$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix} = \left\{ (i\cdot) -= (1\cdot),\ i = \overline{2,n} \right\} =$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1-1 & 0-1 & 1-1 & \cdots & 1-1 \\ 1-1 & 1-1 & 0-1 & \cdots & 1-1 \\ \vdots \\ 1-1 & 1-1 & 1-1 & \cdots & 0-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 1 \\ 0 & 0 & -1 & \cdots & 1 \\ \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix} =$$

$$= 1 \prod_{i=2}^{n} (-1) = 1 \cdot (-1)^{n-2+1} = 1 \cdot (-1)^{n-1} = (-1)^{n-1}$$

$$\begin{vmatrix} a_1 & x & x & \dots & x \\ x & a_2 & x & \dots & x \\ x & x & a_3 & \dots & x \\ \dots \\ x & x & x & \dots & a_n \end{vmatrix} = \left\{ (i\cdot) -= (1\cdot),\ i=\overline{2,n} \right\} =$$

$$= \begin{vmatrix} a_1 & x & x & \dots & x \\ x-a_1 & a_2-x & x-x & \dots & x-x \\ x-a_1 & x-x & a_3-x & \dots & x-x \\ \dots \\ x-a_1 & x-x & x-x & \dots & a_n-x \end{vmatrix} = \begin{vmatrix} a_1 & x & x & \dots & x \\ x-a_1 & a_2-x & 0 & \dots & 0 \\ x-a_1 & 0 & a_3-x & \dots & 0 \\ \dots \\ x-a_1 & 0 & 0 & \dots & a_n-x \end{vmatrix} =$$

$$= \begin{vmatrix} a_1 & x & x & \dots & x \\ -(a_1-x) & a_2-x & 0 & \dots & 0 \\ -(a_1-x) & 0 & a_3-x & \dots & 0 \\ \dots \\ -(a_1-x) & 0 & 0 & \dots & a_n-x \end{vmatrix} = \left\{ (\cdot j) = \frac{(\cdot j)}{a_j-x},\ j=\overline{1,n} \right\} =$$

$$= \prod_{j=1}^{n} (a_j-x) \begin{vmatrix} \frac{a_1}{a_1-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \dots & \frac{x}{a_n-x} \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \dots \\ -1 & 0 & & \dots & 1 \end{vmatrix} = \left\{ (\cdot 1) += \sum_{j=2}^{n} (\cdot j) \right\} =$$

$$= \prod_{k=1}^{n} (a_k - x) \begin{vmatrix} \frac{a_1}{a_1-x} + \sum_{j=2}^{n} \frac{x}{a_j-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ \cancel{-1} + \cancel{1} + \sum_{j=3}^{n} 0 & 1 & 0 & \cdots & 0 \\ \cancel{-1} + 0 + \cancel{1} + \sum_{j=4}^{n} 0 & 0 & 1 & \cdots & 0 \\ \cdots \\ \cancel{-1} + \sum_{j=2}^{n-1} 0 + \cancel{1} & 0 & 0 & \cdots & 1 \end{vmatrix} = \Big\{$$

$$\frac{a_1}{a_1-x} = \frac{(a_1-x)+x}{a_1-x} = 1 + \frac{x}{a_1-x}$$

$$\frac{a_1}{a_1-x} + \sum_{k=2}^{n} \frac{x}{a_k-x} = 1 + \frac{x}{a_1-x} + \sum_{k=2}^{n} \frac{x}{a_k-x} = 1 + \sum_{k=1}^{n} \frac{x}{a_k-x}$$

$$\Big\} = \prod_{k=1}^{n} (a_k - x) \begin{vmatrix} 1 + \sum_{k=1}^{n} \frac{x}{a_k-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} =$$

$$= \prod_{k=1}^{n} (a_k - x) \left(1 + \sum_{k=1}^{n} \frac{x}{a_k-x}\right) \prod_{j=2}^{n} 1 =$$

$$= \prod_{k=1}^{n} (a_k - x) \left(1 + \sum_{k=1}^{n} \frac{x}{a_k-x}\right)$$

-2- once again, but in more detail

$$\det A_n = \begin{vmatrix} a_1 & x & x & \cdots & x \\ x & a_2 & x & \cdots & x \\ x & x & a_3 & \cdots & x \\ \vdots \\ x & x & x & \cdots & x \end{vmatrix} = \{$$

$\hat{a}_{1\cdot} = a_{1\cdot}$; $a_{1\cdot} = \hat{a}_{1\cdot}$

$\forall i \in \{2,3,\dots,n\}$ $\hat{a}_{i\cdot} = a_{i\cdot} - a_{1\cdot}$; $a_{i\cdot} = \hat{a}_{i\cdot} + a_{1\cdot} = \hat{a}_{i\cdot} + \hat{a}_{1\cdot}$

$$\det A_n = \det \begin{pmatrix} a_{1\cdot} \\ a_{2\cdot} \\ \vdots \\ a_{i\cdot} \\ \vdots \\ a_{n\cdot} \end{pmatrix} = \det \begin{pmatrix} \hat{a}_{1\cdot} \\ \hat{a}_{2\cdot} + \hat{a}_{1\cdot} \\ \vdots \\ \hat{a}_{i\cdot} + \hat{a}_{1\cdot} \\ \vdots \\ \hat{a}_{n\cdot} + \hat{a}_{1\cdot} \end{pmatrix} = \det \begin{pmatrix} \hat{a}_{1\cdot} \\ \hat{a}_{2\cdot} \\ \vdots \\ \hat{a}_{i\cdot} \\ \vdots \\ \hat{a}_{n\cdot} \end{pmatrix} = \det \hat{A}_n$$

$$= \begin{vmatrix} a_1 & x & x & \cdots & x \\ x-a_1 & a_2-x & x-x & \cdots & x-x \\ x-a_1 & x-x & a_3-x & \cdots & x-x \\ \vdots & \vdots & \vdots \\ x-a_1 & x-x & x-x & \cdots & x-x \end{vmatrix} = \begin{vmatrix} a_1 & x & x & \cdots & x \\ -(a_1-x) & a_2-x & 0 & \cdots & 0 \\ -(a_1-x) & 0 & a_3-x & \cdots & 0 \\ \vdots \\ -(a_1-x) & 0 & 0 & \cdots & 0 \end{vmatrix} = \{$$

