Дано:

$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Найти:

$\lambda, X \qquad AX = \lambda X$

Решение:

$$\det(A - \lambda E) = \det \begin{pmatrix} \frac{\sqrt{3}}{2} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} - \lambda \end{pmatrix} =$$

$$= \left(\frac{\sqrt{3}}{2} - \lambda\right)\left(\frac{\sqrt{3}}{2} - \lambda\right) - \left(-\frac{1}{2}\right)\frac{1}{2} = \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \left(\frac{1}{2}\right)^2$$

$$\det(A - \lambda E) = 0 \Leftrightarrow \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \left(\frac{1}{2}\right)^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 = -\left(\frac{1}{2}\right)^2 = i^2\left(\frac{1}{2}\right)^2 = \left(\frac{i}{2}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} \frac{\sqrt{3}}{2} - \lambda = \frac{i}{2} \\ \frac{\sqrt{3}}{2} - \lambda = -\frac{i}{2} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \lambda = \frac{\sqrt{3}}{2} - \frac{i}{2} \\ \lambda = \frac{\sqrt{3}}{2} + \frac{i}{2} \end{array}\right.$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$AX = \lambda X \Leftrightarrow (A - \lambda E)X = 0 \Leftrightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left(\frac{\sqrt{3}}{2} - \lambda\right)x_1 - \frac{1}{2}x_2 = 0 \\ \frac{1}{2}x_1 + \left(\frac{\sqrt{3}}{2} - \lambda\right)x_2 = 0 \end{cases}$$

$$\lambda = \frac{\sqrt{3}}{2} - \frac{i}{2} \Leftrightarrow \frac{\sqrt{3}}{2} - \lambda = \frac{i}{2}$$

$$\begin{cases} \frac{i}{2}x_1 - \frac{1}{2}x_2 = 0 \\ \frac{1}{2}x_1 + \frac{i}{2}x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} ix_1 - x_2 = 0 \\ x_1 + ix_2 = 0 \end{cases} \overset{(1.1)}{\Leftrightarrow}$$

$$x_1 + ix_2 = 0 \Leftrightarrow i(x_1 + ix_2) = i \cdot 0 \Leftrightarrow ix_1 + i^2x_2 = 0 \Leftrightarrow ix_1 - x_2 = 0$$

$$\overset{(1.1)}{\Leftrightarrow} \begin{cases} ix_1 - x_2 = 0 \\ ix_1 - x_2 = 0 \end{cases} \Leftrightarrow ix_1 - x_2 = 0 \Leftrightarrow x_2 = ix_1$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ ix_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}}{2} + \frac{i}{2} \Leftrightarrow \frac{\sqrt{3}}{2} - \lambda = -\frac{i}{2}$$

$$\begin{cases} -\frac{i}{2}x_1 - \frac{1}{2}x_2 = 0 \\ \frac{1}{2}x_1 - \frac{i}{2}x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} ix_1 + x_2 = 0 \\ x_1 - ix_2 = 0 \end{cases} \overset{(1.2)}{\Leftrightarrow}$$

$$x_1 - ix_2 = 0 \Leftrightarrow i(x_1 - ix_2) = i \cdot 0 \Leftrightarrow ix_1 - i^2x_2 = 0 \Leftrightarrow ix_1 + x_2 = 0$$

$$\overset{(1.2)}{\Leftrightarrow} \begin{cases} ix_1 + x_2 = 0 \\ ix_1 + x_2 = 0 \end{cases} \Leftrightarrow ix_1 + x_2 = 0 \Leftrightarrow x_2 = -ix_1$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -ix_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Πηλερνη:

$$\lambda = \frac{\sqrt{3}}{2} - \frac{i}{2}, \quad x = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$Ax = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \cdot 1 - \frac{1}{2} i \\ \frac{1}{2} \cdot 1 + \frac{\sqrt{3}}{2} i \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} - \frac{i}{2} \\ \frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix} =$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) \begin{pmatrix} 1 \\ \dfrac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} - \frac{i}{2}} \end{pmatrix} = (2.1)$$

$$\frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} - \frac{i}{2}} = \frac{\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)}{\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)} = \frac{\frac{1}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{i}{2} + i\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} + i\frac{\sqrt{3}}{2}\frac{i}{2}}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{i}{2}\right)^2} =$$

$$= \frac{\frac{\sqrt{3}}{4} + \frac{i}{4} + i\frac{3}{4} + i^2\frac{\sqrt{3}}{4}}{\frac{3}{4} - \frac{i^2}{4}} = \frac{\frac{\sqrt{3}}{4} + i - \frac{\sqrt{3}}{4}}{\frac{3}{4} + \frac{1}{4}} = \frac{i}{1} = i$$

$$(2.1) = \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) \begin{pmatrix} 1 \\ i \end{pmatrix} = \lambda x$$

$$\lambda = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad x = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$Ax = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \cdot 1 - \frac{1}{2}(-i) \\ \frac{1}{2} \cdot 1 + \frac{\sqrt{3}}{2}(-i) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} + \frac{i}{2} \\ \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} =$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \begin{pmatrix} 1 \\ \dfrac{\frac{1}{2} - i\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} + \frac{i}{2}} \end{pmatrix} = (2.2)$$

$$\frac{\frac{1}{2} - i\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} + \frac{i}{2}} = \frac{\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)}{\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)} = \frac{\frac{1}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{i}{2}\right) + \left(-i\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-i\frac{\sqrt{3}}{2}\right)\left(-\frac{i}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{i}{2}\right)^2}$$

$$= \frac{\frac{\sqrt{3}}{4} - \frac{i}{4} - i\frac{3}{4} + i^2\frac{\sqrt{3}}{2}}{\frac{3}{4} - \frac{i^2}{4}} = \frac{\frac{\sqrt{3}}{2} - i - \frac{\sqrt{3}}{2}}{\frac{3}{4} + \frac{1}{4}} = \frac{-i}{1} = -i$$

$$(2.2) = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\begin{pmatrix} 1 \\ -i \end{pmatrix} = \lambda X$$

Obtem:

$$\boxed{\begin{array}{l} \lambda = \frac{\sqrt{3}}{2} - \frac{i}{2}, \quad X = \begin{pmatrix} 1 \\ i \end{pmatrix} \\ \lambda = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad X = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{array}}$$