$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Δ

$$A = \begin{pmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{pmatrix} = (a_{\cdot 1} \; a_{\cdot 2} \; a_{\cdot 3}).$$

$$B = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (b_{\cdot 1} \; b_{\cdot 2} \; b_{\cdot 3})$$

$a_{\cdot 1} = b_{\cdot 1}$; $a_{\cdot 2} = b_{\cdot 2}$

$\det A = \det(a_{\cdot 1} \; a_{\cdot 2} \; a_{\cdot 3}) = \{\forall \alpha \wedge \forall \beta\} = \det(a_{\cdot 1} \; a_{\cdot 2} \; a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2})$

$\forall \gamma \quad \gamma \det B = \gamma \det(b_{\cdot 1} \; b_{\cdot 2} \; b_{\cdot 3}) = \det(b_{\cdot 1} \; b_{\cdot 2} \; \gamma b_{\cdot 3})$

$\det A = \gamma \det B$

$\det(a_{\cdot 1} \; a_{\cdot 2} \; a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2}) = \det(b_{\cdot 1} \; b_{\cdot 2} \; \gamma b_{\cdot 3}) \iff$

$a_{\cdot 1} = b_{\cdot 1} \wedge a_{\cdot 2} = b_{\cdot 2} \wedge a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} = \gamma b_{\cdot 3}$

$$a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} = \begin{pmatrix} a^3 \\ b^3 \\ c^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a^3 + \alpha + \beta a \\ b^3 + \alpha + \beta b \\ c^3 + \alpha + \beta c \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha + a\beta + a^3 \\ \alpha + b\beta + b^3 \\ \alpha + c\beta + c^3 \end{pmatrix} =$$

$$= \gamma b_{\cdot 3} = \gamma \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} = \begin{pmatrix} \gamma a^2 \\ \gamma b^2 \\ \gamma c^2 \end{pmatrix} = \begin{pmatrix} a^2 \gamma \\ b^2 \gamma \\ c^2 \gamma \end{pmatrix} \iff$$

$$\begin{cases} \alpha + a\beta + a^3 = a^2\gamma \\ \alpha + b\beta + b^3 = b^2\gamma \\ \alpha + c\beta + c^3 = c^2\gamma \end{cases} \Leftrightarrow$$

$$\begin{cases} \alpha + a\beta - a^2\gamma + a^3 = 0 \\ \alpha + b\beta - b^2\gamma + b^3 = 0 \\ \alpha + c\beta - c^2\gamma + c^3 = 0 \end{cases} \Leftrightarrow$$

(1)-(2);

$$(\cancel{\alpha} + a\beta - a^2\gamma + a^3) - (\cancel{\alpha} + b\beta - b^2\gamma + b^3) = 0 - 0 \Leftrightarrow$$

$$(a-b)\beta - (a^2-b^2)\gamma + (a^3-b^3) = 0 \Leftrightarrow$$

$$(a-b)\beta - (a-b)(a+b)\gamma + (a-b)(a^2+ab+b^2) = 0 \Leftrightarrow$$

$$\beta - (a+b)\gamma + a^2+ab+b^2 = 0$$

(2)-(3);

$$(\cancel{\alpha} + b\beta - b^2\gamma + b^3) - (\cancel{\alpha} + c\beta - c^2\gamma + c^3) = 0 - 0 \Leftrightarrow$$

$$(b-c)\beta - (b^2-c^2)\gamma + b^3 - c^3 = 0 \Leftrightarrow$$

$$(b-c)\beta - (b-c)(b+c)\gamma + (b-c)(b^2+bc+c^2) = 0 \Leftrightarrow$$

$$\beta - (b+c)\gamma + b^2+bc+c^2 = 0$$

$$\Leftrightarrow \begin{cases} \beta - (a+b)\gamma + a^2+ab+b^2 = 0 \\ \beta - (b+c)\gamma + b^2+bc+c^2 = 0 \\ \alpha + c\beta - c^2\gamma + c^3 = 0 \end{cases} \Leftrightarrow$$

