Fast Fibonacci algorithms

Definition: The Fibonacci sequence is defined as $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n - 1) + F(n - 2)$ for $n \geq 2$. So the sequence (starting with $F(0)$) is 0, 1, 1, 2, 3, 5, 8, 13, 21, ….

If we want to compute a single term in the sequence (e.g. $F(n)$), there are a couple of algorithms to do so. Some algorithms are *much* faster than others.

Algorithms

- **Textbook recursive (extremely slow)**

  Naively, we can directly execute the recurrence as given in the mathematical definition of the Fibonacci sequence. Unfortunately, it's hopelessly slow: It uses $\Theta(n)$ stack space and $\Theta(\varphi^n)$ arithmetic operations, where $\varphi = \frac{\sqrt{5}+1}{2}$ (the golden ratio). In other words, the number of operations to compute $F(n)$ is proportional to the final numerical answer, which grows exponentially.

- **Dynamic programming (slow)**

  It should be clear that if we already computed $F(k - 2)$ and $F(k - 1)$, then we can add them to get $F(k)$. Next, we add $F(k - 1)$ and $F(k)$ to get $F(k + 1)$. We repeat until we reach $k = n$. Most people notice this algorithm automatically, especially when computing Fibonacci by hand. This algorithm takes $\Theta(1)$ space and $\Theta(n)$ operations.

- **Matrix exponentiation (fast)**

  The algorithm is based on this innocent-looking identity (which can be proven by mathematical induction):

  $$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{bmatrix}.$$

  It is important to use exponentiation by squaring with this algorithm, because otherwise it degenerates into the dynamic programming algorithm. This algorithm takes

$\Theta(1)$ space and $\Theta(\log n)$ operations. (Note: We are counting the number of bigint arithmetic operations, not fixed-width word operations.)

- **Fast doubling (faster)**

  Given $F(k)$ and $F(k+1)$, we can calculate these:

  $$F(2k) = F(k)\left[2F(k+1) - F(k)\right],$$
  $$F(2k+1) = F(k+1)^2 + F(k)^2.$$

  These identities can be extracted from the matrix exponentiation algorithm. In a sense, this algorithm is the matrix exponentiation algorithm with the redundant calculations removed. It should be a constant factor faster than matrix exponentiation, but the asymptotic time complexity is still the same.

Summary: The two fast Fibonacci algorithms are matrix exponentiation and fast doubling, each having an asymptotic complexity of $\Theta(\log n)$ bigint arithmetic operations. Both algorithms use multiplication, so they become even faster when Karatsuba multiplication is used. The other two algorithms are slow; they only use addition and no multiplication.

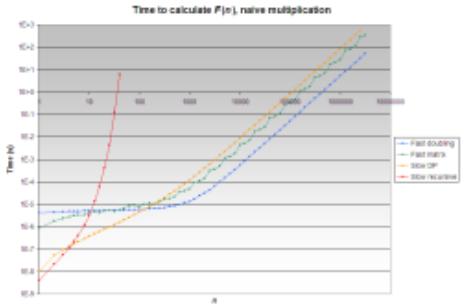
Source code

Implementations are available in multiple languages:

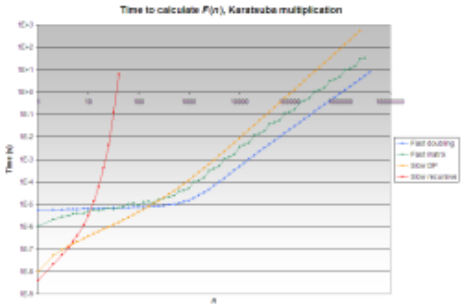
- Java: FastFibonacci.java (all 3 algorithms, timing benchmark, runnable main program)
- Python: fastfibonacci.py (fast doubling function only)
- JavaScript: fastfibonacci.js (fast doubling function only)
- Haskell: fastfibonacci.hs (fast doubling function only)
- C#: FastFibonacci.cs (fast doubling only, runnable main program)
  (requires .NET Framework 4.0 or above; compile with `csc /r:System.Numerics.dll fastfibonacci.cs`)

Benchmarks

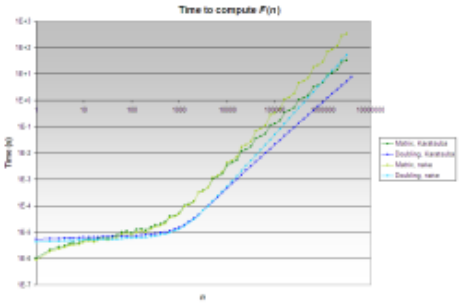
Graphs



All algorithms, naive multiplication



All algorithms, Karatsuba multiplication



Fast algorithms, both multiplication algorithms

(Note: The graphs have logarithmic scales on the *x* and *y* axes.)

