}\overline{v}) \times \mathbb{C}v \times \mathbb{C}\overline{v}.$$

(iii) *The (Catlin's) multitype at p equals*

$$(1, 2, \ldots, 2, 4, \ldots, 4),$$

where the number of 2's equals the Levi rank at p. In particular, the multitype is determined by the Levi rank.

For proofs and more detailed and general statements, see the respective sections below. Part (6) can be used to define the submodule sheaves $\mathcal{S}^{10}(q)$, see §4.2. Then the ideal sheaves $\mathcal{I}(q)$ are defined in Part (5), see also §6. In particular, local sections in $\mathcal{I}(q)$ vanish at points of Levi rank q by Corollary 6.2. The quartic tensor τ^4 is constructed §4.3. In view of Proposition 6.6, the intersection of real kernels of differentials df for $f \in \mathsf{Re}\,\mathcal{I}(q)$ coincides with $\mathsf{Re}\ker\tau^4_p$. Hence we can choose functions f^j satisfying (4) and (2). The normal form in (3) follows from Proposition 4.15.

When M is of finite type 4, Proposition 5.1 implies that τ^4 has no holomorphic kernel, and therefore its (real) kernel as in (2) is totally real, as stated in (i). Statement (ii) is also part of Proposition 5.1. Finally, statement (iii) about the multitype is contained in §5.4.

The following simple example illustrates one of the differences between functions in the ideal sheaf $\mathcal{I}(q)$ and boundary systems (as defined in [C84b, §2], see also §7 below).

Example 2.2. **(transv-f)** Consider the hypersurface $M \subset \mathbb{C}^2_{w,z}$ given by

$$2\mathsf{Re}\, w = \varphi(z, \bar z, \mathsf{Im}\, w), \quad \varphi(z, \bar z, u) := |z|^4 + u^2|z|^2,$$

which is pseudoconvex and of finite type 4. Then a boundary system $\{L_2; r_2\}$ defines the 2-dimensional submanifold $S := \{r_2 = 0\} \subset M$ to contain all points of Levi rank 0. However, since r_2 is of the form

$$\text{(r2)}\ r_2 = \mathsf{Re}\, L^3\langle\theta, [L^2, L^1]\rangle \tag{2.3}$$

(in the setup of Theorem 2.1, part (5)), its differential at 0 is given by

$$dr_2(v) = \mathsf{Re}\, \tau_0^4(v, L_0^3, L_0^2, L_0^1),$$

which vanishes on the transversal space $\{dz = 0\}$. Consequently, any S defined by a boundary system function r_2 must be tangent to $\{dz = 0\}$.

On the other hand, in the ideal sheaf $\mathcal{I}(0)$ we can choose a function given by (2.3) with *transversal* L^3. That will allow to reduce S down to only the origin $z = w = 0$, which, in fact, is the set of points of Levi rank 0.

3. Invariant cubic tensors

We begin by investigating the 3rd order invariants without the pseudoconvexity assumption.

3.1. **Double Lie brackets.** In presence of a nontrivial Levi kernel K_p^{10}, it is natural to look for cubic tensors arising from double Lie brackets with one of the vector fields having its value inside the Levi kernel:

$$\text{(triple)}\ \langle\theta, [L^3, [L^2, L^1]]\rangle, \quad \theta \in \Omega^0, \quad L^3, L^2 \in H^{10},\ L^1 \in \overline{H^{10}}, \quad L_p^1 \in \overline{K_p^{10}}. \tag{3.1}$$

However, the following simple example shows that (3.1) does not define a tensor in general:

Example 3.1. **(one-ker)** Let $M \subset \mathbb{C}^3_{z_1,z_2,w}$ be the degenerate quadric

$$(\mathbf{deg-qua})\ \rho = -(w + \bar w) + z_1\bar z_1 = 0, \tag{3.2}$$

and consider the $(1,0)$ vector fields

$$L^3 := \partial_{z_2}, \quad L^2 := \partial_{z_2} + cz_2\overline{L^1}, \quad L^1 := \partial_{\bar z_1} + z_1\partial_{\bar w}.$$

Then

$$[L^3, [L^2, L^1]] = c[L^1, \overline{L^1}] = c(\partial_w - \partial_{\bar w}),$$

and hence for a contact form $\theta \in \Omega^0$, the value

$$\langle\theta, [L^3, [L^2, L^1]]\rangle_p$$

depends on c, even though all values L_p^j are independent of c. Note that both L_0^2 and L_0^3 (but not $\overline{L_0^1}$) are inside the Levi kernel K_0^{10}. Hence the double Lie bracket does not define any tensor $K_p^{10} \times K_p^{10} \times \overline{H_p^{10}} \to \mathbb{C}Q_p$.

The same example also shows that neither the Levi form derivative $L^3\langle\theta, [L^2, L^1]\rangle$ considered below behaves tensorially on the same spaces.

In contrast, we do get an invariant tensor when *both vector fields* inside the inner Lie bracket have their values in the Levi kernel at the reference point. We write τ^{31} for the corresponding tensor, reflecting the fact that it will become a restriction of the full tensor τ^3 below.

Lemma 3.2. (bracket-tensor) *The double Lie bracket $[L^3, [L^2, L^1]]$ defines an invariant tensor*

$$\tau_p^{31} \colon \mathbb{C}H_p \times K_p^{10} \times \overline{K_p^{10}} \to \mathbb{C}Q_p, \tag{3.3}$$

i.e. there exists an unique τ as above satisfying

$$\tau_p^{31}(L_p^3, L_p^2, L_p^1) = -i[L^3, [L^2, L^1]]_p \mod \mathbb{C}H_p, \quad L^3, L^2, \overline{L^1} \in H^{10}, \quad L_p^2, \overline{L_p^1} \in K_p^{10}.$$

Furthermore, τ_p^{31} is symmetric on $K_p^{10} \times K_p^{10} \times \overline{K_p^{10}}$ in its K^{10}-arguments, and on $\overline{K_p^{10}} \times K_p^{10} \times \overline{K_p^{10}}$ in its $\overline{K^{10}}$-arguments, and satisfies the reality condition

$$(\mathbf{30-sym}) \ \overline{\tau_p^{31}(v^3, v^2, v^1)} = \tau_p^{31}(\overline{v}^3, \overline{v}^1, \overline{v}^2). \tag{3.4}$$

Proof. It suffices to show that

$$(\mathbf{van}) \ [L^3, [L^2, L^1]]_p \in \mathbb{C}H_p \tag{3.5}$$

holds whenever any of the values L_p^j is 0. Since any such L^j can be written as linear combination $\sum a_k L_k$ with $a_k(p) = 0$ and L_k being in the same bundle, it suffices to assume $L^j = a\widetilde{L}^j$ with $a(p) = 0$, in which case (3.5) is easy to verify. The reality condition is straightforward and the symmetries follow from the Jacobi identity. □

A closely related construction was proposed by Webster [W95].

