Parameter estimation of partial differential equations via neural networks

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We are interested in applying neural networks to the problem of parameter estimation of partial differential equations from given observations. Precisely, we are given data $\{D_i\} = \{t_i, x_i, u_i\}, i = 1, \dots, N$ that are observed from the solution of partial differential equation of the form:

$$u_t + \mathscr{N}(u) = 0, \tag{1}$$

where $u = u(t, x)$ is the solution of the equation, subscript t denotes differentiation with respect to time, $\mathscr{N}(u; \lambda)$ is a nonlinear algebraic-differential operator, λ is a vector of parameters. The goal is to estimate λ from the observations $\{D_i\}$ via simultaneous training of neural networks $u(x, t)$ and

$$f(t, x; \lambda) = u_t + \mathscr{N}(u) \tag{2}$$

via training procedure defined by minimizing the mean square error (MSE)

$$\mathrm{MSE} = \mathrm{MSE}_u + \mathrm{MSE}_f, \tag{3}$$

where

$$\mathrm{MSE}_u = \frac{1}{N} \sum_{i=1}^{N} \left[u(t_i, x_i) - u_i\right]^2, \quad \mathrm{MSE}_f = \frac{1}{N} \sum_{i=1}^{N} \left[f(t_i, x_i)\right]^2. \tag{4}$$

As a concrete example, we consider linear heat equation

$$u_t - \lambda u_{xx} - g(t, x) = 0 \tag{5}$$

with one scalar sought-for parameter λ and given source function $g(t, x)$.

As a more difficult example, we consider viscous Burgers' equation

$$u_t + \lambda_1 u u_x - \lambda_2 u_{xx} = 0, \quad x \in [-1; 1], t \in [0, 1] \tag{6}$$

with sought-for parameter $\lambda = (\lambda_1, \lambda_2)^{\mathrm{T}}$. This equation is nonlinear and serves as a prototype of the governing equations of fluid dynamics. It is known that the solutions of this equation may develop sharp gradients in finite time even for smooth initial condition.