Advanced Policy Gradient

Reinforcement Learning

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March 24, 2025

Bias-Variance trade-off

Reminder: Policy Gradient

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Critic: an approximation $V_\phi(s) \approx V^\pi(s)$.

Important: it is a **biased** estimation.

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$Q^{\pi_\theta}(s_t,a_t) \approx y_Q = r + \gamma V_\phi(s_{t+1})$ is also biased.

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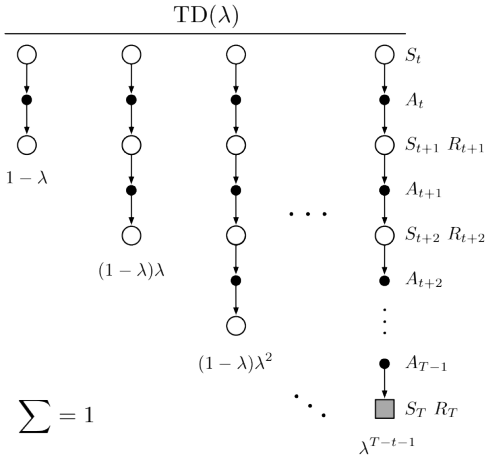
Problem: hard to choose N.

Reminder: TD(λ)

Can't choose? Use them all.

Estimation	Weight
$\Psi_{(1)}(s,a)$	$1-\lambda$
$\Psi_{(2)}(s,a)$	$(1-\lambda)\lambda$
$\Psi_{(3)}(s,a)$	$(1-\lambda)\lambda^2$
$\vdots$	$\vdots$



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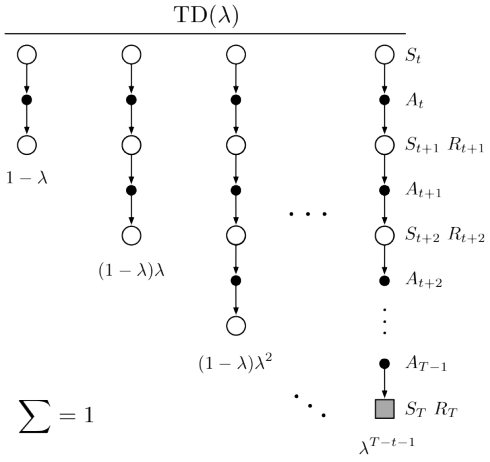
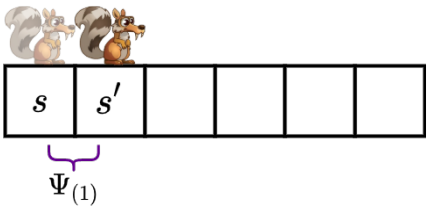


Discounted sum of TD-errors

$$\overbrace{r + \gamma V(s') - V(s)}^{\Psi_{(1)}(s,a)} +$$



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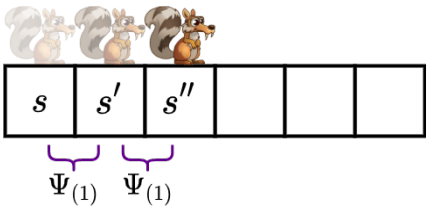

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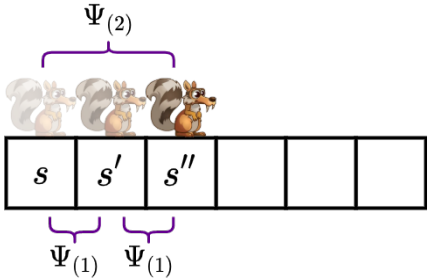

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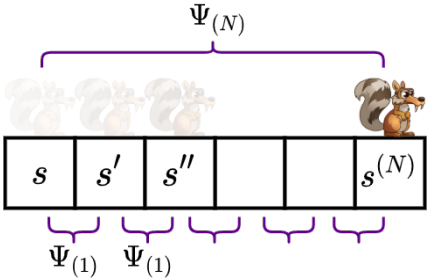

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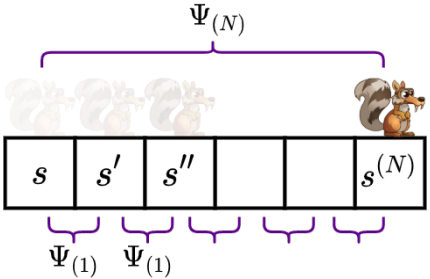

Rule of thumb: discounted sum of discounted sums is discounted sum =D

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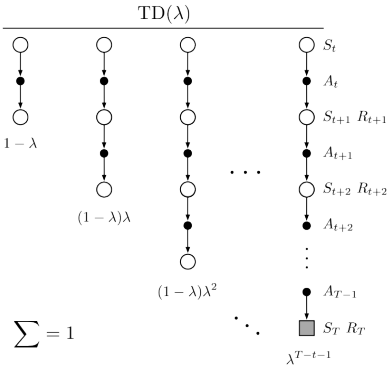

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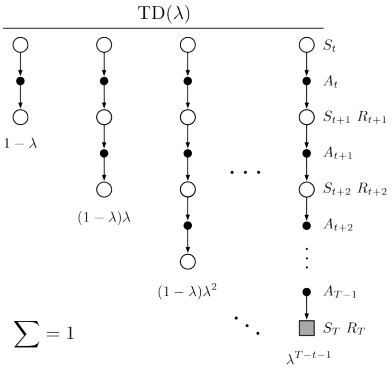

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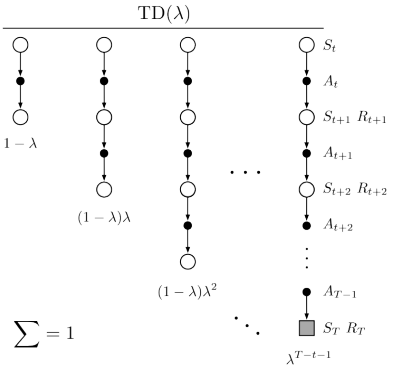