$\forall j \in \{1,2,\dots,n\}$ $\hat{a}_{\cdot j} = \dfrac{a_{\cdot j}}{(a_j - x)}$; $a_{\cdot j} = (a_j - x)\,\hat{a}_{\cdot j}$

$$\det A_n = \det \begin{pmatrix} a_{\cdot 1} & a_{\cdot 2} & \cdots & a_{\cdot j} & \cdots & a_{\cdot n} \end{pmatrix} =$$

$$= \det \begin{pmatrix} (a_1-x)\hat{a}_{\cdot 1} & (a_2-x)\hat{a}_{\cdot 2} & \cdots & (a_j-x)\hat{a}_{\cdot j} & \cdots & (a_n-x)\hat{a}_{\cdot n} \end{pmatrix} =$$

$$= (a_1-x)(a_2-x)\cdots(a_j-x)\dots(a_n-x)\det\begin{pmatrix} \hat{a}_{\cdot 1} & \hat{a}_{\cdot 2} & \cdots & \hat{a}_{\cdot j} & \cdots & \hat{a}_{\cdot n} \end{pmatrix} =$$

$$= \prod_{j=1}^{n} (a_j - x) \det \hat{A}_n$$

$$\Big\} = \prod_{k=1}^{n} (a_j - x) \begin{vmatrix} \frac{a_1}{a_1-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ \frac{-(a_1-x)}{a_1-x} & \frac{a_2-x}{a_2-x} & \frac{0}{a_2-x} & \cdots & \frac{0}{a_n-x} \\ \frac{-(a_1-x)}{a_1-x} & \frac{0}{a_2-x} & \frac{a_3-x}{a_3-x} & \cdots & \frac{0}{a_n-x} \\ \cdots \\ \frac{-(a_1-x)}{a_1-x} & \frac{0}{a_2-x} & \frac{0}{a_3-x} & \cdots & \frac{a_n-x}{a_n-x} \end{vmatrix} =$$

$$= \prod_{k=1}^{n} (a_k - x) \begin{vmatrix} \frac{a_1}{a_1-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \cdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} = \Big\{$$

$\forall j \in \{2,3,\dots,n\}\ \hat{a}_{\cdot j} = a_{\cdot j}\ ;\ a_{\cdot j} = \hat{a}_{\cdot j}$

$j = 1\quad \hat{a}_{\cdot j} = a_{\cdot 1} - \sum_{j=2} a_{\cdot j}\ ;\ a_{\cdot 1} = \hat{a}_{\cdot 1} - \sum_{j=2}^{n} a_{\cdot j} = \hat{a}_{\cdot 1} - \sum_{j=2}^{n} \hat{a}_{\cdot j} =$

$\det A_n = \det\left(a_{\cdot 1}\ \ a_{\cdot 2}\ \cdots\ a_{\cdot n} \right) =$

$= \det\left(\hat{a}_{\cdot 1} - \sum_{j=2}^{n} \hat{a}_{\cdot j}\ \ \hat{a}_{\cdot 2}\ \cdots\ \hat{a}_{\cdot n} \right) = \det\left(\hat{a}_{\cdot 1}\ \hat{a}_{\cdot 2} \cdots \hat{a}_{\cdot n} \right) = \det \hat{A}_n$

$$\Big\} = \prod_{k=1}^{n} (a_k - x) \begin{vmatrix} \frac{a_1}{a_1-x} + \frac{x}{a_2-x} + \frac{x}{a_3-x} + \dots + \frac{x}{a_n-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ \cancel{-1} + \cancel{1} + 0 + \dots + 0 & 1 & 0 & \cdots & 0 \\ \cancel{-1} + 0 + \cancel{1} + \dots + 0 & 0 & 1 & \cdots & 0 \\ \cdots \\ \cancel{-1} + 0 + 0 + \dots + \cancel{1} & & & \cdots & 1 \end{vmatrix} = \Big[$$

$$\frac{a_1}{a_1-x}+\frac{x}{a_2-x}+\frac{x}{a_3-x}+\dots+\frac{x}{a_n-x}=\frac{a_1}{a_1-x}+\sum_{j=2}^{n}\frac{x}{a_j-x}=$$

$$=\left\{\frac{a_1}{a_1-x}=\frac{(a_1-x)+x}{a_1-x}=1+\frac{x}{a_1-x}\right\}=1+\frac{x}{a_1-x}+\sum_{j=2}^{n}\frac{x}{a_j-x}=$$

$$=1+\sum_{j=1}^{n}\frac{x}{a_j-x}$$

$$\Big\}=\prod_{k=1}^{n}(a_k-x)\begin{vmatrix} 1+\sum_{k=1}^{n}\frac{x}{a_k-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}=$$

$$=\prod_{k=1}^{n}(a_k-x)\left(\left(1+\sum_{k=1}^{n}\frac{x}{a_k-x}\right)\prod_{k=2}^{n}1\right)=$$

$$=\prod_{k=1}^{n}(a_k-x)\left(1+\sum_{k=1}^{n}\frac{x}{a_k-x}\right)$$