3.2. **The Levi form derivative.** As alternative to the double Lie bracket tensor, one can differentiate the Levi form after pairing with a contact form, which is similar to the approach employed by Catlin in his boundary system construction:

$$(\mathbf{der}) \ L^3\langle\theta, [L^2, L^1]\rangle. \tag{3.6}$$

Again Example 3.1 shows that (3.6) does not define a tensor if both L_p^2, L_p^3 are in the Levi kernel if L_p^1 is not. On the other hand, if *both vector fields L^1, L^2 inside the Lie brackets have their value at p contained in the Levi kernel*, we do obtain a tensor even when the outside vector field L^3 is not necessarily contained in $\mathbb{C}H$:

Lemma 3.3. (levi-der) *There exists an unique cubic tensor*

$$\tau_p^3 \colon \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \to \mathbb{C}Q_p, \tag{3.7}$$

satisfying

$$\langle\theta_p, \tau_p^3(L_p^3, L_p^2, L_p^1)\rangle = -i(L^3\langle\theta, [L^2, L^1]\rangle)_p, \quad \theta \in \Omega^0, \quad L^3 \in \mathbb{C}T, \quad L^2, \overline{L^1} \in H^{10}, \quad L_p^2, \overline{L_p^1} \in K_p^{10}.$$

Furthermore, τ_p^3 satisfies the reality condition

$$(\mathbf{3-sym}) \ \overline{\tau_p^3(v^3, v^2, v^1)} = \tau_p^3(\overline{v}^3, \overline{v}^1, \overline{v}^2). \tag{3.8}$$

Proof. The proof is similar to that of Lemma 3.2, and the symmetry follows directly from the definition. □

3.3. **A normal form of order 3 and complete set of cubic invariants. (norm-f)** To compare tensors τ^{31} and τ^3, it is convenient to use a partial normal form for the cubic terms. In the sequel we write $\varphi_{j_1\cdots j_m}(x^1,\dots x^m)$ for a polynomial of the multi-degree $(j_1,\dots,j_m)$ in its corresponding variables. We also write $z_k=(z_{k1},\dots,z_{km})\in\mathbb{C}^m$ for coordinate vectors and their components.

Proposition 3.4. (3-normal) *For every real hypersurface M in $\mathbb{C}^n$ and point $p\in M$ of Levi rank q, there exist local holomorphic coordinates*

$$(w,z)=(w,z_2,z_3)\in\mathbb{C}\times\mathbb{C}^q\times\mathbb{C}^{n-q-1},$$

vanishing at p, where M takes the form

$$w+\bar w=\varphi(z,\bar z,i(w-\bar w)),\quad \varphi(z,\bar z,u)=\varphi_2(z,\bar z,u)+\varphi_3(z,\bar z,u)+O(4),$$

where

$$\varphi_2(z,\bar z,u)=\varphi_{11}(z_2,\bar z_2),\quad \varphi_3(z,\bar z,u)=2\mathsf{Re}\,\varphi_{21}(z,\bar z_3)+\varphi_{111}(z_3,\bar z_3,u),$$

and $O(4)$ stands for all terms of total order at least 4.

Proof. It is well-known that the quadratic term φ_2 can be transformed into $\varphi_{11}(z_2,\bar z_2)$ representing the nondegenerate part of the Levi form. Furthermore, as customary, we may assume that the cubic term φ_3 has no harmonic terms.

Next, by suitable polynomial transformations

$$(z,w)\mapsto(z+\sum_{j=1}^r z_{2j}h_j(z,w),w),$$

we can eliminate all cubic monomials of the form $\bar z_{2j}h(z,u)$ and their conjugates, where $h(z,w)$ is any holomorphic quadratic monomial. The proof is completed by inspecting the remaining cubic monomials. □

Next we use the convenient $(1,0)$ vector fields with obvious notation:

Lemma 3.5. (vf-norm) *For a real hypersurface $M\subset\mathbb{C}^n$ given by*

$$\text{(3.9)}\qquad \textbf{(graph)}\ w+\bar w=\varphi(z,\bar z,i(w-\bar w)),\quad (w,z)\in\mathbb{C}\times\mathbb{C}^{n-1},$$

the subbundle H^{10} of $(1,0)$ vector fields is spanned by

$$L_j:=\partial_{z_j}+\frac{\varphi_{z_j}}{1-i\varphi_u}\partial_w,\quad j=1,\dots,n-1.$$

More generally, H^{10} is spanned by all vector fields of the form

$$\text{(3.10)}\qquad \textbf{(lv)}\ L_v:=\partial_v+\frac{\varphi_v}{1-i\varphi_u}\partial_w,\quad v\in\{0\}\times\mathbb{C}^{n-1},$$

where the subscript v denotes the differentiation in the direction of v.

Calculating with special vector fields from Lemma 3.5, we obtain:

Corollary 3.6. (3-calc) *Let M be in the normal form given by Proposition 3.4. Then tensors τ_p^{31} and τ_p^3 defined in Lemmas 3.2 and 3.3 respectively satisfy*

$$(\mathbf{3-dif})\ \langle\theta_0, \tau_0^{31}(v^3, v^2, v^1)\rangle = \langle\theta_0, \tau_0^3(v^3, v^2, v^1)\rangle = \partial_{v^3}\partial_{v^2}\partial_{v^1}\varphi_3, \tag{3.11}$$

where

$$v^3, v^2, \overline{v}^1 \in K_0^{10} \cong \{0\} \times \{0\} \times \mathbb{C}^{n-q-1}, \quad \theta = i\partial\rho, \quad \rho = -2\mathsf{Re}\, w + \varphi.$$

Furthermore, the second identity in (3.11) *still holds for $v^3 \in \mathbb{C}H_0$.*

In particular, τ^{31} is a restriction of τ^3 to $\mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}}$, explaining the notation.

Remark 3.7. The term φ_{21} in Proposition 3.4 represents, up to a nonzero constant multiple, the cubic invariant tensor ψ_3 introduced by Ebenfelt [E]. It follows from Proposition 3.4 that ψ_3 and τ^3 coincide (up to a constant) on their common set of definition and together constitute the full set of cubic invariants of M at p.

3.4. **Symmetric extensions.** As consequence Corollary 3.6, τ^3 is symmetric in K^{10}- or in $\overline{K^{10}}$-vectors whenever two of them occur in any two arguments. This property leads to a natural symmetric extension:

Lemma 3.8. (3-symm) *The restriction*

$$\tau_p^{30} \colon \mathbb{C}K_p \times K_p^{10} \times \overline{K^{10}} \to \mathbb{C}Q_p$$

of the cubic tensor τ_p^3 admits an unique symmetric extension

$$\widetilde{\tau}_p^{30} \colon \mathbb{C}K_p \times \mathbb{C}K_p \times \mathbb{C}K_p \to \mathbb{C}Q_p,$$

satisfying

$$\langle\theta_0, \widetilde{\tau}_0^{30}(v^3, v^2, v^1)\rangle = \partial_{v^3}\partial_{v^2}\partial_{v^1}\varphi_3,$$

whenever M is in a normal form $\rho = -2\mathsf{Re}\, w + \varphi = 0$ as in Proposition 3.4 and $\theta = i\partial\rho$.

Remark 3.9. Note that since φ_3 has no harmonic terms in a normal form, the extension tensor $\widetilde{\tau}^{30}$ always vanishes whenever its arguments are either all in K^{10} or all in $\overline{K^{10}}$.

Example 3.10. In contrast to τ^{30}, the full cubic tensor τ^3 does not in general have any invariant extension to $\mathbb{C}T \times \mathbb{C}K \times \mathbb{C}K$. Indeed, consider the cubic $M \subset \mathbb{C}^2$ given by

$$\rho := -2\mathsf{Re}\, w + 2\mathsf{Re}\,(z^2\bar{z}) = 0.$$

Then $\partial_{\bar{w}}\partial_z\partial_z\varphi_3 = 0$. Now consider a change of coordinates with linear part $(w, z) \mapsto (w, z + iw)$ transforming $\partial_{\bar{w}}$ into $\partial_{\bar{w}} - i\partial_{\bar{z}}$. Then, after removing harmonic terms, the new cubic term takes form

$$\varphi_3 = 2\mathsf{Re}\,(z^2\bar{z}) - 4\mathsf{Im}\, wz\bar{z}.$$

But then $(\partial_{\bar{w}} - i\partial_{\bar{z}})\partial_z\partial_z\varphi_3 \neq 0$, i.e. the 3rd derivatives of φ_3 do not transform as tensor when passing to another normal form.

