2020 Oct. 19 ROB 101

Summary: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}$

$$\text{range}(A) = \text{colspan}(A) = \text{span}\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} \right\} \subset \mathbb{R}^n$$

$$\text{null}(A) = \{x \in \mathbb{R}^m \mid Ax = 0\} = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \middle| \alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + \alpha_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

$\subset \mathbb{R}^m$

$\text{rank}(A) := \dim \text{range}(A)$ Number indep columns/vectors

$\text{nullity}(A) := \dim \text{null}(A)$ Number dependent cols/vectors

Useful Properties of Rank and Nullity

For an $n \times m$ matrix A, the following are true:

(a) $\text{rank}(A) + \text{nullity}(A) = m$, the number of columns in A.

$A\alpha = 0 \iff A^\top A\alpha = 0$

(b) $\text{nullity}(A^\top \cdot A) = \text{nullity}(A)$.

(c) $\text{rank}(A^\top \cdot A) = \text{rank}(A)$.

(d) $\text{rank}(A \cdot A^\top) = \text{rank}(A^\top)$.

(e) For any $m \times k$ matrix B, $\text{rank}(A \cdot B) \leq \text{rank}(A)$.

$\boxed{\text{(f) } \text{rank}(A^\top) = \text{rank}(A).}$

(g) $\text{rank}(A^\top \cdot A) = \text{rank}(A \cdot A^\top)$.

(h) $\text{nullity}(A^\top) + m = \text{nullity}(A) + n$.

Because $\text{rank}(A) = \text{rank}(A^T)$ we have
$\text{rank}(A) \le \min(n, m)$

Def. $V \subset \mathbb{R}^n$ a subspace. A set $\{v_1, \dots, v_k\}$ is a basis for V if

(a) $\{v_1, \dots, v_k\}$ linearly indep. (not too big)

(b) $\text{span}\{v_1, \dots, v_k\} = V$ (not too small)

Not too big: all dependent vectors have been removed

Not too small: can generate everything in V through linear combinations

(c) dim of V is number of vectors in a basis.

$$A = \begin{bmatrix} 1 & 4 & 7 & 10 & 13 \\ 2 & 5 & 8 & 11 & 14 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix}$$

$\text{rank}(A) \le \min(3, 5) \le 3$

Fun Geometry in $\mathbb{R}^n$

Hard work is done! Let's reap our rewards?!

<u>Def.</u> Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$ and $v \in \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. The

dot product of u and v is

$$u \cdot v := \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Remark: $u^T \cdot v = \underbrace{[u_1 \ u_2 \ \cdots \ u_n]}_{1\times n} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{n\times 1} = \quad (1\times 1)$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\boxed{\therefore \ u \cdot v = u^T v}$$

Another name for the dot product of two vectors is inner product

Another notation is $\langle u, v \rangle = u \cdot v$

Compute the dot product of

$u = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

$u \cdot v = (1)(2) + (0)(4) + (3)(1) = 5$

Compute the inner product of

$u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

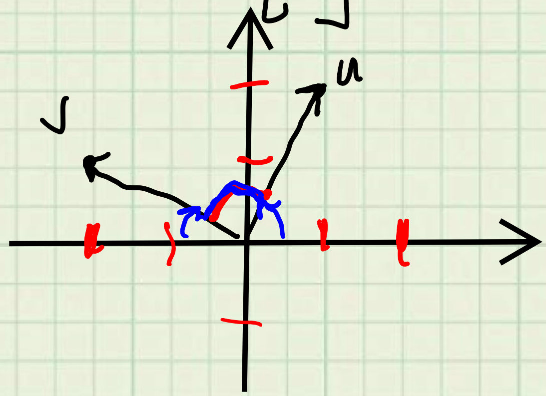
$u^T v = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = (1)(0) + (0)(1) + (-1)(0) + (0)(1) = 0$

<u>Def.</u> Two vectors u and v in $\mathbb{R}^n$ are orthogonal if $u \cdot v = 0$

Notation: $u \perp v$ (read as u is perpendicular to v)

or (read as u is orthogonal to v)

Example: $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



$u \cdot v = (1)(-2) + (2)(1) = 0$

Fact: In $\mathbb{R}^2$, $u \perp v \iff u$ and v form a right angle or at least one is the zero vector.

This is used to generalize angles to $\mathbb{R}^n$, but we'll not do that.

Properties

- $(u+v) \cdot w = u \cdot w + v \cdot w$

why: $(u+v)^T w = (u^T + v^T) w = u^T w + v^T w$
$= u \cdot w + v \cdot w$

- $\alpha \in \mathbb{R}$ $(\alpha u) \cdot v = \alpha \, u \cdot v$

Where we are headed? Orthogonal matrices: Let Q be an $n \times n$ matrix. Then Q is an orthogonal matrix if $Q^T \cdot Q = I_n$

$$\boxed{\text{Yes, this means } Q^{-1} = Q^T.}$$

That has to be a useful property.

Fact: Every $\overbrace{\underbrace{\text{non-singular}}_{\text{invertible}}}^{\det(A) \neq 0}$ matrix A can be factored as

$\boxed{A = Q \cdot R}$, where Q is orthogonal and R is upper triangular.

Consequence: $Ax = b$, $A = Q \cdot R$

$QRx = b$

$$\underbrace{Q^T Q}_{I_n} R x = Q^T b$$

$$\boxed{Rx = Q^T b \quad \text{back substitution}}$$

What does $Q^T Q = I_n$ really mean?

$$Q = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}_{3\times 3} \qquad v_i \in \mathbb{R}^3$$

$$Q^T = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

$$Q^T \cdot Q = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$