$n \in \mathbb{N} \wedge n \geq 2 \quad D = \{(x_i, y_i) \mid i = \overline{1,n}\}$

$\alpha, \beta \in \mathbb{R} \quad y = f(x \mid \alpha, \beta) = \alpha x + \beta$

$i = \overline{1,n} \quad \varepsilon_i(\alpha, \beta) = y_i - f(x_i \mid \alpha, \beta) = y_i - (\alpha x_i + \beta)$

$U(\alpha, \beta) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} [y_i - f(x_i \mid \alpha, \beta)]^2 = \sum_{i=1}^{n} [y_i - (\alpha x_i + \beta)]^2$

$U(\alpha, \beta) \to \min\limits_{(\alpha,\beta) \in \mathbb{R}^2}$

$\boxed{1}$

$dU(\alpha, \beta) = \frac{\partial U}{\partial \alpha} d\alpha + \frac{\partial U}{\partial \beta} d\beta$

$\frac{\partial U}{\partial \alpha} = \frac{\partial}{\partial \alpha} \sum_{i=1}^{n} [y_i - (\alpha x_i + \beta)]^2 = \sum_{i=1}^{n} \frac{\partial}{\partial \alpha} [y_i - (\alpha x_i + \beta)]^2 = (1.1)$

$\frac{\partial}{\partial \alpha} [y_i - (\alpha x_i + \beta)]^2 = 2[y_i - (\alpha x_i + \beta)] \frac{\partial}{\partial \alpha} [y_i - (\alpha x_i + \beta)] =$

$= 2[y_i - (\alpha x_i + \beta)](-x_i) = 2x_i(\alpha x_i + \beta - y_i) = 2(\alpha x_i^2 + \beta x_i - x_i y_i)$

$(1.1) = \sum_{i=1}^{n} 2(\alpha x_i^2 + \beta x_i - x_i y_i) = 2\left(\alpha \sum_{i=1}^{n} x_i^2 + \beta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i y_i\right)$

$\frac{\partial U}{\partial \beta} = \frac{\partial}{\partial \beta} \sum_{i=1}^{n} [y_i - (\alpha x_i + \beta)]^2 = \sum_{i=1}^{n} \frac{\partial}{\partial \beta} [y_i - (\alpha x_i + \beta)]^2 = (1.2)$

$\frac{\partial}{\partial \beta} [y_i - (\alpha x_i + \beta)]^2 = 2[y_i - (\alpha x_i + \beta)] \frac{\partial}{\partial \beta} [y_i - (\alpha x_i + \beta)] =$

$= 2[y_i - (\alpha x_i + \beta)](-1) = 2(\alpha x_i + \beta - y_i)$

$(1.2) = \sum_{i=1}^{n} 2(\alpha x_i + \beta - y_i) = 2\left(\alpha \sum_{i=1}^{n} x_i + \beta \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} y_i\right) =$

$= 2\left(\alpha \sum_{i=1}^{n} x_i + \beta n - \sum_{i=1}^{n} y_i\right)$

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial \beta} = 0 \Leftrightarrow du = \frac{\partial u}{\partial \alpha} d\alpha + \frac{\partial u}{\partial \beta} d\beta = 0\,d\alpha + 0\,d\beta = 0$$

$$\begin{cases} \frac{\partial u}{\partial \alpha} = 2\left(\alpha \sum_{i=1}^{n} x_i^2 + \beta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i y_i\right) = 0 \\ \frac{\partial u}{\partial \beta} = 2\left(\alpha \sum_{i=1}^{n} x_i + \beta n - \sum_{i=1}^{n} y_i\right) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha \sum_{i=1}^{n} x_i^2 + \beta \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \\ \alpha \sum_{i=1}^{n} x_i + \beta n = \sum_{i=1}^{n} y_i \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

$$\Delta = \det \begin{pmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{pmatrix} = \sum_{i=1}^{n} x_i^2 \cdot n - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i =$$

$$= n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 =$$

$$= n^2 \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right) - n^2 \left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)^2 = n^2\, \overline{x^2} - n^2\, \overline{x}^2 =$$