3.5. **Cubic tensors vanishing for pseudoconvex hypersurfaces.** If M is pseudoconvex, the Levi form $\langle\theta, [L, \overline{L}]\rangle$ does not change sign, and therefore the cubic tensor τ^3 must vanish identically. We obtain:

Lemma 3.11. (psc-vanish) *Let M be a pseudoconvex hypersurface and $p \in M$. Then the cubic tensor τ^3_p (and therefore its restriction τ^{31}_p) vanishes identically. Equivalently, the cubic normal form in Proposition 3.4 satisfies*

$$(3.12) \qquad (\mathbf{psc-cubic})\ \varphi_{21}(z, \bar{z}_3) = \sum_{jkl} c_{jkl} z_{2j} z_{2k} \bar{z}_{3l}, \quad \varphi_{111}(z_3, \bar{z}_3, u) = 0.$$

The remaining cubic terms in (3.12) can be absorbed into higher weight terms as follows. We write $o_w(m)$ for terms of weights heigher than m.

Corollary 3.12. (pse-nf-cor) *A pseudoconvex hypersurface M in suitable holomorphic coordinates is given by*

$$(3.13) \qquad (\mathbf{psc-cubic-red})\ w + \bar{w} = \varphi(z, \bar{z}, i(w - \bar{w})), \quad \varphi = \sum_j |z_{2j}|^2 + o_w(1),$$

where o_w is calculated for w, z_{2j}, z_{3k}, and their conjugates, having weights 1, $\frac{1}{2}$, $\frac{1}{3}$ *respectively. In particular, M cannot be of finite type* 3.

The last statement follows directly from (3.13), since the contact orders with lines in z_3 directions are at least 4.

4. Invariant quartic tensors

If the tensor τ^3 (or τ^{30}) vanishes, it is natural to look for higher order invariants by taking iterated Lie brackets or higher order derivatives of Levi form. However, in contrast to the statements of Lemmas 3.2 and 3.3, we don't obtain a tensor even when all vector field values are in the kernel in the strongest sense, as demonstrated by the following example.

Example 4.1. **(ex-4)** Let $M \subset \mathbb{C}^3_{z_1,z_2,w}$ be the degenerate quadric from Example 3.1, and set

$$L := \partial_{z_2} + c z_2(\partial_{z_1} + \bar{z}_1 \partial_w).$$

Then

$$[L, \overline{L}] = |c z_2|^2 (\partial_{\bar{w}} - \partial_w),$$

and both $\langle\theta, [L, [\overline{L}, [L, \overline{L}]]]\rangle_0$ and $(L\bar{L}\langle\theta, [L, \bar{L}]\rangle)_0$ depends on c, even though the value L_0 is contained in the Levi kernel K^{10}_0, is independent of c, and the cubic tensor τ^3_0 identically vanishes.

4.1. **Vector fields that are in the Levi kernel up to order 1.** In view of Example 4.1, in order to obtain a tensor, we need to restrict the choice of the vector fields. This motivates the following definition:

Definition 4.2. (ker-1) Let L be a $(1,0)$ vector field. We say that L *is in the Levi kernel up to order* 1 *at* p if, for any vector fields $L^1 \in H^{10}$, $L^2 \in \mathbb{C}T$, and any contact form θ, the following holds:

$$(\mathbf{1-ker})\ (\langle\theta, [L^1, \overline{L}]\rangle)_p = (L^2\langle\theta, [L^1, \overline{L}]\rangle)_p = 0. \tag{4.1}$$

More generally, for a fixed tangent vector $v \in \mathbb{C}T_p$, we say that L *is in the Levi kernel up to v-order* 1 *at* p if (4.1) holds whenever $L^2_p = v$. (The latter property obviously depends only on v rather than its vector field extension L^2.) Finally, if the above property holds for all v in a vector subspace $V \subset \mathbb{C}T_p$, we say that L *is in the Levi kernel up to V-order* 1 *at* p.

It is straightforward to see that for a

Lemma 4.3. (1-jet-dep) *For any* $(0,1)$ *vector field* $\overline{L}$ *with* $\overline{L}_p \in \overline{K^{10}_p}$*, the expression*

$$(L^2\langle\theta, [L^1, \overline{L}]\rangle)_p$$

only depends on the values L^2_p, L^1_p *and* θ_p*, as well as the* 1*-jet of* L *at* p*. In particular,* L *being in the Levi kernel up to order* 1 *is a linear condition on the* 1*-jet of* L *at* p*.*

Example 4.4. In the setting of Example 4.1, choosing $L^1 := \partial_{z_1}$, we compute

$$(L\langle\theta, [L^1, \overline{L}]\rangle)_0 \neq 0,$$

which shows that L is not in the Levi kernel of tangent order 1, even if its value at 0 is contained in the Levi kernel.

Remark 4.5. **(bracket-restrict)** Using a normal form as in Proposition 3.4 and calculating with vector fields (3.10), we can obtain a condition equivalent to (4.1) with L^2 in $\mathbb{C}H$ (rather than in $\mathbb{C}T$), which can be stated in terms of double Lie bracket:

$$(\mathbf{1-ker'})\ \langle\theta_p, [L^1, \overline{L}]_p\rangle = \langle\theta_p, [L^2, [L^1, \overline{L}]]_p\rangle = \langle\theta_p, [\overline{L}^2, [L^1, \overline{L}]]_p\rangle) = 0. \tag{4.2}$$

A priori, it is not at all clear that vector fields as in Definition 4.2 exist. The following lemma provides an easy way of constructing them.

Lemma 4.6. (1-ker-def) *Let* M *be have Levi form of rank* r *at* $p \in M$*, with Levi kernel* K^{10}_p*. Assume that* L *is in the Levi kernel up to order* 1 *at* p*, as per Definition 4.2. Then* $L_p \in K^{10}_p$ *and*

$$(\mathbf{3-vanish})\ \tau^3_p(L^2_p, L^1_p, \overline{L}_p) = 0, \quad L^2, L^1 \in H^{10}, \quad L^1_p \in K^{10}_p. \tag{4.3}$$

must hold for all L^2, L^1*. (Equivalently,* $\overline{L}_p$ *is contained in the kernel of* τ^3_p *in the last argument).*

Vice versa, assume that $L_p \in K^{10}_p$ *and* (4.3) *hold. Let*

$$(\widetilde{L}^1, \dots, \widetilde{L}^q)$$

be a Levi-orthonormal system of $(1,0)$ *vector fields at* p*, such that* L *is Levi-orthogonal to each* $\widetilde{L}^j$*,* $j = 1, \ldots, q$*, in a neighborhood of* p*. Then* L *is in the Levi kernel up to order* 1 *at* p*.*

Proof. The first part follows directly from the definitions.

Vice versa, since the Levi form has rank r in p, and L_p is Levi-orthogonal to each $\widetilde{L}^j$, it follows that L_p is in the Levi kernel, i.e. the first expression in (4.1) must vanish.

Next, (4.3) implies that the second expression in (4.1) vanishes whenever $L^1_p \in K^{10}_p$. Similarly, in view of the symmetry (3.8), also the third expression vanishes under the same assumption.

Finally, to a general L^1 with $L^1_p \notin K^{10}_p$, we can always add a linear combination of $\widetilde{L}^j$ to achieve the inclusion of the value at p in the Levi kernel. Since L is Levi-orthogonal to each $\widetilde{L}^j$ identically in a neighborhood of p, this does not change (4.1), completing the proof. $\square$

In particular, in view of Lemma 3.11 we obtain:

Corollary 4.7. (gen-pt) *Let* M *be pseudoconvex. Then every* $v \in K^{10}_p$ *extends to a* $(1,0)$ *vector field, which is in the Levi kernel up to order* 1 *at* p*.*

Remark 4.8. More generally, a similar result can be obtained without pseudoconvexity for a $v \in K^{10}_p$ whose conjugate $\overline{v}$ is in the (right) kernel of τ^3, i.e. satisfying

$$\tau^3_p(L^3_p, L^2_p, \overline{v}) = 0$$

for all L^3, L^2. Then there exists a $(0,1)$ vector field $\overline{L}$ extension of $\overline{v}$, which is in the Levi kernel up to order 1 at p.

