$\boxed{4}$ Дано:

$X \sim N(\mu, \sigma^2)$

$\mu = 174$

$\sigma = 8$

Найти:

1. $p(X > 182)$
2. $p(X > 190)$
3. $p(166 \leq X \leq 190)$
4. $p(166 \leq X \leq 182)$
5. $p(158 \leq X \leq 190)$
6. $p(\neg(150 < X) \vee \neg(X < 190))$
7. $p(\neg(150 < X) \vee \neg(X < 198))$
8. $p(X < 166)$

Решение:

$X \sim N(\mu, \sigma^2)$

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x|\mu=0, \sigma^2=1)$$

$$F(x|\mu,\sigma^2) = \int_{-\infty}^{x} f(z|\mu,\sigma^2)\,dz = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right] = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\Phi(x) := \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = F(x|\mu=0, \sigma^2=1)$$

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz$$

1.1

$$P(X \le x) = \int_{-\infty}^{x} f(z|\mu\,\sigma^2)\,dz = F(x|\mu,\sigma^2) =$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi(z)$$

1.2

$$P(x \le X) = \int_{x}^{+\infty} f(z|\mu\,\sigma^2)\,dz =$$

$$= \int_{-\infty}^{x} f(z|\mu\,\sigma^2)\,dz + \int_{x}^{+\infty} f(z|\mu\,\sigma^2)\,dz - \int_{-\infty}^{x} f(z|\mu,\sigma^2)\,dz =$$

$$= \int_{-\infty}^{+\infty} f(z|\mu\,\sigma^2)\,dz - \int_{-\infty}^{x} f(z|\mu,\sigma^2)\,dz =$$

$$= 1 - F(x|\mu\,\sigma^2) =$$

$$= 1 - \Phi\left(\frac{x-\mu}{\sigma}\right) = 1 - \Phi(z)$$

$$p(x \le X) = p(\neg\neg(x < X)) = p(\neg(X < x)) = 1 - p(X < x) =$$
$$= 1 - F(x \mid \mu, \sigma^2)$$

1.3.

$$p(X \le x_1 \vee x_2 < X) = \int_{-\infty}^{x_1} f(\xi \mid \mu, \sigma^2)\, d\xi + \int_{x_2}^{+\infty} f(\xi \mid \mu, \sigma^2)\, d\xi =$$
$$= \int_{-\infty}^{x_1} f(\xi \mid \mu, \sigma^2)\, d\xi + \int_{-\infty}^{x_2} f(\xi \mid \mu, \sigma^2)\, d\xi + \int_{x_2}^{+\infty} f(\xi \mid \mu, \sigma^2)\, d\xi - \int_{-\infty}^{x_2} f(\xi \mid \mu, \sigma^2)\, d\xi =$$
$$= \int_{-\infty}^{x_1} f(\xi \mid \mu, \sigma^2)\, d\xi + \int_{-\infty}^{+\infty} f(\xi \mid \mu, \sigma^2)\, d\xi - \int_{-\infty}^{x_2} f(\xi \mid \mu, \sigma^2)\, d\xi =$$
$$= F(x_1 \mid \mu, \sigma^2) + \left[1 - F(x_2 \mid \mu, \sigma^2)\right] =$$
$$= \Phi\left(\frac{x_1 - \mu}{\sigma}\right) + \left[1 - \Phi\left(\frac{x_2 - \mu}{\sigma}\right)\right] = \Phi(z_1) + \left[1 - \Phi(z_2)\right]$$

$$p(X \le x_1 \vee x_2 \le X) = p(X \le x_1) + p(x_2 \le X) =$$
$$= F(x_1 \mid \mu, \sigma^2) + \left[1 - F(x_2 \mid \mu, \sigma^2)\right]$$

1.4

$$p(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(\xi \mid \mu, \sigma^2)\, d\xi =$$
$$= \int_{-\infty}^{x_2} f(\xi \mid \mu, \sigma^2)\, d\xi - \int_{-\infty}^{x_1} f(\xi \mid \mu, \sigma^2)\, d\xi =$$
$$= F(x_2 \mid \mu, \sigma^2) - F(x_1 \mid \mu, \sigma^2) =$$
$$= \Phi\left(\frac{x_2 - \mu}{\sigma}\right) - \Phi\left(\frac{x_1 - \mu}{\sigma}\right) = \Phi(z_2) - \Phi(z_1)$$

$$p(x_1 \le X \le x_2) = p(x_1 \le X \wedge X \le x_2) = p(\neg\neg(x_1 \le X \wedge X \le x_2)) =$$

$$= p(\neg(\neg(x_1 \le X) \vee \neg(X \le x_2))) = p(\neg(X < x_1 \vee x_2 < X)) =$$

$$= 1 - p(X < x_1 \vee x_2 < X) =$$

$$= 1 - \left[F(x_1 | \mu, \sigma^2) + (1 - F(x_2 | \mu, \sigma^2))\right] =$$

$$= F(x_2 | \mu, \sigma^2) - F(x_1 | \mu, \sigma^2)$$

1.5

$$\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-\xi^2} d\xi = $$

$$= -\frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-\xi^2} (-d\xi) = -\frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-(-\xi)^2} d(-\xi) =$$

