ROB 101 - Fall 2021

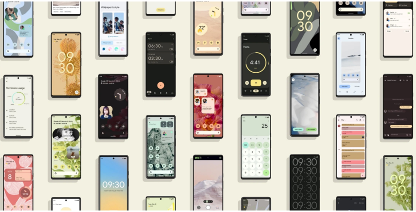
Introduction to Systems of Linear Equations

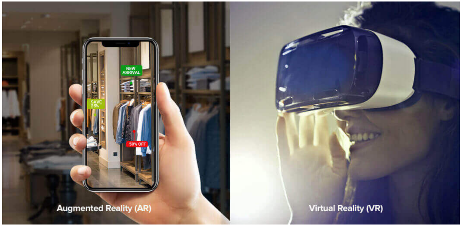
August 30, 2021

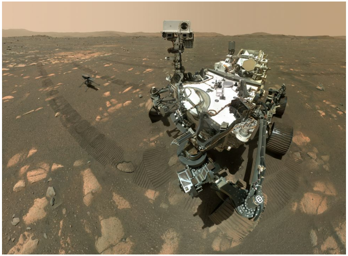
UNIVERSITY OF MICHIGAN

Why (Computational) Linear Algebra?

Why (Computational) Linear Algebra?







Why (Computational) Linear Algebra?

- Building Rome in a Day
  https://grail.cs.washington.edu/rome/
- Matrices and Art
  https://people.engr.tamu.edu/davis/matrices.html
- ...

Why (Computational) Linear Algebra?

- Linear algebra is the workhorse of modern science and engineering.
- Often when our problems are reduced to linear algebra, we can actually solve them (problem formulation is half of the job!).
- Using a computational view of linear algebra, we can solve large-scale real life problems.

What makes you attractive (as a freshman)?

What makes you attractive (as a freshman)?

What makes you attractive (as a freshman)?

- Infinite potential!

What makes you attractive (as a freshman)?

- Infinite potential!
- Time is your biggest asset. Then linear algebra and programming!

Today's Learning Objectives

- An introduction to refresh your knowledge about systems of linear equations;
- Set the stage for cool things to come.

Outcomes

- Examples of systems of linear equations with two unknowns.

- Show that three things are possible when seeking a solution:
  1. there is one and only one solution (one says there is a unique solution, which is shorthand for "there is a solution and it is unique");
  2. there is no solution (at all);
  3. there are an infinite number of solutions.

- Notation that will allow us to have as many unknowns as we'll need.

Review: You Have Done Algebra Before

- Quadratic equation $ax^2 + bx + c = 0$;
- Discriminant $\Delta = b^2 - 4ac$;
- $\Delta > 0 \Leftrightarrow$ 2 distinct real roots (solutions);
- $\Delta = 0 \Leftrightarrow$ repeated roots;
- $\Delta < 0 \Leftrightarrow$ 2 complex roots;
- $x^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Linear Systems, Solutions

A system of linear equations can have

- No solution;
- Unique solution (one and only one solution);
- Infinite number of solutions.

We first look into the case where there are the same number of equations as unknowns, and there is a unique solution.

Consider the following example:

$$\begin{aligned} x + y &= 4 \\ 2x - y &= -1 \end{aligned}$$

Linear Systems of Equations: Case I

$$x + y = 4$$
$$2x - y = -1$$

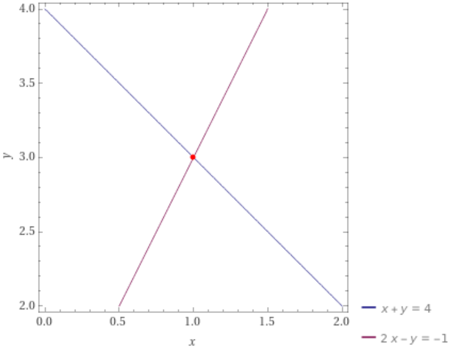


Figure: Graphical solution. Unique solution $\Leftrightarrow$ intersecting lines!

