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Найти: условный экстремум $u(x,y) = 3 - 8x + 6y$
при условии $x^2 + y^2 = 36$

Решение:

$u(x,y) = 3 - 8x + 6y$; $D(u) = \mathbb{R}^2$

$x^2 + y^2 = 36 \Leftrightarrow x^2 + y^2 - 36 \Leftrightarrow f(x,y) = 0 \quad | f(x,y) := x^2 + y^2 - 36$

$f(x,y) = x^2 + y^2 - 36$; $D(f) = \mathbb{R}^2$

$L(x,y,\lambda) = u(x,y) + \lambda f(x,y)$; $D(L) = \{(x,y,\lambda) \in \mathbb{R}^3 \mid \lambda \neq 0\}$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3 - 8x + 6y) = -8 \frac{d}{dx}(x) = -8 \cdot 1 = -8$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3 - 8x + 6y) = 6 \frac{d}{dy}(y) = 6 \cdot 1 = 6$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 - 36) = \frac{d}{dx}(x^2) = 2x$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 - 36) = \frac{d}{dy}(y^2) = 2y$

$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}(u + \lambda f) = \frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x} = -8 + \lambda \cdot 2x = 2\lambda x - 8 = 2(\lambda x - 4)$

$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}(u + \lambda f) = \frac{\partial u}{\partial y} + \lambda \frac{\partial f}{\partial y} = 6 + \lambda \cdot 2y = 2\lambda y + 6 = 2(\lambda y + 3)$

$\frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial \lambda}(u + \lambda f) = f \frac{d}{d\lambda}(\lambda) = f \cdot 1 = f = x^2 + y^2 - 36$

$$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 2(\lambda x - 4) = 0 \\ 2(\lambda y + 3) = 0 \\ x^2 + y^2 - 36 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda x - 4 = 0 \\ \lambda y + 3 = 0 \\ x^2 + y^2 = 36 \end{cases} \overset{(1)}{\Leftrightarrow}$$

$\lambda x - 4 = 0 \Leftrightarrow \lambda x = 4$

$\lambda x = 4 \mid \lambda \neq 0 \Leftrightarrow x = \frac{4}{\lambda}$

$\lambda y + 3 = 0 \Leftrightarrow \lambda y = -3$

$\lambda y = -3 \mid \lambda \neq 0 \Leftrightarrow y = \frac{-3}{\lambda} = -\frac{3}{\lambda}$

$x^2 + y^2 \Big|_{x=\frac{4}{\lambda}, y=-\frac{3}{\lambda}} = \left(\frac{4}{\lambda}\right)^2 + \left(-\frac{3}{\lambda}\right)^2 = \frac{4^2}{\lambda^2} + \frac{3^2}{\lambda^2} = \frac{4^2+3^2}{\lambda^2} =$

$= \frac{16+9}{\lambda^2} = \frac{25}{\lambda^2}$

$x^2 + y^2 = 36 \mid x = \frac{4}{\lambda}, y = -\frac{3}{\lambda} \Leftrightarrow \frac{25}{\lambda^2} = 36 \Leftrightarrow$

$\Leftrightarrow \lambda^2 = \frac{25}{36} = \frac{5^2}{6^2} = \left(\frac{5}{6}\right)^2 \Leftrightarrow$

$\Leftrightarrow \begin{cases} \lambda = \frac{5}{6} \\ \lambda = -\frac{5}{6} \end{cases}$

(1) $\Leftrightarrow$

$\begin{cases} \lambda = \frac{5}{6} \\ x = \frac{4}{\lambda} = 4\left(\frac{5}{6}\right)^{-1} = 4\frac{6}{5} = \frac{24}{5} \\ y = -\frac{3}{\lambda} = -3\left(\frac{5}{6}\right)^{-1} = -3\frac{6}{5} = -\frac{18}{5} \end{cases}$

$\begin{cases} \lambda = -\frac{5}{6} \\ x = \frac{4}{\lambda} = 4\left(-\frac{5}{6}\right)^{-1} = -4\frac{6}{5} = -\frac{24}{5} \\ y = -\frac{3}{\lambda} = -3\left(-\frac{5}{6}\right)^{-1} = 3\frac{6}{5} = \frac{18}{5} \end{cases}$

$P_1 = (x_1, y_1, \lambda_1) = \left(\frac{24}{5}, -\frac{18}{5}, \frac{5}{6}\right)$

$P_2 = (x_2, y_2, \lambda_2) = \left(-\frac{24}{5}, \frac{18}{5}, -\frac{5}{6}\right)$

$f(x,y) = 0 \Rightarrow df = 0 \iff$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$\iff \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \iff$