$$= -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-\xi^2} d\xi = -\operatorname{erf}(x)$$

$$\Phi(-x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{-x}{\sqrt{2}}\right)\right] =$$

$$= \frac{1}{2}\left[1 + \operatorname{erf}\left(-\frac{x}{\sqrt{2}}\right)\right] = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = \frac{1}{2}\left[2 - \left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)\right] =$$

$$= 1 - \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = 1 - \Phi(x)$$

$$\Phi(x) - \Phi(-x) = \Phi(x) - [1 - \Phi(x)] = 2\Phi(x) - 1 \Leftrightarrow$$

$$\Leftrightarrow 2\Phi(x) = 1 + \Phi(x) - \Phi(-x) \Leftrightarrow$$

$$\Leftrightarrow \Phi(x) = \frac{1}{2} + \frac{\Phi(x) - \Phi(-x)}{2}$$

21. $P(X > 182) =$

$= P(X \geq x) = 1 - F(x \mid \mu, \sigma^2) =$

$= 1 - \Phi\left(\frac{x-\mu}{\sigma}\right) = 1 - \Phi(z) =$

$z = \frac{x-\mu}{\sigma} = \frac{182-174}{8} = \frac{8}{8} = 1$

$= 1 - \Phi(1) =$

$= 1 - \left[\frac{1}{2} + \frac{\Phi(1) - \Phi(-1)}{2}\right] = \frac{1}{2} - \frac{\Phi(1) - \Phi(-1)}{2} =$

$= \frac{1}{2}\left[1 - (\Phi(1) - \Phi(-1))\right] \simeq$

$\simeq \frac{0.317}{2} = 0.1585 \simeq 0.159$

22. $P(X > 190) =$

$= P(X \geq x) = 1 - F(x \mid \mu, \sigma^2) =$

$= 1 - \Phi\left(\frac{x-\mu}{\sigma}\right) = 1 - \Phi(z) =$

$z = \frac{x-\mu}{\sigma} = \frac{190-174}{8} = \frac{16}{8} = 2$

$= 1 - \Phi(2) =$

$= 1 - \left[\frac{1}{2} + \frac{\Phi(2) - \Phi(-2)}{2}\right] = \frac{1}{2} - \frac{\Phi(2) - \Phi(-2)}{2} =$

$= \frac{1}{2}\left[1 - (\Phi(2) - \Phi(-2))\right] \simeq$

$\simeq \frac{0.0455}{2} = 0.02275 \simeq 0.0228$

2.3 $P(166 \leq X \leq 190) =$

$= P(x_1 \leq X \leq x_2) = F(x_2|\mu,\sigma^2) - F(x_1|\mu,\sigma^2) =$

$= \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right) = \Phi(z_2) - \Phi(z_1) =$

$z_1 = \frac{x_1-\mu}{\sigma} = \frac{166-174}{8} = \frac{-8}{8} = -1$

$z_2 = \frac{x_2-\mu}{\sigma} = \frac{190-174}{8} = \frac{16}{8} = 2$

$= \Phi(2) - \Phi(-1) =$

$= \Phi(2) - [1-\Phi(1)] = \Phi(1) + \Phi(2) - 1 =$

$= \left[\frac{1}{2} + \frac{\Phi(1)-\Phi(-1)}{2}\right] + \left[\frac{1}{2} + \frac{\Phi(2)-\Phi(-2)}{2}\right] - 1 =$

$= \frac{1}{2}\left[(\Phi(1)-\Phi(-1)) + (\Phi(2)-\Phi(-2))\right] \simeq$

$\simeq \frac{1}{2}(0.683 + 0.954) = \frac{1.637}{2} = 0.8185 \simeq 0.819$

2.4 $P(166 \leq X \leq 182) =$

$= P(x_1 \leq X \leq x_2) = F(x_2|\mu,\sigma^2) - F(x_1|\mu,\sigma^2) =$

$= \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right) = \Phi(z_2) - \Phi(z_1) =$

$z_1 = \frac{x_1-\mu}{\sigma} = \frac{166-174}{8} = \frac{-8}{8} = -1$

$z_2 = \frac{x_2-\mu}{\sigma} = \frac{182-174}{8} = \frac{8}{8} = 1$

$= \Phi(1) - \Phi(-1) \simeq$

$\simeq 0.683$

2.5. $P(158 \le X \le 190) =$

$= P(x_1 \le X \le x_2) = F(x_2 | \mu, \sigma^2) - F(x_1 | \mu, \sigma^2) =$

$= \Phi\left(\frac{x_2 - \mu}{\sigma}\right) - \Phi\left(\frac{x_1 - \mu}{\sigma}\right) = \Phi(z_2) - \Phi(z_1) =$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{190 - 174}{8} = \frac{16}{8} = 2$

$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{158 - 174}{8} = \frac{-16}{8} = -2$

