ROB 101 - Computational Linear Algebra
HW #5

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Due 9 PM = 21:00 ET on Thurs, Oct 21, 2021

There are six (6) HW problems plus a *jupyter notebook* to complete and turn in.

1. Read the remainder of Chapter 7 and (all of) Chapter 8 of our ROB 101 Booklet, `Notes for Computational Linear Algebra`. Based on your reading of the material, summarize in your own words:

   (a) Summarize the purpose of either Chapter 7 or 8 (you do not need to do both);

   (b) Two things you found the most challenging in these Chapters or the most interesting.

2. Determine whether the given sets of vectors are linearly independent or dependent. In each case, document your reasoning, which means to give enough details that you, a year from now, would understand what you did! For this problem, it is recommended that you check independence via the definition: writing out the system of linear equations and checking if the set of linear equations has a non-trivial solution or not. However, you are welcome to use any method we have covered in ROB 101.

   (a) $v_1 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$

   (b) $v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   (c) $v_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

3. For each of the following matrices, determine whether or not their columns are linearly independent. In each case, document your reasoning, which means to give enough details that you, a year from now, would understand what you did!

   (a) $A_1 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}.$

   (b) $A_2 = A_1^\top$, where A_1 is given in part (a).

   (c) A_3 has LU Factorization $P \cdot A = L \cdot U$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; L = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.5 & 1.0 & 0.0 \\ -0.2 & 0.9 & 1.0 \end{bmatrix}, \; U = \begin{bmatrix} 2.0 & -1.0 & 11.0 \\ 0.0 & 0.0 & 2.0 \\ 0.0 & 0.0 & -12.0 \end{bmatrix}$$

4. For each of the given systems of linear equations, analyze the existence and uniqueness of a solution. You may use any method you wish.

   (a) $\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}}_{b}$

(b) $\underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 6 \\ 3 & 9 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}}_{b}$

(c) $\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b}$

(d) $\underbrace{\begin{bmatrix} -2 & 3 & -1 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$

5. Find a least squares solution for the system of linear equations $\underbrace{\begin{bmatrix} 6 & 1 \\ 5 & 1 \\ 11 & 2 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$. The problem is "small" so that you can do all of the computations by hand. If applicable, it is fine to use the formula for inverting a $2 \times 2 matrix$. For larger problems, as you know, the matrix inverse is not an efficient method for solving linear equations.

6. This problem uses the data in Table 1. The problem is once again made deliberately "'small" so that you can do all of the computations by hand, but if you want to do them in Julia, please report some of the intermediate calculations. If applicable, it is fine to use the formula for inverting a 2×2 matrix. For larger problems, as you know, the matrix inverse is not an efficient method for solving linear equations.

   It's your choice if you want to do the plots with Julia or turn in sketches made by hand. Equal points will be awarded for either method.

   (a) Fit a line, $\hat{y} = mx + b$, to the data in Table 1. Provide a graph of your result.

   (b) Fit a line *without offset*, $\hat{y} = mx$, to the data in Table 1 (note, there is no b term). Provide a graph of your result.

   (c) Fit a *quadratic plus constant*, $\hat{y} = ax^2 + c$, to the data in Table 1 (note, there is no bx term). Provide a graph of your result.

   (d) Thought problem: provide a written answer. You have four data points. Suppose you were to fit $\hat{y} = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ to the data. Would there be any error? Explain. Now, suppose you attempted to fit $\hat{y} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$ to the data. What would happen? Explain.

Table 1: Data for problem 6.

i	x_i	y_i
1	-10.0	7.73
2	-6.0	1.79
3	0.0	-4.55
4	8.0	-10.35

This is the end of the drill problems. The second part of the HW set is once again a jupyter notebook. **Please go to** the course Canvas Site and complete the assignment titled "juliahw5".

Hints

Hints: Prob. 2 There are two ways to approach the solution

- **By the definition:** form a linear combination of the vectors to check whether or not there are non-trivial solutions. If there is any non-trivial solution, the set of vectors is linearly dependent. Otherwise, it is linearly independent.
- **Pro Tip!** We place the vectors in a matrix $A = [v_1 \;\; v_2 \;\; \cdots \;\; v_m]$ and check whether or not $\det(A^\top \cdot A) = 0$.

Hints: Prob. 3-(c) Note that A_3 is square. Go back and review how to compute the determinant of a square matrix using an LU Factorization:

$$P \cdot A = L \cdot U.$$

Hints: Prob. 4 Recall the necessary and sufficient condition for $Ax = b$ to have a solution: a solution exists if, and only if, b is a linear combination of the columns of A. Another way to say this is: $Ax = b$ has a solution if, and only if, the matrices A and $[A \;\; b]$ have the same number of linearly independent columns. Review our **semi-Pro Tip!** for how to check it. For uniqueness of solutions, we need to check whether or not the columns of A are linearly independent.

Hints: Prob. 6 (a) Is identical to the example worked in lecture. (b) There is just a single parameter to find; set up the equations and go for it! (c) This is not really different than part (a) because there are two unknown coefficients. (d) Think about the conditions for the solvability of a least squares problem.