(1) − (2):

$$\left(\cancel{\beta} - (a+b)\gamma + a^2+ab+b^2\right) - \left(\cancel{\beta} - (b+c)\gamma + b^2+bc+c^2\right) = 0-0 \iff$$

$$-\left(\left(a+\cancel{b}\right) - \left(\cancel{b}+c\right)\right)\gamma + \left(\left(a^2+ab+\cancel{b^2}\right) - \left(\cancel{b^2}+bc+c^2\right)\right) = 0 \iff$$

$$-(a-c)\gamma + (a^2-c^2) + (ab-bc) = 0 \iff$$

$$-(a-c)\gamma + (a-c)(a+c) + b(a-c) = 0 \iff$$

$$-\gamma + a+c+b = 0 \iff$$

$$-\gamma + a+b+c$$

$$\iff \begin{cases} -\gamma + a+b+c = 0 \\ \beta - (b+c)\gamma + b^2+bc+c^2 = 0 \\ \alpha + c\beta - c^2\gamma + c^3 = 0 \end{cases}$$

$$-\gamma + a+b+c = 0 \iff$$

$$\gamma = a+b+c$$

$$\beta - (b+c)\gamma + b^2+bc+c^2 = 0 \iff$$

$$\beta = (b+c)\gamma - (b^2+bc+c^2) = \{\gamma = a+b+c\} =$$

$$= (b+c)(a+b+c) - (b^2+bc+c^2) =$$

$$= (ba + b^2 + bc + ca + cb + c^2) - (b^2+bc+c^2) =$$

$$= (\cancel{b^2} + \cancel{c^2} + ab + 2bc + ca) - (\cancel{b^2} + \cancel{c^2} + \cancel{bc}) =$$

$$= ab+bc+ca$$

$$\alpha + c\beta - c^2\gamma + c^3 = 0 \iff$$

$$\alpha = -c\beta + c^2\gamma - c^3 = \{\beta = ab + bc + ca;\ \gamma = a + b + c\} =$$

$$= -c(ab + bc + ca) + c^2(a + b + c) - c^3 =$$

$$= -(cab + bc^2 + c^2a) + (c^2a + c^2b + c^3) - c^3 =$$

$$= \cancel{(c^3 + ac^2 + bc^2)} - (\cancel{c^3} + \cancel{ac^2} + \cancel{bc^2} + abc) = abc$$

$$\iff \begin{cases} \alpha = abc \\ \beta = ab + bc + ca \\ \gamma = a + b + c \end{cases}$$

$$\det A = \det(a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3}) = \{\alpha = abc \wedge \beta = ab + bc + ca\} =$$

$$= \det(a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2}) = \{\gamma = a + b + c\} =$$

$$= \gamma \det(b_{\cdot 1}\ b_{\cdot 2}\ b_{\cdot 3}) = \gamma \det B$$

$$\det A = (a + b + c)\det B$$

$\square$

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

▷

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \{ (\cdot 3) += abc(\cdot 1) + (ab+bc+ca)(\cdot 2) \} =$$

$$= \begin{vmatrix} 1 & a & a^3 + abc + (ab+bc+ca)a \\ 1 & b & b^3 + abc + (ab+bc+ca)b \\ 1 & c & c^3 + abc + (ab+bc+ca)c \end{vmatrix} = \{$$

$$a^3 + abc + (ab+bc+ca)a = a^3 + abc + a^2b + bca + ca^2 =$$

$$= a^3 + a^2b + ca^2 + 2abc = a^2(a+b+c) + 2abc$$

$$b^3 + abc + (ab+bc+ca)b = b^3 + abc + ab^2 + b^2c + cab =$$

$$= ab^2 + b^3 + b^2c + 2abc = b^2(a+b+c) + 2abc$$

$$c^3 + abc + (ab+bc+ca)c = c^3 + abc + abc + bc^2 + c^2a =$$

$$= c^2a + bc^2 + c^3 + 2abc = c^2(a+b+c) + 2abc$$

$$\} = \begin{vmatrix} 1 & a & a^2(a+b+c) + 2abc \\ 1 & b & b^2(a+b+c) + 2abc \\ 1 & c & c^2(a+b+c) + 2abc \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & a & a^2(a+b+c) \\ 1 & b & b^2(a+b+c) \\ 1 & c & c^2(a+b+c) \end{vmatrix} + \begin{vmatrix} 1 & a & 2abc \\ 1 & b & 2abc \\ 1 & c & 2abc \end{vmatrix} =$$

$$= (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + 2abc \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} =$$

$$= (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$\square$

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$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