$$= n^2 \left(\overline{x^2} - \overline{x}^2\right)$$

$$\Delta \neq 0$$

$$\Delta_1 = \det\begin{pmatrix} \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} y_i & n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i \cdot n - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i =$$

$$= n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i =$$

$$= n^2\left(\frac{1}{n}\sum_{i=1}^{n} x_i y_i\right) - n^2\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = n^2\,\overline{xy} - n^2\,\overline{x}\,\overline{y} =$$

$$= n^2\left(\overline{xy} - \overline{x}\,\overline{y}\right)$$

$$\Delta_2 = \det\begin{pmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i \end{pmatrix} = \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} x_i =$$

$$= n^2\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) - n^2\left(\frac{1}{n}\sum_{i=1}^{n} x_i y_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) =$$

$$= n^2\,\overline{x^2}\,\overline{y} - n^2\,\overline{xy}\,\overline{x} = n^2\left(\overline{x^2}\,\overline{y} - \overline{xy}\,\overline{x}\right)$$

$$\alpha = \frac{\Delta_1}{\Delta} = \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i} =$$

$$= \frac{n^2\left(\overline{xy} - \overline{x}\,\overline{y}\right)}{n^2\left(\overline{x^2} - \overline{x}^2\right)} = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\overline{x^2} - \overline{x}^2}$$

$$\beta = \frac{\Delta_2}{\Delta} = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} x_i}{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i} =$$

$$= \frac{n^2\left(\overline{x^2}\,\overline{y} - \overline{xy}\,\overline{x}\right)}{n^2\left(\overline{x^2} - \overline{x}^2\right)} = \frac{\overline{x^2}\,\overline{y} - \overline{xy}\,\overline{x}}{\overline{x^2} - \overline{x}^2}$$

$\boxed{2}$

$$\mu_{x_i} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\sigma^2_{x_i} = \overline{(x-\bar{x})^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 =$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right) = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \bar{x}^2\sum_{i=1}^{n} 1\right) =$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \bar{x}^2 n\right) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - 2\bar{x}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \bar{x}^2 =$$

$$= \overline{x^2} - 2\bar{x}\bar{x} + \bar{x}^2 = \overline{x^2} - 2\bar{x}^2 + \bar{x}^2 = \overline{x^2} - \bar{x}^2$$

$$\mu_{x_i} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\mu_{y_i} = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$cov_{xy} = \overline{(x-\bar{x})(y-\bar{y})} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) =$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(x_iy_i - x_i\bar{y} - \bar{x}y_i + \bar{x}\bar{y}\right) =$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_iy_i - \bar{y}\sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n} y_i + \bar{x}\bar{y}\sum_{i=1}^{n} 1\right) =$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_iy_i - \bar{y}\sum_{i=1}^{n} x_i - \bar{x}\sum_{i=1}^{n} y_i + \bar{x}\bar{y}n\right) =$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} x_iy_i\right) - \bar{y}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \bar{x}\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) + \bar{x}\bar{y} =$$

$$= \overline{xy} - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y} = \overline{xy} - 2\bar{x}\bar{y} + \bar{x}\bar{y} = \overline{xy} - \bar{x}\bar{y}$$

$$\Delta = n^2\left(\overline{x^2} - \bar{x}^2\right) = n^2 \sigma_x^2$$

$$\Delta \neq 0 \land n \in \mathbb{N} \iff \sigma_x \neq 0$$

$$\alpha = \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - \bar{x}^2} =$$

$$= \left(\sigma_x \neq 0\right) = \frac{\mathrm{cov}_{xy}}{\sigma_x^2} =$$

$$= \left(\sigma_y \neq 0\right) = \frac{\mathrm{cov}_{xy}}{\sigma_x \sigma_y}\,\frac{\sigma_y}{\sigma_x} = r_{xy}\frac{\sigma_y}{\sigma_x}$$

$$\beta = \frac{\overline{x^2}\,\bar{y} - \overline{xy}\,\bar{x}}{\overline{x^2} - \bar{x}^2} =$$

$$= \frac{\overline{x^2}\,\bar{y} - \bar{x}^2\bar{y} - \overline{xy}\,\bar{x} + \bar{x}\,\bar{y}\,\bar{x}}{\overline{x^2} - \bar{x}^2} =$$

$$= \frac{\bar{y}\left(\overline{x^2} - \bar{x}^2\right) - \bar{x}\left(\overline{xy} - \bar{x}\,\bar{y}\right)}{\overline{x^2} - \bar{x}^2} =$$

$$= \bar{y} - \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - \bar{x}^2}\,\bar{x} = \bar{y} - \alpha\bar{x}$$

