$\boxed{2}$ Найти: $\int e^{2x}\cos 3x\,dx$

Решение:

$J_1(x,\alpha,\beta) := \int e^{\alpha x}\sin(\beta x)dx$

$J_2(x,\alpha,\beta) := \int e^{\alpha x}\cos(\beta x)dx$

$J_1 = \int e^{\alpha x}\sin(\beta x)dx = \int \sin(\beta x)(e^{\alpha x}dx) = \int u\,dv = (1)$

$u=\sin(\beta x) \Rightarrow du = d\sin(\beta x) = \cos(\beta x)d(\beta x) = \cos\beta x(\beta dx) = \beta\cos(\beta x)dx$

$dv = e^{\alpha x}dx = e^{\alpha x}\frac{1}{\alpha}(\alpha dx) = \frac{1}{\alpha}(e^{\alpha x}d(\alpha x)) = \frac{1}{\alpha}de^{\alpha x} = d\left(\frac{e^{\alpha x}}{\alpha}\right) \Rightarrow v = \frac{e^{\alpha x}}{\alpha}$

$uv = \sin(\beta x)\frac{e^{\alpha x}}{\alpha} = \frac{e^{\alpha x}}{\alpha}\sin(\beta x)$

$v\,du = \frac{e^{\alpha x}}{\alpha}(\beta\cos(\beta x)dx) = \frac{\beta}{\alpha}e^{\alpha x}\cos(\beta x)dx$

$(1) = uv - \int v\,du = \frac{e^{\alpha x}}{\alpha}\sin(\beta x) - \int\frac{\beta}{\alpha}e^{\alpha x}\cos(\beta x)dx =$

$= \frac{e^{\alpha x}}{\alpha}\sin(\beta x) - \frac{\beta}{\alpha}\int e^{\alpha x}\cos(\beta x)dx = \frac{e^{\alpha x}}{\alpha}\sin(\beta x) - \frac{\beta}{\alpha}J_2$

$J_2 = \int e^{\alpha x}\cos(\beta x)dx = \int\cos(\beta x)(e^{\alpha x}dx) = \int u\,dv = (2)$

$u=\cos(\beta x) \Rightarrow du = d\cos(\beta x) = -\sin(\beta x)d(\beta x) = -\sin(\beta x)\beta dx = -\beta\sin(\beta x)dx$

$dv = e^{\alpha x}dx = e^{\alpha x}\frac{1}{\alpha}(\alpha dx) = \frac{1}{\alpha}(e^{\alpha x}d(\alpha x)) = \frac{1}{\alpha}de^{\alpha x} = d\left(\frac{e^{\alpha x}}{\alpha}\right) \Rightarrow v = \frac{e^{\alpha x}}{\alpha}$

$uv = \cos(\beta x)\frac{e^{\alpha x}}{\alpha} = \frac{e^{\alpha x}}{\alpha}\cos(\beta x)$

$v\,du = \frac{e^{\alpha x}}{\alpha}(-\beta\sin(\beta x)dx) = -\frac{\beta}{\alpha}e^{\alpha x}\sin(\beta x)dx$

$(2) = uv - \int v\,du = \frac{e^{\alpha x}}{\alpha}\cos(\beta x) - \int\left(-\frac{\beta}{\alpha}\right)e^{\alpha x}\sin(\beta x)dx =$

$= \frac{e^{\alpha x}}{\alpha}\cos(\beta x) + \frac{\beta}{\alpha}\int e^{\alpha x}\sin(\beta x)dx = \frac{e^{\alpha x}}{\alpha}\cos(\beta x) + \frac{\beta}{\alpha}J_1$