$\frac{\partial f}{\partial x} = 2x \; ; \; \frac{\partial f}{\partial y} = 2y$

$\iff 2x dx + 2y dy = 0 \iff 2(x dx + y dy) = 0 \iff$

$\iff x dx + y dy = 0$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(-8) = 0$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(-8) = 0$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(6) = 0 = \frac{\partial^2 f}{\partial x \partial y}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(6) = 0$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2\frac{\partial}{\partial x}(x) = 2$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x) = 0$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0 = \frac{\partial^2 f}{\partial x \partial y}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2$

$$\frac{\partial^2 L}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \lambda \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} + \lambda \frac{\partial^2 \varphi}{\partial x^2} =$$

$$= 0 + 1 \cdot 2 = 2\lambda$$

$$\frac{\partial^2 L}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) + \lambda \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} + \lambda \frac{\partial^2 \varphi}{\partial y \partial x} =$$

$$= 0 + \lambda \cdot 0 = 0$$

$$\frac{\partial^2 L}{\partial \lambda \partial x} = \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial x}\right) = \frac{\partial}{\partial \lambda}\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial \lambda}(\lambda) = \frac{\partial}{\partial x} \cdot 1 = \frac{\partial}{\partial x} =$$

$$= 2x$$

$$\frac{\partial^2 L}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \lambda \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} + \lambda \frac{\partial^2 \varphi}{\partial x \partial y} =$$

$$= 0 + 1 \cdot 0 = 0 = \frac{\partial^2 L}{\partial y \partial x}$$

$$\frac{\partial^2 L}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \lambda \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} + \lambda \frac{\partial^2 \varphi}{\partial y^2} =$$

$$= 0 + \lambda \cdot 2 = 2\lambda$$

$$\frac{\partial^2 L}{\partial y \partial \lambda} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}\right) = \frac{\partial}{\partial y}\frac{\partial}{\partial \lambda}(\lambda) = \frac{\partial}{\partial y} \cdot 1 = \frac{\partial}{\partial y} =$$

$$= 2y$$

$$\frac{\partial^2 L}{\partial \lambda \partial x} = \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial x}\right) = \frac{\partial}{\partial x} = 2x = \frac{\partial^2 L}{\partial x \partial \lambda}$$

$$\frac{\partial^2 L}{\partial \lambda \partial y} = \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial y}\right) = \frac{\partial}{\partial y} = 2y = \frac{\partial^2 L}{\partial y \partial \lambda}$$

$$\frac{\partial^2 L}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda} = 0$$

$$d^2L = \frac{\partial^2 L}{\partial x^2}dx^2 + \frac{\partial^2 L}{\partial y^2}dy^2 + \frac{\partial^2 L}{\partial \lambda^2}d\lambda^2 +$$

$$+ 2\frac{\partial^2 L}{\partial x \partial y}dxdy + 2\frac{\partial^2 L}{\partial y \partial \lambda}dyd\lambda + 2\frac{\partial^2 L}{\partial \lambda \partial x}d\lambda dx =$$

$$= 2\lambda dx^2 + 2\lambda dy^2 + 0d\lambda^2 +$$

$$+ 2\cdot 0\, dxdy + 2\cdot 2y\, dyd\lambda + 2\cdot 2x\, d\lambda dx =$$

$$= 2\lambda dx^2 + 2\lambda dy^2 + 2\cdot 2y\, dyd\lambda + 2\cdot 2x\, d\lambda dx =$$

$$= 2\left(\lambda(dx^2+dy^2) + 2d\lambda(xdx + ydy)\right) =$$

$$= \left[xdx + ydy = 0\right] = 2\left(\lambda(dx^2+dy^2) + 2d\lambda\cdot 0\right) =$$

$$= 2\lambda(dx^2+dy^2)$$

$d^2L > 0 \Leftrightarrow 2\lambda(dx^2+dy^2) > 0 \Leftrightarrow \lambda(dx^2+dy^2) > 0 \Leftrightarrow$

$\Leftrightarrow \lambda > 0 \wedge (dx \neq 0 \vee dy \neq 0)$

$d^2L < 0 \Leftrightarrow 2\lambda(dx^2+dy^2) < 0 \Leftrightarrow \lambda(dx^2+dy^2) < 0 \Leftrightarrow$

$\Leftrightarrow \lambda < 0 \wedge (dx \neq 0 \vee dy \neq 0)$

$l_1 = (x_1, y_1, \lambda_1) = \left(\frac{24}{5}, -\frac{18}{5}, \frac{5}{6}\right)$

$\frac{\partial L}{\partial x}(l_1) = \frac{\partial L}{\partial y}(l_1) = \frac{\partial L}{\partial \lambda}(l_1) = 0 \Leftrightarrow dL(l_1) = 0$

$\lambda_1 = \frac{5}{6} > 0 \Leftrightarrow d^2L(l_1) > 0 \mid dx \neq 0 \vee dy \neq 0$

$(x_1, y_1) = \left(\frac{24}{5}, -\frac{18}{5}\right)$ – точка условного минимума

$l_2 = (x_2, y_2, \lambda_2) = \left(-\frac{24}{5}, \frac{18}{5}, -\frac{5}{6}\right)$

$\frac{\partial L}{\partial x}(l_2) = \frac{\partial L}{\partial y}(l_2) = \frac{\partial L}{\partial \lambda}(l_2) = 0 \Leftrightarrow dL(l_2) = 0$

$\lambda_2 = -\frac{5}{6} < 0 \Leftrightarrow d^2L(l_1) < 0 \mid dx \neq 0 \vee dy \neq 0$

$(x_2, y_2) = \left(-\frac{24}{5}, \frac{18}{5}\right)$ – точка условного максимума