

[2] Aufgabe: $\int e^{2x} \cos 3x dx$

Permutation:

$$J_1(x, \alpha, \beta) := \int e^{\alpha x} \sin(\beta x) dx$$

$$J_2(x, \alpha, \beta) := \int e^{\alpha x} \cos(\beta x) dx$$

$$J_1 = \int e^{\alpha x} \sin(\beta x) dx = \int \sin(\beta x) (e^{\alpha x} dx) = \int u dv = (1)$$

$$u = \sin(\beta x) \rightarrow du = d \sin(\beta x) = \cos(\beta x) d(\beta x) = \cos \beta x (\beta dx) = \beta \cos(\beta x) dx$$

$$dv = e^{\alpha x} dx = e^{\alpha x} \frac{1}{\alpha} (d dx) = \frac{1}{\alpha} (e^{\alpha x} d dx) = \frac{1}{\alpha} d e^{\alpha x} = d \left(\frac{e^{\alpha x}}{\alpha} \right) \Rightarrow v = \frac{e^{\alpha x}}{\alpha}$$

$$uv = \sin(\beta x) \frac{e^{\alpha x}}{\alpha} = \frac{e^{\alpha x}}{\alpha} \sin(\beta x)$$

$$v du = \frac{e^{\alpha x}}{\alpha} (\beta \cos(\beta x) dx) = \frac{\beta}{\alpha} e^{\alpha x} \cos(\beta x) dx$$

$$(1) = uv - \int v du = \frac{e^{\alpha x}}{\alpha} \sin(\beta x) - \int \frac{\beta}{\alpha} e^{\alpha x} \cos(\beta x) dx =$$

$$= \frac{e^{\alpha x}}{\alpha} \sin(\beta x) - \frac{\beta}{\alpha} \int e^{\alpha x} \cos(\beta x) dx = \frac{e^{\alpha x}}{\alpha} \sin(\beta x) - \frac{\beta}{\alpha} J_2$$

$$J_2 = \int e^{\alpha x} \cos(\beta x) dx = \int \cos(\beta x) (e^{\alpha x} dx) = \int u dv = (2)$$

$$u = \cos(\beta x) \rightarrow du = d \cos(\beta x) = -\sin(\beta x) d(\beta x) = -\sin(\beta x) \beta dx = -\beta \sin(\beta x) dx$$

$$dv = e^{\alpha x} dx = e^{\alpha x} \frac{1}{\alpha} (d dx) = \frac{1}{\alpha} (e^{\alpha x} d dx) = \frac{1}{\alpha} d e^{\alpha x} = d \left(\frac{e^{\alpha x}}{\alpha} \right) \Rightarrow v = \frac{e^{\alpha x}}{\alpha}$$

$$uv = \cos(\beta x) \frac{e^{\alpha x}}{\alpha} = \frac{e^{\alpha x}}{\alpha} \cos(\beta x)$$

$$v du = \frac{e^{\alpha x}}{\alpha} (-\beta \sin(\beta x) dx) = -\frac{\beta}{\alpha} e^{\alpha x} \sin(\beta x) dx$$

$$(2) = uv - \int v du = \frac{e^{\alpha x}}{\alpha} \cos(\beta x) - \int \left(-\frac{\beta}{\alpha} \right) e^{\alpha x} \sin(\beta x) dx =$$

$$= \frac{e^{\alpha x}}{\alpha} \cos(\beta x) + \frac{\beta}{\alpha} \int e^{\alpha x} \sin(\beta x) dx = \frac{e^{\alpha x}}{\alpha} \cos(\beta x) + \frac{\beta}{\alpha} J_1$$

$$\begin{cases} J_1 = \frac{e^{ix}}{x} \sin(px) - \frac{1}{x} J_2 \\ J_2 = \frac{e^{ix}}{x} \cos(px) + \frac{1}{x} J_1 \end{cases} \Leftrightarrow \begin{cases} J_1 + \frac{1}{x} J_2 = \frac{e^{ix}}{x} \sin(px) \\ -\frac{1}{x} J_1 + J_2 = \frac{e^{ix}}{x} \cos(px) \end{cases} \Leftrightarrow$$

