2.2

$$X \sim N(\mu, \sigma^2)$$

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x|\mu=0, \sigma^2=1)$$

$$F(x|\mu,\sigma^2) = \int_{-\infty}^{x} f(\xi|\mu,\sigma^2)\,d\xi = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right] = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\Phi(x) := \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = F(x|\mu=0, \sigma^2=1)$$

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\,d\xi$$

1.

$$P(X \le x) = \int_{-\infty}^{x} f(\xi|\mu,\sigma^2)\,d\xi = F(x|\mu,\sigma^2) =$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right) = \left[z = \frac{x-\mu}{\sigma}\right] = \Phi(z)$$

2.

$$P(x \le X) = \int_{x}^{+\infty} f(\xi|\mu,\sigma^2)\,d\xi =$$

$$= \int_{-\infty}^{x} f(\xi|\mu,\sigma^2)\,d\xi + \int_{x}^{+\infty} f(\xi|\mu,\sigma^2)\,d\xi - \int_{-\infty}^{x} f(\xi|\mu,\sigma^2)\,d\xi =$$

$$= \int_{-\infty}^{+\infty} f(\xi|\mu,\sigma^2)\,d\xi - \int_{-\infty}^{x} f(\xi|\mu,\sigma^2)\,d\xi =$$

$$= 1 - \int_{-\infty}^{x} f(\xi|\mu,\sigma^2)\,d\xi = 1 - F(x|\mu,\sigma^2) =$$

$$= 1 - \Phi\left(\frac{x-\mu}{\sigma}\right) = \left[z = \frac{x-\mu}{\sigma}\right] = 1 - \Phi(z)$$

$$P(X \geq x) = P(\neg(X < x)) = 1 - P(X < x) =$$
$$= 1 - F(x|\mu,\sigma^2) = 1 - \Phi\left(\frac{x-\mu}{\sigma}\right) =$$
$$= \left[z = \frac{x-\mu}{\sigma}\right] = 1 - \Phi(z)$$

3.

$$P(X \leq x_1 \lor x_2 \leq X) = \int_{-\infty}^{x_1} f(\xi|\mu,\sigma^2)\,d\xi + \int_{x_2}^{+\infty} f(\xi|\mu,\sigma^2)\,d\xi =$$
$$= \int_{-\infty}^{x_1} f(\xi|\mu,\sigma^2)\,d\xi + \int_{-\infty}^{x_2} f(\xi|\mu,\sigma^2)\,d\xi + \int_{x_2}^{+\infty} f(\xi|\mu,\sigma^2)\,d\xi - \int_{-\infty}^{x_2} f(\xi|\mu,\sigma^2)\,d\xi =$$
$$= \int_{-\infty}^{x_1} f(\xi|\mu,\sigma^2)\,d\xi + \int_{-\infty}^{+\infty} f(\xi|\mu,\sigma^2)\,d\xi - \int_{-\infty}^{x_2} f(\xi|\mu,\sigma^2)\,d\xi =$$
$$= F(x_1|\mu,\sigma^2) + 1 - F(x_2|\mu,\sigma^2) = \Phi\left(\frac{x_1-\mu}{\sigma}\right) + 1 - \Phi\left(\frac{x_2-\mu}{\sigma}\right) =$$
$$= \left[z_1 = \frac{x_1-\mu}{\sigma}, z_2 = \frac{x_2-\mu}{\sigma}\right] = \Phi(z_1) + 1 - \Phi(z_2)$$

$$P(X \leq x_1 \lor x_2 \leq X) = P(X \leq x_1) + P(x_2 \leq X) =$$
$$= F(x_1|\mu,\sigma^2) + 1 - F(x_2|\mu,\sigma^2) = \Phi\left(\frac{x_1-\mu}{\sigma}\right) + 1 - \Phi\left(\frac{x_2-\mu}{\sigma}\right) =$$
$$= \left[z_1 = \frac{x_1-\mu}{\sigma}, z_2 = \frac{x_2-\mu}{\sigma}\right] = \Phi(z_1) + 1 - \Phi(z_2)$$

4.

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(\xi|\mu,\sigma^2)\,d\xi =$$
$$= \int_{-\infty}^{x_1} f(\xi|\mu,\sigma^2)\,d\xi + \int_{x_1}^{x_2} f(\xi|\mu,\sigma^2)\,d\xi - \int_{-\infty}^{x_1} f(\xi|\mu,\sigma^2)\,d\xi =$$
$$= \int_{-\infty}^{x_2} f(\xi|\mu,\sigma^2)\,d\xi - \int_{-\infty}^{x_1} f(\xi|\mu,\sigma^2)\,d\xi =$$
$$= F(x_2|\mu,\sigma^2) - F(x_1|\mu,\sigma^2) = \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right) =$$
$$= \left[z_1 = \frac{x_1-\mu}{\sigma}, z_2 = \frac{x_2-\mu}{\sigma}\right] = \Phi(z_2) - \Phi(z_1)$$

$$p(x_1 < X \le x_2) = p(x_1 \le X \wedge X \le x_2) = p(\neg\neg(x_1 \le X \wedge X \le x_2)) =$$

$$= p(\neg(\neg(x_1 < X) \vee \neg(X \le x_2))) = p(\neg(X < x_1 \vee x_2 < X)) =$$

$$= 1 - p(X < x_1 \vee x_2 < X) = 1 - [F(x_1 | \mu, \sigma^2) + 1 - F(x_2 | \mu, \sigma^2)] =$$

$$= F(x_2 | \mu, \sigma^2) - F(x_1 | \mu, \sigma^2) = \Phi\left(\frac{x_2 - \mu}{\sigma}\right) - \Phi\left(\frac{x_1 - \mu}{\sigma}\right) =$$

$$= \left[z_1 = \frac{x_1 - \mu}{\sigma}, z_2 = \frac{x_2 - \mu}{\sigma}\right] = \Phi(z_2) - \Phi(z_1)$$

$$\boxed{\begin{aligned}
&p(X \le x) = F(x | \mu, \sigma^2) = \\
&= \left[z = \frac{x - \mu}{\sigma}\right] = \Phi(z) \\
&p(x \le X) = 1 - F(x | \mu, \sigma^2) = \\
&= \left[z = \frac{x - \mu}{\sigma}\right] = 1 - \Phi(z) \\
&p(X \le x_1 \vee x_2 \le X) = F(x_1 | \mu, \sigma^2) + 1 - F(x_2 | \mu, \sigma^2) = \\
&= \left[z_1 = \frac{x_1 - \mu}{\sigma}, z_2 = \frac{x_2 - \mu}{\sigma}\right] = \Phi(z_1) + 1 - \Phi(z_2) \\
&p(x_1 \le X \le x_2) = F(x_2 | \mu, \sigma^2) - F(x_1 | \mu, \sigma^2) = \\
&= \left[z_1 = \frac{x_1 - \mu}{\sigma}, z_2 = \frac{x_2 - \mu}{\sigma}\right] = \Phi(z_2) - \Phi(z_1)
\end{aligned}}$$