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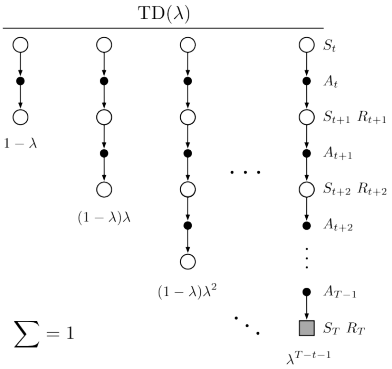

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GAE in Advantage Actor-Critic

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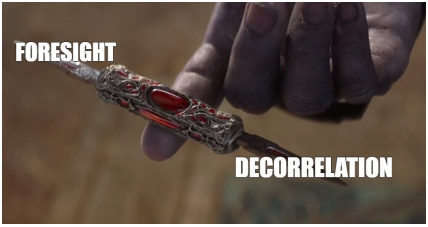

In A2C rollouts are usually short, so $\lambda = 1$ is common choice.
(sometimes called **max-trace** estimation)

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Value-based (DQN+)

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✗ trains $Q^*(s, a)$;
(complicated intermediate stage)

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✓ trains policy directly;
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✓ off-policy;
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✗ on-policy;
(all data is useless after each SGD step)

Proximal Policy Optimization (PPO)

Off-policy Policy Gradient?

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May be we can somehow express $J(\pi_\theta)$ using $A^{\pi^{\text{old}}}(s,a)$?

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So, for arbitrary trajectory $\mathcal{T} \coloneqq s_0, a_0, s_1, a_1, \dots$ and arbitrary policy π:

$$-V^{\pi}(s_0) = \sum_{t\geqslant 0} \left[\gamma^{t+1} V^{\pi}(s_{t+1}) - \gamma^t V^{\pi}(s_t)\right] \tag{1}$$

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\begin{aligned}
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Consider the following *local approximation* (**«surrogate objective»**):

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  - optimizing θ with fixed *data* (fixed $\Psi(s,a) \approx A^{\pi^{\text{old}}}(s,a)$) will learn $\pi_\theta(a \mid s) = 1$ if $\Psi(s,a) > 0$, $\pi_\theta(a \mid s) = 0$ otherwise.

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$$C := \frac{4\gamma \max_{s,a} |A^{\pi^{\text{old}}}(s,a)|}{(1-\gamma)^2}$$

Minorization-maximization algorithm

We discovered a **variational lower bound** for our objective:

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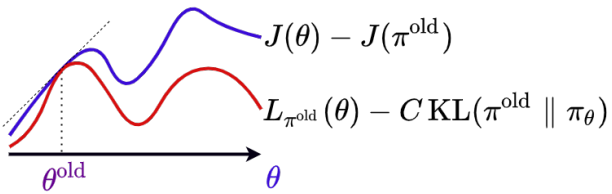

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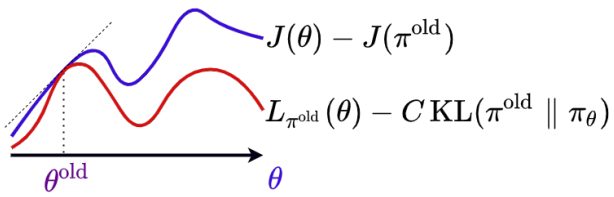

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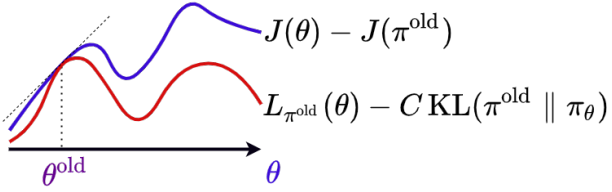

✓ guarantees monotonic improvement!

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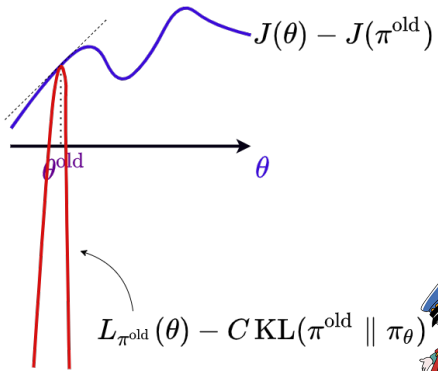

✗ impractical: real C is too huge!

Trust-Region optimization

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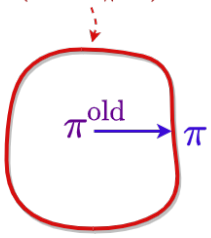

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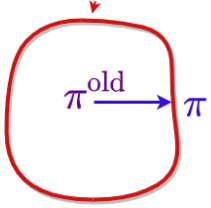

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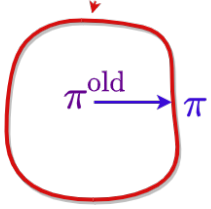

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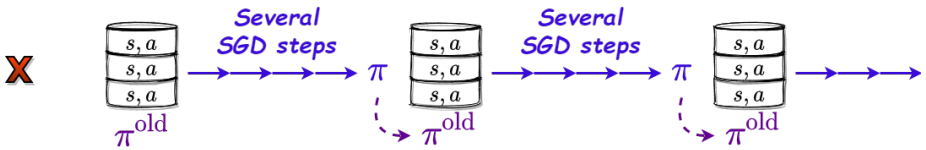
Algorithm **TRPO** attempts to solve this conditioned task approximately. *Complicated implementation*.

