6. Исследовать на экстремум функцию:

$$u(x,y) = x^2y + \frac{y^3}{3} + 2x^2 + 3y^2 - 1$$

Решение:

$D(u) = \{(x,y) \in \mathbb{R}^2\}$

1. Необходимое условие

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2y + \frac{y^3}{3} + 2x^2 + 3y^2 - 1\right) = y\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x^2) =$$

$$= y \cdot 2x + 2 \cdot 2x = 2xy + 4x,$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2y + \frac{y^3}{3} + 2x^2 + 3y^2 - 1\right) = x^2\frac{d}{dy}(y) + \frac{1}{3}\frac{d}{dy}(y^3) + 3\frac{d}{dy}(y^2) =$$

$$= x^2 \cdot 1 + \frac{1}{3} \cdot 3y^2 + 3 \cdot 2y = x^2 + y^2 + 6y$$

$$\begin{cases} \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2xy + 4x = 0 \\ x^2 + y^2 + 6y = 0 \end{cases} \overset{(1)}{\Leftrightarrow}$$

$$2xy + 4x = 0 \Leftrightarrow 2x(y+2) = 0 \Leftrightarrow x(y+2) = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y + 2 = 0 \end{cases} \Leftrightarrow \left[\begin{array}{l} x = 0 \\ y = -2 \end{array}\right.$$

$$x^2 + y^2 + 6y = 0 \Leftrightarrow x^2 + (y^2 + 2y \cdot 3 + 3^2) - 3^2 = 0 \Leftrightarrow x^2 + (y+3)^2 - 3^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x^2 + (y+3)^2 = 3^2$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} x = 0 \\ y = -2 \end{array}\right. \\ x^2 + (y+3)^2 = 3^2 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x = 0 \\ x^2 + (y+3)^2 = 3^2 \end{cases} \\ \begin{cases} y = -2 \\ x^2 + (y+3)^2 = 3^2 \end{cases} \end{array}\right. \overset{(2)}{\Leftrightarrow}$$

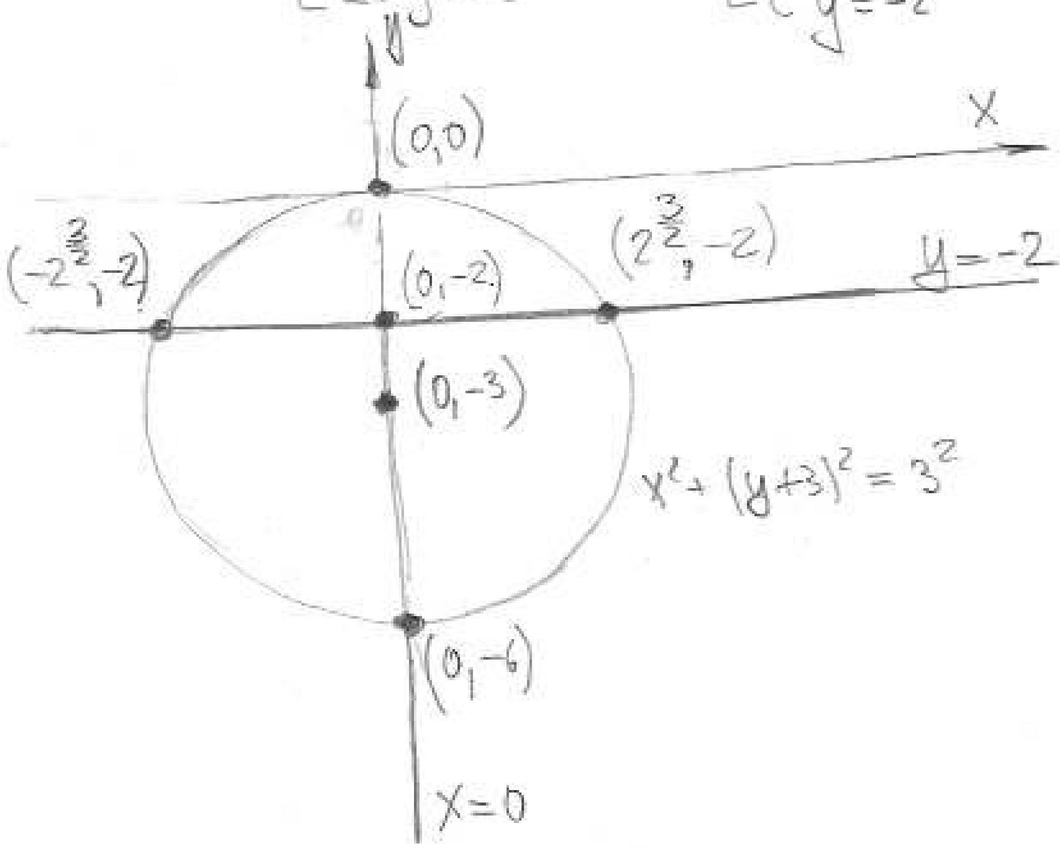
$$x^2 + (y+3)^2 = 3^2 \mid x = 0 \Leftrightarrow 0^2 + (y+3)^2 = 3^3 \Leftrightarrow (y+3)^2 = 3^2 \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} y + 3 = 3 \\ y + 3 = -3 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} y = 0 \\ y = -6 \end{array}\right.$$