4.2. **Invariant submodule sheaves of vector fields. (submodules)** The notion of the *Levi kernel inclusion up to order* 1 has been defined pointwise in Definition 4.2. In order to have a uniform control for Levi kernels in nearby points, we shall need to define corresponding sheafs of vector field submodules as follows.

Definition 4.9. (q-sheaf) Let $M \subset \mathbb{C}^n$ be a real hypersurface. Denote by $\mathcal{T}^{10}$ the sheaf of all $(1,0)$ vector fields on M. For every $q \leq n-1$, define $\widetilde{\mathcal{S}}^{10}(q) \subset \mathcal{T}^{10}$ to be the submodule sheaf consisting of all vector fields on M which are contained in the Levi kernel up to order 1 at every point of Levi rank $\leq q$.

As a direct consequence of Lemma 4.6, we obtain the following strengthening of Corollary 4.7:

Corollary 4.10. (q-sheaf-sect) *Let* M *be a pseudoconvex hypersurface. Then for every* q*, local sections of* $\widetilde{\mathcal{S}}^{10}(q)$ *span the Levi kernel* K^{10}_x *at every point* $x \in M$ *of Levi rank* q*.*

Note that to guarantee the existence of sufficiently many sections as in Corollary 4.10, it is important to restrict the property underlying Definition 4.9 only to points of Levi rank $\leq q$. Without that restriction, the sheaf would become trivial e.g. for any manifold M that is generically Levi-nondegenerate (which is the case for any M of finite type).

Definition 4.9 requires to check the condition at eery point of Levi rank $\leq q$, which may be difficult to deal with in practice. Analysing the proof we arrive at the smaller submodule sheaf $\mathcal{S}^{10}(q)$ (as defined in Theorem 2.1, part (6)):

Definition 4.11. (vanish-level) Define the submodule sheaf $\mathcal{S}^{10}(q)$ to be generated by all $(1,0)$ vector fields L satisfying $L \in V_L^\perp$, with $V_L \subset H^{10}$ a Levi-nondegenerate subbundle of rank q in a neighborhood of p.

Then Lemma 4.6 yelds all the needed properties:

Corollary 4.12. (span-ker) *Under the same assumptions as in Corollary 4.10, for every q, $\mathcal{S}^{10}(q)$ is a submodule sheaf of $\widetilde{\mathcal{S}}^{10}(q)$, which has its local sections also span the Levi kernel K_x^{10} at every point $x \in M$ of Levi rank q.*

4.3. **Construction of the quartic tensor. (quartic)** Equipped with special vector fields as in Definition 4.2, we can now define an invariant quartic tensor via second order derivatives of the Levi form:

Lemma 4.13. (d2) *Let M be such that the cubic tensor τ_p^3 vanishes for some $p \in M$. Then there exists an unique tensor*

$$\tau_p^4 \colon \mathbb{C}T_p \times \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \to \mathbb{C}Q_p,$$

such that for any $(1,0)$ vector fields $\overline{L^1}, L^2 \in H^{10}$ that are in the Levi kernel up to order 1 at p, any vector fields $L^3, L^4 \in \mathbb{C}T$, and any contact form $\theta \in \Omega^0$,

$$(\mathbf{1-levi'})\ \langle \theta_p, \tau_p^4(L_p^4, L_p^3, L_p^2, L_p^1)\rangle = -i(L^4L^3\langle\theta, [L^2, L^1]\rangle)_p. \tag{4.4}$$

More generally, (4.4) still holds whenever both $\overline{L}^1$ and L^2 are in the Levi kernel up to L_p^j-order 1 at p, for $j = 3, 4$.

Proof. Similar to the proof of Lemma 3.2, it suffices to prove that the right-hand side of (4.4) vanishes whenever either $L^k = a\widetilde{L}^k$ for some $k = 1, 2, 3, 4$, or $\theta = a\widetilde{\theta}$, where a is a smooth function vanishing at p. In the following $\widetilde{a}$ will denote either a or the conjugate $\bar{a}$ and we assume (without loss of generality) that each of L^3, L^4 is contained in either H^{10} or H^{01}.

Now the vanishing of the right-hand side in (4.4) is obvious for $k = 4$. For $k = 3$, it takes the form

$$(L^4\widetilde{a})_p(\widetilde{L}^3\langle\theta, [L^2, L^1]\rangle)_p,$$

which must vanish in view of Definition 4.2. For $k = 1$, we obtain

$$(L^4\bar{a})_p(L^3\langle\theta, [L^2, \widetilde{L}^1]\rangle)_p + (L^3\bar{a})_p(L^4\langle\theta, [L^2, \widetilde{L}^1]\rangle)_p + (L^4L^3\bar{a})_p(\langle\theta, [L^2, \widetilde{L}^1]\rangle)_p,$$

which again vanishes in view of Definition 4.2. For $k = 2$, the proof follows from the case $k = 1$ by exchanging L^2 and L^1 and conjugating. Finally, for $\theta = a\widetilde{\theta}$, we obtain

$$(L^4a)_p(L^3\langle\widetilde{\theta}, [L^2, L^1]\rangle)_p + (L^3a)_p(L^4\langle\widetilde{\theta}, [L^2, L^1]\rangle)_p + (L^4L^3a)_p(\langle\widetilde{\theta}, [L^2, L^1]\rangle)_p,$$

which vanishes by the same argument. ☐

Remark 4.14. In higher generality, when the cubic tensor τ_p^3 may not vanish completely, a quartic tensor τ_p^4 can still be constructed via (4.4) along certain kernels of τ_p^3. We will not pursue this direction as our focus here is on the pseudoconvex case when τ_p^3 always vanishes identically.

4.4. **A normal form up to weight** $1/4$. Since the cubic normal form for pseudoconvex hypersurfaces (3.13) is in some sense lacking nondegenerate terms, we extend it by lowering the weight of z_3 from $1/3$ to $1/4$ (an renaming z_3 to z_4):

Proposition 4.15. (4-normal) *For every pseudoconvex real hypersurface M in $\mathbb{C}^n$ and point $p \in M$ of Levi rank q, there exist local holomorphic coordinates*

$$(w, z) = (w, z_2, z_3) \in \mathbb{C} \times \mathbb{C}^q \times \mathbb{C}^{n-q-1},$$

vanishing at p, where M takes the form

$$\textbf{(phi24)}\ w + \bar{w} = \varphi(z, \bar{z}, i(w - \bar{w})), \quad \varphi = \varphi_2 + \varphi_4 + o_w(1), \tag{4.5}$$

where

$$\varphi_2(z, \bar{z}, u) = \sum_{j=1}^{q} |z_{2j}|^2, \quad \varphi_4(z, \bar{z}, u) = 2\mathsf{Re}\, \varphi_{31}(z_4, \bar{z}_4) + \varphi_{22}(z_4, \bar{z}_4),$$

such that the weight estimate o_w is calculated for u, z_{2j}, z_{4k}, and their conjugates, having weights 1, $\frac{1}{2}$, $\frac{1}{4}$ respectively. Here each polynomial φ_{jk} is bihomogenous of bidegree (j, k) in its arguments.
Furthermore, the following hold:

(1) *For every $v \in K_0^{10} \cong \{0\} \times \{0\} \times \mathbb{C}^{n-q-1}$, the vector field L_v given by (3.10) is in the Levi kernel up to v^0-order 1 at 0 for any $v^0 \in \mathbb{C}K_0$.*
(2) *For $v^4, v^3 \in \mathbb{C}K_0$ and $v^2, \overline{v}^1 \in K_0^{10}$, we have*

$$\textbf{(phi}-\textbf{diff)}\ \tau_p^4(v^4, v^3, v^2, v^1) = \partial_{v^4}\partial_{v^3}\partial_{v^2}\partial_{v^1}\varphi_4. \tag{4.6}$$