Proximal Policy Optimization (PPO): Pipeline

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\frac{\pi_\theta(a\mid s)}{\pi^{\text{old}}(a\mid s)}A^{\pi^{\text{old}}}(s,a) - C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta) \to \max_\theta$$



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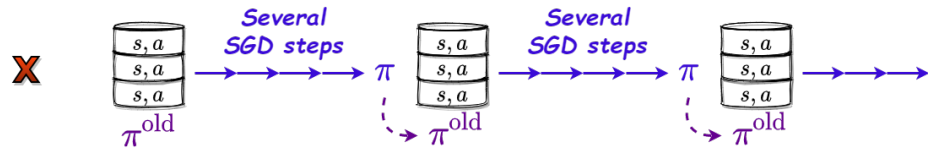


Clipping Objective

$$L_{\pi^{\text{old}}}(\theta) - C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta) \to \max_\theta$$

Default surrogate function:

$$\rho(\theta) := \frac{\pi_\theta(a \mid s)}{\pi^{\text{old}}(a \mid s)}$$

$$L_{\pi^{\text{old}}}(\theta) := \mathbb{E}_{s,a}\rho(\theta)A^{\pi^{\text{old}}}(s,a)$$

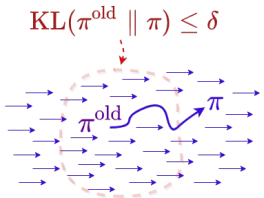
Clipping Objective

$$L_{\pi^{\text{old}}}(\theta) - C\,\mathrm{KL}(\pi^{\text{old}} \,\|\, \pi_\theta) \to \max_\theta$$

Default surrogate function:

$$\rho(\theta) := \frac{\pi_\theta(a \mid s)}{\pi^{\text{old}}(a \mid s)}$$

$$L_{\pi^{\text{old}}}(\theta) := \mathbb{E}_{s,a}\rho(\theta)A^{\pi^{\text{old}}}(s,a)$$



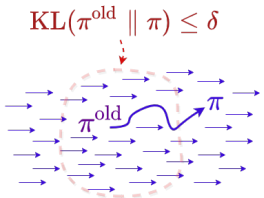
Clipping Objective

$$L_{\pi^{\text{old}}}(\theta) - C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta) \to \max_\theta$$

Default surrogate function:

$$\rho(\theta) := \frac{\pi_\theta(a \mid s)}{\pi^{\text{old}}(a \mid s)}$$

$$L_{\pi^{\text{old}}}(\theta) := \mathbb{E}_{s,a}\rho(\theta)A^{\pi^{\text{old}}}(s,a)$$



Clipped surrogate function:

$$\rho^{\text{clip}}(\theta) := \mathrm{clip}(\rho(\theta), 1-\epsilon, 1+\epsilon)$$

$$L^{\text{clip}}_{\pi^{\text{old}}}(\theta) := \mathbb{E}_{s,a}\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)$$

Clipping Objective

$$L_{\pi^{\text{old}}}(\theta) - C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta) \to \max_\theta$$

Default surrogate function:

$$\rho(\theta) := \frac{\pi_\theta(a \mid s)}{\pi^{\text{old}}(a \mid s)}$$

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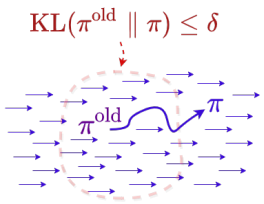

Clipped surrogate function:

$$\rho^{\text{clip}}(\theta) := \mathrm{clip}(\rho(\theta), 1-\epsilon, 1+\epsilon)$$

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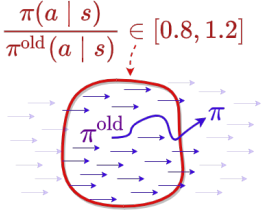

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\substack{\text{term with clipped}\\ \text{importance sampling weight}}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{term with clipped importance sampling weight}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Advantage Sign	Direction	Bad ratio case	Gradient
$A^{\pi^{\text{old}}}(s,a) \geqslant 0$			

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\substack{\text{term with clipped}\\ \text{importance sampling weight}}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Advantage Sign	Direction	Bad ratio case	Gradient
$A^{\pi^{\text{old}}}(s,a) \geqslant 0$	$\pi_\theta(a \mid s) \Uparrow$		

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{term with clipped importance sampling weight}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Advantage Sign	Direction	Bad ratio case	Gradient
$A^{\pi^{\text{old}}}(s,a) \geqslant 0$	$\pi_\theta(a \mid s) \Uparrow$	$\rho(\theta) > 1.2$	

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{term with clipped importance sampling weight}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Advantage Sign	Direction	Bad ratio case	Gradient
$A^{\pi^{\text{old}}}(s,a) \geqslant 0$	$\pi_\theta(a \mid s) \Uparrow$	$\rho(\theta) > 1.2$	0

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{term with clipped importance sampling weight}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Advantage Sign	Direction	Bad ratio case	Gradient
$A^{\pi^{\text{old}}}(s,a) \geqslant 0$	$\pi_\theta(a \mid s) \Uparrow$	$\rho(\theta) > 1.2$ $\rho(\theta) < 0.8$	0

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{term with clipped importance sampling weight}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

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$A^{\pi^{\text{old}}}(s,a) \geqslant 0$	$\pi_\theta(a \mid s) \Uparrow$	$\rho(\theta) > 1.2$ $\rho(\theta) < 0.8$	0 same

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\substack{\text{term with clipped}\\ \text{importance sampling weight}}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Advantage Sign	Direction	Bad ratio case	Gradient
$A^{\pi^{\text{old}}}(s,a) \geqslant 0$	$\pi_\theta(a \mid s) \Uparrow$	$\rho(\theta) > 1.2$	0
		$\rho(\theta) < 0.8$	same
$A^{\pi^{\text{old}}}(s,a) < 0$	$\pi_\theta(a \mid s) \Downarrow$	$\rho(\theta) > 1.2$	
		$\rho(\theta) < 0.8$	

Recalling lower bound intuition

$$\mathbb{E}_{s\sim d_{\pi^{\text{old}}}(s)}\mathbb{E}_{a\sim\pi^{\text{old}}(a|s)}\min(\overbrace{\rho(\theta)A^{\pi^{\text{old}}}(s,a)}^{\text{original term}},\ \overbrace{\rho^{\text{clip}}(\theta)A^{\pi^{\text{old}}}(s,a)}^{\substack{\text{term with clipped}\\ \text{importance sampling weight}}}) - \overbrace{C\,\mathrm{KL}(\pi^{\text{old}} \parallel \pi_\theta)}^{\text{«regularization»}} \to \max_\theta$$