$$x^2+(y+3)^2=3^2 \mid y=-2 \Leftrightarrow x^2+(-2+3)^2=3^2 \Leftrightarrow x^2+1^2=3^2 \Leftrightarrow$$

$$\Leftrightarrow x^2=3^2-1=9-1=8=2^3 \Leftrightarrow \left[\begin{array}{l} x=(2^3)^{\frac{1}{2}} \\ x=-(2^3)^{\frac{1}{2}} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=2^{\frac{3}{2}} \\ y=-2^{\frac{3}{2}} \end{array}\right.$$

$$\overset{(2)}{\Leftrightarrow} \left[\begin{array}{l} \left\{\begin{array}{l} x=0 \\ \left[\begin{array}{l} y=0 \\ y=-6 \end{array}\right. \end{array}\right. \\ \left\{\begin{array}{l} y=-2 \\ \left[\begin{array}{l} x=2^{\frac{3}{2}} \\ x=-2^{\frac{3}{2}} \end{array}\right. \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \left[\begin{array}{l} \left\{\begin{array}{l} x=0 \\ y=0 \end{array}\right. \\ \left\{\begin{array}{l} x=0 \\ y=-6 \end{array}\right. \end{array}\right. \\ \left[\begin{array}{l} \left\{\begin{array}{l} x=2^{\frac{3}{2}} \\ y=-2 \end{array}\right. \\ \left\{\begin{array}{l} x=-2^{\frac{3}{2}} \\ y=-2 \end{array}\right. \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \left\{\begin{array}{l} x=0 \\ y=0 \end{array}\right. \\ \left\{\begin{array}{l} x=0 \\ y=-6 \end{array}\right. \\ \left\{\begin{array}{l} x=2^{\frac{3}{2}} \\ y=-2 \end{array}\right. \\ \left\{\begin{array}{l} x=-2^{\frac{3}{2}} \\ y=-2 \end{array}\right. \end{array}\right.$$



② достаточное условие (критерий Сильвестра)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy + 4x\right) = 2y\frac{d}{dx}(x) + 4\frac{d}{dx}(x) = 2y\cdot 1 + 4\cdot 1 = 2y + 4$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + 4x\right) = 2x\frac{d}{dy}(y) = 2x\cdot 1 = 2x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 + y^2 + 6y\right) = \frac{d}{dx}(x^2) = 2x = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 + y^2 + 6y\right) = \frac{d}{dy}(y^2) + \frac{d}{dy}(6y) = 2y + 6$$

$$\Delta_1 = \frac{\partial^2 u}{\partial x^2} = 2y + 4 = 2(y+2)$$

$$\Delta_2 = \det\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 u}{\partial y \partial x} =$$

$$= \left[\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right] = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 =$$

$$= (2y+4)(2y+6) - (2x)^2 = 2(y+2)2(y+3) - 2^2x^2 =$$

$$= 4\left((y+2)(y+3) - x^2\right)$$

$(0,0)$

$$\Delta_1(0,0) = 2(y+2)\Big|_{x=0,y=0} = 2(0+2) = 2\cdot2 = 4 > 0$$

$$\Delta_2(0,0) = 4\left((y+2)(y+3) - x^2\right)\Big|_{x=0,y=0} = 4\left((0+2)(0+3) - 0^2\right) = 4(2\cdot3 - 0) = 4\cdot6 = 24 > 0$$

$\Delta_1(0,0) > 0 \wedge \Delta_2(0,0) > 0 \Leftrightarrow (0,0)$ – точка минимума

$(0,-6)$

$$\Delta_1(0,-6) = 2(y+2)\Big|_{x=0,y=-6} = 2(-6+2) = 2(-4) = -8 < 0$$

$$\Delta_2(0,-6) = 4\left((y+2)(y+3) - x^2\right)\Big|_{x=0,y=-6} = 4\left((-6+2)(-6+3) - 0^2\right) = 4\left((-4)(-3) - 0\right) = 4\cdot12 = 48 > 0$$

$\Delta_1(0,-6) \wedge \Delta_2(0,-6) \Leftrightarrow (0,-6)$ – точка максимума

$(\pm 2^{\frac{3}{2}}, -2)$

$$\Delta_1\Big|_{y=-2} = 2(y+2)\Big|_{y=-2} = 2(-2+2) = 2\cdot0 = 0$$

Ответ: по критерию Сильвестра

$(0,0)$ – точка минимума

$(0,-6)$ – точка максимума