ROB 101 - Fall 2021

The Vector Space $\mathbb{R}^n$: Part 1

September 29, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- Instead of working with individual vectors, we will work with a collection of vectors.

- Our first encounter with some of the essential concepts in Linear Algebra that go beyond systems of equations.

Outcomes

- Vectors as n-tuples of real numbers
- $\mathbb{R}^n$ as the collection of all n-tuples of real numbers
- Linear combinations of vectors
- Linear independence of vectors
- Relation of these concepts to the existence and uniqueness of solutions to $Ax = b$.
- LU Factorization to check the linear independence of a set of vectors, and LDLT to check if one vector is a linear combination of a set of vectors.

Uniqueness of Solutions to $Ax = b$

- We have a matrix equation, $Ax = b$, we understand that $\bar{x}$ is a solution to the equation if, and only if, $A\bar{x} - b = 0$.

Uniqueness of Solutions to $Ax = b$

- We have a matrix equation, $Ax = b$, we understand that $\bar{x}$ is a solution to the equation if, and only if, $A\bar{x} - b = 0$.

- Now suppose there exists a solution $\alpha \neq 0$ for the matrix equation $A\alpha = 0$.

Uniqueness of Solutions to $Ax = b$

- We have a matrix equation, $Ax = b$, we understand that $\bar{x}$ is a solution to the equation if, and only if, $A\bar{x} - b = 0$.

- Now suppose there exists a solution $\alpha \neq 0$ for the matrix equation $A\alpha = 0$.

- By summing two equations,

$$A\bar{x} + A\alpha - b = 0$$
$$A(\bar{x} + \alpha) - b = 0$$

Uniqueness of Solutions to $Ax = b$

- We have a matrix equation, $Ax = b$, we understand that $\bar{x}$ is a solution to the equation if, and only if, $A\bar{x} - b = 0$.

- Now suppose there exists a solution $\alpha \neq 0$ for the matrix equation $A\alpha = 0$.

- By summing two equations,

$$A\bar{x} + A\alpha - b = 0$$
$$A(\bar{x} + \alpha) - b = 0$$

- We conclude that $\bar{x} + \alpha$ must be a solution too!

Remark

Hence, we're motivated to study the solutions of $A\alpha = 0$. The system of linear equations $A\alpha = 0$ is called homogeneous.

Remark

Hence, we're motivated to study the solutions of $A\alpha = 0$. The system of linear equations $A\alpha = 0$ is called homogeneous.

$\alpha = 0$ is always a solution of $A\alpha = 0$. If there is a solution such that $\alpha \neq 0$, then it is called a nontrivial solution.

Consider $Ax = b$, where A is an $n \times m$ matrix and x is an $m \times 1$ vector.

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \right].$$

Using our column times row method for matrix multiplication, we have

$$Ax = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_m \end{array}\right]$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} x_m.$$

Linear Combinations

$$
\begin{aligned}
Ax &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \\
&= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} x_m \\
&= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix}.
\end{aligned}
$$

For purely psychological reasons, let's replace x_i with α_i so we can think of them as numerical values instead of variables.

$$\begin{aligned} A\alpha &= A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + \alpha_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix}. \end{aligned}$$

Definition

The following sum of scalars times vectors,

$$\alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + \alpha_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix},$$

is called a *linear combination* of the columns of A.

Linear Combination of the Columns of A

Let's set $A\alpha = b$, and turn it around as $b = A\alpha$.

Linear Combination of the Columns of A

Let's set $A\alpha = b$, and turn it around as $b = A\alpha$.

Fact

A vector $\alpha \in \mathbb{R}^m$ is a solution to $Ax = b$ (that is, $A\alpha = b$) if, and only if

$$b = \alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + \alpha_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix},$$

that is, b can be expressed as a linear combination of the columns of A.

A vector $v \in \mathbb{R}^n$ is a *linear combination of* $\{u_1, u_2, \ldots, u_m\} \subset \mathbb{R}^n$ if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m.$$

A vector $v \in \mathbb{R}^n$ is a *linear combination of* $\{u_1, u_2, \ldots, u_m\} \subset \mathbb{R}^n$ if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m.$$

Remark

$A \subset B$ means A is a subset of B. This simply means that B includes or contains A. For example, $\{1,2\} \subset \{1,2,3\}$.