In particular, the restriction

$$\textbf{(t40)}\ \tau_p^{40} \colon \mathbb{C}K_p \times \mathbb{C}K_p \times K_p^{10} \times \overline{K_p^{10}} \to \mathbb{C}Q_p \tag{4.7}$$

of τ_p^4 is symmetric whenever its arguments can be interchanged and satisfies the reality condition

$$\overline{\tau_p^4(v^4, v^3, v^2, v^1)} = \tau_p^4(\overline{v}^4, \overline{v}^3, \overline{v}^1, \overline{v}^2).$$

Proof. The existence of the desired normal form is a direct consequence of Lemma 3.11. A direct calculation shows the special vector fields in (3.10) with $v \in \{0\} \times \{0\} \times \mathbb{C}^{n-q-1}$ are in the Levi kernel up to tangential order 1 as claimed. The remaining properties are straightforward. □

Similarly to Corollary 3.6, one can also show the following quartic Lie bracket representation of the restriction (4.7):

Lemma 4.16. *The restriction τ_q^{40} satisfies*

$$\langle \theta_p, \tau_p^{41}(L_p^4, L_p^3, L_p^2, L_p^1) \rangle = -i \langle \theta_p, [L^4, [L^3, [L^2, L^1]]]_p \rangle$$

whenever $L^2, \overline{L}^1 \in H^{10}$ are in the Levi kernel up to $\mathbb{C}K$-order 1 at p, $L^3, L^4 \in \mathbb{C}H$, and $\theta \in \Omega^0$ is any contact form.

Remark 4.17. It is easy to see that pseudoconvexity of M implies that the quartic polynomial $\varphi_4(z_3, \bar z_3)$ in (4.5) is plurisubharmonic. Conversely, every plurisubharmonic φ_4 appears in a normal form of some pseudoconvex hypersurface, e.g. the model hypersurface

$$w + \bar w = \sum_{j=1}^{q} |z_{2j}|^2 + \varphi_4(z_4, \bar z_4).$$

4.5. **Symmetric extension.** Similarly to Lemma 3.8, we obtain a symmetric extension for the Levi kernel restriction of τ^4:

Lemma 4.18. (4-symm) *The restriction*

$$\tau_p^{40} \colon \mathbb{C}K_p \times \mathbb{C}K_p \times K_p^{10} \times \overline{K_p^{10}} \to \mathbb{C}Q_p$$

of the quartic tensor τ_p^4 admits an unique symmetric extension

$$\widetilde\tau_p^{40} \colon \mathbb{C}K_p \times \mathbb{C}K_p \times \mathbb{C}K_p \times \mathbb{C}K_p \to \mathbb{C}Q_p,$$

satisfying

$$\text{(4.8)} \qquad (\mathbf{tensor-dif})\ \langle \theta_0, \widetilde\tau_0^{40}(v^4, v^3, v^2, v^1)\rangle = \partial_{v^4}\partial_{v^3}\partial_{v^2}\partial_{v^1}\varphi_4,$$

whenever M is in a normal form $\rho = -2\mathsf{Re}\, w + \varphi = 0$ as in Proposition 4.15 and $\theta = i\partial\rho$. In fact, (4.8) *holds whenever φ satisfies $d\varphi_0 = 0$ and $\partial_{v^j}\partial_{v^2}\partial_{v^1}\varphi_3 = 0$ for $j = 3, 4$.*

5. Applications and properties of the quartic tensor

5.1. **Relation with the D'Angelo finite type.** The quartic tensor τ^4 can be used to completely characterize the finite type up to 4 in the sense of D'Angelo [D82] (the last property is related to D'Angelo's "Property P", see [D82, Definition 5.1]):

Proposition 5.1. (type-quartic) *Let M be a pseudoconvex hypersurface with nontrivial Levi kernel at p. Then M is of D'Angelo type 4 at p if and only if for every nonzero vector $v \in K_p^{10}$, the tensor τ_p^4 does not vanish when restricted to*

$$\text{(5.1)} \qquad (\mathbf{v-res})\ (\mathbb{C}v + \mathbb{C}\overline{v}) \times (\mathbb{C}v + \mathbb{C}\overline{v}) \times \mathbb{C}v \times \mathbb{C}\overline{v}.$$

In fact, the latter property implies the following stronger nonvanishing conclusion:

$$\tau_p^4(v, \overline{v}, v, \overline{v}) \neq 0.$$

Proof. We may assume M is put into its normal form as in Proposition 4.15.

If the restriction of τ_p^4 vanishes on (5.1) for some $v \neq 0$, we may assume $v = \partial_{z_{31}}$, where $z_3 = (z_{31}, \ldots, z_{3,n-r})$. Then it follows from the normal form that the line $\mathbb{C}v$ has order of contact with M higher than 4, hence the D'Angelo type at p is also higher than 4.

On the other hand, suppose the restriction of τ_p^4 to (5.1) does not vanish for any $v \neq 0$. Assume by contradiction, there exists a nontrivial holomorphic curve

$$\gamma \colon (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0), \quad \gamma(t) = \sum_{k \ge k_0} a_k t^k, \quad a_{k_0} \neq 0,$$

whose contact order with M at 0 is higher than 4. Recall that the contact order is given by

$$\frac{\nu(\rho \circ \gamma)}{\nu(\gamma)},$$

where ρ is any defining function of M and ν is the vanishing order at 0, in particular, $\nu(\gamma) = k_0 \geq 1$. Taking $\rho := -2\mathsf{Re}\, w + \varphi$, we must have $a_{k_0} \in \{0\} \times \mathbb{C}^n$, otherwise the contact order would be 1. Similarly, expanding $\rho \circ \gamma$, it follows by induction that

$$a_l \in \{0\} \times \mathbb{C}^n, \quad l < 4k_0,$$

and

$$a_k \in \{0\} \times \{0\} \times \mathbb{C}^{n-r}, \quad k < 2k_0,$$

for otherwise the contact order would be less than 4. Finally collecting terms of order $4k_0$ and using our assumption that the contact order is greater than 4, we obtain

$$\textbf{(4k)}\ \varphi_2(a_{2k_0}t^{2k_0}, \overline{a}_{2k_0}\overline{t}^{2k_0}) + \varphi_4(a_{k_0}t^{k_0}, \overline{a}_{k_0}\overline{t}^{k_0}) = 0. \tag{5.2}$$

In particular, it follows that

$$\varphi_4(a_{k_0}\xi, \overline{a}_{k_0}\overline{\xi}) = c\xi^2\overline{\xi}^2.$$

Since φ_4 is plurisubharmonic, we must have $c \geq 0$. Hence both terms in (5.2) are nonnegative, and therefore must vanish. In particular, $ca_{k_0}^2\overline{a}_{k_0}^2 = 0$, implying

$$\varphi_4(a_{k_0}\xi, \overline{a}_{k_0}\overline{\xi}) = 0,$$

which is in contradiction with our nonvanishing assumption on τ_p^4. Hence the D'Angelo type is 4 completing the proof of the converse direction.

Finally, the last statement follows from the plurisubharmonicity of φ_4 in any normal form. $\square$

The pseudoconvexity assumption in Proposition 5.1 cannot be dropped:

Example 5.2. Let $M \subset \mathbb{C}^3_{w,z_1,z_2}$ be given by

$$2\mathsf{Re}\, w = |z_1|^2 - |z_2|^4.$$

Then M contains the image of the curve $t \mapsto (0, t^2, t)$ and is hence of infinite type at 0. On the other hand, M is in the normal form (4.5) and hence $\tau_0^4(v, \overline{v}, v, \overline{v}) \neq 0$ for any $v \neq 0 \in K_0^{10}$.