$$\begin{pmatrix} 1 & \frac{1}{x} \\ -\frac{1}{x} & 1 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} \frac{e^{ix}}{x} \sin(px) \\ \frac{e^{ix}}{x} \cos(px) \end{pmatrix}$$

$$\Delta = \det \begin{pmatrix} 1 & \frac{1}{x} \\ -\frac{1}{x} & 1 \end{pmatrix} = 1 \cdot 1 - \frac{1}{x} \left(-\frac{1}{x}\right) = 1 + \left(\frac{1}{x}\right)^2 = \frac{x^2 + 1}{x^2}$$

$$\Delta \neq 0 \quad \begin{pmatrix} 1 & \frac{1}{x} \\ -\frac{1}{x} & 1 \end{pmatrix}^{-1} = \Delta^{-1} \begin{pmatrix} (-1)^{1+1} \cdot 1 & (-1)^{1+2} \left(-\frac{1}{x}\right) \\ (-1)^{2+1} \frac{1}{x} & (-1)^{2+2} \cdot 1 \end{pmatrix} =$$

$$= \Delta^{-1} \begin{pmatrix} 1 \cdot 1 & (-1) \left(-\frac{1}{x}\right) \\ (-1) \cdot \frac{1}{x} & 1 \cdot 1 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 1 & \frac{1}{x} \\ -\frac{1}{x} & 1 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 1 & \frac{1}{x} \\ \frac{1}{x} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{1}{x} \\ -\frac{1}{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{x} \\ \frac{1}{x} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{x} \\ \frac{1}{x} & 1 \end{pmatrix} \Delta^{-1} \begin{pmatrix} 1 & \frac{1}{x} \\ \frac{1}{x} & 1 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 1 & \frac{1}{x} \\ \frac{1}{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{x} \\ \frac{1}{x} & 1 \end{pmatrix} =$$

$$= \Delta^{-1} \begin{pmatrix} 1 \cdot 1 + \frac{1}{x} \cdot \frac{1}{x} & 1 \cdot \left(-\frac{1}{x}\right) + \frac{1}{x} \cdot 1 \\ \left(-\frac{1}{x}\right) \cdot 1 + 1 \cdot \frac{1}{x} & \left(-\frac{1}{x}\right) \left(-\frac{1}{x}\right) + 1 \cdot 1 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 1 + \left(\frac{1}{x}\right)^2 & -\frac{1}{x} + \frac{1}{x} \\ -\frac{1}{x} + \frac{1}{x} & \left(\frac{1}{x}\right)^2 + 1 \end{pmatrix} =$$

$$= \Delta^{-1} \begin{pmatrix} 1 + \left(\frac{1}{x}\right)^2 & 0 \\ 0 & 1 + \left(\frac{1}{x}\right)^2 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} = \Delta^{-1} \Delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{e^{\alpha x}}{\alpha} \sinh(\beta x) \\ \frac{e^{\alpha x}}{\alpha} \cosh(\beta x) \end{pmatrix} =$$

$$= \Delta^{-1} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} \frac{e^{\alpha x}}{\alpha} \begin{pmatrix} \sinh(\beta x) \\ \cosh(\beta x) \end{pmatrix} = \frac{e^{\alpha x}}{\alpha \Delta} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \sinh(\beta x) \\ \cosh(\beta x) \end{pmatrix} =$$

$$= \frac{e^{\alpha x}}{\alpha \Delta} \frac{1}{\alpha} \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \sinh(\beta x) \\ \cosh(\beta x) \end{pmatrix} = \frac{e^{\alpha x}}{\alpha^2 \Delta} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \sinh(\beta x) \\ \cosh(\beta x) \end{pmatrix} =$$

$$\alpha^2 \Delta = \alpha^2 \frac{\alpha^2 + \beta^2}{\alpha^2} = \alpha^2 + \beta^2$$

$$= \frac{e^{\alpha x}}{\alpha^2 + \beta^2} \begin{pmatrix} \alpha \sinh(\beta x) - \beta \cosh(\beta x) \\ \beta \sinh(\beta x) + \alpha \cosh(\beta x) \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z_1 = \frac{e^{\alpha x}}{\alpha^2 + \beta^2} (\alpha \sinh(\beta x) - \beta \cosh(\beta x)) \\ z_2 = \frac{e^{\alpha x}}{\alpha^2 + \beta^2} (\beta \sinh(\beta x) + \alpha \cosh(\beta x)) \end{cases}$$