Because

$$\underbrace{\begin{bmatrix}-3\\-5\\-7\end{bmatrix}}_{v} = 2\underbrace{\begin{bmatrix}3\\2\\1\end{bmatrix}}_{u_1} -9\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{u_2},$$

we have that v is a linear combination of $\{u_1, u_2\}$.

Example

Is the vector $v = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ a linear combination of

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}?$$

- We need to develop a way to check if a vector is, or is not, a linear combination of other vectors! (Why?)

Existence of Solutions to $Ax = b$

- We need to develop a way to check if a vector is, or is not, a linear combination of other vectors! (Why?)
- Because the equation $Ax = b$ has a solution iff, b can be written as a linear combination of the columns of A.

Linear Independence of a Set of Vectors

- The set of vectors $\{v_1, v_2, ..., v_m\} \subset \mathbb{R}^n$ is *linearly dependent* if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ *not all zero* yielding a linear combination of vectors that adds up to the zero vector,

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0_{n\times 1}.$$

Linear Independence of a Set of Vectors

- On the other hand, the vectors $\{v_1, v_2, ..., v_m\}$ are *linearly independent* if the *only* real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ yielding a linear combination of vectors that adds up to the zero vector,

  $$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0_{n\times 1},$$

  are $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_m = 0.$

Concise Definition of Linear Independence

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0_{n\times 1} \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0_{m\times 1}.$$

Remark

We note that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0_{n\times 1}$ *corresponds to*

$$A\alpha = 0.$$

Linear Independence through the Lens of $Ax = 0$

Remark

We note that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0_{n\times 1}$ *corresponds to*

$$A\alpha = 0.$$

Where $A := \begin{bmatrix} v_1 & v_2 & \ldots & v_m \end{bmatrix}_{n\times m}$, *and* $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}_{m\times 1}$.

Example

By applying the definition, determine if the set of vectors

$$v_1 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

is linearly independent or dependent.

$$v_1 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

The definition says $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{3\times 1}$.

$$v_1 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

The definition says $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{3\times 1}$.

We form $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 4 & 3 \\ 0 & 7 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$.

We wish to fine the nontrivial solutions to $A\alpha = 0$.

$$A = \begin{bmatrix} \sqrt{2} & 4 & 3 \\ 0 & 7 & 1 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}.$$

Fortunately, A is in the upper-triangular form. We can use back substitution.

$$A\alpha = \begin{bmatrix} \sqrt{2} & 4 & 3 \\ 0 & 7 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0.$$

It is clear that the only solution to the bottom equation is $\alpha_3 = 0$, the only solution to the middle equation is then $\alpha_2 = 0$, and finally, the only solution to the top equation is $\alpha_1 = 0$. Hence, the only solution to

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

is $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$, and hence the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent.

$$A\alpha = \begin{bmatrix} \sqrt{2} & 4 & 3 \\ 0 & 7 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0.$$

A Pro Tip for Checking Linear Independence

Consider $A_{n\times m} \cdot \alpha_{m\times 1} = 0_{n\times 1}$.

A Pro Tip for Checking Linear Independence

Consider $A_{n\times m} \cdot \alpha_{m\times 1} = 0_{n\times 1}$.

- We multiply both sides from left by A^T.

A Pro Tip for Checking Linear Independence

Consider $A_{n \times m} \cdot \alpha_{m \times 1} = 0_{n \times 1}$.

- We multiply both sides from left by A^{T}.
- We get $A^{\mathsf{T}} A \alpha = 0$.

A Pro Tip for Checking Linear Independence

Consider $A_{n\times m} \cdot \alpha_{m\times 1} = 0_{n\times 1}$.

- We multiply both sides from left by A^T.
- We get $A^\mathsf{T} A\alpha = 0$.
- We note that $A^\mathsf{T} A$ is an $m \times m$ (square) matrix.