D

$$A = \begin{pmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{pmatrix} = (a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3})$$

$$B = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (b_{\cdot 1}\ b_{\cdot 2}\ b_{\cdot 3})$$

$$a_{\cdot 1} = b_{\cdot 1}\ ;\ a_{\cdot 2} = b_{\cdot 3}$$

$$\det A = \det(a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3}) = \{\forall \alpha \wedge \forall \beta\} =$$

$$= \det(a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2})$$

$$\det B = \det(b_{\cdot 1}\ b_{\cdot 2}\ b_{\cdot 3}) = (-1)\det(b_{\cdot 1}\ b_{\cdot 3}\ b_{\cdot 2})$$

$$\forall \gamma \quad \gamma \det B = \gamma(-1)\det(b_{\cdot 1}\ b_{\cdot 3}\ b_{\cdot 2}) = -\gamma \det(b_{\cdot 1}\ b_{\cdot 3}\ b_{\cdot 2}) =$$

$$= \det(b_{\cdot 1}\ b_{\cdot 3}\ -\gamma b_{\cdot 2})$$

$$\det A = \gamma \det B \iff$$

$$\det(a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2}) = \det(b_{\cdot 1}\ b_{\cdot 3}\ -\gamma b_{\cdot 2}) \iff$$

$$a_{\cdot 1} = b_{\cdot 1} \wedge a_{\cdot 2} = b_{\cdot 3} \wedge \ a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} = -\gamma b_{\cdot 2}$$

$a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} =$

$$= \begin{pmatrix} a^3 \\ b^3 \\ c^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} = \begin{pmatrix} a^3 + \alpha + \beta a^2 \\ b^3 + \alpha + \beta b^2 \\ c^3 + \alpha + \beta c^2 \end{pmatrix} = \begin{pmatrix} \alpha + a^2\beta + a^3 \\ \alpha + b^2\beta + b^3 \\ \alpha + c^2\beta + c^3 \end{pmatrix} =$$

$$= -\gamma b_2 = -\gamma \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\gamma a \\ -\gamma b \\ -\gamma c \end{pmatrix} = \begin{pmatrix} -a\gamma \\ -b\gamma \\ -c\gamma \end{pmatrix} \iff$$

$$\begin{cases} \alpha + a^2\beta + a^3 = -a\gamma \\ \alpha + b^2\beta + b^3 = -b\gamma \\ \alpha + c^2\beta + c^3 = -c\gamma \end{cases} \iff$$

$$\begin{cases} \alpha + a^2\beta + a\gamma + a^3 = 0 \\ \alpha + b^2\beta + b\gamma + b^3 = 0 \\ \alpha + c^2\beta + c\gamma + c^3 = 0 \end{cases} \iff$$

$(1) - (2)$:

$$(\cancel{\alpha} + a^2\beta + a\gamma + a^3) - (\cancel{\alpha} + b^2\beta + b\gamma + b^3) = 0 - 0 \iff$$

$$(a^2 - b^2)\beta + (a - b)\gamma + (a^3 - b^3) = 0 \iff$$

$$(a-b)(a+b)\beta + (a-b)\gamma + (a-b)(a^2 + ab + b^2) = 0 \iff$$

$$(a+b)\beta + \gamma + a^2 + ab + b^2 = 0$$

$(2)-(3)$:

$$(\cancel{\alpha}+b^2\beta+b\gamma+b^3)-(\cancel{\alpha}+c^2\beta+c\gamma+c^3)=0-0 \iff$$

$$(b^2-c^2)\beta+(b-c)\gamma+(b^3-c^3)=0 \iff$$

$$(b-c)(b+c)\beta+(b-c)\gamma+(b-c)(b^2+bc+c^2)=0 \iff$$

$$(b+c)\beta+\gamma+b^2+bc+c^2=0$$

$$\iff \begin{cases} (a+b)\beta+\gamma+a^2+ab+b^2=0 \\ (b+c)\beta+\gamma+b^2+bc+c^2=0 \\ \alpha+c^2\beta+c\gamma+c^3=0 \end{cases} \iff$$