[3]

$$d^2u(\alpha,\beta) = \frac{\partial}{\partial\alpha}(du)\,d\alpha + \frac{\partial}{\partial\beta}(du)\,d\beta =$$

$$= \frac{\partial}{\partial\alpha}\left(\frac{\partial u}{\partial\alpha}d\alpha + \frac{\partial u}{\partial\beta}d\beta\right)d\alpha + \frac{\partial}{\partial\beta}\left(\frac{\partial u}{\partial\alpha}d\alpha + \frac{\partial u}{\partial\beta}d\beta\right)d\beta =$$

$$= \left[\frac{\partial}{\partial\alpha}\left(\frac{\partial u}{\partial\alpha}\right)d\alpha + \frac{\partial}{\partial\alpha}\left(\frac{\partial u}{\partial\beta}\right)d\beta\right]d\alpha + \left[\frac{\partial}{\partial\beta}\left(\frac{\partial u}{\partial\alpha}\right)d\alpha + \frac{\partial}{\partial\beta}\left(\frac{\partial u}{\partial\beta}\right)d\beta\right]d\beta =$$

$$= \left(\frac{\partial^2 u}{\partial\alpha^2}d\alpha + \frac{\partial^2 u}{\partial\alpha\partial\beta}d\beta\right)d\alpha + \left(\frac{\partial^2 u}{\partial\beta\partial\alpha}d\alpha + \frac{\partial^2 u}{\partial\beta^2}d\beta\right)d\beta =$$

$$= \frac{\partial^2 u}{\partial\alpha^2}d\alpha^2 + \left(\frac{\partial^2 u}{\partial\alpha\partial\beta} + \frac{\partial^2 u}{\partial\beta\partial\alpha}\right)d\alpha d\beta + \frac{\partial^2 u}{\partial\beta^2}d\beta^2 =$$

$$= \left(\frac{\partial^2 u}{\partial\alpha\partial\beta} = \frac{\partial^2 u}{\partial\beta\partial\alpha}\right) = \frac{\partial^2 u}{\partial\alpha^2}d\alpha^2 + 2\frac{\partial^2 u}{\partial\alpha\partial\beta}d\alpha d\beta + \frac{\partial^2 u}{\partial\beta^2}d\beta^2$$

$$\frac{\partial^2 u}{\partial\alpha^2} = \frac{\partial}{\partial\alpha}\left(\frac{\partial u}{\partial\alpha}\right) = \frac{\partial}{\partial\alpha}\,2\left(\alpha\sum_{i=1}^n x_i^2 + \beta\sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i\right) = 2\sum_{i=1}^n x_i^2$$

$$\frac{\partial^2 u}{\partial\beta\partial\alpha} = \frac{\partial}{\partial\beta}\left(\frac{\partial u}{\partial\alpha}\right) = \frac{\partial}{\partial\beta}\,2\left(\alpha\sum_{i=1}^n x_i^2 + \beta\sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i\right) = 2\sum_{i=1}^n x_i$$

$$\frac{\partial^2 u}{\partial\alpha\partial\beta} = \frac{\partial}{\partial\alpha}\left(\frac{\partial u}{\partial\beta}\right) = \frac{\partial}{\partial\alpha}\,2\left(\alpha\sum_{i=1}^n x_i + \beta n - \sum_{i=1}^n y_i\right) = 2\sum_{i=1}^n x_i = \frac{\partial^2 u}{\partial\beta\partial\alpha}$$

