Review

- Lower Triangular System & Matrix

$$\begin{aligned} 2x_1 &= 4 \\ 3x_1 - 2x_2 &= 6 \\ x_1 - 3x_2 + 5x_3 &= 7 \end{aligned} \quad \longleftrightarrow \quad \begin{bmatrix} 2 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} x_3 = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} & 0 \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}}_{A}$$

- Zero above the diagonal
- $\det(A) = a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}$

$$\boxed{a_{ij} = 0 \quad \text{for } j > i}$$

$$\begin{bmatrix} 1 & \boxed{4} & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \boxed{7} \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$$

(a) No (b) Yes (c) No

Which are lower triangular?

Forward Substitution & Lower Triangular Systems

$$\begin{aligned} 2x_1 &= 2 \\ -x_1 - 3x_2 &= -7 \\ 6x_1 + 7x_2 + 8x_3 &= 44 \end{aligned} \quad \longleftrightarrow \quad \begin{bmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 44 \end{bmatrix}$$

Lower triangular &
det $\neq 0$

Isolate x_1, x_2, x_3:

$$2x_1 = 2 \quad \Rightarrow \quad x_1 = 1$$

$$-3x_2 = -7 + \boxed{x_1} \quad \Rightarrow \quad x_2 = \frac{-7+(1)}{-3} = 2$$

$$8x_3 = 44 - 6x_1 - 7x_2 \quad \Rightarrow \quad x_3 = \frac{44 - 6(1) - 7(2)}{8}$$

$$= 3$$

The general form of a lower triangular system with a non-zero determinant is

$$\begin{aligned} a_{11}x_1 &= b_1 \quad (a_{11} \neq 0) \\ a_{21}x_1 + a_{22}x_2 &= b_2 \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \quad (a_{nn} \neq 0) \end{aligned}$$

and the solution proceeds from top to bottom, like this

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \quad (a_{11} \neq 0) \\ x_2 &= \frac{b_2 - a_{21}x_1}{a_{22}} \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ x_n &= \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}}{a_{nn}} \quad (a_{nn} \neq 0). \end{aligned}$$

When can forward substitution go wrong?

Ans When $\det(A) = 0$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & \boxed{0} & 0 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$2x_1 = 4 \Rightarrow x_1 = 2$

$0\,x_2 = 5 - 3x_1 \Rightarrow x_2 = \dfrac{5-3(2)}{0}$

Julia goes nuts

Check diag(A) does not contain zero.

Upper Triangular

$$
\begin{aligned}
2x_1 + 3x_2 + 4x_3 &= 20 \\
-x_2 + 2x_3 &= 4 \\
-2x_3 &= -6
\end{aligned}
$$

$\Updownarrow$

$$
\underbrace{\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \\ -6 \end{bmatrix}
$$

All entries below the diagonal are zero.

- $\square$ det(A) = product of all elements on the diagonal

$$\det(A) = 2 \cdot (-1)(-2) \neq 0$$

Solution Method: Called back or backward substitution:

start at the bottom and work to the top!

Isolate x_3, x_2, x_1

$$-2x_3 = -6 \Rightarrow x_3 = 3$$

$$-x_2 = 4 - 2x_3 \Rightarrow x_2 = \frac{4-2(3)}{-1} = 2$$

$$2x_1 = 20 - 3x_2 - 4x_3 \Rightarrow x_1 = \frac{20-3(2)-4(3)}{2} = 1$$

The general form of an upper triangular system with a non-zero determinant is

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \quad (a_{11} \neq 0) \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \quad (a_{22} \neq 0) \\
a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \quad (a_{33} \neq 0) \\
\vdots &= \vdots \\
a_{nn}x_n &= b_n \quad (a_{nn} \neq 0),
\end{aligned}
$$

and the solution proceeds from bottom to top, like this,

Back substitution ✓

$$
\begin{aligned}
x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} && (a_{11} \neq 0) \\
x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} && (a_{22} \neq 0) \\
\vdots &= \vdots && \vdots \\
x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} && (a_{n-1,n-1} \neq 0) \\
x_n &= \frac{b_n}{a_{nn}} && (a_{nn} \neq 0),
\end{aligned}
$$

Matrix Multiplication

Learning Objectives

- How to partition matrices into rows and columns
- How to multiply two matrices
- How to swap rows of a matrix

Outcomes

- Multiplying a row vector by a column vector.
- Recognizing the rows and columns of a matrix
- Standard definition of matrix multiplication $A \cdot B$ using the rows of A and the columns of B
- Size restrictions when multiplying matrices
- Examples that work and those that don't because the sizes are wrong
- Introduce a second way of computing the product of two matrices using the columns of A and the rows of B. This will later be used to compute the LU decomposition in a very simple way.
- Permutation matrices

Multiplying a Row Vector times a Column Vector

$a^{row} = [a_1 \quad a_2 \quad \cdots \quad a_k]$ $\quad 1 \times k$ matrix

$$b^{col} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$$ $\quad k \times 1$ matrix

with the same number of elements.

The product $a^{row} \cdot b^{col}$ is defined as follows

$$a^{row} \cdot b^{col} := \sum_{i=1}^{k} a_i b_i := a_1 b_1 + a_2 b_2 + \cdots + a_k b_k$$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} := a_1 b_1 + a_2 b_2 + \cdots a_k b_k$$

Examples

$a^{row} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$, $b^{col} = \begin{bmatrix} -1 \\ +2 \\ -3 \end{bmatrix}$

$a^{row} \cdot b^{col} = 3(-1) + 2(2) + 1(-3) = -2$

$a^{row} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$; $b^{col} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$a^{row} \cdot b^{col} =$ undefined (error) (nonsense!)

END.