$= \Phi(2) - \Phi(-2) \approx$

≈ 0.954

2.6 $P(\neg(150 < X) \vee \neg(X < 190)) =$

$= P(X \le 150 \vee 190 \le X) =$

$= P(X \le x_1 \vee x_2 \le X) = F(x_1 | \mu, \sigma^2) + [1 - F(x_2 | \mu, \sigma^2)] =$

$= \Phi\left(\frac{x_1 - \mu}{\sigma}\right) + \left[1 - \Phi\left(\frac{x_2 - \mu}{\sigma}\right)\right] = \Phi(z_1) + [1 - \Phi(z_2)] =$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{150 - 174}{8} = \frac{-24}{8} = -3$

$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{190 - 174}{8} = \frac{16}{8} = 2$

$= \Phi(-3) + [1 - \Phi(2)] =$

$= [1 - \Phi(3)] + [1 - \Phi(2)] = [1 - \Phi(2)] + [1 - \Phi(3)] =$

$= \left[1 - \left(\frac{1}{2} + \frac{\Phi(2) - \Phi(-2)}{2}\right)\right] + \left[1 - \left(\frac{1}{2} + \frac{\Phi(3) - \Phi(-3)}{2}\right)\right] =$

$= \left[\frac{1}{2} - \frac{\Phi(2) - \Phi(-2)}{2}\right] + \left[\frac{1}{2} - \frac{\Phi(3) - \Phi(-3)}{2}\right] =$

$= \frac{1}{2}\left[1 - (\Phi(2) - \Phi(-2)) + 1 - (\Phi(3) - \Phi(-3))\right] \approx$

$\approx \frac{1}{2}(0.0455 + 0.00270) = \frac{0.0482}{2} = 0.0241$

2.7. $p(\neg(150 < X \wedge \neg(X < 198))) =$

$= p(X \leq 150 \vee 198 \leq X) =$

$= p(X \leq x_1 \vee x_2 \leq X) = F(x_1 | \mu, \sigma^2) + [1 - F(x_2 | \mu, \sigma^2)] =$

$= \Phi\left(\frac{x_1 - \mu}{\sigma}\right) + \left[1 - \Phi\left(\frac{x_2 - \mu}{\sigma}\right)\right] = \Phi(z_1) + [1 - \Phi(z_2)] =$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{150 - 174}{8} = \frac{-24}{8} = -3$

$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{198 - 174}{8} = \frac{24}{8} = 3$

$= \Phi(-3) + [1 - \Phi(3)] =$

$= 1 - [\Phi(3) - \Phi(-3)] \simeq$

$\simeq 0.00270$

2.8. $p(X < 166) =$

$= p(X < x) = F(x | \mu, \sigma^2) =$

$= \Phi\left(\frac{x - \mu}{\sigma}\right) = \Phi(z) =$

$z = \frac{x - \mu}{\sigma} = \frac{166 - 174}{8} = \frac{-8}{8} = -1$

$= \Phi(-1) =$

$= 1 - \Phi(1) =$

$= 1 - \left[\frac{1}{2} + \frac{\Phi(1) - \Phi(-1)}{2}\right] = \frac{1}{2} - \frac{\Phi(1) - \Phi(-1)}{2} =$

$= \frac{1}{2}\left[1 - (\Phi(1) - \Phi(-1))\right] \simeq$

$\simeq \frac{0.317}{2} = 0.1585 \simeq 0.159$

Dobleni:

1. $p(X > 182) = 1 - \Phi(1) =$

$= \frac{1}{2}\left[1 - \left(\Phi(1) - \Phi(-1)\right)\right] \simeq 0.159$

2. $p(X > 190) = 1 - \Phi(2) =$

$= \frac{1}{2}\left[1 - \left(\Phi(2) - \Phi(-2)\right)\right] \simeq 0.0228$

3. $p(166 \leq X \leq 190) = \Phi(2) - \Phi(-1) =$

$= \frac{1}{2}\left[\left(\Phi(1) - \Phi(-1)\right) + \left(\Phi(2) - \Phi(-2)\right)\right] \simeq 0.819$

4. $p(166 \leq X \leq 182) = \Phi(1) - \Phi(-1) \simeq 0.683$

5. $p(158 \leq X \leq 190) = \Phi(2) - \Phi(-2) \simeq 0.954$

6. $p(\neg(150 < X) \vee \neg(X < 190)) = \Phi(-3) + \left[1 - \Phi(2)\right] =$

$= \frac{1}{2}\left[1 - \left(\Phi(2) - \Phi(-2)\right) + 1 - \left(\Phi(3) - \Phi(-3)\right)\right] \simeq 0.0241$

7. $p(\neg(150 < X) \vee \neg(X < 198)) = \Phi(-3) + \left[1 - \Phi(3)\right] =$

$= 1 - \left[\Phi(3) - \Phi(-3)\right] \simeq 0.0027$

8. $p(X < 166) = \Phi(-1) =$

$= \frac{1}{2}\left[1 - \left(\Phi(1) - \Phi(-1)\right)\right] \simeq 0.159$