Advantage Sign	Direction	Bad ratio case	Gradient
$A^{\pi^{\text{old}}}(s,a) \geqslant 0$	$\pi_\theta(a \mid s) \Uparrow$	$\rho(\theta) > 1.2$	0
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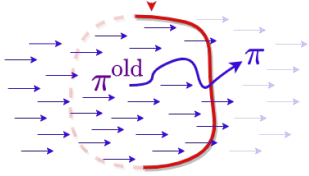
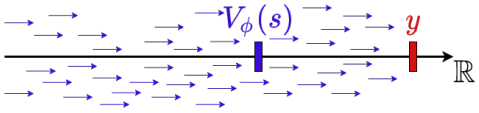
$A^{\pi^{\text{old}}}(s,a) < 0$	$\pi_\theta(a \mid s) \Downarrow$	$\rho(\theta) > 1.2$	same
		$\rho(\theta) < 0.8$	0

$$\frac{\pi(a \mid s)}{\pi^{\text{old}}(a \mid s)} \in [0.8, 1.2]$$



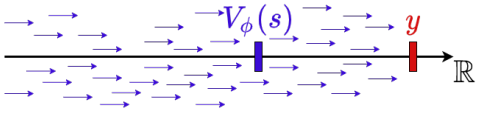
Clipped Critic Loss

$\text{Loss}(\phi) := (y - V(\phi))^2 =$



Clipped Critic Loss

$$\text{Loss}(\phi) := (y - V(\phi))^2 =$$
$$= (y - V^{\text{old}} + V^{\text{old}} - V(\phi))^2$$



Clipped Critic Loss

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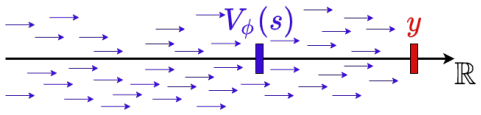
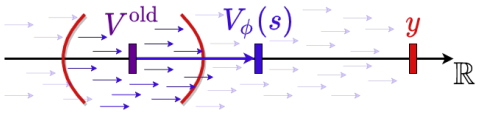


$$\text{Loss}^{\text{clip}}(\phi) := (y - V^{\text{old}} + \text{clip}(V^{\text{old}} - V(\phi), -\hat{\epsilon}, \hat{\epsilon}))^2$$



Clipped Critic Loss

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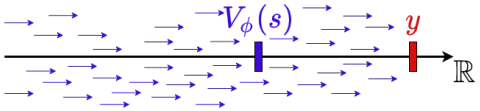
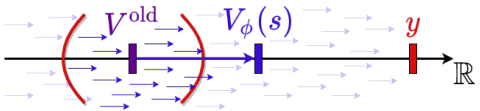
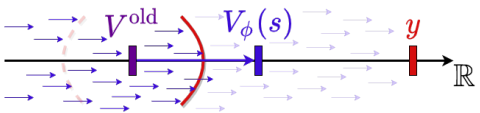
$$= (y - V^{\text{old}} + V^{\text{old}} - V(\phi))^2$$



$$\text{Loss}^{\text{clip}}(\phi) := (y - V^{\text{old}} + \text{clip}(V^{\text{old}} - V(\phi), -\hat{\epsilon}, \hat{\epsilon}))^2$$



$$\max(\text{Loss}(\phi), \text{Loss}^{\text{clip}}(\phi))$$



Proximal Policy Optimization: implementation matters

Key elements:

- ✓ Clipped policy loss
- ✓ Clipped critic loss
- ✓ GAE

Pipeline details:

- ! Advantage normalization in mini-batches
- No KL regularization
- Entropy loss

Other hacks:

- ! Reward normalization[1] and clipping
- Observations normalization and clipping[2]
- Orthogonal initialization of layers
- ϵ (clipping parameter) annealing

Standard tricks:

- Adam, learning rate annealing
- Tanh activation functions
- ! Gradient clipping

[1] divided by running std of collected cumulative rewards
[2] can be critical in continuous control

Full Pipeline: pt.I

Proximal Policy Optimization (PPO)

Initialize $\pi(a \mid s, \theta), V_\phi(s)$;

Full Pipeline: pt.I

Proximal Policy Optimization (PPO)

Initialize $\pi(a \mid s, \theta), V_\phi(s)$;

for $k = 0, 1, 2 \ldots$

- collect several rollouts $s_0, a_0, r_0, s_1, \mathrm{done}_1, a_1 \ldots s_N, \mathrm{done}_N$ using $\pi(a \mid s, \theta)$;
  store probabilities of selected actions as $\pi^{\mathrm{old}}(a_t \mid s_t) := \pi(a_t \mid s_t, \theta)$
  store critic output as $V^{\mathrm{old}}(s_t) := V_\phi(s_t)$

Full Pipeline: pt.I

Proximal Policy Optimization (PPO)

Initialize $\pi(a \mid s, \theta), V_\phi(s)$;

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  store probabilities of selected actions as $\pi^{\mathrm{old}}(a_t \mid s_t) := \pi(a_t \mid s_t, \theta)$
  store critic output as $V^{\mathrm{old}}(s_t) := V_\phi(s_t)$
- compute 1-step errors: $\Psi_{(1)}(s_t, a_t) := r_t + \gamma(1 - \mathrm{done}_{t+1}) V_\phi(s_{t+1}) - V_\phi(s_t)$

Full Pipeline: pt.I

Proximal Policy Optimization (PPO)

