$\boxed{2}$ Найти: $\int e^{2x}\cos 3x\,dx$

Решение:

$$\int e^{2x}\cos 3x\,dx = \int \cos 3x\left(e^{2x}dx\right) = \int u\,dv = (1)$$

$$u=\cos 3x \Rightarrow du = d\cos 3x = -\sin 3x\,d(3x) = -\sin 3x\;3dx = -3\sin 3x\,dx$$

$$dv = e^{2x}dx = e^{2x}\frac{1}{2}(2dx) = \frac{1}{2}e^{2x}d(2x) = \frac{1}{2}de^{2x} = d\left(\frac{e^{2x}}{2}\right) \Rightarrow v = \frac{e^{2x}}{2}$$

$$(1) = uv - \int v\,du = \cos 3x\,\frac{e^{2x}}{2} - \int\frac{e^{2x}}{2}\left(-3\sin 3x\,dx\right) = (2)$$

$$-\int\frac{e^{2x}}{2}\left(-3\sin 3x\,dx\right) = -\int\left(-\frac{3}{2}\right)e^{2x}\sin 3x\,dx = -\left(-\frac{3}{2}\right)\int e^{2x}\sin 3x\,dx = \frac{3}{2}\int e^{2x}\sin 3x\,dx$$

$$(2) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int\sin 3x\left(e^{2x}dx\right) =$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int u\,dv = (3)$$

$$u = \sin 3x \Rightarrow du = d\sin 3x = \cos 3x\,d(3x) = \cos 3x\;3dx = 3\cos 3x\,dx$$

$$dv = e^{2x}dx = e^{2x}\frac{1}{2}(2dx) = \frac{1}{2}e^{2x}d(2x) = \frac{1}{2}de^{2x} = d\left(\frac{e^{2x}}{2}\right) \Rightarrow v = \frac{e^{2x}}{2}$$

$$(3) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(uv - \int v\,du\right) =$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\sin 3x\,\frac{e^{2x}}{2} - \int\frac{e^{2x}}{2}\left(3\cos 3x\,dx\right)\right) = (4)$$

$$-\int\frac{e^{2x}}{2}\left(3\cos 3x\,dx\right) = -\int\frac{3}{2}e^{2x}\cos 3x\,dx = -\frac{3}{2}\int e^{2x}\cos 3x\,dx$$

$$(4) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right) =$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{1}{2}\cdot\frac{3}{2}e^{2x}\sin 3x - \left(\frac{3}{2}\right)^2\int e^{2x}\cos 3x\,dx =$$

$$= \frac{1}{2}e^{2x}\left(\cos 3x + \frac{3}{2}\sin 3x\right) - \left(\frac{3}{2}\right)^2\int e^{2x}\cos 3x\,dx$$

$$J := \int e^{2x} \cos 3x \, dx$$

$$J = \frac{1}{2} e^{2x} \left(\cos 3x + \frac{3}{2} \sin 3x \right) - \left(\frac{3}{2}\right)^2 J \iff$$

$$\iff J + \left(\frac{3}{2}\right)^2 J = \frac{1}{2} e^{2x} \left(\cos 3x + \frac{3}{2} \sin 3x \right) \iff$$

$$\iff \left(1 + \left(\frac{3}{2}\right)^2\right) J = \frac{1}{2} e^{2x} \left(\cos 3x + \frac{3}{2} \sin 3x \right) \iff$$

$$\iff J = \frac{1}{2} \left(1 + \left(\frac{3}{2}\right)^2\right)^{-1} e^{2x} \left(\cos 3x + \frac{3}{2} \sin 3x \right) = (5)$$

$$\left(1 + \left(\frac{3}{2}\right)^2\right)^{-1} = \left(1 + \frac{3^2}{2^2}\right)^{-1} = \left(\frac{2^2 + 3^2}{2^2}\right)^{-1} = \frac{2^2}{2^2+3^2}$$

$$\cos 3x + \frac{3}{2} \sin 3x = \frac{2\cos 3x + 3 \sin 3x}{2}$$

$$(5) = \frac{1}{2} \frac{2^2}{2^2+3^2} e^{2x} \frac{1}{2} \left(2 \cos 3x + 3 \sin 3x\right) =$$

$$= \frac{e^{2x}}{2^2+3^2} \left(2 \cos 3x + 3 \sin 3x\right) \Rightarrow$$

$$\Rightarrow \int e^{2x} \cos 3x \, dx = \frac{e^{2x}}{2^2+3^2} \left(2 \cos 3x + 3 \sin 3x\right) + C$$

Проверка:

$$\frac{d}{dx}\left(\frac{e^{2x}}{2^2+3^2}\left(2\cos 3x+3\sin 3x\right)+C\right)=$$

$$=\frac{1}{2^2+3^2}\frac{d}{dx}\left(e^{2x}\left(2\cos 3x+3\sin 3x\right)\right)=$$

$$=\frac{1}{2^2+3^2}\left(\frac{d}{dx}\left(e^{2x}\right)\left(2\cos 3x+3\sin 3x\right)+e^{2x}\frac{d}{dx}\left(2\cos 3x+3\sin 3x\right)\right)=(6)$$

$$\frac{d}{dx}\left(e^{2x}\right)=e^{2x}\frac{d}{dx}(2x)=e^{2x}\,2=2e^{2x}$$

$$\frac{d}{dx}\left(2\cos 3x+3\sin 3x\right)=2\frac{d}{dx}(\cos 3x)+3\frac{d}{dx}(\sin 3x)=$$

$$=2(-\sin 3x)\frac{d}{dx}(3x)+3\cos 3x\frac{d}{dx}(3x)=2(-\sin 3x)\,3+3\cos 3x\cdot 3=$$

$$=-2\cdot 3\sin 3x+3^2\cos 3x$$

$$(6)=\frac{1}{2^2+3^2}\left(2e^{2x}\left(2\cos 3x+3\sin 3x\right)+e^{2x}\left(-2\cdot 3\sin 3x+3^2\cos 3x\right)\right)=$$

$$=\frac{e^{2x}}{2^2+3^2}\left(2^2\cos 3x+2\cdot 3\sin 3x-2\cdot 3\sin 3x+3^2\cos 3x\right)=$$

$$=\frac{e^{2x}}{2^2+3^2}\left(2^2+3^2\right)\cos 3x=e^{2x}\cos 3x$$

Ответ: $\boxed{\int e^{2x}\cos 3x=\frac{e^{2x}}{2^2+3^2}\left(2\cos 3x+3\sin 3x\right)+C}$