2020 Sept 18 ROB 101

Review: (Like Alice down the rabbit hole!)

1)

General Case of Matrix Multiplication using Columns of A and Rows of B

Suppose that A is $n \times k$ and B is $k \times m$ so that the two matrices are compatible for matrix multiplication. Then

$$A \cdot B = \sum_{i=1}^{k} a_i^{\text{col}} \cdot b_i^{\text{row}},$$

the "sum of the columns of A multiplied by the rows of B". A more precise way to say it would be "the sum over i of the i-th column of A times the i-th row of B." In HW, we'll play with this idea enough that it will become ingrained into your subconscious!

$$A \cdot B = \underbrace{\begin{bmatrix} \boxed{\begin{matrix}1\\2\\3\end{matrix}} & \boxed{\begin{matrix}4\\5\\6\end{matrix}} \end{bmatrix}}_{3\times 2} \cdot \underbrace{\begin{bmatrix} \boxed{7 \quad 8 \quad 9} \\ \boxed{1 \quad 0 \quad 1} \end{bmatrix}}_{2\times 3} = [3\times 3]$$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}7 & 8 & 9\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix}\begin{bmatrix}1 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 7 & 8 & 9 \\ 14 & 16 & 18 \\ 21 & 24 & 27 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 4 \\ 5 & 0 & 5 \\ 6 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 & 13 \\ 19 & 16 & 23 \\ 27 & 24 & 33 \end{bmatrix}$$

(fixed, 20 Sept.)

2) $Ax = b$ suppose $A = L \cdot U$

$\therefore L \cdot \underbrace{Ux}_{y} = b \iff Ly = b$ & $Ux = y$

$\Updownarrow$

$Ly = b$ where $y = Ux$

Solve first: $Ly = b$ via forward substitution

Solve second: $Ux = y$ via back substitution

$\left\{ \begin{array}{c} \text{One hard problem} \\ Ax = b \end{array} \right\} \iff \left\{ \begin{array}{l} \text{Two easy problems} \\ \bullet \text{ Lower triangular system} \\ \bullet \text{ Upper triangular system} \end{array} \right\}$

Fact: Scales to huge problems

<u>Today</u> LU Factorization

Given $A_{n \times n}$, how to find

$L_{n \times n}$ lower triang. & $U_{n \times n}$ upper triang.

such that $A = L \cdot U$

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

Seek $L = \begin{bmatrix} \quad \end{bmatrix}$ $\quad U = \begin{bmatrix} \quad \end{bmatrix}$

such that $A = L \cdot U$

Observation

Define $R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$ and

$$C = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \frac{1}{a_{11}}$$

$$C \cdot R = \begin{bmatrix} 1 \\ a_{21}/a_{11} \\ a_{31}/a_{11} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & * & * \\ a_{31} & * & * \end{bmatrix}$$

$*$ don't care

$$A - CR = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & * & * \\ a_{31} & * & * \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{\text{Stuff}}_{2\times 2}} \\ 0 & & \end{bmatrix}_{3\times 3}$$

$$L = \begin{bmatrix} C \frac{1}{C(1)} \end{bmatrix}_{3\times 1}, \quad U = \begin{bmatrix} R \end{bmatrix}_{1\times 3}$$

$A - LC \Rightarrow$ almost 2×2

LU Factorization of

$$M = \begin{bmatrix} -2 & -4 & -6 \\ -2 & 1 & -4 \\ -2 & 11 & -4 \end{bmatrix}_{3\times 3}$$

Initialize

Temp = copy(m)

L = [empty matrix]

U = [empty matrix]

k = 1

$$C = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$$

$$\text{pivot} = C[k] = C[1] = -2$$

$$C = C \frac{1}{\text{pivot}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$R = \begin{bmatrix} -2 & -4 & -6 \end{bmatrix}$$

$$\text{Temp} = \text{Temp} - C * R$$

$$= \begin{bmatrix} -2 & -4 & -6 \\ -2 & 1 & -4 \\ -2 & 11 & -4 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 & -4 & -6 \end{bmatrix}}_{\begin{bmatrix} -2 & -4 & -6 \\ -2 & -4 & -6 \\ -2 & -4 & -6 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 15 & 2 \end{bmatrix}$$

$$L = [L \quad C] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$U = [U \; ; \; R] = \begin{bmatrix} -2 & -4 & -6 \end{bmatrix}$$