Table

n	Fast doubling, Karatsuba multiplication	Fast matrix, Karatsuba multiplication	Fast doubling, naive multiplication	Fast matrix, naive multiplication	Slow dynamic programming	Slow recursive
1	5 414	1 042	4 197	887	10	4
2	5 638	2 092	4 442	1 822	53	22
3	5 708	2 740	4 509	2 342	92	56
4	5 945	3 027	4 733	2 660	133	114
5	5 989	3 677	4 787	3 147	172	219
6	5 972	3 956	4 765	3 371	211	400
8	6 191	3 972	4 969	3 428	289	1 161
10	6 283	4 952	5 022	4 154	370	3 113
13	6 307	5 610	5 046	4 667	488	13 480
16	6 479	4 955	5 177	4 210	605	57 300
20	6 542	5 923	5 234	4 985	763	394 000
25	6 632	6 565	5 263	5 479	964	4 373 000
32	6 794	5 887	5 388	4 908	1 235	127 500 000
40	6 818	6 880	5 433	5 715	1 552	5 980 000 000
50	6 806	7 742	5 486	6 446	2 023	
63	6 931	10 180	5 589	8 339	2 598	
79	7 162	11 090	5 753	9 187	3 396	
100	7 279	9 225	5 904	7 717	4 472	

n	Fast doubling, Karatsuba multiplication	Fast matrix, Karatsuba multiplication	Fast doubling, naive multiplication	Fast matrix, naive multiplication	Slow dynamic programming	Slow recursive
126	7 427	12 410	6 059	10 220	5 866	
158	7 600	13 090	6 141	10 900	7 888	
200	8 006	11 700	6 556	9 969	10 640	
251	8 146	15 660	6 672	13 060	14 280	
316	8 597	18 810	7 089	16 530	19 610	
398	9 501	20 550	8 078	18 120	27 650	
501	9 964	24 050	8 492	21 340	38 970	
631	11 070	38 790	9 510	35 720	55 540	
794	13 020	41 810	11 520	39 380	80 280	
1 000	14 660	50 870	13 130	48 230	118 000	
1 259	18 640	99 020	16 990	95 640	175 300	
1 585	25 300	113 500	23 660	110 800	263 000	
1 995	32 360	148 100	30 770	144 700	397 500	
2 512	45 540	314 800	43 980	311 400	608 800	
3 162	67 800	372 200	66 250	369 000	937 200	
3 981	98 560	491 500	96 780	488 100	1 457 000	
5 012	143 500	1 050 000	145 900	1 132 000	2 269 000	
6 310	214 100	1 284 000	227 700	1 357 000	3 546 000	
7 943	320 300	1 662 000	351 300	1 821 000	5 547 000	
10 000	466 400	3 519 000	538 400	4 382 000	8 700 000	
12 589	691 100	4 303 000	851 700	5 254 000	13 640 000	
15 849	1 007 000	5 481 000	1 310 000	7 079 000	21 440 000	
19 953	1 493 000	11 800 000	2 081 000	17 260 000	33 620 000	
25 119	2 185 000	13 620 000	3 296 000	20 710 000	53 030 000	
31 623	3 205 000	17 570 000	5 159 000	27 860 000	83 310 000	
39 811	4 637 000	36 800 000	8 109 000	68 540 000	131 500 000	
50 119	6 750 000	42 430 000	12 910 000	82 230 000	207 700 000	
63 096	9 913 000	54 770 000	20 410 000	110 600 000	326 900 000	
79 433	14 450 000	113 300 000	32 300 000	275 100 000	517 100 000	
100 000	20 800 000	130 600 000	51 640 000	330 700 000	819 700 000	
125 893	30 380 000	168 900 000	81 150 000	445 200 000	1 296 000 000	
158 489	44 090 000	346 800 000	129 200 000	1 103 000 000	2 058 000 000	
199 526	63 260 000	405 400 000	205 100 000	1 325 000 000	3 249 000 000	
251 189	92 330 000	517 300 000	325 100 000	1 766 000 000	5 153 000 000	
316 228	133 700 000	1 055 000 000	515 700 000	4 413 000 000	8 161 000 000	

