ROB 101 - Fall 2021

The Vector Space $\mathbb{R}^n$: Part 3
(Range, Null Space, Rank and Nullity)

October 27, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- Learn how to define coordinates in a subspace of $\mathbb{R}^n$ and understand how many coordinates you need.
- An introduction to eigenvalues and eigenvectors of square matrices.
- Applying to matrices some of the essential concepts in Linear Algebra.

Outcomes

- Range of a matrix and its relation to column span and null space.
- Handy matrix properties dealing with rank and nullity.

- A function (or a map) view of a matrix defines two subspaces:
  1. its *null space* and
  2. its *range*.

A Function View of a Matrix

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- Let A be an $n \times m$ matrix.

- A function (or a map) view of a matrix defines two subspaces:
  1. its *null space* and
  2. its *range*.
- Let A be an $n \times m$ matrix.
- We can then define a function $f : \mathbb{R}^m \to \mathbb{R}^n$ by, for each $x \in \mathbb{R}^m$

$$f(x) := Ax \in \mathbb{R}^n.$$

The following subsets are naturally motivated by the function view of a matrix.

Definition

1. $\mathrm{null}(A) := \{x \in \mathbb{R}^m \mid Ax = 0_{n\times 1}\}$ is the *null space* of A.
2. $\mathrm{range}(A) := \{y \in \mathbb{R}^n \mid y = Ax \text{ for some } x \in \mathbb{R}^m\}$ is the *range* of A.

Example

Find the null space of

$$A_1 = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & -1 & 1 \end{array} \right].$$

Example

$$A_1 x = 0 \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

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$$\iff \begin{bmatrix} x_1 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x = \begin{bmatrix} 0 \\ \alpha \\ \alpha \end{bmatrix},$$

for $\alpha \in \mathbb{R}$.

$$A_1 x = 0 \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$
$$\iff \begin{bmatrix} x_1 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x = \begin{bmatrix} 0 \\ \alpha \\ \alpha \end{bmatrix},$$

for $\alpha \in \mathbb{R}$.

Hence,

$$\mathrm{null}(A_1) = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}.$$

Find the null space of

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A_2 x = 0 \iff \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
A_2 x = 0 \iff & \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\iff & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
A_2x = 0 &\iff \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&\iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Hence,

$$
\mathrm{null}(A_2) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.
$$

Example

Find the ranges of

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } A_4 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We note that A_3 is 2×3. Hence,

$$\begin{aligned} \operatorname{range}(A_3) &= \left\{ A_3 x \mid x \in \mathbb{R}^3 \right\} \\ &= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \right\} \end{aligned}$$

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&= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}
\end{aligned}$$

Example

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&= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\} \\
&= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}
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&= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R} \right\},
\end{aligned}
$$

where the third column of A_3 was eliminated because it is linearly dependent on the first two columns; in fact, it is the negative of the second column.

We note that A_4 is 3×3. Hence,

$$
\begin{aligned}
\operatorname{range}(A_4) &= \left\{ A_4 x \mid x \in \mathbb{R}^3 \right\} \\
&= \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \right\} \\
&= \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}
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&= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}
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&= \left\{\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; \alpha_1, \alpha_3 \in \mathbb{R}\right\},
\end{aligned}$$

where the second column was eliminated because it is dependent on the first column (in fact, it is twice the first column).

Range of A Equals Column Span of A

Let A be an $n \times m$ matrix, its columns are vectors in $\mathbb{R}^n$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} =: \begin{bmatrix} a_1^{\mathrm{col}} & a_2^{\mathrm{col}} & \ldots & a_m^{\mathrm{col}} \end{bmatrix}$$

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Then

$$\boxed{\text{range}(A) := \{Ax \mid x \in \mathbb{R}^m\} = \text{span}\{a_1^{\rm col}, a_2^{\rm col}, \ldots, a_m^{\rm col}\} =: \text{col span}\{A\}.}$$

Range of A Equals Column Span of A

Remark

$\{Ax \mid x \in \mathbb{R}^m\} = \{x_1 a_1^{\text{col}} + x_2 a_2^{\text{col}} + \cdots + x_m a_m^{\text{col}} \mid (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m\} =: \text{col span}\{A\}.$

Show that both (a) the null space and (b) range of an $n \times m$ matrix A are subspaces.

(a):

1. We suppose that v_1 and v_2 are in $\mathrm{null}(A)$. Hence, $Av_1 = 0$ and $Av_2 = 0$.

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3. For the linear combination to be in $\mathrm{null}(A)$, we must have that A multiplying $\alpha_1 v_1 + \alpha_2 v_2$ yields zero. So we check

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2 = 0 + 0 = 0.$$

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$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2 = 0 + 0 = 0.$$

Hence, $\mathrm{null}(A)$ is closed under linear combinations and it is therefore a subspace.

(b):

1. We suppose that v_1 and v_2 are in $\mathrm{range}(A)$. Hence, there exists u_1 and u_2 such that $Au_1 = v_1$ and $Au_2 = v_2$.