$$\begin{cases} J_1 = \frac{e^{\alpha x}}{\alpha}\sin(\beta x) - \frac{\beta}{\alpha} J_2 \\ J_2 = \frac{e^{\alpha x}}{\alpha}\cos(\beta x) + \frac{\beta}{\alpha} J_1 \end{cases} \Leftrightarrow \begin{cases} J_1 + \frac{\beta}{\alpha} J_2 = \frac{e^{\alpha x}}{\alpha}\sin(\beta x) \\ -\frac{\beta}{\alpha} J_1 + J_2 = \frac{e^{\alpha x}}{\alpha}\cos(\beta x) \end{cases} \Leftrightarrow$$

$$\begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} \frac{e^{\alpha x}}{\alpha}\sin(\beta x) \\ \frac{e^{\alpha x}}{\alpha}\cos(\beta x) \end{pmatrix}$$

$$\Delta = \det \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix} = 1\cdot 1 - \frac{\beta}{\alpha}\left(-\frac{\beta}{\alpha}\right) = 1 + \left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^2+\beta^2}{\alpha^2}$$

$$\Delta \neq 0 \quad \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix}^{-1} = \Delta^{-1} \begin{pmatrix} (-1)^{1+1}\cdot 1 & (-1)^{1+2}\cdot\left(-\frac{\beta}{\alpha}\right) \\ (-1)^{2+1}\cdot\frac{\beta}{\alpha} & (-1)^{2+2}\cdot 1 \end{pmatrix}^T =$$

$$= \Delta^{-1} \begin{pmatrix} 1\cdot 1 & (-1)\left(-\frac{\beta}{\alpha}\right) \\ (-1)\cdot\frac{\beta}{\alpha} & 1\cdot 1 \end{pmatrix}^T = \Delta^{-1} \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix}^T = \Delta^{-1} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix} \Delta^{-1} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} =$$

$$= \Delta^{-1} \begin{pmatrix} 1\cdot 1 + \frac{\beta}{\alpha}\cdot\frac{\beta}{\alpha} & 1\cdot\left(-\frac{\beta}{\alpha}\right) + \frac{\beta}{\alpha}\cdot 1 \\ \left(-\frac{\beta}{\alpha}\right)\cdot 1 + 1\cdot\frac{\beta}{\alpha} & \left(\frac{\beta}{\alpha}\right)\left(-\frac{\beta}{\alpha}\right) + 1\cdot 1 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 1 + \left(\frac{\beta}{\alpha}\right)^2 & -\frac{\beta}{\alpha} + \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} + \frac{\beta}{\alpha} & \left(\frac{\beta}{\alpha}\right)^2 + 1 \end{pmatrix} =$$

$$= \Delta^{-1} \begin{pmatrix} 1 + \left(\frac{\beta}{\alpha}\right)^2 & 0 \\ 0 & 1 + \left(\frac{\beta}{\alpha}\right)^2 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} = \Delta^{-1}\Delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ -\frac{\beta}{\alpha} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{e^{\alpha x}}{\alpha} \sin(\beta x) \\ \frac{e^{\alpha x}}{\alpha} \cos(\beta x) \end{pmatrix} =$$

$$= \Delta^{-1} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} \frac{e^{\alpha x}}{\alpha} \begin{pmatrix} \sin(\beta x) \\ \cos(\beta x) \end{pmatrix} = \frac{e^{\alpha x}}{\alpha \Delta} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \sin(\beta x) \\ \cos(\beta x) \end{pmatrix} =$$

$$= \frac{e^{\alpha x}}{\alpha \Delta} \frac{1}{\alpha} \alpha \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \sin(\beta x) \\ \cos(\beta x) \end{pmatrix} = \frac{e^{\alpha x}}{\alpha^2 \Delta} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \sin(\beta x) \\ \cos(\beta x) \end{pmatrix} =$$

$$\alpha^2 \Delta = \alpha^2 \frac{\alpha^2+\beta^2}{\alpha^2} = \alpha^2+\beta^2$$

$$= \frac{e^{\alpha x}}{\alpha^2+\beta^2} \begin{pmatrix} \alpha \sin(\beta x) - \beta \cos(\beta x) \\ \beta \sin(\beta x) + \alpha \cos(\beta x) \end{pmatrix} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} z_1 = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha \sin(\beta x) - \beta \cos(\beta x)\right) \\ z_2 = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\beta \sin(\beta x) + \alpha \cos(\beta x)\right) \end{cases}$$