A Pro Tip for Checking Linear Independence

Let $y = A\alpha$. We then note the following chain of implications

$$
\begin{aligned}
(A\alpha = 0) \implies \left(A^\top \cdot A\alpha = 0\right) &\implies \left(\alpha^\top A^\top \cdot A\alpha = 0\right) \\
&\implies \left((A\alpha)^\top \cdot (A\alpha) = 0\right) \implies (A\alpha = 0)\,,
\end{aligned}
$$

where the last implication follows from

$$y = 0_{n\times 1} \iff y^\top y = 0.$$

A Pro Tip for Checking Linear Independence

Let $y = A\alpha$. We then note the following chain of implications

$$
\begin{aligned}
(A\alpha = 0) \implies \big(A^\top \cdot A\alpha = 0\big) &\implies \big(\alpha^\top A^\top \cdot A\alpha = 0\big) \\
&\implies \Big((A\alpha)^\top \cdot (A\alpha) = 0\Big) \implies (A\alpha = 0)\,,
\end{aligned}
$$

where the last implication follows from

$$y = 0_{n\times 1} \iff y^\top y = 0.$$

From logic, we know that when we have

$$(a) \implies (b) \implies (c) \implies (d) \implies (a),$$

a chain of implications that begins and ends with the same proposition, then we deduce that

$$(a) \iff (b) \iff (c) \iff (d).$$

A Pro Tip for Checking Linear Independence

Corollary

$$\boxed{A\alpha = 0 \iff \left(A^\top A\right)\alpha = 0.}$$

Summary of the Pro Tip for Checking Linear Independence

Consider the vectors in $\mathbb{R}^n$,

$$\left\{ v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, ..., v_m = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} \right\},$$

Summary of the Pro Tip for Checking Linear Independence

Consider the vectors in $\mathbb{R}^n$,

$$\left\{ v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, ..., v_m = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} \right\},$$

and use them as the columns of a matrix that we call A,

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{array} \right].$$

Summary of the Pro Tip for Checking Linear Independence

The following statements are equivalent:

- The set of vectors $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

Summary of the Pro Tip for Checking Linear Independence

The following statements are equivalent:

- The set of vectors $\{v_1, v_2, \dots, v_m\}$ is linearly independent.
- The $m \times m$ matrix $A^\mathsf{T} \cdot A$ is invertible.

Summary of the Pro Tip for Checking Linear Independence

The following statements are equivalent:

- The set of vectors $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.
- The $m \times m$ matrix $A^\mathsf{T} \cdot A$ is invertible.
- $\det(A^\mathsf{T} \cdot A) \neq 0$.

Summary of the Pro Tip for Checking Linear Independence

The following statements are equivalent:

- The set of vectors $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.
- The $m \times m$ matrix $A^\mathsf{T} \cdot A$ is invertible.
- $\det(A^\mathsf{T} \cdot A) \neq 0$.
- For any LU Factorization $P \cdot (A^\mathsf{T} \cdot A) = L \cdot U$ of $A^\mathsf{T} A$, the $m \times m$ upper triangular matrix U has no zeros on its diagonal.

By applying the definition, determine if the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 6 \\ 2 \\ -3 \end{bmatrix}$$

are linearly independent or dependent.

We use the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 6 \\ 2 \\ -3 \end{bmatrix}$$

and form the matrix

$$A := \begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 6 \\ 3 & 4 & 2 \\ 1 & 5 & -3 \end{bmatrix}.$$

We go to Julia and compute that

$$A^\top \cdot A = \begin{bmatrix} 15.0 & 13.0 & 17.0 \\ 13.0 & 45.0 & -19.0 \\ 17.0 & -19.0 & 53.0 \end{bmatrix},$$

and that its LU Factorization is $P \cdot \left(A^\top \cdot A\right) = L \cdot U$, where

$$P = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}, \quad L = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.8 & 1.0 & 0.0 \\ 0.9 & 0.5 & 1.0 \end{bmatrix},$$

and

$$U = \begin{bmatrix} 17.0 & -19.0 & 53.0 \\ 0.0 & 59.5 & -59.5 \\ 0.0 & 0.0 & \boxed{0.0} \end{bmatrix}.$$

and that its LU Factorization is $P \cdot \left(A^\top \cdot A\right) = L \cdot U$, where

$$P = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}, \quad L = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.8 & 1.0 & 0.0 \\ 0.9 & 0.5 & 1.0 \end{bmatrix},$$

and

$$U = \begin{bmatrix} 17.0 & -19.0 & 53.0 \\ 0.0 & 59.5 & -59.5 \\ 0.0 & 0.0 & \boxed{0.0} \end{bmatrix}.$$

We observe that U has a zero on its diagonal and hence the set $\{v_1, v_2, v_3\}$ is linearly dependent.

Next Time

- The Vector Space $\mathbb{R}^n$: Part 1 (To be continued ...)
- Read Chapter 7 of ROB 101 Book