Initialize $\pi(a \mid s, \theta), V_\phi(s)$;

for $k = 0, 1, 2 \ldots$

- collect several rollouts $s_0, a_0, r_0, s_1, \mathrm{done}_1, a_1 \ldots s_N, \mathrm{done}_N$ using $\pi(a \mid s, \theta)$;
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- compute 1-step errors: $\Psi_{(1)}(s_t, a_t) := r_t + \gamma(1 - \mathrm{done}_{t+1}) V_\phi(s_{t+1}) - V_\phi(s_t)$
- compute GAE advantage estimations: $\Psi^{\mathrm{GAE}}(s_{N-1}, a_{N-1}) := \Psi_{(1)}(s_{N-1}, a_{N-1})$
- for t from $N-2$ to 0:
  - $\Psi^{\mathrm{GAE}}(s_t, a_t) :=$

Full Pipeline: pt.I

Proximal Policy Optimization (PPO)

Initialize $\pi(a \mid s, \theta), V_\phi(s)$;

for $k = 0, 1, 2 \ldots$

- collect several rollouts $s_0, a_0, r_0, s_1, \mathrm{done}_1, a_1 \ldots s_N, \mathrm{done}_N$ using $\pi(a \mid s, \theta)$;
  store probabilities of selected actions as $\pi^{\mathrm{old}}(a_t \mid s_t) := \pi(a_t \mid s_t, \theta)$
  store critic output as $V^{\mathrm{old}}(s_t) := V_\phi(s_t)$
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- compute GAE advantage estimations: $\Psi^{\mathrm{GAE}}(s_{N-1}, a_{N-1}) := \Psi_{(1)}(s_{N-1}, a_{N-1})$
- for t from $N-2$ to 0:
  - $\Psi^{\mathrm{GAE}}(s_t, a_t) := \Psi_{(1)}(s_t, a_t) + \lambda\gamma(1 - \mathrm{done}_{t+1}) \Psi^{\mathrm{GAE}}(s_{t+1}, a_{t+1})$

Full Pipeline: pt.I

Proximal Policy Optimization (PPO)

Initialize $\pi(a \mid s, \theta), V_\phi(s)$;

for $k = 0, 1, 2 \ldots$

- collect several rollouts $s_0, a_0, r_0, s_1, \mathrm{done}_1, a_1 \ldots s_N, \mathrm{done}_N$ using $\pi(a \mid s, \theta)$;
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- compute critic targets: $y(s_t) := \Psi^{\mathrm{GAE}}(s_t, a_t) + V_\phi(s_t)$

Full Pipeline: pt.I

Proximal Policy Optimization (PPO)

Initialize $\pi(a \mid s, \theta), V_\phi(s)$;

for $k = 0, 1, 2 \ldots$

- collect several rollouts $s_0, a_0, r_0, s_1, \mathrm{done}_1, a_1 \ldots s_N, \mathrm{done}_N$ using $\pi(a \mid s, \theta)$;
  store probabilities of selected actions as $\pi^{\mathrm{old}}(a_t \mid s_t) := \pi(a_t \mid s_t, \theta)$
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- for t from $N-2$ to 0:
  - $\Psi^{\mathrm{GAE}}(s_t, a_t) := \Psi_{(1)}(s_t, a_t) + \lambda\gamma(1 - \mathrm{done}_{t+1})\Psi^{\mathrm{GAE}}(s_{t+1}, a_{t+1})$
- compute critic targets: $y(s_t) := \Psi^{\mathrm{GAE}}(s_t, a_t) + V_\phi(s_t)$
- construct dataset of $(s_t, a_t, \Psi^{\mathrm{GAE}}(s_t, a_t), y(s_t), \pi^{\mathrm{old}}(a_t \mid s_t), V^{\mathrm{old}}(s_t))$

Full Pipeline: pt.II

Proximal Policy Optimization (PPO) -- cont.

- go through dataset $\mathrm{n\_epochs}$ times, sampling mini-batches of size B; for each mini-batch:

Full Pipeline: pt.II

Proximal Policy Optimization (PPO) -- cont.

- go through dataset $\mathrm{n\_epochs}$ times, sampling mini-batches of size B; for each mini-batch:
  - normalize $\Psi^{\mathrm{GAE}}(s, a)$ in the batch by subtracting mean and dividing by std

Full Pipeline: pt.II

Proximal Policy Optimization (PPO) -- cont.

- go through dataset $\mathrm{n\_epochs}$ times, sampling mini-batches of size B; for each mini-batch:
  - normalize $\Psi^{\mathrm{GAE}}(s, a)$ in the batch by subtracting mean and dividing by std
  - compute importance sampling weights:

$$\rho(s, a, \theta) := \frac{\pi(a \mid s, \theta)}{\pi^{\mathrm{old}}(a \mid s)}, \qquad \rho^{\mathrm{clip}}(s, a, \theta) = \mathrm{clip}(\rho(s, a, \theta), 1 - \epsilon, 1 + \epsilon)$$

Full Pipeline: pt.II

Proximal Policy Optimization (PPO) -- cont.

- go through dataset $\mathrm{n\_epochs}$ times, sampling mini-batches of size B; for each mini-batch:
  - normalize $\Psi^{\mathrm{GAE}}(s,a)$ in the batch by subtracting mean and dividing by std
  - compute importance sampling weights:

$$\rho(s,a,\theta) := \frac{\pi(a \mid s,\theta)}{\pi^{\mathrm{old}}(a \mid s)}, \qquad \rho^{\mathrm{clip}}(s,a,\theta) = \mathrm{clip}(\rho(s,a,\theta), 1-\epsilon, 1+\epsilon)$$

  - update actor:

$$L_1(s,a,\theta) := \rho(s,a,\theta)\Psi^{\mathrm{GAE}}(s,a), \qquad L_2(s,a,\theta) := \rho^{\mathrm{clip}}(s,a,\theta)\Psi^{\mathrm{GAE}}(s,a)$$

$$\theta \leftarrow \theta + \alpha \nabla_\theta \frac{1}{B} \sum_{s,a} \min(L_1(s,a,\theta), L_2(s,a,\theta))$$

Full Pipeline: pt.II

Proximal Policy Optimization (PPO) -- cont.