$$\frac{\partial^2 u}{\partial\beta^2} = \frac{\partial}{\partial\beta}\left(\frac{\partial u}{\partial\beta}\right) = \frac{\partial}{\partial\beta}\,2\left(\alpha\sum_{i=1}^n x_i + \beta n - \sum_{i=1}^n y_i\right) = 2n$$

$$d^2u = \frac{\partial^2 u}{\partial\alpha^2}d\alpha^2 + 2\frac{\partial^2 u}{\partial\alpha\partial\beta}d\alpha d\beta + \frac{\partial^2 u}{\partial\beta^2}d\beta^2 =$$

$$= 2\sum_{i=1}^n x_i^2\,d\alpha^2 + 2\cdot 2\sum_{i=1}^n x_i\,d\alpha d\beta + 2n\cdot d\beta^2 =$$

$$= 2\left(\sum_{i=1}^n x_i^2\,d\alpha^2 + 2\sum_{i=1}^n x_i\,d\alpha d\beta + n\,d\beta^2\right) =$$

$$= 2n\left[\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right)d\alpha^2 + 2\left(\frac{1}{n}\sum_{i=1}^n x_i\right)d\alpha d\beta + d\beta^2\right] =$$

$$= 2n\left(\overline{x^2}\,d\alpha^2 + 2\bar{x}\,d\alpha d\beta + d\beta^2\right) =$$

$$= 2n\left(\overline{x^2}d\alpha^2 - \overline{x}^2 d\alpha^2 + \overline{x}^2 d\alpha^2 + 2\overline{x}\,d\alpha\, d\beta + d\beta^2\right) =$$

$$= 2n\left[\left(\overline{x^2} - \overline{x}^2\right)d\alpha^2 + \left(\overline{x}\,d\alpha + d\beta\right)^2\right] =$$

$$= 2n\left[\sigma_x^2 d\alpha^2 + \left(\mu_x d\alpha + d\beta\right)^2\right].$$

$$\sigma_x \neq 0 \Rightarrow d^2u = 2n\left[\sigma_x^2 d\alpha^2 + \left(\mu_x d\alpha + d\beta\right)^2\right] > 0$$

$\boxed{4}$

$$\frac{\partial u}{\partial x}(\alpha,\beta) = \frac{\partial u}{\partial y}(\alpha,\beta) = 0 \wedge d^2u(\alpha,\beta) > 0 \Leftrightarrow$$

$$\Leftrightarrow (\alpha,\beta) = \underset{(\alpha,\beta)\in\mathbb{R}^2}{\arg\min}\,(u)$$

$n \in \mathbb{N} \wedge n \geq 2 \quad D = \{(x_i, y_i) \mid i = \overline{1,n}\}$; выборка

$\alpha, \beta \in \mathbb{R} \quad y = f(x \mid \alpha, \beta) = \alpha x + \beta$; регрессионная модель

$i = \overline{1,n} \quad \varepsilon_i = y_i - f(x_i \mid \alpha, \beta) = y_i - (\alpha x_i + \beta)$

$U(\alpha, \beta) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} [y_i - f(x_i \mid \alpha, \beta)]^2 = \sum_{i=1}^{n} [y_i - (\alpha x_i + \beta)]^2$;

функция ошибок

$U(\alpha, \beta) \to \min_{(\alpha,\beta) \in \mathbb{R}^2}$; критерий

$\sigma_x \neq 0$

$$(\alpha, \beta) = \underset{(\alpha,\beta) \in \mathbb{R}^2}{\operatorname{argmin}} (U)$$

$$\alpha = \frac{n \sum_{i=1}^{n} x^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i} = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\overline{x^2} - \overline{x}^2} =$$

$$= \frac{\operatorname{cov}_{xy}}{\sigma_x^2} = (\sigma_y \neq 0) = r_{xy} \frac{\sigma_y}{\sigma_x}$$

$$\beta = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} x_i}{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i} = \frac{\overline{x^2}\,\overline{y} - \overline{xy}\,\overline{x}}{\overline{x^2} - \overline{x}^2} =$$

$$= \overline{y} - \alpha \overline{x}$$