$\underline{k=2}$

$$C = \begin{bmatrix} 0 \\ 5 \\ 15 \end{bmatrix}$$

$$\text{pivot} = C[k] = C[2] = 5$$

$$C = C \frac{1}{\text{pivot}} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$R = [0 \quad 5 \quad 2]$$

$$\text{Temp} - C * R =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 15 & 2 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} [0 \quad 5 \quad 2]}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 15 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 15 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Inserted 20 September

Finish the example here; will also include in Monday's notes

$$L = [L \;\; C] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$U = [U \, ; R] = \begin{bmatrix} -2 & -4 & -6 \\ 0 & 5 & 2 \end{bmatrix}$$

k=3

$$\text{Temp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$C = \text{Temp}[:,k] = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \; ; R = \text{Temp}[k,:]' = [0 \;\; 0 \;\; -4]$$

$$\text{pivot} = C[k] = -4$$

$$C = C \frac{1}{\text{pivot}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\text{Temp} = \text{Temp} - C * R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [0 \;\; 0 \;\; -4]}_{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}}$$

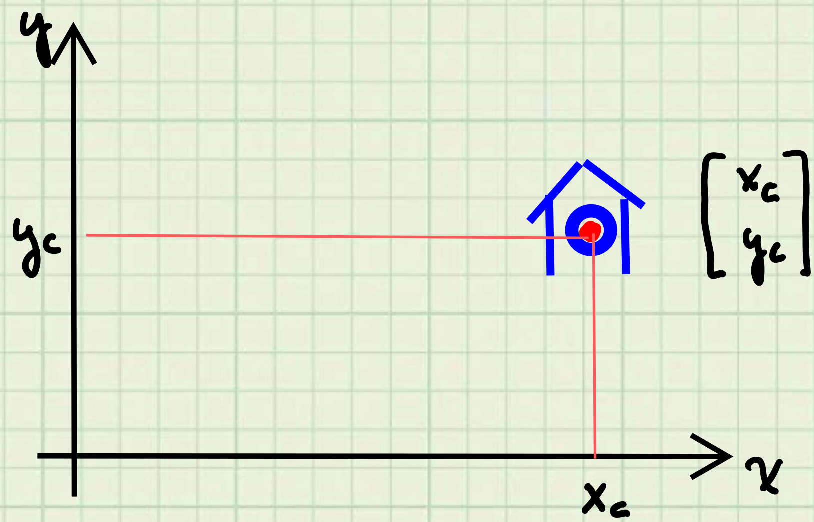
$$L = [L, C] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$U = [U; R] = \begin{bmatrix} -2 & -4 & -6 \\ 0 & 5 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

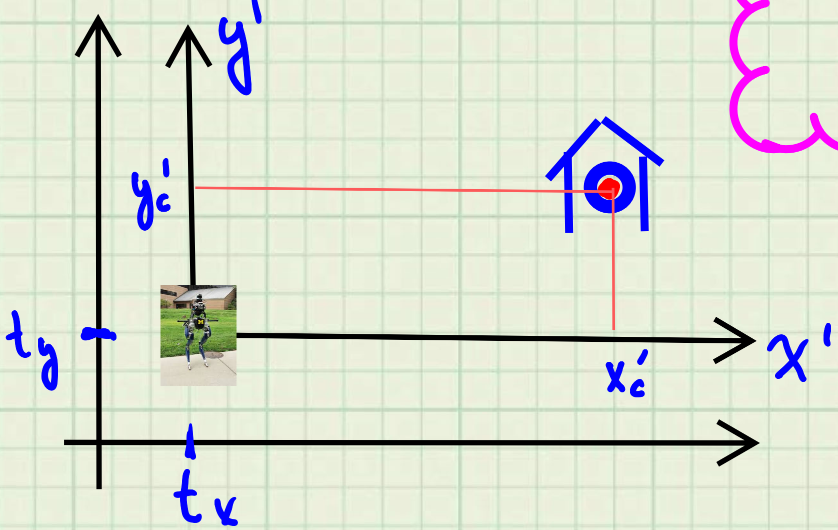
$$\therefore \quad M = L \cdot U$$

$$\begin{bmatrix} -2 & -4 & -6 \\ -4 & 1 & -4 \\ -6 & 11 & -4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} -2 & -4 & -6 \\ 0 & 5 & 2 \\ 0 & 0 & -4 \end{bmatrix}}_{U}$$

