$\boxed{1}$ Найти производную $\frac{dy}{dx}$ функции $y(x)$, заданной неявно $\operatorname{arctg}\left(\frac{y}{x}\right) = \ln\sqrt{x^2+y^2}$

Решение:

$\operatorname{arctg}\left(\frac{y}{x}\right) = \ln\sqrt{x^2+y^2}$

$D(y) = \{ x \in \mathbb{R} \mid x \neq 0 \}$

$\operatorname{arctg}\left(\frac{y}{x}\right) = \ln\sqrt{x^2+y^2} \Rightarrow \frac{d}{dx}\operatorname{arctg}\left(\frac{y}{x}\right) = \frac{d}{dx}\ln\sqrt{x^2+y^2}$

$$\frac{d}{dx}\operatorname{arctg}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\frac{d}{dx}(y)x - y\frac{d}{dx}(x)}{x^2} =$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\frac{dy}{dx}x - y\cdot 1}{x^2} = \frac{x\frac{dy}{dx} - y}{x^2\left(1+\frac{y^2}{x^2}\right)} =$$

$$= [x \neq 0] = \frac{x\frac{dy}{dx} - y}{x^2+y^2}$$

$$\frac{d}{dx}\ln\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{d}{dx}\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\frac{d}{dx}(x^2+y^2) =$$

$$= \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\left(\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2)\right) = \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\left(2x + 2y\frac{dy}{dx}\right) =$$

$$= \frac{2\left(y\frac{dy}{dx} + x\right)}{2\left(\sqrt{x^2+y^2}\right)^2} = \frac{y\frac{dy}{dx} + x}{x^2+y^2}$$

$$\frac{x\frac{dy}{dx} - y}{x^2+y^2} = \frac{y\frac{dy}{dx} + x}{x^2+y^2} \Leftrightarrow x\frac{dy}{dx} - y = y\frac{dy}{dx} + x \Leftrightarrow$$

$$\Leftrightarrow (x-y)\frac{dy}{dx} = x+y \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \;\Big|\; x-y \neq 0$$

$x - y \neq 0 \Leftrightarrow y \neq x$

$\neg(y \neq x) \Leftrightarrow y = x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x) = 1$

Ответ: $\left[\begin{array}{ll} \frac{dy}{dx} = \frac{x+y}{x-y} & \mid y \neq x \\ \frac{dy}{dx} = 1 & \mid y = x \end{array}\right.$

Вариант 2:

$$\operatorname{arctg}\left(\frac{y}{x}\right) = \ln\sqrt{x^2+y^2} \Leftrightarrow \operatorname{arctg}\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} = 0 \Leftrightarrow$$

$$\Leftrightarrow F(x,y) = 0 \;\Big|\; F(x,y) := \operatorname{arctg}\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2}$$

$$D(F) = \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; x \neq 0\right\}$$

$$F(x,y) = 0 \Rightarrow \frac{dF}{dx} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \Leftrightarrow \frac{\partial F}{\partial y}\frac{dy}{dx} = -\frac{\partial F}{\partial x} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F}{\partial x}\left(\frac{\partial F}{\partial y}\right)^{-1} \;\Big|\; \frac{\partial F}{\partial y} \neq 0$$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\operatorname{arctg}\left(\frac{x}{y}\right) - \ln\sqrt{x^2+y^2}\right) = \frac{\partial}{\partial x}\operatorname{arctg}\left(\frac{y}{x}\right) - \frac{\partial}{\partial x}\ln\sqrt{x^2+y^2} = (1)$$

$$\frac{\partial}{\partial x}\operatorname{arctg}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\, y\frac{d}{dx}\left(\frac{1}{x}\right) =$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\, y\left(-\frac{1}{x^2}\right) = -\frac{y}{x^2\left(1+\frac{y^2}{x^2}\right)} =$$

$$= \left[x \neq 0\right] = -\frac{y}{x^2+y^2}$$

$$\frac{\partial}{\partial x}\ln\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial x}\left(x^2+y^2\right) =$$

$$= \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\frac{d}{dx}\left(x^2\right) = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\, 2x =$$

$$= \frac{2x}{2\left(\sqrt{x^2+y^2}\right)^2} = \frac{x}{x^2+y^2}$$

$$(1) = -\frac{y}{x^2+y^2} - \frac{x}{x^2+y^2} = -\frac{x+y}{x^2+y^2}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\operatorname{arctg}\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2}\right) = \frac{\partial}{\partial y}\operatorname{arctg}\left(\frac{y}{x}\right) - \frac{\partial}{\partial y}\ln\sqrt{x^2+y^2} = (2)$$

$$\frac{\partial}{\partial y}\operatorname{arctg}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}\frac{d}{dy}(y) =$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}\,1 = \frac{1}{x\left(1+\left(\frac{y}{x}\right)^2\right)} =$$

$$= [x \neq 0] = \frac{x}{x^2\left(1+\frac{y^2}{x^2}\right)} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\ln\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial y}(x^2+y^2) =$$

$$= \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\frac{d}{dy}(y^2) = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\,2y =$$

$$= \frac{2y}{2\left(\sqrt{x^2+y^2}\right)^2} = \frac{y}{x^2+y^2}$$

$$(2) = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{x-y}{x^2+y^2}$$

$$\frac{\partial F}{\partial y} = 0 \Leftrightarrow \frac{x-y}{x^2+y^2} = 0 \Leftrightarrow x-y=0 \Leftrightarrow y=x \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

$$\frac{\partial F}{\partial y} \neq 0 \Leftrightarrow \neg\left(\frac{\partial F}{\partial y}=0\right) \Leftrightarrow \neg(y=x) \Leftrightarrow y \neq x$$

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x}\left(\frac{\partial F}{\partial y}\right)^{-1} = -\left(-\frac{x+y}{x^2+y^2}\right)\left(\frac{x-y}{x^2+y^2}\right)^{-1} = \frac{x+y}{x-y}$$

Ответ:

$$\left[\begin{array}{ll} \frac{dy}{dx} = \frac{x+y}{x-y} & \Big|\; y \neq x \\ \frac{dy}{dx} = 1 & \Big|\; y = x \end{array}\right.$$