ROB 101 - Fall 2021

Solutions of Nonlinear Equations (Bisection and Newton's Methods)

November 3, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- Extend our horizons from linear equations to nonlinear equations.
- Appreciate the power of using algorithms to iteratively construct approximate solutions to a problem.
- Accomplish all of this without assuming a background in Calculus.

- Learn that a root is a solution of an equation of the form $f(x) = 0$.
- Learn two methods for finding roots of real-valued functions of a real variable, that is for $f : \mathbb{R} \to \mathbb{R}$, namely the Bisection Method and Newton's Method
- Become comfortable with the notion of a "local slope" of a function at a point and how to compute it numerically.

Root of an Equation

- We will limit our notion of a solution to the set of real numbers or real vectors.

- For example, $x^2 + 1 = 0$, has no real solutions because its discriminant is $\Delta = b^2 - 4ac = -4 < 0$.

- Nevertheless, many interesting problems in Engineering and Science can be formulated and solved in terms of "real solutions" to systems of equations.

Root of an Equation

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then $f(x) = 0$ defines an equation.

- A solution to the equation is also called a *root* that is $x^* \in \mathbb{R}^n$ is a root of $f(x) = 0$ if

$$f(x^*) = 0.$$

- Just as with quadratic equations, it is possible to have multiple real solutions or no real solutions.

- What about $f(x) = \pi$?

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- Define a new function, $\bar{f}(x) := f(x) - \pi$, then

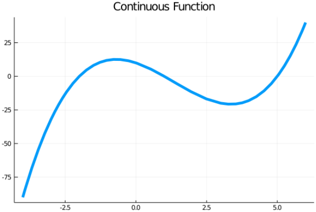
$$\bar{f}(x^*) = 0 \iff f(x^*) - \pi = 0 \iff f(x^*) = \pi.$$

- x^* is a root of our new function $\bar{f}(x)$.

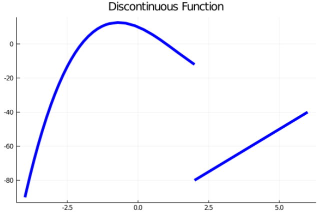
- Informally, a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if you can draw the graph of $y = f(x)$ on a sheet of paper without lifting your pencil (from the paper).

- Also, a function is valid, if for a given $x \in \mathbb{R}$, there can be only one value of $y \in \mathbb{R}$ such that $y = f(x)$.

Continuous Functions



(a)



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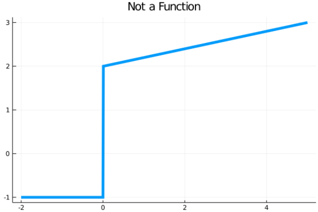


(c)

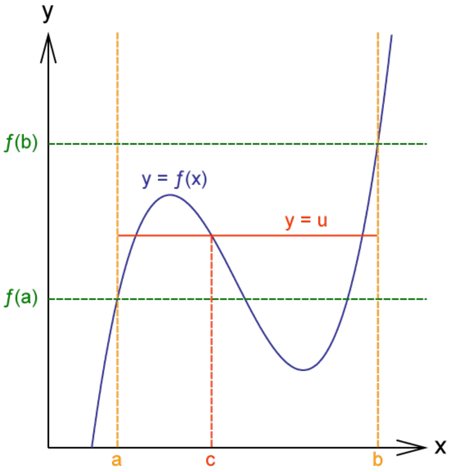
Theorem

Assume that f is a continuous real valued function and you know two real numbers $a < b$ such that $f(a) \cdot f(b) < 0$. Then there exists a real number c such that

- $a < c < b$ *(c is between a and b), and*
- $f(c) = 0$ *(c is a root).*

The values a and b are said to bracket the root, c.

Intermediate Value Theorem



Let's switch to the Julia notebooks!

Next Time

- Vector-valued Functions and Newton-Raphson Method
- Read Chapter 11 of ROB 101 Book