$(1)-(2)$:

$$((a+b)\beta+\cancel{\gamma}+a^2+ab+b^2)-((b+c)\beta+\cancel{\gamma}+b^2+bc+c^2)=0-0 \iff$$

$$((a+\cancel{b})-(\cancel{b}+c))\beta+(a^2+ab+\cancel{b^2})-(\cancel{b^2}+bc+c^2)=0 \iff$$

$$(a-c)\beta+(a^2-c^2)+(ab-bc)=0 \iff$$

$$(a-c)\beta+(a-c)(a+c)+b(a-c)=0 \iff$$

$$\beta+a+c+b=0 \iff$$

$$\beta+a+b+c=0$$

$$\iff \begin{cases} \beta+a+b+c=0 \\ (b+c)\beta+\gamma+b^2+bc+c^2=0 \\ \alpha+c^2\beta+c\gamma+c^3=0 \end{cases}$$

(1):

$\beta + a + b + c = 0$

$\beta = -(a+b+c)$

(2):

$(b+c)\beta + \gamma + b^2 + bc + c^2 = 0$

$\gamma = -(b+c)\beta - (b^2+bc+c^2) = \{\beta = -(a+b+c)\} =$

$= -(b+c)(-1)(a+b+c) - (b^2+bc+c^2) =$

$= (ba + b^2 + bc + ca + cb + c^2) - (b^2+bc+c^2) =$

$= (\cancel{b^2} + \cancel{c^2} + ab + \cancel{2}bc + ca) - (\cancel{b^2} + \cancel{c^2} + \cancel{bc}) =$

$= ab + bc + ca$

(3):

$\alpha + c^2\beta + c\gamma + c^3 = 0$

$\alpha = -c^2\beta - c\gamma - c^3 = \{\beta = -(a+b+c);\ \gamma = ab + bc + ca\} =$

$= -c^2(-1)(a+b+c) - c(ab+bc+ca) - c^3 =$

$= (c^2a + c^2b + c^3) - (cab + bc^2 + c^2a) - c^3 =$

$= (\cancel{c^3} + \cancel{bc^2} + \cancel{ac^2}) - (\cancel{c^3} + \cancel{bc^2} + \cancel{c^2a} + abc) =$

$= -abc$

$\det A = \det(a_{.1}\ a_{.2}\ a_{.3}) = \{\alpha = -abc \wedge \beta = -(a+b+c)\} =$

$= \det(a_{.1}\ a_{.2}\ \ a_{.3} + \alpha a_{.1} + \beta a_{.2}) = \{\gamma = ab + bc + ca\} =$

$= \gamma \det B \Rightarrow$

$\det A = (ab + bc + ca)\det B$

$\square$

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

D

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \left\{ (\cdot 3) - = -abc(\cdot 1) + (a+b+c)(\cdot 2) \right\} =$$

$$= \begin{vmatrix} 1 & a^2 & a^3 - abc - (a+b+c)a^2 \\ 1 & b^2 & b^3 - abc - (a+b+c)b^2 \\ 1 & c^2 & c^3 - abc - (a+b+c)c^2 \end{vmatrix} = \{$$

(13): $a^3 - abc - (a+b+c)a^2 = \cancel{a^3} - abc - (\cancel{a^3} + a^2b + ca^2) =$

$= -(a^2b + abc + ca^2) = -a(ab+bc+ca) =$

(23): $b^3 - abc - (a+b+c)b^2 = \cancel{b^3} - abc - (ab^2 + \cancel{b^3} + b^2c) =$

$= -(ab^2 + b^2c + cba) = -b(ab+bc+ca)$

(33): $c^3 - abc - (a+b+c)c^2 = \cancel{c^3} - abc - (c^2a + bc^2 + \cancel{c^3}) =$

$= -(abc + bc^2 + c^2a) = -c(ab+bc+ca)$

$$\} = \begin{vmatrix} 1 & a^2 & -a(ab+bc+ca) \\ 1 & b^2 & -b(ab+bc+ca) \\ 1 & c^2 & -c(ab+bc+ca) \end{vmatrix} = -(ab+bc+ca) \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} =$$

$$= -(ab+bc+ca)(-1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (ab+bc+ca) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$\square$