prove:

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

proof:

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \{ (\cdot 2) \mathrel{-}= (\cdot 1) \,;\, (\cdot 3) \mathrel{-}= (\cdot 1) \} \doteq$$

$$= \begin{vmatrix} 1 & 1-1 & 1-1 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} = \{$$

$$b^2-a^2 = (b-a)(b+a) \,;\, (b^3-a^3) = (b-a)(b^2+ba+a^2)$$

$$c^2-a^2 = (c-a)(c+a) \,;\, c^3-a^3 = (c-a)(c^2+ca+a^2)$$

$$\} = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \\ a^3 & (b-a)(b^2+ba+a^2) & (c-a)(c^2+ca+a^2) \end{vmatrix} = \left[\begin{matrix} (\cdot 2) = (b-a)^{-1}(\cdot 2) \\ (\cdot 3) = (c-a)^{-1}(\cdot 3) \end{matrix} \right] =$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b+a & c+a \\ a^3 & b^2+ba+a^2 & c^2+ca+a^2 \end{vmatrix} = \{ (\cdot 2) \mathrel{-}= (\cdot 3) \} =$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0-0 & 0 \\ a^2 & (b+a)-(c+a) & c+a \\ a^3 & (b^2+ba+a^2)-(c^2+ca+a^2) & c^2+ca+a^2 \end{vmatrix} = \{$$

$$(b+a)-(c+a)=b-c$$

$$(b^2+ba+a^2)-(c^2+ca+a^2)=(b^2+ba)-(c^2+ca)=$$

$$=(b^2-c^2)+(ba-ca)=(b-c)(b+c)+a(b-c)=(b-c)(b+c+a)=$$

$$=(b-c)(a+b+c)$$

$$\Big\} = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & b-c & c+a \\ a^3 & (b-c)(a+b+c) & a^2+ca+a^2 \end{vmatrix} = \Big\{ (\cdot 2) = (b-c)^{-1}(2) \Big\} =$$

$$=(b-a)(c-a)(b-c)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & 1 & c+a \\ a^3 & a+b+c & c^2+ca+a^2 \end{vmatrix} = \Big\{ (\cdot 3) - = (c+a)(\cdot 3) \Big\} =$$

$$=(b-a)(c-a)(b-c)\begin{vmatrix} 1 & 0 & 0-(c+a)\cdot 0 \\ a^2 & 1 & (c+a)-(c+a)\cdot 1 \\ a^3 & a+b+c & (c^2+ca+a^2)-(c+a)(a+b+c) \end{vmatrix} = \Big\{$$

$$(c+a)(a+b+c)=ca+cb+c^2+a^2+ab+ac=a^2+c^2+ab+bc+2ca$$

$$(c^2+ca+a^2)-(c+a)(a+b+c)=(a^2+c^2+ca)-(a^2+c^2+ab+bc+2ca)=$$

$$=-(ab+bc+ca)$$

$$\Big\} = (b-a)(c-a)(b-c)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & 1 & 0 \\ a^3 & a+b+c & -(ab+bc+ca) \end{vmatrix} = \Big\{$$

$$\begin{vmatrix} \lambda & 0 & 0 \\ * & \lambda_2 & 0 \\ * & * & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3 = \{\lambda_1=\lambda_2=1\} = 1\cdot 1\cdot\lambda_3 = \lambda_3$$

$$\Big\} = (b-a)(c-a)(b-c)(-1)(ab+bc+ca) = (a-b)(b-c)(c-a)(ab+bc+ca)$$

prove:

$$\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$$

proof:

$$\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = \{(2.) -= (1.);\ (3.) -= (1.)\} =$$

$$= \begin{vmatrix} 1 & a & a^4 \\ 1-1 & b-a & b^4-a^4 \\ 1-1 & c-a & c^4-a^4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^4 \\ 0 & b-a & b^4-a^4 \\ 0 & c-a & c^4-a^4 \end{vmatrix} = \{$$

