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Найти: условный экстремум $u(x,y) = 3 - 8x + 6y$

при условии $x^2+y^2=36$

Решение:

$u(x,y) = 3-8x+6y$; $D(u) = \mathbb{R}^2$

$x^2+y^2=36 \Leftrightarrow x^2+y^2-36 \Leftrightarrow f(x,y) = 0$ | $f(x,y) := x^2+y^2-36$

$f(x,y) = x^2+y^2-36$; $D(f) = \mathbb{R}^2$

$L(x,y,\lambda) = u(x,y) + \lambda f(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3-8x+6y) = -8\frac{d}{dx}(x) = -8\cdot 1 = -8$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3-8x+6y) = 6\frac{d}{dy}(y) = 6\cdot 1 = 6$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2-36) = \frac{d}{dx}(x^2) = 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2-36) = \frac{d}{dy}(y^2) = 2y$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}(u+\lambda f) = \frac{\partial u}{\partial x} + \lambda\frac{\partial f}{\partial x} = -8 + \lambda\cdot 2x = 2\lambda x - 8 = 2(\lambda x - 4)$$

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}(u+\lambda f) = \frac{\partial u}{\partial y} + \lambda\frac{\partial f}{\partial y} = 6 + \lambda\cdot 2y = 2\lambda y + 6 = 2(\lambda y + 3)$$

$$\frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial \lambda}(u+\lambda f) = f\frac{d}{d\lambda}(\lambda) = f\cdot 1 = f = x^2+y^2-36$$

$$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 2(\lambda x - 4) = 0 \\ 2(\lambda y + 3) = 0 \\ x^2+y^2-36=0 \end{cases} \Leftrightarrow \begin{cases} \lambda x - 4 = 0 \\ \lambda y + 3 = 0 \\ x^2+y^2=36 \end{cases} \overset{(1)}{\Leftrightarrow}$$

$\lambda x - 4 = 0 \Leftrightarrow \lambda x = 4$

$\lambda x = 4 \wedge \lambda \neq 0 \Leftrightarrow x = \frac{4}{\lambda}$

$\lambda y + 3 = 0 \Leftrightarrow \lambda y = -3$

$\lambda y = -3 \wedge \lambda \neq 0 \Leftrightarrow y = \frac{-3}{\lambda} = -\frac{3}{\lambda}$

$$x^2 + y^2 \Big|_{x = \frac{4}{\lambda},\, y = -\frac{3}{\lambda}} = \left(\frac{4}{\lambda}\right)^2 + \left(-\frac{3}{\lambda}\right)^2 = \frac{4^2}{\lambda^2} + \frac{3^2}{\lambda^2} = \frac{4^2 + 3^2}{\lambda^2} =$$

$$= \frac{16 + 9}{\lambda^2} = \frac{25}{\lambda^2}$$

$$x^2 + y^2 = 36 \;\Big|\; x = \frac{4}{\lambda} \wedge y = -\frac{3}{\lambda} \Leftrightarrow \frac{25}{\lambda^2} = 36 \Leftrightarrow$$

$$\Leftrightarrow \lambda^2 = \frac{25}{36} = \frac{5^2}{6^2} = \left(\frac{5}{6}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} \lambda = \frac{5}{6} \\ \lambda = -\frac{5}{6} \end{array}\right.$$

$$\overset{(1)}{\Leftrightarrow} \left[\begin{array}{l} \begin{cases} \lambda = \frac{5}{6} \\ x = \frac{4}{\lambda} = 4\left(\frac{5}{6}\right)^{-1} = 4\,\frac{6}{5} = \frac{24}{5} \\ y = -\frac{3}{\lambda} = -3\left(\frac{5}{6}\right)^{-1} = -3\,\frac{6}{5} = -\frac{18}{5} \end{cases} \\ \begin{cases} \lambda = -\frac{5}{6} \\ x = \frac{4}{\lambda} = 4\left(-\frac{5}{6}\right)^{-1} = -4\,\frac{6}{5} = -\frac{24}{5} \\ y = -\frac{3}{\lambda} = -3\left(-\frac{5}{6}\right)^{-1} = 3\,\frac{6}{5} = \frac{18}{5} \end{cases} \end{array}\right.$$

$$\boxed{\begin{array}{l} \left(\lambda = \frac{5}{6} \wedge x = \frac{24}{5} \wedge y = -\frac{18}{5}\right) \vee \left(\lambda = -\frac{5}{6} \wedge x = -\frac{24}{5} \wedge y = \frac{18}{5}\right) \\ \frac{\partial L}{\partial x}(x, y, \lambda) = 0 \wedge \frac{\partial L}{\partial y}(x, y, \lambda) = 0 \wedge \frac{\partial L}{\partial \lambda}(x, y, \lambda) = 0 \end{array}}$$

