# ROB 101 - Computational Linear Algebra
# HW #4

Profs Grizzle and Ghaffari

**Due 9 PM = 21:00 ET on Thurs, Oct 07, 2021**

There are six (6) HW problems plus a *jupyter notebook* to complete and turn in.

1. Read Chapter 6 and the first three sections of Chapter 7 of our ROB 101 Booklet, `Notes for Computational Linear Algebra`. Based on your reading of the material, summarize in your own words:

   (a) Summarize Chapter 6 and its purpose;

   (b) Two things you found the most challenging in Chapter 6 or the first three sections of Chapter 7.

2. Read about LU Factorizations with Row Permutations in Sections 5.5 and 5.6 of Chapter 5, `Notes for Computational Linear Algebra`. We will NOT ask you to perform the algorithm by hand. This problem asks you instead to demonstrate your understanding of how to use a factorization of the form $P \cdot A = L \cdot U$ to solve $Ax = b$.

   You are given that

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} 2 & -1 & 2 \\ 6 & -3 & 9 \\ 2 & -3 & 6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 6 & -3 & 9 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

   Using the LU Factorization, find the solution to $Ax = b$, for

$$b = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

   Show clearly the steps of your work.

3. Compute (by hand) the determinants of the following matrices. You may use freely any facts from lecture or the book that make the computations easier. You'll be doing yourself a favor if you note those facts as you use them, though you are not required to do so.

   (a) $A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$. Give the determinant of $A$.

   (b) $A = B^{-1}$, where $\det(B) = \frac{2}{7}$. Give the determinant of $A$.

   (c) Find $\det(A)$ if the matrix $A$ has LU Factorization $P \cdot A = L \cdot U$, where

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \det(L) = 1, \ \mathrm{diag}(U) = \begin{bmatrix} 0.01 & 0.1 \end{bmatrix}$$

4. Compute the inverse of each of the following matrices or explain why the matrix is not invertible.

   (a) $A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

   (b) $A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(c) $A_3 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ and you are given that $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}$ Give your answer as a product of two matrices.

(d) $A_4 = \begin{bmatrix} 6 & -2 & 4 & 4 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & -3 & 18 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

5. For each of the matrices[1] given in Problem 4, compute the matrix transpose. If the matrix is given as a product of two matrices, also give the transpose as a product of two matrices. You can just write down the answers. Their is no real work to show!

6. (a) For $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$, compute the linear combination $w = 3u - 4v$

(b) For $x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, compute the linear combination $z = -x + 2y$.

(c) For $v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ determine if $v_3 = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$ is a linear combination of $v_1$ and $v_2$, and if it is, find the coefficients $a_1$ and $a_2$ so that $v_3 = a_1 v_1 + a_2 v_2$

(d) Determine if the system of linear equations

$$\begin{aligned} -x_1 - 2x_2 &= 3 \\ 2x_1 + 5x_2 &= -9 \\ x_1 + x_2 &= 0 \end{aligned}$$

has a solution. If it has a solution, please provide one. If it does not have a solution, explain why not.

**Hint:** Write the system of linear equations in matrix form and work part (c) before doing part (d).

**This is the end of the drill problems.** The second part of the HW set is once again a jupyter notebook. **Please go to** the course Canvas Site and complete the assignment titled "juliahw4".

[1] Before you compute the inverse.

**Hints**

**Hints: Prob. 2** If $Ax = b$, and $P \cdot A = L \cdot U$, then $P \cdot Ax = P \cdot b$ and $L \cdot Ux = Pb$. Hence, $L \cdot Ux = \bar{b}$, where $\bar{b} := P \cdot b$. The problem can then be solved as follows:

- We introduce a dummy variable $y := Ux$ which transforms the equation $L \cdot Ux = \bar{b}$ into two equations

$$\begin{aligned} Ly &= \bar{b} \\ Ux &= y \end{aligned}$$

- We first solve $Ly = \bar{b}$ for the dummy variable $y$ by forward substitution, because $L$ is lower triangular.
- We then solve $Ux = y$ for our unknown $x$ by back substitution, because $U$ is upper triangular.
- $x$ then solves the original equation because

$$(Ly = \bar{b}) \;\&\; (Ux = y) \implies L \cdot Ux = \bar{b} = Pb \implies P \cdot Ax = Pb \implies P^\top \cdot P \cdot Ax = P^\top \cdot Pb \implies Ax = b,$$

  where we have used $P \cdot A = L \cdot U$ and $P^\top \cdot P = I$.

**Hints: Prob. 3-(c)** Recall your facts about the determinant of a product of matrices.

**Hints: Prob. 5** You are computing the transposes of the original matrices and not the transposes of the inverse matrices!

**Hints: Prob. 6-(c)** See the examples in the booklet.

**Hints: Prob. 6-(d)** Is "$b$" a linear combination of the columns of "$A$"?