5.2. **Uniformity of the quartic tensor.** The sheaves $\mathcal{S}^{10}(q)$ introduced in Definition 4.11 can be used to obtain a uniform behavior of τ_p^4 as p varies over the set of nearby points of bounded Levi rank. In fact, as direct consequence from the definition and Corollary 4.12, we obtain that τ_p^4 can be calculated using local sections of $\mathcal{S}^{10}(q)$:

Corollary 5.3. (uni-t4) *For every vector fields $L^4, L^3 \in \mathbb{C}T$ and $L^2, \overline{L}^1 \in \mathcal{S}^{10}(q)$ defined in an open set $U \subset M$, the identity* (4.4) *holds simultaneously for all points $p \in U$ of Levi rank q.*

Remark 5.4. In the context of Corollary 5.3, it is essential to require the vector fields L^2, L^1 to be contained in Levi kernels up order 1 (rather than merely contained there). In fact, for a higher order perturbation of Examples 4.1 where 0 is the only Levi-degenerate point, choosing vector fields L^j as higher order perturbations of the vector field L in the example or its conjugate would violate (4.4).

It is important to note that the conclusion of Corollary 5.3 may not hold for points $p \in U$ of Levi rank $> q$ when $L_p^2, \overline{L}_p^1 \in K_p^{10}$. In fact, τ_p^4 may not even be continuous e.g. may vanish for p of higher Levi rank even when $\tau_{p_0}^4$ does not vanish on any line for p_0 of Levi rank q. This is illustrated by D'Angelo's celebrated example where the finite type is not upper-semicontinuous [D80, D82]:

Example 5.5 (J. P. D'Angelo). **(da-ex)** Let $M \subset \mathbb{C}^3_{w,z_1,z_2}$ be given by

$$2\mathsf{Re}\, w = |z_1^2 - wz_2|^2 + |z_2|^4.$$

Then M is of Levi rank 0 and finite type 4 at 0 and hence τ_0^4 does not vanish on the lines products (5.1) in view of Proposition 5.1. In fact, M is in its normal form as in Proposition 4.15 with $\varphi_4 = |z_1|^4 + |z_2|^4$, and hence

$$\tau_0^4(v, \overline{v}, v, \overline{v}) = 4(|v_1|^4 + |v_2|^4), \quad v \in K_0^{10} \cong \{0\} \times C^2_{z_1,z_2}.$$

On the other hand, at every $p = (it, 0, 0)$ on the imaginary axis with $t \neq 0$, the Levi rank is 1, and M can be locally transformed into a normal form (4.5) with vanishing φ_4 implying $\tau_p^4(v, \overline{v}, v, \overline{v}) = 0$ for any $v \in K_p^{10}$. Thus $\tau_p^4(v, \overline{v}, v, \overline{v})$ cannot be continuous for any $v = v(p)$ converging to any $v(0) \neq 0$ as $p \to 0$.

Of course, this phenomenon is closely related to the lack of upper-semicontinuity of the type demonstrated by D'Angelo.

5.3. **Kernels of quartic tensors.** For any homogenous polynomial, consider the following notion of holomorphic kernel:

Definition 5.6. The *holomorphic kernel of a homogeneous polynomial* $P(z, \bar{z})$, $z \in \mathbb{C}^n$, is defined to be the subspace of all $(1, 0)$ vectors v such that

$$(\mathbf{kernel-def})\ \partial_v P(z, \bar{z}) \equiv \partial_{\overline{v}} P(z, \bar{z}) \equiv 0. \tag{5.3}$$

Equivalently, the holomorphic kernel is the space of all v such that both v and $\overline{v}$ belong to the kernel of the polarization of p.

It is straightforward to see the following simple characterization of the kernel:

Lemma 5.7. (ker-coor) *The holomorphic kernel of* p *is the maximal subspace* V *such that, there exists a linear change of coordinates such that*

$$V = \oplus_{j=1}^l (\mathbb{C}\partial_{z_j} \oplus \mathbb{C}\partial_{\bar{z}_j})$$

and $P(z, \bar{z})$ *is independent of the variables* $z_1, \ldots, z_l$ *and their conjugates.*

Definition 5.8. (rank) The *rank* of P is $n-d$, where d is the dimension of the holomorphic kernel.

Also separating bihomogeneous components in (5.3), we obtain:

Lemma 5.9. *Let*
$$P(z,\bar z)=\sum P_{kl}(z,\bar z)$$
be a decomposition into components P_{kl} of bidegree (k,l) in $(z,\bar z)$. Then the holomorphic kernel of P equals the intersection of kernels of P_{kl} for all k,l.

Next we compare the holomorphic kernel of the polynomial φ_4 in the normal form given by Proposition 4.15 and the restriction
$$\tau_p^{40}\colon \mathbb{C}K_p\times\mathbb{C}K_p\times K_p^{10}\times\overline{K_p^{10}}\to\mathbb{C}Q_p.$$
of the quartic tensor τ_p^4.

Definition 5.10. The *holomorphic kernel* of τ_p^{40} is $V\cap\overline{V}$, where
$$V=\ker\tau_p^{40}=\{v\in\mathbb{C}K_p:\tau_p^{40}(v,v^3,v^2,v^1)=0\text{ for all }v^3,v^2,v^1\}.$$

First of all, remark that without pseudoconvexity assumption, the holomorphic kernel of τ_p^{40} may get larger than that of φ_4:

Example 5.11. **(diff-kernels)** Let $M\subset\mathbb{C}^3_{w,z_1,z_2}$ be given by
$$2\mathsf{Re}\,w=\varphi_4(z,\bar z):=2\mathsf{Re}\,(z_1^3\bar z_2).$$
Then the arguments in the proof of Proposition 4.15 can be used to show that (4.6) still holds, implying that ∂_{z_2} and $\partial_{\bar z_2}$ are in the kernel of τ_0^{40} in the 1st and 2nd arguments but not in the 3rd one.

On the other hand, in presence of pseudoconvexity, both kernels must coincide as the following lemma shows. As a matter of convention, for a multilinear function $f(v^1,\dots,v^m)$, we call its kernel in the kth argument the space of all v^k such that $f(v^1,\dots,v^m)=0$ holds for all v^j with $j\neq k$.

Lemma 5.12. (ker-rel) *Let M be in its normal form given by Proposition 4.15, and assume that M is pseudoconvex. Then both holomorphic kernels of τ^{40} in the 1st and 2nd arguments coincide with holomorphic kernel V of φ_4. Furthermore, the kernels of τ^{40} in the 3rd and 4th arguments coincide respectively with V and $\overline{V}$.*

Proof. As direct consequence of (4.6) we obtain that the holomorphic kernel of φ_4 is contained in the kernel of τ_p^{40} in each argument.

Vice versa, let v be $(1,0)$ vector in the holomoprhic kernel of τ_p^{40} (in the 1st argument). We write $\xi=z_4$ for brevity. After a linear change of coordinates we may assume $v=\partial_{\xi_1}$, where ξ_1 is the first component of ξ in the notation of Proposition 4.15. Then it follows from (4.6) that $\partial_{\xi_1}\varphi_4$ is harmonic. Since φ_4 has no harmonic terms, it must have the form
$$\varphi_4=2\mathsf{Re}\,(\bar\xi_1 h(\xi))+R,$$

where h is holomorphic and R is independent of ξ_1. Now since M is pseudoconvex, φ_4 is plurisubharmonic, in particular,

$$(5.4)\qquad (\mathbf{t})\ (\partial_{\xi_1} + t\partial_{\xi_j})(\partial_{\bar\xi_1} + t\partial_{\bar\xi_j})\varphi_4 \geq 0$$

holds for all $t \in \mathbb{R}$. Then for $t = 0$, we obtain $\partial_{\xi_1} h \geq 0$. Since h is holomorphic, we must have $\partial_{\xi_1} h \equiv 0$. Hence the linear part of (5.4) must be ≥ 0 and therefore equal to 0, since t is any real number. But this means $h \equiv 0$, and hence $v = \partial_{\xi_1}$ is in the holomorphic kernel of φ_4 as claimed.