$$b^4-a^4 = (b^2)^2-(a^2)^2 = (b^2-a^2)(b^2+a^2) = (b-a)(b+a)(b^2+a^2)$$

$$c^4-a^4 = (c^2)^2-(a^2)^2 = (c^2-a^2)(c^2+a^2) = (c-a)(c+a)(c^2+a^2)$$

$$\} = \begin{vmatrix} 1 & a & a^4 \\ 0 & b-a & (b-a)(b+a)(b^2+a^2) \\ 0 & c-a & (c-a)(c+a)(c^2+a^2) \end{vmatrix} = \{(2.) = (b-a)^{-1}(2.);\ (3.) = (c-a)^{-1}(3.)\} =$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^4 \\ 0 & 1 & (b+a)(b^2+a^2) \\ 0 & 1 & (c+a)(c^2+a^2) \end{vmatrix} = \{(3.) -= (2.)\} =$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^4 \\ 0 & 1 & (b+a)(b^2+a^2) \\ 0-0 & 1-1 & (c+a)(c^2+a^2)-(b+a)(b^2+a^2) \end{vmatrix} = \{$$

$$(c+a)(c^2+a^2)-(b+a)(b^2+a^2)=$$

$$=\left(c(c^2+a^2)+a(c^2+a^2)\right)-\left(b(b^2+a^2)+a(b^2+a^2)\right)=$$

$$=c(c^2+a^2)-b(b^2+a^2)+a(c^2+\cancel{a^2})-a(b^2+\cancel{a^2})=$$

$$=c^3-b^3+a^2(c-b)+a(c^2-b^2)=$$

$$=(c-b)(c^2+cb+b^2)+a^2(c-b)+a(c-b)(c+b)=$$

$$=(c-b)\left(c^2+cb+b^2+a^2+a(c+b)\right)=(c-b)(a^2+b^2+c^2+ab+bc+ca)$$

$$\Big\}=(b-a)(c-a)\begin{vmatrix} 1 & a & a^4 \\ 0 & 1 & (b+a)(b^2+a^2) \\ 0 & 0 & (c-b)(a^2+b^2+c^2+ab+bc+ca) \end{vmatrix}=\Big\{$$

$$\begin{vmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{vmatrix}=\lambda_1\lambda_2\lambda_3=\{\lambda_1=\lambda_2=1\}=\lambda_3$$

$$\Big\}=(b-a)(c-a)(c-b)(a^2+b^2+c^2+ab+bc+ca)=\Big\{$$

$$(b-a)(c-a)(c-b)=(-1)(a-b)(-1)(b-c)(c-a)=(a-b)(b-c)(c-a)$$

$$\Big\}=(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$$

$$A = \begin{pmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{pmatrix} ; \quad B = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

prove: $\det A = \det B$

proof:

$$A = \begin{pmatrix} a_{\cdot 1} & a_{\cdot 2} & a_{\cdot 3} \end{pmatrix} ; \quad a_{\cdot 1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; \quad a_{\cdot 2} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} ; \quad a_{\cdot 3} = \begin{pmatrix} bc \\ ca \\ ab \end{pmatrix}$$

$$B = \begin{pmatrix} b_{\cdot 1} & b_{\cdot 2} & b_{\cdot 3} \end{pmatrix} ; \quad b_{\cdot 1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; \quad b_{\cdot 2} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} ; \quad b_{\cdot 3} = \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix}$$

$a_{\cdot 1} = b_{\cdot 1}$; $a_{\cdot 2} = b_{\cdot 2}$

$$\det A = \det \begin{pmatrix} a_{\cdot 1} & a_{\cdot 2} & a_{\cdot 3} \end{pmatrix} = \left[\forall \alpha \wedge \forall \beta \right] =$$

$$= \det \begin{pmatrix} a_{\cdot 1} & a_{\cdot 2} & a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} \end{pmatrix} =$$

$$= \det B = \det \begin{pmatrix} b_{\cdot 1} & b_{\cdot 2} & b_{\cdot 3} \end{pmatrix} \Leftrightarrow$$

$$a_{\cdot 1} = b_{\cdot 1} \wedge a_{\cdot 2} = b_{\cdot 2} \wedge a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} = b_{\cdot 3}$$