- go through dataset $\mathrm{n\_epochs}$ times, sampling mini-batches of size B; for each mini-batch:
  - normalize $\Psi^{\mathrm{GAE}}(s,a)$ in the batch by subtracting mean and dividing by std
  - compute importance sampling weights:

$$\rho(s,a,\theta) := \frac{\pi(a \mid s,\theta)}{\pi^{\mathrm{old}}(a \mid s)}, \qquad \rho^{\mathrm{clip}}(s,a,\theta) = \mathrm{clip}(\rho(s,a,\theta), 1-\epsilon, 1+\epsilon)$$

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$$L_1(s,a,\theta) := \rho(s,a,\theta)\Psi^{\mathrm{GAE}}(s,a), \qquad L_2(s,a,\theta) := \rho^{\mathrm{clip}}(s,a,\theta)\Psi^{\mathrm{GAE}}(s,a)$$

$$\theta \leftarrow \theta + \alpha \nabla_\theta \frac{1}{B} \sum_{s,a} \min(L_1(s,a,\theta), L_2(s,a,\theta))$$

  - update critic:

$$\mathrm{Loss}_1(s,\phi) := (y(s) - V_\phi(s))^2$$

$$\mathrm{Loss}_2(s,\phi) := \left(y(s) - V^{\mathrm{old}}(s) - \mathrm{clip}(V_\phi(s) - V^{\mathrm{old}}(s), \hat{\epsilon}, -\hat{\epsilon})\right)^2$$

$$\phi \leftarrow \phi - \alpha \nabla_\phi \frac{1}{B} \sum_{s} \max(\mathrm{Loss}_1(s,\phi), \mathrm{Loss}_2(s,\phi))$$

Go-to Model Free Algorithms

	Discrete Action Space	*Continuous Action Space*
On-policy	**PPO**	**PPO**
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Hometask 5:

Continuous control; you can choose PPO, TD3 or SAC.

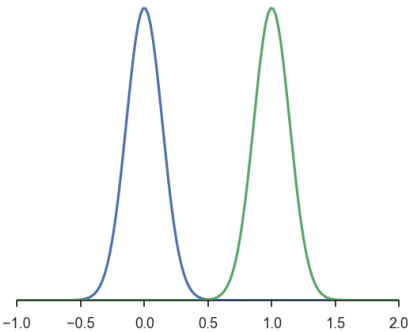
Literature

- High-Dimensional Continuous Control Using Generalized Advantage Estimation;
- Trust Region Policy Optimization;
- Proximal Policy Optimization Algorithms;
- Implementation Matters in Deep Policy Gradients: A Case Study on PPO and TRPO;
- OpenAI Five;

Appendix: Trust-Region Policy Optimization (TRPO)

Parameter space vs distribution space

Two gaussians: $\mathcal{N}(0, 0.2), \mathcal{N}(1, 0.2)$



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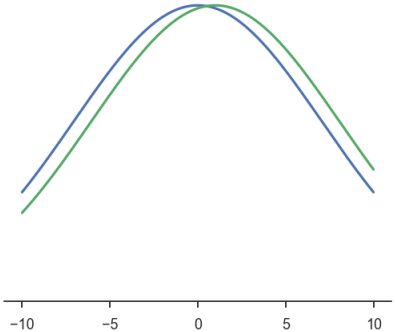

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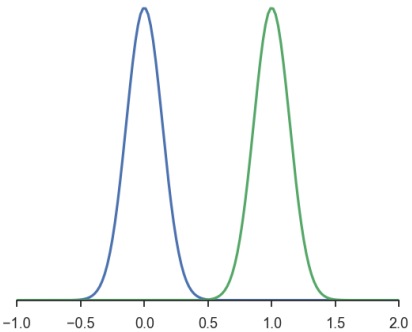

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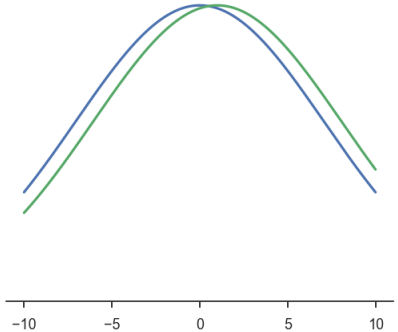

Euclidean distance: $\sqrt{(\mu_1 - \mu_2)^2 + (\sigma_1 - \sigma_2)^2}$

Statistical distance: e.g. $\mathrm{KL}(\mathcal{N}(\mu_1, \sigma_1) \parallel \mathcal{N}(\mu_2, \sigma_2))$

Step size issue: riding from the cliff

$$\theta_{k+1} = \theta_k + \alpha \hat{g}_k$$

If step size too large:

- step too far leads to bad policy;
- we start *collecting data* with bad policy;
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We need optimization method that measures distance in **distribution space**, not in **parameter space**!

A view on optimization methods

Consider the following optimization task:

$$G(\theta) := G(q_\theta(x)) \to \min_\theta$$

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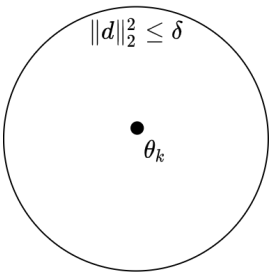
  - seek θ_{k+1} as solution of
  $$\begin{cases} m_k(\theta) \to \min\limits_\theta \\ r(\theta, \theta_k) \leqslant \delta \end{cases}$$

Natural gradient descent

$$G(\theta) := G(q_\theta(x)) \to \min_\theta$$

Gradient Descent

$$\begin{cases} G(\theta_k + d) \approx G(\theta_k) + \nabla_\theta G(\theta_k)^T d \to \min\limits_d \\ \|d\|_2 \leqslant \delta \end{cases}$$



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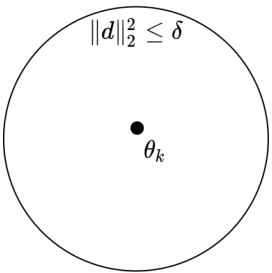