$$\begin{aligned}
 \frac{dI_1}{dx} &= \frac{d}{dx} \left(\frac{e^{dx}}{d^2 + \beta^2} (d \sin(\beta x) - \beta \cos(\beta x)) \right) = \\
 &= \frac{1}{d^2 + \beta^2} \left(\frac{d}{dx} (e^{dx}) (d \sin(\beta x) - \beta \cos(\beta x)) + e^{dx} \left(d \frac{d}{dx} (\sin(\beta x)) - \beta \frac{d}{dx} (\cos(\beta x)) \right) \right) = \\
 &= \frac{1}{d^2 + \beta^2} \left(d e^{dx} (d \sin(\beta x) - \beta \cos(\beta x)) + e^{dx} (d \cos(\beta x) \beta - \beta (-\sin(\beta x)) \beta) \right) = \\
 &= \frac{e^{dx}}{d^2 + \beta^2} \left(d (d \sin(\beta x) - \beta \cos(\beta x)) + (d \beta \cos(\beta x) + \beta^2 \sin(\beta x)) \right) = \\
 &= \frac{e^{dx}}{d^2 + \beta^2} (d^2 \sin(\beta x) + \beta^2 \sin(\beta x)) = \frac{e^{dx}}{d^2 + \beta^2} (d^2 + \beta^2) \sin(\beta x) = \\
 &= e^{dx} \sin(\beta x)
 \end{aligned}$$

$$\begin{aligned}
 \frac{dI_2}{dx} &= \frac{d}{dx} \left(\frac{e^{dx}}{d^2 + \beta^2} (\beta \sin(\beta x) + d \cos(\beta x)) \right) = \\
 &= \frac{1}{d^2 + \beta^2} \left(\frac{d}{dx} (e^{dx}) (\beta \sin(\beta x) + d \cos(\beta x)) + e^{dx} \left(\beta \frac{d}{dx} (\sin(\beta x)) + d \frac{d}{dx} (\cos(\beta x)) \right) \right) = \\
 &= \frac{1}{d^2 + \beta^2} \left(d e^{dx} (\beta \sin(\beta x) + d \cos(\beta x)) + e^{dx} (\beta \cos(\beta x) \beta + d (-\sin(\beta x)) \beta) \right) = \\
 &= \frac{e^{dx}}{d^2 + \beta^2} \left(d (\beta \sin(\beta x) + d \cos(\beta x)) + (\beta^2 \cos(\beta x) - d \beta \sin(\beta x)) \right) = \\
 &= \frac{e^{dx}}{d^2 + \beta^2} (d^2 \cos(\beta x) + \beta^2 \cos(\beta x)) = \frac{e^{dx}}{d^2 + \beta^2} (d^2 + \beta^2) \cos(\beta x) = \\
 &= e^{dx} \cos(\beta x)
 \end{aligned}$$

$$\int e^{2x} \cos 3x \, dx = I_2(x, 2, 3) =$$

$$= \frac{e^{2x}}{2^2 + 3^2} \left(\beta \sin(\beta x) + \alpha \cos(\beta x) \right) \Big|_{\substack{\alpha=2 \\ \beta=3}} =$$

$$= \frac{e^{2x}}{2^2 + 3^2} \left(3 \sin 3x + 2 \cos 3x \right) \Rightarrow$$

$$\Rightarrow \int e^{2x} \cos 3x \, dx = \frac{e^{2x}}{2^2 + 3^2} \left(3 \sin 3x + 2 \cos 3x \right) + C$$

Omben:

$$\int e^{ax} \sin(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} \left(a \sin(bx) - b \cos(bx) \right)$$

$$\int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} \left(b \sin(bx) + a \cos(bx) \right) \Rightarrow$$

$$\Rightarrow \int e^{2x} \cos 3x \, dx = \frac{e^{2x}}{2^2 + 3^2} \left(3 \sin 3x + 2 \cos 3x \right) + C$$