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(b):

1. We suppose that v_1 and v_2 are in $\mathrm{range}(A)$. Hence, there exists u_1 and u_2 such that $Au_1 = v_1$ and $Au_2 = v_2$.

2. We form a linear combination $\alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}^n$ and check if it is also in $\mathrm{range}(A)$.

3. For the linear combination to be in $\mathrm{range}(A)$, we must produce a $u \in \mathbb{R}^m$ such that $Au = \alpha_1 v_1 + \alpha_2 v_2$. We propose $u = \alpha_1 u_1 + \alpha_2 u_2$ and check that

$$Au = A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A v_2 = \alpha_1 v_1 + \alpha_2 v_2.$$

(b):

1. We suppose that v_1 and v_2 are in $\mathrm{range}(A)$. Hence, there exists u_1 and u_2 such that $Au_1 = v_1$ and $Au_2 = v_2$.

2. We form a linear combination $\alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}^n$ and check if it is also in $\mathrm{range}(A)$.

3. For the linear combination to be in $\mathrm{range}(A)$, we must produce a $u \in \mathbb{R}^m$ such that $Au = \alpha_1 v_1 + \alpha_2 v_2$. We propose $u = \alpha_1 u_1 + \alpha_2 u_2$ and check that

$$Au = A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A v_2 = \alpha_1 v_1 + \alpha_2 v_2.$$

Hence $\alpha_1 v_1 + \alpha_2 v_2 \in \mathrm{range}(A)$. Because it is closed under linear combinations, $\mathrm{range}(A)$ is therefore a subspace.

Relation of Null Space and Range to Solutions of Linear Equations

Suppose that A is $n \times m$. Here are the key relations between solutions of $Ax = b$ and the null space and range of A.

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Relation of Null Space and Range to Solutions of Linear Equations

Suppose that A is $n \times m$. Here are the key relations between solutions of $Ax = b$ and the null space and range of A.

2. If $Ax = b$ has a solution, then it is unique if, and only if, $\mathrm{null}(A) = \{0_{m\times 1}\}$.

Relation of Null Space and Range to Solutions of Linear Equations

Suppose that A is $n \times m$. Here are the key relations between solutions of $Ax = b$ and the null space and range of A.

3 Suppose that $\tilde{x}$ is a solution of $Ax = b$, so that $A\tilde{x} = b$. Then the set of all solutions is

$$\begin{aligned} \{x \in \mathbb{R}^m \mid Ax = b\} &= \tilde{x} + \mathrm{null}(A) \\ &:= \{\hat{x} \in \mathbb{R}^m \mid \hat{x} = \tilde{x} + \eta, \eta \in \mathrm{null}(A)\}. \end{aligned}$$

Relation of Null Space and Range to Solutions of Linear Equations

Suppose that A is $n \times m$. Here are the key relations between solutions of $Ax = b$ and the null space and range of A.

4. $Ax = b$ has a unique solution if, and only if $b \in \mathrm{range}(A)$ and $\mathrm{null}(A) = \{0_{m\times 1}\}$.

Relation of Null Space and Range to Solutions of Linear Equations

Suppose that A is $n \times m$. Here are the key relations between solutions of $Ax = b$ and the null space and range of A.

5. When $b = 0_{n\times 1}$, then it is always true that $b \in \mathrm{range}(A)$. Hence we deduce that $Ax = 0_{n\times 1}$ has a unique solution if, and only if, $\mathrm{null}(A) = \{0_{m\times 1}\}$.

Definition

For an $n \times m$ matrix A,

1. $\mathrm{rank}(A) := \dim \mathrm{range}(A)$.
2. $\mathrm{nullity}(A) := \dim \mathrm{null}(A)$.

Because $\mathrm{range}(A) \subset \mathbb{R}^n$, we see that $\mathrm{rank}(A) \leq n$.

Theorem

For an $n \times m$ matrix A, we have the property

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = m \quad \textit{number of columns of } A.$$

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Theorem

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$$\mathrm{rank}(A) + \mathrm{nullity}(A) = m \quad \textit{number of columns of } A.$$

- *Since $\mathrm{rank}(A)$ is equal to the number of linearly independent columns of A, it follows that $\mathrm{nullity}(A)$ is counting the number of linearly dependent columns of A.*

- *If all of the columns of A are linearly independent, then none are dependent, and hence $\mathrm{null}(A) = \{0_{m\times 1}\}$.*

Proof.

See Chapter 10.6 of ROB 101 Book. □

Example

Verify the Rank-Nullity Theorem for

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3} \quad \text{and} \quad A_4 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}.$$

We have that $\mathrm{rank}(A_3) = 2$ and $\mathrm{rank}(A_4) = 2$. Also, $\mathrm{nullity}(A_3) = 1$ and thus $2 + 1 = 3$, the number of columns of A_3.

A quick calculation gives that

$$\mathrm{null}(A_4) = \mathrm{span}\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\}.$$

Hence, $\mathrm{nullity}(A_4) = 1$ and $2 + 1 = 3$, the number of columns of A_4.

Next Time

- Recap
- Chapters 1-10 of ROB 101 Book