$$f(x,y) = 0 \Rightarrow df = 0 \Leftrightarrow$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Leftrightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \Leftrightarrow$$

$$\frac{\partial f}{\partial x} = 2x \; ; \; \frac{\partial f}{\partial y} = 2y$$

$$\Leftrightarrow 2x dx + 2y dy = 0 \Leftrightarrow 2(x dx + y dy) = 0 \Leftrightarrow$$

$$\Leftrightarrow x dx + y dy = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(-8) = 0$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(-8) = 0$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(6) = 0 = \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(6) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2\frac{d}{dx}(x) = 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0 = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2$$

$$\frac{\partial^2 L}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \lambda \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} + \lambda \frac{\partial^2 f}{\partial x^2} =$$

$$= 0 + \lambda \cdot 2 = 2\lambda$$

$$\frac{\partial^2 L}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + \lambda \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} + \lambda \frac{\partial^2 f}{\partial y \partial x} =$$

$$= 0 + \lambda \cdot 0 = 0$$

$$\frac{\partial^2 L}{\partial \lambda \partial x} = \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial x}\right) = \frac{\partial}{\partial \lambda}\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial x} \frac{d}{d\lambda}(\lambda) = \frac{\partial f}{\partial x} \cdot 1 = \frac{\partial f}{\partial x} =$$

$$= 2x$$

$$\frac{\partial^2 L}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \lambda \frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + \lambda \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} + \lambda \frac{\partial^2 f}{\partial x \partial y} =$$

$$= 0 + \lambda \cdot 0 = 0 = \frac{\partial^2 L}{\partial y \partial x}$$

$$\frac{\partial^2 L}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \lambda \frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \lambda \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} + \lambda \frac{\partial^2 f}{\partial y^2} =$$

$$= 0 + \lambda \cdot 2 = 2\lambda$$

$$\frac{\partial^2 L}{\partial \lambda \partial y} = \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial y}\right) = \frac{\partial}{\partial \lambda}\left(\frac{\partial u}{\partial y} + \lambda \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial y} \frac{d}{d\lambda}(\lambda) = \frac{\partial f}{\partial y} \cdot 1 = \frac{\partial f}{\partial y} =$$

$$= 2y$$

$$\frac{\partial^2 L}{\partial x \partial \lambda} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial f}{\partial x} = 2x = \frac{\partial^2 L}{\partial \lambda \partial x}$$

$$\frac{\partial^2 L}{\partial y \partial \lambda} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial f}{\partial y} = 2y = \frac{\partial^2 L}{\partial \lambda \partial y}$$

$$\frac{\partial^2 L}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial f}{\partial \lambda} = 0$$

$$d^2L = \frac{\partial^2 L}{\partial x^2}dx^2 + \frac{\partial^2 L}{\partial y^2}dy^2 + \frac{\partial^2 L}{\partial \lambda^2}d\lambda^2 +$$

$$+ 2\frac{\partial^2 L}{\partial x \partial y}dx\,dy + 2\frac{\partial^2 L}{\partial y \partial \lambda}dy\,d\lambda + 2\frac{\partial^2 L}{\partial \lambda \partial x}d\lambda\,dx =$$

$$= 2\lambda dx^2 + 2\lambda dy^2 + 0\,d\lambda^2 +$$

$$+ 2\cdot 0\,dx\,dy + 2\cdot 2y\,dy\,d\lambda + 2\cdot 2x\,d\lambda\,dx =$$

$$= 2\left(\lambda(dx^2+dy^2) + 2(x\,dx + y\,dy)d\lambda\right) =$$

$x\,dx + y\,dy = 0$

$$= 2\left(\lambda(dx^2+dy^2) + 2\cdot 0\,d\lambda\right) =$$

$$= 2\lambda(dx^2+dy^2)$$

$$\boxed{\begin{matrix} \forall \lambda > 0 \ \ \forall (x,y)\in \mathbb{R}^2 \ \forall (dx,dy)\in \{\mathbb{R}^2/(0,0)\} \ \ d^2L(x,y,\lambda) > 0 \\ \forall \lambda < 0 \ \ \forall (x,y)\in \mathbb{R}^2 \ \forall (dx,dy)\in \{\mathbb{R}^2/(0,0)\} \ \ d^2L(x,y,\lambda) < 0 \end{matrix}}$$

Ответ: $(x,y) = \left(\frac{24}{5}, -\frac{18}{5}\right)$ – точка условного минимума

$(x,y) = \left(-\frac{24}{5}, \frac{18}{5}\right)$ – точка условного максимума