The claimed statements for kernels in other arguments of τ_p^{40} are obtained by repeating the same proof. □

In view of Lemma 5.12, we simply refer to the *holomorphc kernel of* τ^{40} for its kernel in the 1st (and, equivalently, in the 2nd) argument, Also the *rank of* τ^{40} is $\dim K_p^{10} - d$, where d is the dimension of its holomorphic kernel, which coincides with the rank of φ_4 in the sense of Definition 5.8.

5.4. **Relation with Catlin's multitype. (m-type)** Recall that the multitype of $M \subset \mathbb{C}^{n+1}$ at p is the lexicographically maximal

$$\Lambda = (\lambda_1, \dots, \lambda_{n+1}), \quad \lambda_1 \geq \dots \geq \lambda_{n+1},$$

such that a defining function ρ of M satisfies

$$(5.5)\qquad (\mathbf{rho-wt})\ \rho = O_w(1)$$

for a choice of holomorphic coordinates $(z_1, \dots, z_{n+1})$, which together with their conjugates are assigned respectively the weights $(\lambda_1^{-1}, \dots, \lambda_{n+1}^{-1})$.

The main problem with multitype is that for a given coordinate representation, it is difficult to know whether concrete weights actually realize their lexicographic maximum. We now give a simple way of calculating a part of the multitype in terms of the rank of the tensor τ_4. In case τ_4 has trivial kernel, that gives the complete multitype.

Proposition 5.13. (multi-quartic) *Let $M \subset \mathbb{C}^n$ be a pseudoconvex hypersurface, and $p \in M$ a point with Levi form of rank q_2 and the restricted quartic tensor τ^{40} of rank q_4. Then the multitype $\Lambda = (\lambda_1, \dots, \lambda_n)$ of M at p satisfies*

$$(5.6)\qquad (\mathbf{wt-eq})\ \lambda_1 = 1, \quad \lambda_2 = \dots \lambda_{q_2+1} = 2, \quad \lambda_{q_2+2} = \dots = \lambda_{q_2+q_4+1} = 4,$$

and

$$(5.7)\qquad (\mathbf{wt-ineq})\ \lambda_k > 4, \quad k > q_2 + q_4 + 1.$$

In particular, if τ^{40} has only trivial kernel, the multitype is $(1, 2, \dots, 2, 4, \dots, 4)$, where the number of 2's equals the Levi rank.

Proof. By Lemma 5.7, in addition to the normal form in Proposition 4.15, we can make φ_4 independent of the last d coordinates, where d is the dimension of the $(1,0)$ kernel of τ_4. This shows that it is possible to achieve (5.5) with weights satisfying both (5.6) and (5.7).

The actual multitype may only be lexicographically higher, in particular, (5.7) is already satisfied. Assume by contradiction that we have another choice of coordinates with higher weights failing one of the equalities in (5.6). However, we must obviously have $\lambda_1 = 1$ and the Levi form invariance forces the next q_2 weights to be equal 2. Therefore we must have some $\lambda_k > 4$ for $k \le q_2 + q_4 + 1$. In those coordinates, we would have the same normal form as in Proposition 4.15 with φ_4 being independent of z_j at least for $j \ge q_2 + q_4 + 1$. That, however, would mean that the rank of τ_4 is less than q_4, which is a contradiction. Hence the multitype must satisfy all of (5.6) as claimed. □

6. Ideal sheaves for Levi rank level sets

(ideal)
We use the vector field submodule sheaves $\mathcal{S}^{10}(q)$ in Definition 4.11 to define invariant ideal sheaves of smooth functions $\mathcal{I}(q)$ (as in Theorem 2.1, part (5)):

Definition 6.1. Let $M \subset \mathbb{C}^n$ be a pseudoconvex hypersurface. For every q, define $\mathcal{I}(q)$ to be the ideal sheaf generated by all (smooth complex) functions f of the form

$$f = L^3 \langle \theta, [L^2, L^1] \rangle,$$

where $\theta \in \Omega^0$ is a contact form, $L^3 \in \mathbb{C}T$ arbitrary complex vector field, and $L^2, \overline{L}^1 \in \mathcal{S}^{10}(q)$ arbitrary sections.

As direct consequence of Corollary 4.12 and Lemma 3.11, we obtain a general way of constructing submanifolds containing level sets of Levi rank:

Corollary 6.2. (iq-van) *Let M be a pseudoconvex hypersurface. Then every local section in $\mathcal{I}(q)$ vanishes at all points of Levi rank q. In particular, for any collection $f^1, \ldots, f^m$ of real functions from the real part $\mathsf{Re}\,\mathcal{I}(q)$ defined in an open set $U \subset M$ satisfying*

$$df^1 \wedge \ldots \wedge df^m \neq 0,$$

the submanifold

$$S = \{f^1 = \ldots = f^m = 0\}$$

contains the set of all points of Levi rank q in U.

Remark 6.3. Note that due to our definition of $\mathcal{I}(q)$, any complex multiple of a local section is again a local section. Consequently, it suffices to take only sections in $\mathsf{Re}\,\mathcal{I}(q)$ to define the same set.

We next apply the quartic tensor to describe the differentials of sections in $\mathcal{I}(q)$.

Definition 6.4. For an ideal sheaf $\mathcal{I}$ define it *kernel* at p

$$\ker_p \mathcal{I} \subset \mathbb{C}T_p,$$

to be the intersection of kernels of all differentials df_p, where f is any local section of $\mathcal{I}$ in a neighborhood of p.

Then Corollary 6.2 implies:

Lemma 6.5. *Let $p \in M$ be a point of Levi rank q. Then the kernel of the qth sublevel ideal $\mathcal{I}(q)$ at p coincides with the kernel of the quartic tensor τ^4_p.*

We can now summarize this paragraph's results as follows:

Proposition 6.6. (main0) *Let $M \subset \mathbb{C}^n$ be a pseudoconvex real hypersurface, and $p \in M$ a point of Levi rank q. Then in a neighborhood of p, the set of all points of the same Levi rank q is contained in a real submanifold $S \subset M$ through p such that*

$$T_pS = \ker \tau^4_p,$$

and S is given by the vanishing of local sections

$$f^1, \ldots, f^m \in \mathcal{I}(q), \quad df^1 \wedge \ldots \wedge df^m \neq 0.$$

In particular, when M of finite type 4 at p, the intersection of T_pS with the Levi kernel at p is totally real.

7. Relation with Catlin's boundary systems

(bs)

7.1. **Maximal Levi-nondegenerate subbundle.** Recall that *Catlin's boundary system* construction for a hypersurface M at a point $p \in M$ begins with a maximal collection of $(1,0)$ vector fields $L_2, \ldots, L_{q+1}$ tangent to M such that the Levi form matrix

$$(\langle \theta, [L_j, \overline{L}_k] \rangle)_{2 \le j,k \le q}$$

is nonsingular. In particular, q must be equal to the Levi rank at p.

Invariantly, consider any *maximal Levi-nondegenerate subbundle* through p, i.e. any smooth subbundle $V^{10} \subset H^{10}$ where the restriction of the Levi form is nondegenerate. Then obviously any such V^{10} appears as the span of the first q vector fields in Catlin's boundary system, and vice versa, every such span is a maximal Levi-nondegenerate subbundle.

