[4] Найти: $\int_2^{+\infty} \frac{dx}{x^2+x-2}$

Решение:

$$x^2+x-2 = \left(x^2 + 2\cdot x\cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2\right) - \left(\frac{1}{2}\right)^2 - 2 = \left(x+\frac{1}{2}\right)^2 - \left(2+\frac{1}{4}\right) =$$

$$= \left(x+\frac{1}{2}\right)^2 - \frac{9}{4} = \left(x+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = \left(x+\frac{1}{2}-\frac{3}{2}\right)\left(x+\frac{1}{2}+\frac{3}{2}\right) =$$

$$= (x-1)(x+2)$$

$\alpha, \beta \in \mathbb{R} : 1 = \alpha(x-1) + \beta(x+2)$

$$\alpha(x-1)+\beta(x+2) = \alpha x - \alpha + \beta x + 2\beta = (\alpha x + \beta x) + (-\alpha + 2\beta) =$$

$$= (\alpha+\beta)x + (-\alpha+2\beta)$$

$$1 = \alpha(x-1) + \beta(x+2) \iff (\alpha+\beta)x + (-\alpha+2\beta) = 1 \iff$$

$$\iff \begin{cases} \alpha+\beta = 0 \\ -\alpha+2\beta = 1 \end{cases} \overset{(1)}{\iff}$$

$\alpha+\beta = 0 \iff \beta = -\alpha$

$-\alpha+2\beta = -\alpha+2(-\alpha) = -3\alpha = 1 \iff \alpha = -\frac{1}{3}$

$\beta = -\alpha = -\left(-\frac{1}{3}\right) = \frac{1}{3}$

$\overset{(1)}{\Longrightarrow} \alpha = -\frac{1}{3} \wedge \beta = \frac{1}{3}$

$$\frac{dx}{x^2+x-2} = \frac{\alpha(x-1)+\beta(x+2)}{(x-1)(x+2)}\,dx =$$

$$= \alpha\frac{x-1}{(x-1)(x+2)}\,dx + \beta\frac{x-2}{(x-1)(x+2)}\,dx = \alpha\frac{dx}{x+2} + \beta\frac{dx}{x-1} =$$

$$= \alpha\, d\ln|x+2| + \beta\, d\ln|x-1| = d\left(\alpha\ln|x+2| + \beta\ln|x-1|\right) =$$

$$= d\ln\left(|x+2|^{\alpha}|x-1|^{\beta}\right) = d\ln\left(|x+2|^{-\frac{1}{3}}|x-1|^{\frac{1}{3}}\right) =$$

$$= d\ln\left(\left(\frac{|x-1|}{|x+2|}\right)^{\frac{1}{3}}\right) = d\ln\left|\frac{x-1}{x+2}\right|^{\frac{1}{3}}$$

$$\int_2^{+\infty} \frac{dx}{x^2+x-2} = \int_2^{+\infty} d\ln\left|\frac{x-1}{x+2}\right|^{\frac{1}{3}} = (2)$$

$x \geq 2 \Leftrightarrow x+2 \geq 2+2 = 4 > 0$

$x \geq 2 \Leftrightarrow x-1 \geq 2-1 = 1 > 0$

$x+2 > 0 \wedge x-1 > 0 \Rightarrow \frac{x-1}{x+2} > 0 \Rightarrow$

$\Rightarrow \left|\frac{x-1}{x+2}\right| = \frac{x-1}{x+2}$

$$(2) = \int_2^{+\infty} d\ln\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}} = \ln\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}}\Bigg|_{x=2}^{x\to+\infty} = (3)$$

$$\lim_{x\to+\infty} \ln\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}} = \lim_{x\to+\infty} \ln\left(\frac{1-\frac{1}{x}}{1+\frac{2}{x}}\right)^{\frac{1}{3}} =$$

$$= \ln\left(\frac{1-\lim\limits_{x\to+\infty}\frac{1}{x}}{1+2\lim\limits_{x\to+\infty}\frac{1}{x}}\right)^{\frac{1}{3}} = \ln\left(\frac{1-0}{1+2 0}\right)^{\frac{1}{3}} = \ln\left(\frac{1}{1}\right)^{\frac{1}{3}} =$$

$= \ln 1 = 0$

$$\ln\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}}\Bigg|_{x=2} = \ln\left(\frac{2-1}{2+2}\right)^{\frac{1}{3}} = \ln\left(\frac{1}{4}\right)^{\frac{1}{3}} =$$

$= \ln\left(2^{-2}\right)^{\frac{1}{3}} = \ln 2^{-\frac{2}{3}} = -\frac{2}{3}\ln 2$

$(3) = 0 - \left(-\frac{2}{3}\ln 2\right) = \frac{2}{3}\ln 2$

Odgovor: $\boxed{\int_2^{+\infty} \frac{dx}{x^2+x-2} = \frac{2}{3}\ln 2}$