n	Fast doubling, Karatsuba multiplication	Fast matrix, Karatsuba multiplication	Fast doubling, naive multiplication	Fast matrix, naive multiplication	Slow dynamic programming	Slow recursive
398 107	191 900 000	1 228 000 000	815 500 000	5 311 000 000	12 930 000 000	
501 187	280 200 000	1 572 000 000	1 297 000 000	7 059 000 000	20 520 000 000	
630 957	404 900 000	3 181 000 000	2 061 000 000	17 570 000 000	32 570 000 000	
794 328	580 700 000	3 691 000 000	3 265 000 000	21 090 000 000	51 650 000 000	
1 000 000	846 100 000	4 724 000 000	5 182 000 000	28 310 000 000	82 000 000 000	
1 258 925	1 221 000 000	9 570 000 000	8 168 000 000	70 280 000 000	130 300 000 000	
1 584 893	1 750 000 000	11 050 000 000	12 970 000 000	84 120 000 000	207 300 000 000	
1 995 262	2 549 000 000	14 230 000 000	20 610 000 000	112 700 000 000	329 700 000 000	
2 511 886	3 676 000 000	28 800 000 000	32 610 000 000	279 900 000 000	525 100 000 000	
3 162 278	5 247 000 000	32 980 000 000	51 600 000 000	335 600 000 000		
3 981 072	7 654 000 000					

All times are given in nanoseconds (ns) with 4 significant figures. All the tests above were performed on Intel Core 2 Quad Q6600 (2.40 GHz) using a single thread, Windows XP SP 3, Java 1.6.0_22.

Proofs

Matrix exponentiation

We will use weak induction to prove this identity.

- **Base case**

  For $n = 1$, clearly $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1 = \begin{bmatrix} F(2) & F(1) \\ F(1) & F(0) \end{bmatrix}$.

- **Induction step**

  Assume for $n \geq 1$ that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{bmatrix}$. Then:

$$
\begin{aligned}
&\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n+1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} F(n+1)+F(n) & F(n+1)+0 \\ F(n)+F(n-1) & F(n)+0 \end{bmatrix} \\
&= \begin{bmatrix} F(n+2) & F(n+1) \\ F(n+1) & F(n) \end{bmatrix}.
\end{aligned}
$$

Fast doubling

We will assume the fact that the matrix exponentiation method is correct for all $n \geq 1$.

$$
\begin{aligned}
&\begin{bmatrix} F(2n+1) & F(2n) \\ F(2n) & F(2n-1) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{2n} \\
&= \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n} \right)^2 \\
&= \begin{bmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{bmatrix}^2 \\
&= \begin{bmatrix} F(n+1)^2+F(n)^2 & F(n+1)F(n)+F(n)F(n-1) \\ F(n)F(n+1)+F(n-1)F(n) & F(n)^2+F(n-1)^2 \end{bmatrix}.
\end{aligned}
$$

Therefore, by equating the cells in the matrix:

$$
\begin{aligned}
F(2n+1) &= F(n+1)^2 + F(n)^2. \\
F(2n) &= F(n)\left[F(n+1) + F(n-1)\right] \\
&= F(n)\left[F(n+1) + (F(n+1) - F(n))\right] \\
&= F(n)\left[2F(n+1) - F(n)\right]. \\
F(2n-1) &= F(n)^2 + F(n-1)^2.
\end{aligned}
$$

More info

- Wikipedia: Fibonacci number
- YouTube: Eric Severson - Secrets of the Fibonacci Tiles

Categories: Programming, Math, Java, JavaScript / TypeScript, Python Last updated: 2023-01-22

Feedback: Question/comment? Contact me