$A \in \mathbb{R}^{3\times 3} \Rightarrow \det A = \det A^T$

$\triangleright\ A = (a_{ij}) \in \mathbb{R}^{3\times 3}$.

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} -$$
$$- a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{32} =$$

$$= a^t_{11} a^t_{22} a^t_{33} + a^t_{21} a^t_{32} a^t_{13} + a^t_{31} a^t_{12} a^t_{23} -$$
$$- a^t_{31} a^t_{22} a^t_{13} - a^t_{21} a^t_{12} a^t_{33} - a^t_{11} a^t_{32} a^t_{23} =$$

$$= a^t_{11} a^t_{22} a^t_{33} + a^t_{13} a^t_{21} a^t_{32} + a^t_{12} a^t_{23} a^t_{31} -$$
$$- a^t_{13} a^t_{22} a^t_{31} - a^t_{12} a^t_{21} a^t_{33} - a^t_{11} a^t_{23} a^t_{32} =$$

$$= a^t_{11} a^t_{22} a^t_{33} + a^t_{12} a^t_{23} a^t_{31} + a^t_{13} a^t_{21} a^t_{32} -$$
$$- a^t_{13} a^t_{22} a^t_{31} - a^t_{12} a^t_{21} a^t_{33} - a^t_{11} a^t_{23} a^t_{32} =$$

$$= \det \begin{pmatrix} a^t_{11} & a^t_{12} & a^t_{13} \\ a^t_{21} & a^t_{22} & a^t_{23} \\ a^t_{31} & a^t_{32} & a^t_{33} \end{pmatrix} = \det A_3^T$$

$A \in \mathbb{R}^{3\times 3} \Rightarrow \det A = \det A^T$

$\square$

- q1 - proof for rows

$A=(a_{ij})\in\mathbb{R}^{3\times3}$ $\exists i\in\{1,2,3\}$ $\forall j\in\{1,2,3\}$ $a_{ij}=0$ $\Rightarrow$ $\det A=0$

D

$A=(a_{ij})\in\mathbb{R}^{3\times3}$

$$\det A=\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{32} & a_{33} \end{pmatrix}=$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-$$
$$-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{33}$$

$\exists i=1$ $\forall j\in\{1,2,3\}$ $a_{ij}=a_{1j}=0$ $\Rightarrow$

$$\Rightarrow \det A=0\,a_{22}a_{33}+0\,a_{23}a_{31}+0\,a_{21}a_{32}-$$
$$-0\,a_{22}a_{31}-0\,a_{21}a_{33}-0\,a_{23}a_{33}=0$$

$\exists i=2$ $\forall j\in\{1,2,3\}$ $a_{ij}=a_{2j}=0$ $\Rightarrow$

$$\Rightarrow \det A=a_{11}\,0\,a_{33}+a_{12}\,0\,a_{31}+a_{13}\,0\,a_{32}-$$
$$-a_{13}\,0\,a_{31}-a_{12}\,0\,a_{33}-a_{11}\,0\,a_{33}=0$$

$\exists i=3$ $j\in\{1,2,3\}$ $a_{ij}=a_{3j}=0$ $\Rightarrow$

$$\Rightarrow \det A=a_{11}a_{22}\,0+a_{12}a_{23}\,0+a_{13}a_{21}\,0-$$
$$-a_{13}a_{22}\,0-a_{12}a_{21}\,0-a_{11}a_{23}\,0=0$$

$A=(a_{ij})\in\mathbb{R}^{3\times3}$ $\exists i\in\{1,2,3\}$ $\forall j\in\{1,2,3\}$ $a_{ij}=0$ $\Rightarrow$ $\det A$

$\square$

- g1 - proof for columns

$A=(a_{ij})\in\mathbb{R}^{3\times3}$ $\exists j\in\{1,2,3\}\ \forall i\in\{1,2,3\}\ a_{ij}=0 \Rightarrow \det A=0$

D

$A=(a_{ij})\in\mathbb{R}^{3\times3}$

$$\det A = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$\exists j=1\ \forall i\in\{1,2,3\}\ a_{ij}=a_{i1}=0 \Rightarrow$

$$\Rightarrow \det A = 0\,a_{22}a_{33} + a_{12}a_{23}0 + a_{13}0\,a_{32} -$$

$$- a_{13}a_{22}0 - a_{12}0\,a_{33} - 0\,a_{23}a_{32} = 0$$

$\exists j=2\ \forall i\in\{1,2,3\}\ a_{ij}=a_{i2}=0 \Rightarrow$

$$\Rightarrow \det A = a_{11}0\,a_{33} + 0\,a_{23}a_{31} + a_{13}a_{21}0 -$$

$$- a_{13}0\,a_{31} - 0\,a_{21}a_{33} - a_{11}a_{23}0 = 0$$

$\exists j=3\ \forall i\in\{1,2,3\}\ a_{ij}=a_{i3}=0 \Rightarrow$

$$\Rightarrow \det A = a_{11}a_{22}0 + a_{12}0\,a_{31} + 0\,a_{21}a_{32} -$$

$$- 0\,a_{22}a_{31} - a_{12}a_{21}0 - a_{11}0\,a_{32} = 0$$

$A=(a_{ij})\in\mathbb{R}^{3\times3}$ $\exists j\in\{1,2,3\}\ \forall i\in\{1,2,3\} \Rightarrow \det A=0$