One way to compute a solution is to solve for x in terms of y in the first equation and then substitute that into the second equation,

$$
\begin{aligned}
x + y = 4 &\implies x = 4 - y \\
2x - y = -1 &\implies 2(4 - y) - y = -1 \\
&\implies -3y = -9 \\
&\implies y = 3
\end{aligned}
$$

going back to the top

$$x = 4 - y \implies x = 1$$

One way to compute a solution is to solve for x in terms of y in the first equation and then substitute that into the second equation,

$$\begin{aligned} x + y = 4 &\implies x = 4 - y \\ 2x - y = -1 &\implies 2(4 - y) - y = -1 \\ &\implies -3y = -9 \\ &\implies y = 3 \\ &\text{going back to the top} \\ x = 4 - y &\implies x = 1 \end{aligned}$$

Remark

You can try on your own solving for y in terms of x and repeating the above steps. You will obtain the same answer, namely $x = 1, y = 3$.

Another "trick" you can try, just to see if we can generate a different answer, is to add the first equation to the second, which will eliminate y,

$$\begin{array}{r} x + y = 4 \\ +\ 2x - y = -1 \\ \hline 3x + 0y = 3 \end{array}$$

$$\implies x = 1$$

Going back to the top and using either of the two equations

$$\begin{aligned} x + y = 4 &\implies y = 4 - x \\ &\implies y = 3 \end{aligned}$$

or

$$\begin{aligned} 2x - y = -1 &\implies -y = -2x - 1 \\ &\implies -y = -3 \\ &\implies y = 3 \end{aligned}$$

gives the same answer as before, namely, $x = 1, y = 3$.

- In fact, the set of equations has one, and only one, solution. Mathematically speaking, we say the set of equations has a *unique solution*.

- Often, we will stack x and y together and write the solution as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Same number of equations as unknowns, and there is no solution:

$$x - y = 1$$
$$2x - 2y = -1$$

Linear Systems of Equations: Case II

Same number of equations as unknowns, and there is no solution:

$$\begin{aligned} x - y &= 1 \\ 2x - 2y &= -1 \end{aligned}$$

The left-hand side of the second equation is twice the left-hand side of the first equation, namely, $2x - 2y = 2(x - y)$, but when we look to the right-hand sides, $-1 \neq 2 \cdot 1$, and hence the equations are *inconsistent*.

Linear Systems of Equations: Case II

$$\begin{aligned} x - y &= 1 \\ 2x - 2y &= -1 \end{aligned}$$

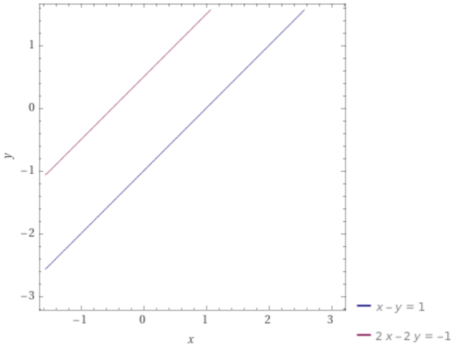


Figure: No solution ⇔ parallel lines!

Let's try to find a solution just as we did for the first set of equations,

$$\begin{aligned} x - y = 1 &\implies x = y + 1 \\ 2x - 2y = -1 &\implies 2(y+1) - 2y = -1 \\ &\implies 2 + 2y - 2y = -1 \\ &\implies 2 = -1. \end{aligned}$$

Let's try to find a solution just as we did for the first set of equations,

$$\begin{aligned} x - y = 1 &\implies x = y + 1 \\ 2x - 2y = -1 &\implies 2(y+1) - 2y = -1 \\ &\implies 2 + 2y - 2y = -1 \\ &\implies 2 = -1. \end{aligned}$$

Hence, trying to solve the equations has led us to a *contradiction*, namely $2 = -1$.

