HW # 07 Solutions: Supplement

1. The notion of basis vectors for a subspace threw me for a loop. Can you help me out?

   Answer: Are you OK with the idea that every vector in $\mathbb{R}^n$ can be expressed as a linear combination of the natural basis vectors $\{e_1, e_2, \ldots, e_n\}$,

   $$(x_1, x_2, \ldots, x_n) \longleftrightarrow x_1e_1 + x_2e_2 + \cdots + x_ne_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}?$$

   If not, come see us! How about the idea that a line angled at 45^o in $\mathbb{R}^2$ is a subspace and it is spanned by $v_1 = e_1 + e_2$. More generally, a line angled at θ radians would be spanned by $v_1 = \cos(\theta)e_1 + \sin(\theta)e_2$, which, for θ not equal to a multiple of $\pi/2$, we could also write as the span of $\widetilde{v}_1 = e_1 + \frac{\sin(\theta)}{\cos(\theta)}e_2$. Not working, come see us!

   Next, we take an $n \times m$ matrix A and define $V :=$ null(A). If the columns of A are not linearly independent, then $V \subset \mathbb{R}^m$ contains an (unaccountably) infinite number of m-vectors. Could we even enter an uncountable number of things into Julia for doing computations? **No way! No how!** Because V is a subset of $\mathbb{R}^m$, its dimension is less than or equal to m. However, by using the notion of basis vectors, we reduce the problem to entering in a list of basis vectors for V. And if we mess up and enter too many vectors from V, we can use LDLT to remove linearly dependent elements and get back to having a basis!

   Bottom line: All subspaces of dimension greater than or equal to one contain an infinite number of vectors. A basis is a finite means of generating a subspace. A basis of a subspace is a *Goldilock's* kind of idea: it is big enough such that its span generates the subspace and small enough that it is linearly independent.

2. How can a column span have fewer elements than there are columns in the matrix?

   Answer: Consider $A = \begin{bmatrix} -15.0 & 1.0 \\ 15.0 & -1.0 \end{bmatrix}$. While A has two columns, they are linearly dependent. We note that

   $$\text{col span}\{A\} = \text{span}\{\begin{bmatrix} -15.0 \\ 15.0 \end{bmatrix}, \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix}\} = \text{span}\{\begin{bmatrix} -15.0 \\ 15.0 \end{bmatrix}\} = \text{span}\{\begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix}\} = \text{span}\{\begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}\}$$

   for all $\alpha \neq 0$.

3. Utility of Range/rank and Null Space/nullity? Why do we need them?

   There are many layers to this question!

   (a) **Answer 1:** In terms of uniqueness of solution to $Ax = b$, where A is $n \times m$:

   If $Ax = b$ has a solution, then the following statements are equivalent:
   - the solution is unique
   - the columns of A are linearly independent
   - rank$(A) = m$, the number of columns of A
   - null$(A) = \{0_{m\times 1}\}$, the zero subspace of $\mathbb{R}^m$
   - nullity$(A) = 0$

   The first time you encounter the fact that rank$(A) = m \iff$ nullity$(A) = 0$, it seems very surprising. However, once you understand that

   $$\alpha_{m\times 1} \in \text{null}(A) \iff \alpha_1 a_1^{\text{col}} + \cdots + \alpha_m a_m^{\text{col}} = 0_{n\times 1},$$

   it all starts to make sense. The null space is giving the set of all linear combinations of the columns of A that yield the zero vector in $\mathbb{R}^n$.

(b) **Answer 2:** In terms of existence of solutions to $Ax = b$, where A is $n \times m$:

The following statements are equivalent:

- $Ax = b$ has a solution for all $b \in \mathbb{R}^n$
- the col span$\{A\} = \mathbb{R}^n$
- the range$(A) = \mathbb{R}^n$
- rank$(A) = n$, the number of rows of A

(c) **Answer 3:** In terms of understanding the set of all solutions to $Ax = b$, where A is $n \times m$:

If $\bar{x} \in \mathbb{R}^m$ is ANY solution to $Ax = b$, then the set of ALL solutions is given by

$$\{ \text{ set of all solutions } \} := \{x \in \mathbb{R}^m \mid Ax = b\} = \bar{x} + \text{null}(A) = \{\bar{x} + x \mid x \in \mathbb{R}^m \text{ and } Ax = 0_{n\times 1}\}$$

4. Why is the Rank Nullity Theorem important?

Answer : The Rank Nullity Theorem states that the columns of A can be divided into a first set of column vectors that are linearly independent and a second set of vectors that are dependent on the first set of vectors. The rank of A is the number of column vectors in the first set, while the nullity of A is the number of column vectors in the second set.