Next Catlin considers the Levi-orthogonal subbundle

$$S^{10} := (V^{10})^{\perp} \subset H^{10}$$

(T^{10}_{q+2} in Catlin's notation). In particular, the subbundle S^{10} *contains all Levi kernels K^{10}_x at all points $x \in M$ near p*, even when $\dim K^{10}_x$ depends on x. That makes the fiber $(V^{10}_x)^{\perp}$ unique whenever the Levi rank at x is the same as p, even when V^{10}_x itself may not be unique. On the other hand, at points x of higher Levi rank, $(V^{10}_x)^{\perp}$ clearly depends on the choice of V^{10}_x.

The rest of Catlin's boundary system construction only depends on the subbundle S^{10} rather than on V^{10} and its chosen basis.

7.2. **Levi kernel inclusion of higher order for S^{10}.** As mentioned before, S^{10} contains the Levi kernel at every point. On the other hand, if M is pseudoconvex, we have shown in Lemma 4.6 that S^{10} is itself *contained in the Levi kernel up to order* 1 *at* p as defined in Definition 4.2. That permits to use arbitrary sections of S^{10} in the calculation of the quartic tensor τ^4:

Corollary 7.1. (s10-tensor) *Let M be a pseudoconvex hypersurface, $V^{10} \subset H^{10}$ a maximal Levi-nondegenerate subbundle at $p \in M$, and S^{10} the Levi-orthogonal complement of V^{10}. Then the quartic tensor τ^4_p defined by Lemma 4.13 satisfies*

$$\langle \theta_p, \tau^4_p(L^4_p, L^3_p, L^2_p, L^1_p)\rangle = -i(L^4 L^3 \langle \theta, [L^2, L^1]\rangle)_p$$

for any $L^4, L^3 \in \mathbb{C}T$, $L^2, \overline{L}^1 \in S^{10}$ and $\theta \in \Omega^0$.

7.3. **Relation with the rest of Catlin's boundary system construction. (bs-2)** The remaining part of Catlin's construction is based on the higher order Levi form derivatives

$$\text{(7.1)} \qquad (\mathbf{L-th})\ \mathcal{L}\theta := L^m \dots L^3 \langle \theta, [L^2, L^1]\rangle, \quad \mathcal{L} = (L^m, \dots, L^1),$$

where $\theta = \partial r$ and r is a defining function of M. Then a boundary system

$$\text{(7.2)} \qquad (\mathbf{bd-sys})\ \mathcal{B} = \{r_1, r_{q_2+2}, \dots, r_\nu; L_2, \dots, L_\nu\}, \quad q_2 + 2 \le \nu \le n,$$

is constructed together with associated weights

$$\alpha_1 = 1 > \alpha_2 = \dots = \alpha_q = 2 > \alpha_{q+1} \ge \dots \ge \alpha_\nu,$$

where $r_1 = r$ is the given defining function, L_j and r_j are respectively smooth $(1,0)$ vector fields and smooth real functions in a neighborhood of p. The construction proceeds by induction as follows. Assuming a boundary system is constructed for given ν, define the next subbundle

$$T^{10}_{\nu+1} := \{L \in T^{10}_{q_2+2} : \partial r_{q_2+2}(L) = \dots = \partial r_\nu(L) = 0\}.$$

Then count all previous L_j and their conjugates with weight α_j, and consider a new vector field $L_{\nu+1} \in T^{10}_{\nu+1}$ and its conjugage, whose weight $\alpha = \alpha_{\nu+1}$ is to be determined. Now look for all lists $\mathcal{L} = (L^m, \dots, L^1)$ with each $L^k \in \{L_{q+2}, \dots, L_{\nu+1}\}$, which are of total weight 1 and *ordered*, i.e. $L_j, \overline{L}_j$ preceed $L_k, \overline{L}_k$ whenever $j > k$, such that

$$\text{(7.3)} \qquad (\mathbf{non-vanish})\ (\mathcal{L}\partial\rho)_p \ne 0.$$

The list must contain the new vector field $L_{\nu+1}$ or its conjugate, and the new weight $\alpha_{\nu+1}$ is chosen to be minimal possible with that property. Finally set either

$$r_{\nu+1} := \mathsf{Re}\, L^{m-1} \dots L^3 \langle \theta, [L^2, L^1]\rangle \text{ or } r_{\nu+1} := \mathsf{Im}\, L^{m-1} \dots L^3 \langle \theta, [L^2, L^1]\rangle$$

such that

$$(L_{\nu+1} r_{\nu+1})_p \ne 0,$$

which is always possible in view of (7.3), since the first vector field in the list, L^m is either $L_{\nu+1}$ or its conjugate. Restating Lemma 4.6 and Corollariy 7.1, we have:

Corollary 7.2. *Let M be pseudoconvex hypersurface with Levi form of rank q at p. Fix a boundary system $\{L_2, \ldots, L_{q+1}\}$ at p. Then $S^{10} = T^{10}_{q+2} = V^\perp$ for $V := \operatorname{span}\{L_2, \ldots, L_{q+1}\}$. Further, for any vector fields $L^4, L^3 \in S^{10} + \overline{S^{10}}$, $L^2 \in S^{10}$, $L^1 \in \overline{S^{10}}$, we have*

$$L^3\langle\theta, [L^2, L^1]\rangle_p = 0,$$
$$L^4 L^3\langle\theta, [L^2, L^1]\rangle_p = \tau^{40}_p(L^4_p, L^3_p, L^2_p, L^1_p).$$

In other words, for lists $\mathcal{L}$ of length 3, the derivative $(\mathcal{L}\theta)_p$ vanishes, whereas for lists of length 4, it only depends on the vector field values at p and is given by the restricted quartic tensor τ_4 (regardless of the choice of the boundary system).

Thus via the quartic tensor restriction τ^{40}_p, the nonvanishing condition in (7.3) is reduced to a purely algebraic property only depending on the vector fields' values at p.

References

[BS16] S. Biard, E. J. Straube. L^2-Sobolev theory for the complex Green operator. Notes of four lectures given by the second author at the January 2016 School in Complex Analysis and Geometry at the Tsinghua Sanya International Mathematics Forum, Sanya, China; https://arxiv.org/abs/1606.00728

[C83] David Catlin. Necessary conditions for subellipticity of the $\bar\partial$-Neumann problem. *Ann. of Math.* (2) 117 (1983), no. 1, 147–171.

[C84a] David Catlin. Global regularity of the $\bar\partial$-Neumann problem. Complex analysis of several variables (Madison, Wis., 1982), 39–49, Proc. Sympos. Pure Math., 41, Amer. Math. Soc., Providence, RI, 1984.

[C84b] David Catlin. Boundary invariants of pseudoconvex domains. *Ann. of Math. (2)*, **120**(3): 529–586, 1984.

[C87] David Catlin. Subelliptic estimates for the $\bar\partial$-Neumann problem on pseudoconvex domains. *Ann. of Math. (2)*, **126** (1): 131–191, 1987.

[D80] John P. D'Angelo. Subelliptic estimates and failure of semicontinuity for orders of contact. *Duke Math. J.* 47 (1980), no. 4, 955–957.

[D82] John P. D'Angelo. Real hypersurfaces, orders of contact, and applications. *Ann. of Math. (2)*, **115** (3): 615–637, 1982.

[E] Ebenfelt. New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy. *J. Differential Geom.* 50 (1998), no. 2, 207–247.

[K64a] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. I. *Ann. of Math.* (2) 78 1963, 112–148.

[K64b] J. J. Kohn. Harmonic integrals on strongly pseudo-convex manifolds. II. *Ann. of Math.* (2) 79 1964, 450–472.

[K79] J. J. Kohn. Subellipticity of the $\bar\partial$-Neumann problem on pseudo-convex domains: sufficient conditions. *Acta Math.* 142 (1979), no. 1-2, 79–122.

[W95] S. M. Webster. The holomorphic contact geometry of a real hypersurface, Modern Methods in Complex Analysis (T. Bloom et al, eds.), Annals of Mathematics Studies 137, Princeton University Press, Princeton, N.J., 1995, pp. 327–342.

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