$$a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} = b_{\cdot 3}$$

$$a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2} = \begin{pmatrix} bc \\ ca \\ ab \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} a \\ b \\ c \end{pmatrix} =$$

$$= \begin{pmatrix} bc + \alpha \cdot 1 + \beta \cdot a \\ ca + \alpha \cdot 1 + \beta \cdot b \\ ab + \alpha \cdot 1 + \beta \cdot c \end{pmatrix} = \begin{pmatrix} \alpha + a\beta + bc \\ \alpha + b\beta + ca \\ \alpha + c\beta + ab \end{pmatrix} =$$

$$= b_{\cdot 3} = \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} \alpha + a\beta + bc = a^2 \\ \alpha + b\beta + ca = b^2 \\ \alpha + c\beta + ab = c^2 \end{cases} \iff \begin{cases} \alpha + a\beta + bc = a^2 \\ \alpha + b\beta + ca = b^2 \\ \alpha + c\beta + ab = c^2 \end{cases}$$

$$\begin{cases} \alpha + a\beta + bc = a^2 \\ \alpha + b\beta + ca = b^2 \end{cases} \iff$$

$$\begin{cases} (\cancel{\alpha} + a\beta + bc) - (\cancel{\alpha} + b\beta + ca) = a^2 - b^2 \\ b(\alpha + \cancel{a\beta} + bc) - a(\alpha + \cancel{b\beta} + ca) = ba^2 - ab^2 \end{cases} \iff$$

$$\begin{cases} (a-b)\beta + c(b-a) = a^2 - b^2 \\ (b-a)\alpha + c(b^2 - a^2) = ab(a-b) \end{cases} \iff$$

$$\begin{cases} (a-b)\beta = a^2 - b^2 + c(a-b) = (a-b)(a+b) + c(a-b) = (a-b)(a+b+c) \\ -(a-b)\alpha = ab(a-b) + c(a^2 - b^2) = ab(a-b) + c(a-b)(a+b) = (a-b)(ab + c(a+b)) = \\ \qquad = (a-b)(ab + bc + ca) \end{cases}$$

$$\iff \begin{cases} \beta = a+b+c \\ -\alpha = ab + bc + ca \end{cases} \iff \begin{cases} \alpha = -(ab + bc + ca) \\ \beta = a+b+c \end{cases}$$

$$\alpha + c\beta + ab = -(ab + bc + ca) + c(a+b+c) + ab =$$

$$= -(ab + bc + ca) + (ab + bc + ca) + c^2 = c^2$$

$$\det A = \det(a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3}) = \{\forall \alpha \wedge \forall \beta\} = \det(a_{\cdot 1}\ a_{\cdot 2}\ a_{\cdot 3} + \alpha a_{\cdot 1} + \beta a_{\cdot 2}) =$$

$$= \{\alpha = -(ab + bc + ca) \wedge \beta = a+b+c\} = \det(b_{\cdot 1}\ b_{\cdot 2}\ b_{\cdot 3}) = \det B$$

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prove:

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

proof:

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \left\{ (\cdot 3) += -(ab+bc+ca)(\cdot 1) + (a+b+c)(\cdot 2) \right\} =$$

$$= \begin{vmatrix} 1 & a & bc - (ab+bc+ca)\cdot 1 + (a+b+c)a \\ 1 & b & ca - (ab+bc+ca)\cdot 1 + (a+b+c)b \\ 1 & c & ab - (ab+bc+ca)\cdot 1 + (a+b+c)c \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & a & bc - (ab+bc+ca) + a^2 + ab + ca \\ 1 & b & ca - (ab+bc+ca) + ab + b^2 + cb \\ 1 & c & ab - (ab+bc+ca) + ac + bc + c^2 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & a & a^2 + \cancel{(ab+bc+ca)} - \cancel{(ab+bc+ca)} \\ 1 & b & b^2 + \cancel{(ab+bc+ca)} - \cancel{(ab+bc+ca)} \\ 1 & c & c^2 + \cancel{(ab+bc+ca)} - \cancel{(ab+bc+ca)} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$