Linear Systems of Equations: Case III

Same number of equations as unknowns, and there is an infinite number of solutions:

$$\begin{aligned} x - y &= 1 \\ 2x - 2y &= 2 \end{aligned}$$

Linear Systems of Equations: Case III

Same number of equations as unknowns, and there is an infinite number of solutions:

$$\begin{aligned} x - y &= 1 \\ 2x - 2y &= 2 \end{aligned}$$

This time the left-hand side of the second equation is twice the left-hand side of the first equation, namely, $2x - 2y = 2(x - y)$.

Linear Systems of Equations: Case III

Same number of equations as unknowns, and there is an infinite number of solutions:

$$\begin{aligned} x - y &= 1 \\ 2x - 2y &= 2 \end{aligned}$$

Remark

When we look to the right-hand side, $2 = 2 \cdot 1$, and hence the two equations are the "same" in the sense that is one equation can be obtained from the other equation by multiplying both sides of it by a non-zero constant.

Linear Systems of Equations: Case III

$$\begin{aligned} x - y &= 1 \\ 2x - 2y &= 2 \end{aligned}$$

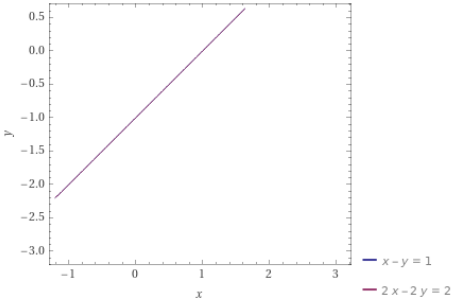


Figure: Infinite number of solutions ⇔ coincident lines!

We solve for x in terms of y in the first equation and then substitute that into the second equation,

$$\begin{aligned} x - y = 1 &\implies x = y + 1 \\ 2x - 2y = 2 &\implies 2(y+1) - 2y = 2 \\ &\implies 2y + 2 - 2y = 2 \\ &\implies 2 = 2. \end{aligned}$$

We solve for x in terms of y in the first equation and then substitute that into the second equation,

$$\begin{aligned} x - y = 1 &\implies x = y + 1 \\ 2x - 2y = 2 &\implies 2(y+1) - 2y = 2 \\ &\implies 2y + 2 - 2y = 2 \\ &\implies 2 = 2. \end{aligned}$$

The conclusion, $2 = 2$, is perfectly correct, but tells us nothing about y.

Linear Systems of Equations: Case III

- In fact, we can view the value of y as an arbitrary *free parameter* and hence the solution is

$$x = y + 1, \quad -\infty < y < \infty.$$

- In fact, we can view the value of y as an arbitrary *free parameter* and hence the solution is

$$x = y + 1, \quad -\infty < y < \infty.$$

- The solution can also be expressed as

$$y = x - 1, \quad -\infty < x < \infty,$$

  which perhaps inspires you to plot the solution as a line in $\mathbb{R}^2$, with slope $m = 1$ and y-intercept $b = -1$.

Consider a set of two equations with two unknowns x and y

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2,$$

and constants $a_{11}, a_{12}, a_{21}, a_{22}$ and b_1, b_2.

Summary

Consider a set of two equations with two unknowns x and y

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2,$$

and constants $a_{11}, a_{12}, a_{21}, a_{22}$ and b_1, b_2.

Depending on the values of the constants, the linear equations can have:

- a unique solution,
- no solution,
- or an infinity of solutions.

- The same is true for n linear equations in n unknowns. What if, e.g., $n = 50$?
- We don't have enough letters in the alphabet to name all variables, x,y,z,w,q, $\ldots$.
- Using xx, xxx, $zzzz$ will be extremely ugly and confusing.

- The same is true for n linear equations in n unknowns. What if, e.g., $n = 50$?
- We don't have enough letters in the alphabet to name all variables, x,y,z,w,q, $\ldots$.
- Using xx, xxx, $zzzz$ will be extremely ugly and confusing.
- Solution: We use indexing; x_1,x_2, $\ldots$, x_n. Problem solved!

Next Time

- Vectors, Matrices, and Matrix Determinant
- Read Chapter 2 of ROB 101 Book