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$X \sim N(\mu, \sigma^2)$ нормальное распределение

$Z \sim N(\mu=0, \sigma^2=1)$ стандартное нормальное распределение

1. плотность вероятности

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$f(z|\mu=0,\sigma^2=1) = \left[\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)\right]_{\substack{\mu=0\\ \sigma=1}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} =: \varphi(z)$$

2.

$$\int_{-\infty}^{+\infty} f(x|\mu,\sigma^2)\,dx = 1 \quad ; \quad \int_{-\infty}^{+\infty} e^{-\xi^2} = \sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} f(z|\mu=0,\sigma^2=1)\,dz = \int_{-\infty}^{+\infty} \varphi(z)\,dz = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{z}{\sqrt{2}}\right)^2} d\left(\frac{z}{\sqrt{2}}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1$$

3. математическое ожидание

$$E[X] = \int_{-\infty}^{+\infty} x f(x|\mu,\sigma^2)\,dx = \mu \quad ; \quad \int_{-\infty}^{+\infty} \xi e^{-\xi^2} d\xi = 0$$

$$E[Z] = \int_{-\infty}^{+\infty} z f(z|\mu=0,\sigma^2=1)\,dz = [\mu]_{\mu=0} = 0$$

$$E[Z] = \int_{-\infty}^{+\infty} z f(z|\mu=0,\sigma^2=1)\,dz = \int_{-\infty}^{+\infty} z \varphi(z)\,dz =$$

$$= \int_{-\infty}^{+\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z e^{-\frac{z^2}{2}}\,dz = \frac{0}{\sqrt{2\pi}} = 0$$

4. дисперсия

$$D[X] = \int_{-\infty}^{+\infty} (x - E[X])^2 f(x|\mu,\sigma^2)\,dx = \sigma^2 \; ; \; \int_{-\infty}^{+\infty} \xi^2 e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2}$$

$$D[Z] = \int_{-\infty}^{+\infty} (z - E[Z])^2 f(z|\mu=0,\sigma^2=1)\,dz = \left[\sigma^2\right]_{\sigma^2=1} = 1$$

$$D[Z] = \int_{-\infty}^{+\infty} (z - E[Z])^2 f(z|\mu=0,\sigma^2=1)\,dz = \int_{-\infty}^{+\infty} z^2 \varphi(z)\,dz = \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz =$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{z^2}{2} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2}} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left(\frac{z}{\sqrt{2}}\right)^2 e^{-\left(\frac{z}{\sqrt{2}}\right)^2} d\left(\frac{z}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \xi^2 e^{-\xi^2} d\xi = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1$$

5. функция распределения

$$p(X \le x) = F(x|\mu,\sigma^2) = \int_{-\infty}^{x} f(\eta|\mu,\sigma^2)\,d\eta = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right] \; ; \; \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$$

$$p(Z \le z) = F(z|\mu=0,\sigma^2=1) = \left[\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z-\mu}{\sqrt{2}\sigma}\right)\right]\right]_{\substack{\mu=0\\ \sigma=1}} = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right] =: \Phi(z)$$

$$p(Z \le z) = F(z|\mu=0,\sigma^2=1) = \int_{-\infty}^{z} f(\eta|\mu=0,\sigma^2=1)\,d\eta = \int_{-\infty}^{z} \varphi(\eta)\,d\eta = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2}} d\eta =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-\frac{\eta^2}{2}} \frac{d\eta}{\sqrt{2}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-\left(\frac{\eta}{\sqrt{2}}\right)^2} d\left(\frac{\eta}{\sqrt{2}}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{z}{\sqrt{2}}} e^{-\xi^2} d\xi =$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{0} e^{-\xi^2} d\xi + \int_0^{\frac{z}{\sqrt{2}}} e^{-\xi^2} d\xi\right] = (5)$$

$$\int_{-\infty}^{0} e^{-\xi^2} d\xi = -\int_0^{-\infty} e^{-\xi^2} d\xi = \int_0^{-\infty} e^{-\xi^2} (-d\xi) = \int_0^{-\infty} e^{-(-\xi)^2} d(-\xi) = \int_0^{+\infty} e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2}$$

$$(5) = \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} + \int_0^{\frac{z}{\sqrt{2}}} e^{-\xi^2} d\xi\right] = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{z}{\sqrt{2}}} e^{-\xi^2} d\xi =$$

$$= \frac{1}{2}\left[1 + \frac{2}{\sqrt{\pi}} \int_0^{\frac{z}{\sqrt{2}}} e^{-\xi^2} d\xi\right] = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right] =: \Phi(z)$$

6.

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = (6.1)$$

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x \mid \mu = 0, \sigma^2 = 1)$$

$$(6.1) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$F(x \mid \mu, \sigma^2) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right] = (6.2)$$

$$\Phi := \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = F(x \mid \mu = 0, \sigma^2 = 1)$$

$$(6.2) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$X \sim N(\mu, \sigma^2)$$

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad -\infty < x < +\infty$$

$$\int_0^{+\infty} e^{-\xi^2} = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{+\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{+\infty} \xi e^{-\xi^2} d\xi = 0$$

$$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \mu$$

$$\int_{-\infty}^{+\infty} \xi^2 e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2}$$

$$D[X] = \int_{-\infty}^{+\infty} (x - E[X])^2 f(x)\,dx = \sigma^2$$

$$p(X \leq x) = F(x|\mu,\sigma^2) = \int_{-\infty}^{x} f(\eta)\,d\eta = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$$

$$X \sim N(\mu, \sigma^2)$$

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\int_{-\infty}^{+\infty} f(x|\mu,\sigma^2) = 1$$

$$E[X] = \int_{-\infty}^{+\infty} x f(x|\mu,\sigma^2) = \mu$$

$$D[X] = \int_{-\infty}^{+\infty} (x - E[x])^2 dx = \sigma^2$$

$$p(X \leq x) = F(x|\mu,\sigma^2) = \int_{-\infty}^{x} f(\eta|\mu,\sigma^2)\,d\eta =$$

$$= \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$

$$f(z|\mu=0,\sigma^2=1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} =: \varphi(z)$$

$$\int_{-\infty}^{+\infty} \varphi(z)\,dz = 1$$

$$E[Z] = \int_{-\infty}^{+\infty} z\,\varphi(z)\,dz = 0$$

$$D[Z] = \int_{-\infty}^{+\infty} z^2 \varphi(z)\,dz = 1$$

$$p(Z \leq z) = F(z|\mu=0,\sigma^2=1) = \int_{-\infty}^{z} f(\eta|\mu=0,\sigma^2=1)\,d\eta =$$

$$= \int_0^z \varphi(\eta)\,d\eta = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right] =: \Phi(z)$$

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$F(x|\mu,\sigma^2) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$