Project #1



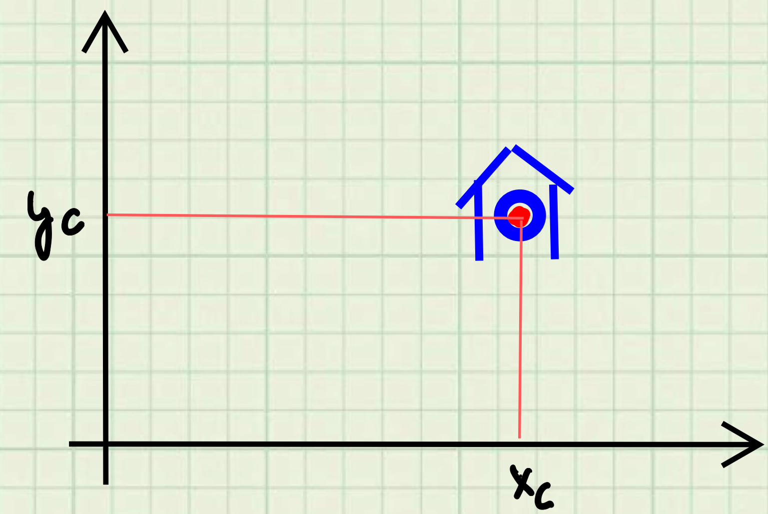
Translation



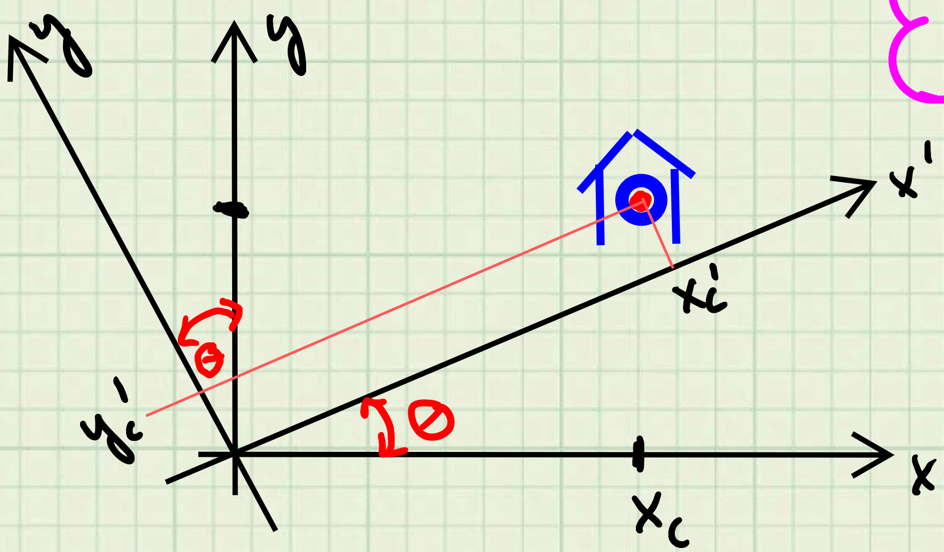
Cassie measures the object at $\begin{bmatrix}x_c'\\y_c'\end{bmatrix}$, but the object is "really" at $\begin{bmatrix}x_c\\y_c\end{bmatrix}$. We know Cassie is at $\begin{bmatrix}t_x\\t_y\end{bmatrix}$.

What is the true position of the object?

$$\begin{bmatrix}x_c\\y_c\end{bmatrix} = \begin{bmatrix}x_c'\\y_c'\end{bmatrix} + \begin{bmatrix}t_x\\t_y\end{bmatrix}.$$



Rotation



Cassie has turned by an angle of θ, and now "sees" the object at $\begin{bmatrix} x_c' \\ y_c' \end{bmatrix}$.

Where is the object in the "world frame"?

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_c' \\ y_c' \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

called a rotation matrix

The sensor [Inertial Measurement Unit or IMU] and algorithm [Invariant Kalman Filter] are amazing in their own right. They are treated in ROB 501 & ROB 530. Robotics faculty would like to offer undergrad versions of that material.