$$\frac{dJ_1}{dx} = \frac{d}{dx}\left(\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha\sin(\beta x) - \beta\cos(\beta x)\right)\right) =$$

$$= \frac{1}{\alpha^2+\beta^2}\left(\frac{d}{dx}\left(e^{\alpha x}\right)\left(\alpha\sin(\beta x) - \beta\cos(\beta x)\right) + e^{\alpha x}\left(\alpha\frac{d}{dx}\left(\sin(\beta x)\right) - \beta\frac{d}{dx}\left(\cos(\beta x)\right)\right)\right) =$$

$$= \frac{1}{\alpha^2+\beta^2}\left(\left(\alpha e^{\alpha x}\right)\left(\alpha\sin(\beta x) - \beta\cos(\beta x)\right) + e^{\alpha x}\left(\alpha\cos(\beta x)\beta - \beta\left(-\sin(\beta x)\right)\beta\right)\right) =$$

$$= \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha\left(\alpha\sin(\beta x) - \beta\cos(\beta x)\right) + \left(\alpha\beta\cos(\beta x) + \beta^2\sin(\beta x)\right)\right) =$$

$$= \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha^2\sin(\beta x) + \beta^2\sin(\beta x)\right) = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha^2+\beta^2\right)\sin(\beta x) =$$

$$= e^{\alpha x}\sin(\beta x)$$

$$\frac{dJ_2}{dx} = \frac{d}{dx}\left(\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\beta\sin(\beta x) + \alpha\cos(\beta x)\right)\right) =$$

$$= \frac{1}{\alpha^2+\beta^2}\left(\frac{d}{dx}\left(e^{\alpha x}\right)\left(\beta\sin(\beta x) + \alpha\cos(\beta x)\right) + e^{\alpha x}\left(\beta\frac{d}{dx}\left(\sin(\beta x)\right) + \alpha\frac{d}{dx}\left(\cos(\beta x)\right)\right)\right) =$$

$$= \frac{1}{\alpha^2+\beta^2}\left(\alpha e^{\alpha x}\left(\beta\sin(\beta x) + \alpha\cos(\beta x)\right) + e^{\alpha x}\left(\beta\cos(\beta x)\beta + \alpha\left(-\sin(\beta x)\right)\beta\right)\right) =$$

$$= \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha\left(\beta\sin(\beta x) + \alpha\cos(\beta x)\right) + \left(\beta^2\cos(\beta x) - \alpha\beta\sin(\beta x)\right)\right) =$$

$$= \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha^2\cos(\beta x) + \beta^2\cos(\beta x)\right) = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha^2+\beta^2\right)\cos(\beta x) =$$

$$= e^{\alpha x}\cos(\beta x)$$

$$\int e^{2x}\cos 3x\,dx = \mathcal{J}_2(x,2,3) =$$

$$= \left.\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\beta\sin(\beta x)+\alpha\cos(\beta x)\right)\right|_{\substack{\alpha=2\\ \beta=3}} =$$

$$= \frac{e^{2x}}{2^2+3^2}\left(3\sin 3x + 2\cos 3x\right) \Rightarrow$$

$$\Rightarrow \int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{2^2+3^2}\left(3\sin 3x+2\cos 3x\right)+C$$

Ответ:

$$\boxed{\begin{aligned}
&\int e^{\alpha x}\sin(\beta x)\,dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\alpha\sin(\beta x)-\beta\cos(\beta x)\right)\\
&\int e^{\alpha x}\cos(\beta x)\,dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left(\beta\sin(\beta x)+\alpha\cos(\beta x)\right) \Rightarrow\\
&\Rightarrow \int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{2^2+3^2}\left(3\sin 3x+2\cos 3x\right)+C
\end{aligned}}$$