$\square$

-92- proof for rows

$A=(a_{ij})\in\mathbb{R}^{3\times 3}$ $B=(b_{ij})\in\mathbb{R}^{3\times 3}$ $\alpha\in\mathbb{R}$

$\exists z\in\{1,2,3\}\ (\forall j\in\{1,2,3\}\ a_{zj}=\alpha b_{zj})\wedge(\forall i\in\{1,2,3\}\setminus\{z\}\ \forall j\in\{1,2,3\}\ a_{ij}=b_{ij})\Rightarrow$

$\Rightarrow \det A=\alpha\det B$

$\triangleright$

$A=(a_{ij})\in\mathbb{R}^{3\times 3}$

$$\det A=\det\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-$$
$$-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}=$$

$B=(b_{ij})\in\mathbb{R}^{3\times 3}$

$$\det B=\det\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}=b_{11}b_{22}b_{33}+b_{12}b_{23}b_{31}+b_{13}b_{21}b_{32}-$$
$$-b_{13}b_{22}b_{31}-b_{12}b_{21}b_{33}-b_{11}b_{23}b_{32}=$$

$\exists z=1$

$(\forall j\in\{1,2,3\}\ a_{zj}=\alpha b_{zj}\Leftrightarrow a_{1j}=\alpha b_{1j})\wedge$

$(\forall i\in\{1,2,3\}\setminus\{z\}=\{2,3\}\ \forall j\in\{1,2,3\}\ a_{ij}=b_{ij})\Rightarrow$

$$\det A=(\alpha b_{11})b_{22}b_{33}+(\alpha b_{12})b_{23}b_{31}+(\alpha b_{13})b_{21}b_{32}-$$
$$-(\alpha b_{13})b_{22}b_{31}-(\alpha b_{12})b_{21}b_{33}-(\alpha b_{11})b_{23}b_{32}=$$
$$=\alpha\Big(b_{11}b_{22}b_{33}+b_{12}b_{23}b_{31}+b_{13}b_{21}b_{32}-$$
$$-b_{13}b_{22}b_{31}-b_{12}b_{21}b_{33}-b_{11}b_{23}b_{32}\Big)=$$
$$=\alpha\det B$$

$\exists\ r = 2$

$\left(\forall j \in \{1,2,3\} \quad a_{2j} = \alpha b_{2j} \Leftrightarrow a_{2j} = \alpha b_{2j}\right) \wedge$

$\wedge \left(\forall i \in \{1,2,3\} \setminus \{2\} = \{1,3\} \quad \forall j \in \{1,2,3\} \quad a_{ij} = b_{ij}\right) \Rightarrow$

$\Rightarrow \det A = b_{11} (\alpha b_{22}) b_{33} + b_{12} (\alpha b_{23}) b_{31} + b_{13} (\alpha b_{21}) b_{32} -$

$- b_{13} (\alpha b_{22}) b_{31} - b_{12} (\alpha b_{21}) b_{23} - b_{11} (\alpha b_{23}) b_{32} =$

$= \alpha \det B$

$\exists\ r = 3$

$\left(\forall j \in \{1,2,3\} \quad a_{2j} = \alpha b_{2j} \Leftrightarrow a_{3j} = \alpha b_{3j}\right) \wedge$

$\wedge \left(\forall i \in \{1,2,3\} \setminus \{2\} = \{1,2\} \quad \forall j \in \{1,2,3\} \quad a_{ij} = b_{ij}\right) \Rightarrow$

$\Rightarrow \det A = b_{11} b_{22} (\alpha b_{33}) + b_{12} b_{23} (\alpha b_{31}) + b_{13} b_{21} (\alpha b_{32}) -$

$- b_{13} b_{22} (\alpha b_{31}) - b_{12} b_{21} (\alpha b_{32}) - b_{11} b_{23} (\alpha b_{32}) =$

$= \alpha \det B$

$A = (a_{ij}) \in \mathbb{R}^{3 \times 3} \quad B = (b_{ij}) \in \mathbb{R}^{3 \times 3} \quad \alpha \in \mathbb{R}$