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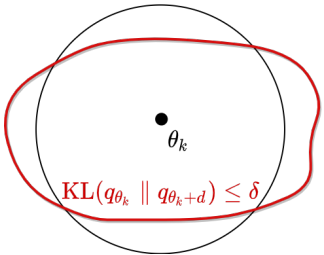

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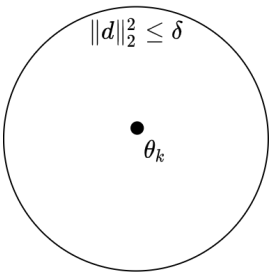

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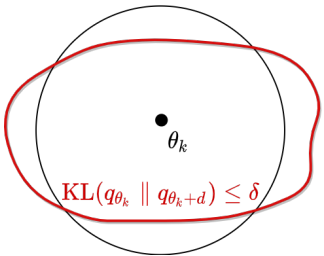

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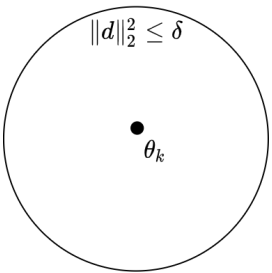

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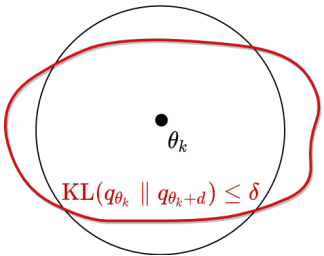

Solution: ?!?

Fisher Matrix

Use Taylor expansion for

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Let $F(\theta_k)$ be the **Fisher matrix** of $q_\theta(x)$ at point θ_k:

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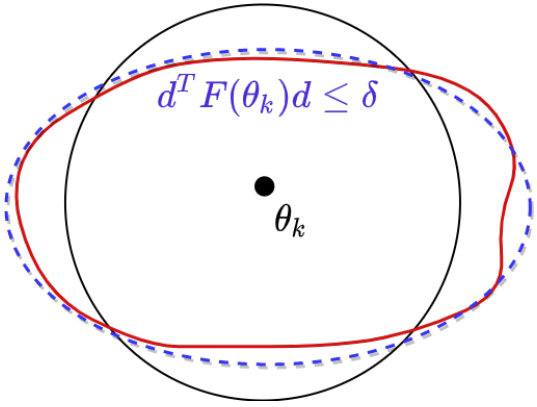
Then Taylor approximation of second order is:

$$\mathrm{KL}(q_{\theta_k}(x) \parallel q_{\theta_k+d}(x)) \approx \frac{1}{2} d^T F(\theta_k) d$$

Direction of natural gradient

$$\begin{cases} G(\theta_k + d) \approx G(\theta_k) + \nabla_\theta G(\theta_k)^T d \to \min\limits_d \\ d^T F(\theta_k) d \leqslant \delta \end{cases}$$



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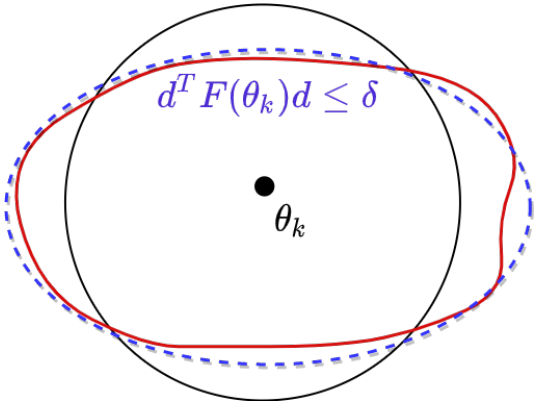

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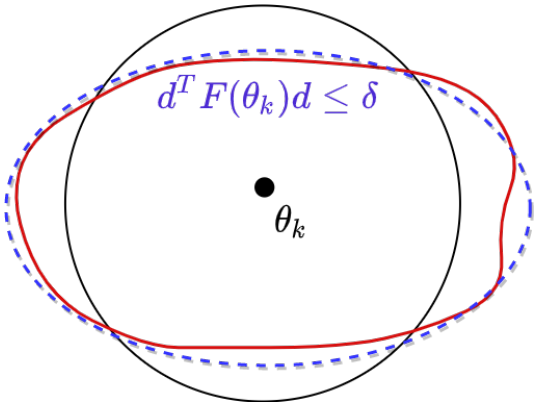
Natural gradient uses the following update:

$$\theta_{k+1} = \theta_k - \alpha F^{-1}(\theta_k)\nabla_\theta G(\theta_k),$$

where α is determined by the choice of δ.



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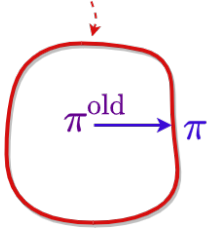

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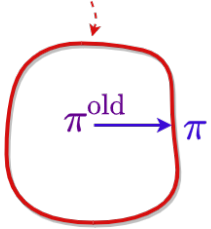

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Use Taylor expansion for both model and condition to solve this system!

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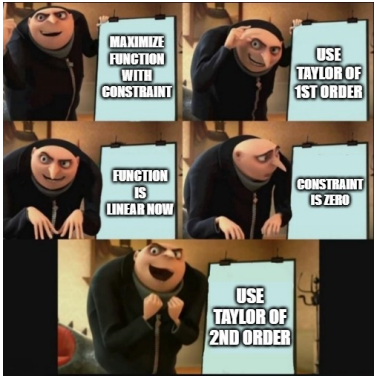

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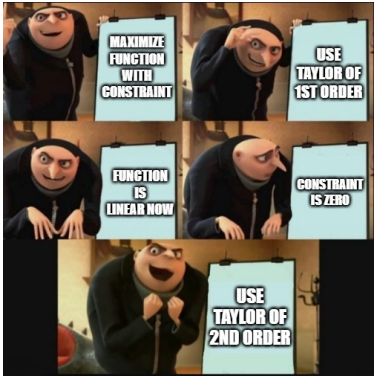

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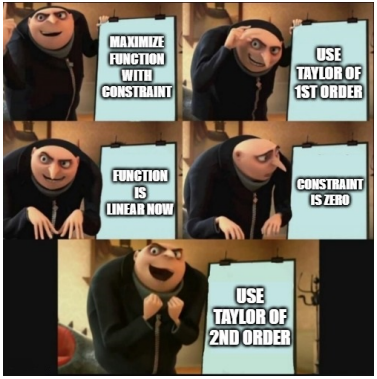

Natural Gradient Ascent

Solution: $d \propto H^{-1} g$

Reality check: can we compute inverse hessian?

$$\theta_{k+1} = \theta_k + \alpha H^{-1} g$$

Danger!

