– 56 –

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} =$$

$$= acb + bac + cba - ccc - bbb - aaa =$$
$$= abc + abc + abc - c^3 - b^3 - a^3 =$$
$$= 3abc - (a^3 + b^3 + c^3)$$

– 57 –

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} =$$

$$= aaa + bbb + ccc - cab - bca - abc =$$
$$= a^3 + b^3 + c^3 - abc - abc - abc =$$
$$= a^3 + b^3 + c^3 - 3abc$$

– 58 –

$$\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix} = 0c0 + ad0 + 0be - 0c0 - ab0 - 0de = 0$$

– 59 –

$$\begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix} =$$

$$= abc + xxx + xxx - xbx - xxc - axx =$$
$$= abc + x^3 + x^3 - bx^2 - cx^2 - ax^2 =$$
$$= 2x^3 - (a+b+c)x^2 + abc$$

$$\begin{vmatrix} a+x & x & x \\ x & b+x & x \\ x & x & c+x \end{vmatrix} =$$

$$= (a+x)(b+x)(c+x) + xxx + xxx -$$

$$- x(b+x)x - xx(c+x) - (a+x)xx =$$

$$= (a+x)(b+x)(c+x) + x^3 + x^3 - \left[x^2(b+x) + x^2(c+x) + x^2(a+x)\right] =$$

$$= (x+a)(x+b)(x+c) + 2x^3 - x^2(3x+a+b+c) = \{$$

$$(x+a)(x+b)(x+c) = (xx + xb + ax + ab)(x+c) = (x^2 + ax + bx + ab)(x+c) =$$

$$= x^2x + axx + bxx + abx + x^2c + axc + bxc + abc =$$

$$= x^3 + ax^2 + bx^2 + abx + cx^2 + cax + bcx + abc =$$

$$= x^3 + ax^2 + bx^2 + cx^2 + abx + bcx + cax + abc =$$

$$= x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$$

$$\} = \left[x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc\right] + 2x^3 -$$

$$- x^2(3x + a + b + c) =$$

$$= \left[\cancel{3x^3} + \cancel{(a+b+c)x^2} + (ab+bc+ca)x + abc\right] -$$

$$- \left[\cancel{3x^3} + \cancel{(a+b+c)x^2}\right] = (ab+bc+ca)x + abc$$

$$\begin{vmatrix} \alpha^2+1 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2+1 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2+1 \end{vmatrix} =$$

$$= (\alpha^2+1)(\beta^2+1)(\gamma^2+1) + (\alpha\beta)(\beta\gamma)(\alpha\gamma) + (\alpha\gamma)(\alpha\beta)(\beta\gamma) -$$

$$- (\alpha\gamma)(\beta^2+1)(\alpha\gamma) - (\alpha\beta)(\alpha\beta)(\gamma^2+1) - (\alpha^2+1)(\beta\gamma)(\beta\gamma) =$$

$$= (\alpha^2+1)(\beta^2+1)(\gamma^2+1) + (\alpha\beta\gamma)^2 + (\alpha\beta\gamma)^2 -$$

$$- \left[(\alpha\gamma)^2(\beta^2+1) + (\alpha\beta)^2(\gamma^2+1) + (\beta\gamma)^2(\alpha^2+1)\right] = \Big\{$$

$$(\alpha^2+1)(\beta^2+1)(\gamma^2+1) = (\alpha^2\beta^2 + \alpha^2 + \beta^2 + 1)(\gamma^2+1) =$$

$$= \alpha^2\beta^2\gamma^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \gamma^2 + \alpha^2\beta^2 + \alpha^2 + \beta^2 + 1 =$$

$$= (\alpha\beta\gamma)^2 + (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + \alpha^2 + \beta^2 + \gamma^2 + 1$$

$$(\alpha\gamma)^2(\beta^2+1) + (\alpha\beta)^2(\gamma^2+1) + (\beta\gamma)^2(\alpha^2+1) =$$

$$= (\alpha\gamma)^2\beta^2 + (\alpha\gamma)^2 + (\alpha\beta)^2\gamma^2 + (\alpha\beta)^2 + (\beta\gamma)^2\alpha^2 + (\beta\gamma)^2 =$$

$$= 3(\alpha\beta\gamma)^2 + (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$$

$$\Big\} = \left[\cancel{(\alpha\beta\gamma)^2} + \cancel{(\alpha\beta)^2} + \cancel{(\beta\gamma)^2} + \cancel{(\gamma\alpha)^2} + \alpha^2 + \beta^2 + \gamma^2 + 1\right] + \cancel{2(\alpha\beta\gamma)^2} -$$

$$- \left[\cancel{3(\alpha\beta\gamma)^2} + \cancel{(\alpha\beta)^2} + \cancel{(\beta\gamma)^2} + \cancel{(\gamma\alpha)^2}\right] =$$

$$= 1 + \alpha^2 + \beta^2 + \gamma^2$$

$$\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ 0 & -\sin\beta & \cos\beta \end{vmatrix} =$$

$$= \cos\alpha(\cos\alpha\cos\beta)\cos\beta + (\sin\alpha\cos\beta)(\cos\alpha\sin\beta)0 + (\sin\alpha\sin\beta)(-\sin\alpha)(-\sin\beta) -$$

$$- (\sin\alpha\sin\beta)(\cos\alpha\cos\beta)0 - (\sin\alpha\cos\beta)(-\sin\alpha)\cos\beta - \cos\alpha(\cos\alpha\sin\beta)(-\sin\beta) =$$

$$= \cos^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta =$$

$$= (\sin^2\alpha\sin^2\beta + \cos^2\alpha\sin^2\beta) + (\sin^2\alpha\cos^2\beta + \cos^2\alpha\cos^2\beta) =$$

$$= \sin^2\beta(\sin^2\alpha + \cos^2\alpha) + \cos^2\beta(\sin^2\alpha + \cos^2\alpha) =$$

$$= \{\sin^2\alpha + \cos^2\alpha = 1\} = \sin^2\beta + \cos^2\beta = 1$$

-63-

$$\begin{vmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{vmatrix} =$$

$$= \sin\alpha\cos\beta\cdot 1 + \cos\alpha\cdot 1\cdot\sin\gamma + 1\cdot\sin\beta\cos\gamma -$$
$$- 1\cos\beta\sin\gamma - \cos\alpha\sin\beta\cdot 1 - \sin\alpha\cdot 1\cdot\cos\gamma =$$

$$= \sin\alpha\cos\beta + \sin\gamma\cos\alpha + \sin\beta\cos\gamma -$$
$$- \cos\beta\sin\gamma - \cos\alpha\sin\beta - \cos\gamma\sin\alpha =$$

$$= (\sin\alpha\cos\beta - \cos\alpha\sin\beta) +$$
$$+ (\sin\beta\cos\gamma - \cos\beta\sin\gamma) +$$
$$+ (\sin\gamma\cos\alpha - \cos\gamma\sin\alpha) = \{$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\} = \sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha) =$$