$\exists\ r \in \{1,2,3\} \quad \left(\forall j \in \{1,2,3\} \quad a_{rj} = \alpha b_{rj}\right) \wedge \left(\forall i \in \{1,2,3\} \setminus \{r\} \quad \forall j \in \{1,2,3\} \quad a_{ij} = b_{ij}\right) \Rightarrow$

$\Rightarrow \det A = \alpha \det B$

$\square$

-92- proof for columns

$A=(a_{ij})\in\mathbb{R}^{3\times3}$ $B=(b_{ij})\in\mathbb{R}^{3\times3}$

$j_1\in\{1,2,3\}$ $\lambda\in\mathbb{R}$

$\left(\forall i\in\{1,2,3\}\ a_{ij_1}=\lambda b_{ij_1}\right)\wedge\left(\forall i\in\{1,2,3\}\ \forall j\in\{1,2,3\}\setminus\{j_1\}\ a_{ij}=b_{ij}\right)$

$\Rightarrow \det A=\lambda\det B$ $\triangleright$

$A=(a_{ij})\in\mathbb{R}^{3\times3}$ $B=(b_{ij})\in\mathbb{R}^{3\times3}$ $\lambda\in\mathbb{R}$

$j_1=1\in\{1,2,3\}$

$\left(\forall i\in\{1,2,3\}\ \left(a_{ij_1}=\lambda b_{ij_1}\Leftrightarrow a_{i1}=\lambda b_{i1}\right)\right)\wedge$

$\wedge\left(\forall i\in\{1,2,3\}\ \forall j\in\{1,2,3\}\setminus\{j_1\}=\{2,3\}\ a_{ij}=b_{ij}\right)$

$$\det A=\det\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}=$$

$$=(\lambda b_{11})b_{22}b_{33}+b_{12}b_{23}(\lambda b_{31})+b_{13}(\lambda b_{21})b_{32}-b_{13}b_{22}(\lambda b_{31})-b_{12}(\lambda b_{21})b_{33}-(\lambda b_{11})b_{23}b_{32}=$$

$$=\lambda\left(b_{11}b_{22}b_{33}+b_{12}b_{23}b_{31}+b_{13}b_{21}b_{32}-b_{13}b_{22}b_{31}-b_{12}b_{21}b_{33}-b_{11}b_{23}b_{32}\right)=$$

$$=\lambda\det\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}=\lambda\det B$$

$j_1 = 2 \in \{1,2,3\}$

$$\left(\forall i \in \{1,2,3\} \ \left(a_{ij_1} = \lambda b_{ij_1} \Leftrightarrow a_{i2} = \lambda b_{i2}\right)\right) \wedge$$
$$\wedge \left(\forall i \in \{1,2,3\} \ \forall j \in \{1,2,3\} \setminus \{j_1\} = \{1,3\} \quad a_{ij} = b_{ij}\right)$$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} =$$
$$= b_{11}(\lambda b_{22})b_{33} + (\lambda b_{12})b_{23}b_{31} + b_{13}b_{21}(\lambda b_{32}) - b_{13}(\lambda b_{22})b_{31} - (\lambda b_{12})b_{21}b_{33} - b_{11}b_{23}(\lambda b_{32}) =$$
$$= \lambda(b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32}) =$$
$$= \lambda \det B$$

$j_1 = 3 \in \{1,2,3\}$

$$\left(\forall i \in \{1,2,3\} \ \left(a_{ij_1} = \lambda b_{ij_1} \Leftrightarrow a_{i3} = \lambda b_{i3}\right)\right) \wedge$$
$$\wedge \left(\forall i \in \{1,2,3\} \ \forall j \in \{1,2,3\} \setminus \{j_1\} = \{1,2\} \quad a_{ij} = b_{ij}\right)$$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} =$$
$$= b_{11}b_{22}(\lambda b_{33}) + b_{12}(\lambda b_{23})b_{31} + (\lambda b_{13})b_{21}b_{32} - (\lambda b_{13})b_{22}b_{31} - b_{12}b_{21}(\lambda b_{33}) - b_{11}(\lambda b_{23})b_{32} =$$
$$= \lambda\left(b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32}\right) =$$
$$= \lambda \det B$$

$A = (a_{ij}) \in \mathbb{R}^{3\times 3} \quad B = (b_{ij}) \in \mathbb{R}^{3\times 3}$

$j_1 \in \{1,2,3\} \qquad \lambda \in \mathbb{R}$

$$\left(\forall i \in \{1,2,3\} \ a_{ij_1} = \lambda b_{ij_1}\right) \wedge \left(\forall i \in \{1,2,3\} \ \forall j \in \{1,2,3\} \setminus \{j_1\} \ a_{ij} = b_{ij}\right)$$
$$\Rightarrow \det A = \lambda \det B$$