Natural gradient descent is similar to second-order methods.

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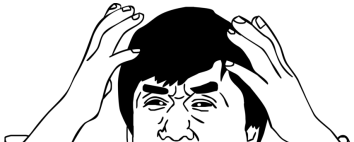

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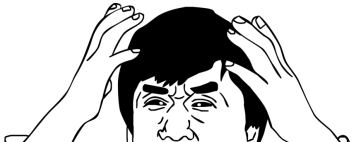

Solve the following linear system:

$$Hd = g$$

using **conjugate gradient** method.

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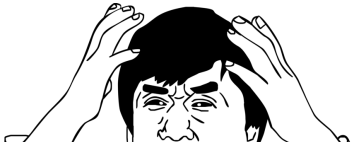

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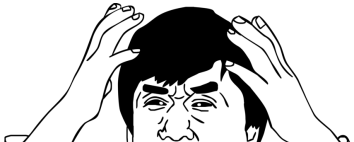

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$$Hd = g$$

using **conjugate gradient** method.

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2. initialize d_0 arbitrary;
3. **for** $i = 0, 1, 2 \ldots M$**:**

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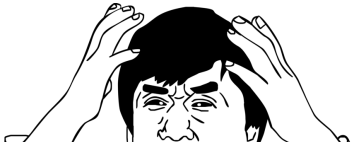

Solve the following linear system:

$$Hd = g$$

using **conjugate gradient** method.

1. compute g (as usual in standard gradient descent);
2. initialize d_0 arbitrary;
3. **for** $i = 0, 1, 2 \ldots M$**:**
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Reality check: can we compute inverse hessian?

$$\theta_{k+1} = \theta_k + \alpha H^{-1} g$$

Danger!

Natural gradient descent is similar to second-order methods.

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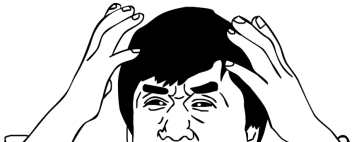

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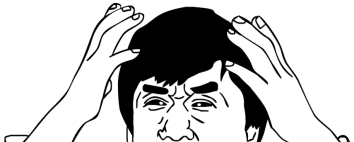

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Line Search

How to determine learning rate α_k?

(it is interconnected with δ in trust region view)

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$$L_{\pi^{\text{old}}}(\theta_{k+1}) > 0, \qquad \mathbb{E}_s \, \mathrm{KL}(\pi^{\text{old}}(\cdot \mid s) \parallel \pi_{\theta_{k+1}}(\cdot \mid s))) \leqslant \delta$$

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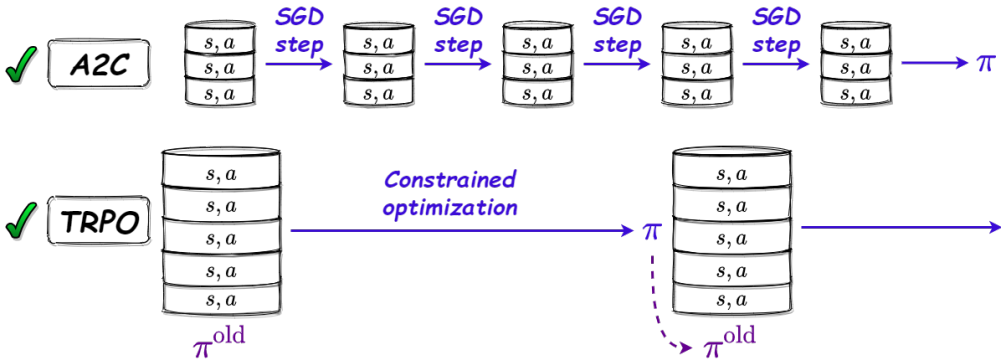
e. g. decrease α_k in half until these constraints are met.

TRPO: pipeline



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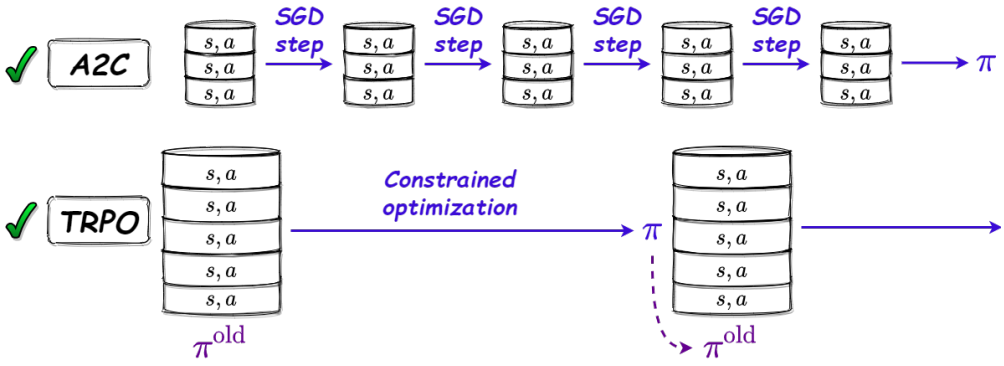

- ✓ better utilization of GAE
- ✓ more sample-efficient than A2C
- ✓ considered quite robust

- ✗ critic has to be trained in separate optimization process
- ✗ computationally expensive
- ✗ complicated :{