$\square$

- g2 - proof for rows

$A = (a_{ij}) \in \mathbb{R}^{3\times3}$ $B = (b_{ij}) \in \mathbb{R}^{3\times3}$

$i_1 \in \{1,2,3\}$ $\lambda \in \mathbb{R}$

$(\forall j \in \{1,2,3\}\ a_{i_1j} = \lambda b_{i_1j}) \wedge (\forall i \in \{1,2,3\} \setminus \{i_1\}\ \forall j \in \{1,2,3\}\ a_{ij} = b_{ij})$

$\Rightarrow \det A = \lambda \det B$ $\triangleright$

$A = (a_{ij}) \in \mathbb{R}^{3\times3}$ $B = (b_{ij}) \in \mathbb{R}^{3\times3}$ $\lambda \in \mathbb{R}$

$i_1 = 1 \in \{1,2,3\}$

$(\forall j \in \{1,2,3\}\ (a_{i_1j} = \lambda b_{i_1j} \Leftrightarrow a_{1j} = \lambda b_{1j})) \wedge$

$\wedge (\forall i \in \{1,2,3\} \setminus \{i_1\} = \{2,3\}\ \forall j \in \{1,2,3\}\ a_{ij} = b_{ij})$

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{31} =$$

$$= (\lambda b_{11})b_{22}b_{33} + (\lambda b_{12})b_{23}b_{31} + (\lambda b_{13})b_{21}b_{32} - (\lambda b_{13})b_{22}b_{31} - (\lambda b_{12})b_{21}b_{32} - (\lambda b_{11})b_{23}b_{31} =$$

$$= \lambda(b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{32} - b_{11}b_{23}b_{31}) =$$

$$= \lambda \det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \lambda \det B$$

$i_1 = 2 \in \{1,2,3\}$

$$\left(\forall j \in \{1,2,3\} \quad \left(a_{i_1 j} = \lambda b_{i_1 j} \Leftrightarrow a_{2j} = \lambda b_{2j}\right)\right) \land$$

$$\land \left(\forall i \in \{1,2,3\} \setminus \{i_1\} = \{1,3\} \quad \forall j \in \{1,2,3\} \quad a_{ij} = b_{ij}\right)$$

$$\det A = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{31} =$$

$$= b_{11} (\lambda b_{22}) b_{33} + b_{12} (\lambda b_{23}) b_{31} + b_{13} (\lambda b_{21}) b_{32} - b_{13} (\lambda b_{22}) b_{31} - b_{12} (\lambda b_{21}) b_{33} - b_{11} (\lambda b_{23}) b_{31} =$$

$$= \lambda \left(b_{11} b_{22} b_{33} + b_{12} b_{23} b_{31} + b_{13} b_{21} b_{32} - b_{13} b_{22} b_{31} - b_{12} b_{21} b_{23} - b_{11} b_{23} b_{31}\right) =$$

$$= \lambda \det B$$

$i_1 = 3 \in \{1,2,3\}$

$$\left(\forall j \in \{1,2,3\} \quad \left(a_{i_1 j} = \lambda b_{i_1 j} \Leftrightarrow a_{3j} = \lambda b_{3j}\right)\right) \land$$

$$\land \left(\forall i \in \{1,2,3\} \setminus \{i_1\} = \{1,2\} \quad \forall j \in \{1,2,3\} \quad a_{ij} = b_{ij}\right)$$

$$\det A = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{31} =$$

$$= b_{11} b_{22} (\lambda b_{33}) + b_{12} b_{23} (\lambda b_{31}) + b_{13} b_{21} (\lambda b_{32}) - b_{13} b_{22} (\lambda b_{31}) - b_{12} b_{21} (\lambda b_{33}) - b_{11} b_{23} (\lambda b_{31}) =$$

$$= \lambda \left(b_{11} b_{22} b_{33} + b_{12} b_{23} b_{21} + b_{13} b_{21} b_{32} - b_{13} b_{22} b_{31} - b_{12} b_{21} b_{33} - b_{11} b_{23} b_{31}\right) =$$

$$= \lambda \det B$$

$\Rightarrow$

$A = (a_{ij}) \in \mathbb{R}^{3 \times 3} \quad B = (b_{ij}) \in \mathbb{R}^{3 \times 3}$

$i_1 \in \{1,2,3\} \qquad \lambda \in \mathbb{R}$

$$\left(\forall j \in \{1,2,3\} \quad a_{i_1 j} = \lambda b_{i_1 j}\right) \land \left(\forall i \in \{1,2,3\} \setminus \{i_1\} \quad \forall j \in \{1,2,3\} \quad a_{ij} = b_{ij}\right)$$

$\Rightarrow \det A = \lambda \det B$ $\square$