2020 October 02 ROB 101

PRO TIP

Summary

- Columns of $A = \begin{array}{cccc} v_1 & v_2 & & v_m \\ \downarrow & \downarrow & & \downarrow \end{array}$ $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$ are linearly independent if, and only if, $\det(A^\top A) \neq 0$

- Write $P \cdot (A^\top A) = L \cdot U$. Then $\det(A^\top A) \neq 0$ if, and only if, $\det(U) \neq 0$ if, and only if, diagonal of U does not have any zero elements.

$$\left\{ v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \ldots, v_m = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} \right\}$$ linearly indep

$\iff$ only solution to $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$

is $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \underset{\substack{\uparrow \\ \text{zero vector}}}{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\iff$ only solution to $\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}}_{A_{n\times m}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}}_{\alpha_{m\times 1}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{0_{n\times 1}}$

is $\alpha = 0_{m\times 1}$

We ended with $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = 0 \iff (y_1)^2 + (y_2)^2 + \cdots + (y_n)^2 = 0$

$$\iff y^T y = \underbrace{[y_1 \ \ y_2 \ \cdots \ y_n]}_{1\times n} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{n\times 1} = 0_{1\times 1}$$

Claim $\underbrace{\underset{n\times m}{A}\,\alpha_{m\times 1} = 0_{n\times 1}}_{\text{Rectangular}} \iff \underbrace{\underbrace{A^T A}_{m\times m}\,\alpha_{m\times 1} = 0_{m\times 1}}_{\text{Square (easier!)}}$

$y := A\alpha$ know $y = 0 \iff y^T y = 0$

$$A\alpha = 0 \iff (\alpha^T A^T)(A\alpha) = 0$$

$$\iff \alpha^T A^T A \alpha = 0$$

$y = 0$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $y^T y = 0$

$\underset{(a)}{A\alpha = 0} \implies \underset{(b)}{A^T A\alpha = 0} \implies \underset{(c)}{\alpha^T A^T A\alpha = 0} \implies$

$\implies \underset{(a)}{A\alpha = 0}$

$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$

$\therefore$ $(a) \iff (b) \iff (c)$

$$\therefore \boxed{A\alpha = 0 \iff \underbrace{A^T A}\alpha = 0}$$

square

$$\left[\therefore A\alpha=0 \text{ has only the trivial solution}\right] \iff \left[A^TA\alpha=0 \text{ has only the trivial solution}\right]$$

$$\iff \det(A^TA)\neq 0.$$

Uniqueness of Solutions to $Ax=b$

$$Ax = \left[a_1^{col}\ a_2^{col} \cdots a_m^{col}\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = b$$

$$\boxed{x_1 a_1^{col} + x_2 a_2^{col} + \cdots + x_m a_m^{col} = b}$$

x a solution if, and only if, holds.

Let's suppose we have two candidate solutions

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_m \end{bmatrix}, \quad \bar{\bar{x}} = \begin{bmatrix} \bar{\bar{x}}_1 \\ \bar{\bar{x}}_2 \\ \vdots \\ \bar{\bar{x}}_m \end{bmatrix}$$

Question: When must we have $\bar{x}=\bar{\bar{x}}$? (Uniqueness)

They are both solutions, hence

$A\bar{x} = b$ and $A\bar{\bar{x}} = b$. $A = n \times m$

$\therefore A\bar{x} - A\bar{\bar{x}} = b - b = 0_{n\times 1}$

$A(\bar{x} - \bar{\bar{x}}) = 0_{n\times 1}$

$$(\bar{x}_1 - \bar{\bar{x}}_1) a_1^{col} + (\bar{x}_2 - \bar{\bar{x}}_2) a_2^{col} + \cdots + (\bar{x}_m - \bar{\bar{x}}_m) a_m^{col} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Only solution to $\alpha_1 a_1^{col} + \cdots + \alpha_m a_m^{col} = 0$ is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0 \iff \{a_1^{col}, \ldots, a_m^{col}\}$ is linearly indep.

If $Ax = b$ has a solution, it is unique $\iff$ columns of A are linearly indep.

Check it via $\det(A^T A) \neq 0$.

$\iff U$ has no zero elements

on the diagonal $\boxed{P\cdot(A^T A) = L\,U}$

Number of Linearly Indep Vectors in a Set?

Given $\{v_1, v_2, \ldots, v_m\} \subset \mathbb{R}^n$, can we define the maximum number of linear indep vectors in the set?

Why is this interesting = important?

When does $Ax = b$ have a solution?

$\Longleftrightarrow$ b is a linear combination of the columns of A.

$\Longleftrightarrow$ b is not linearly independent from the columns of A

$\{a_1^{col}, a_2^{col}, \ldots, a_m^{col}\}$, $\{a_1^{col}, a_2^{col}, \ldots, a_m^{col}, b\}$

these set must have the same number of linearly indep vectors!

How to count number of lin. indep vectors in $\{v_1, v_2, \ldots, v_m\}$?

Start from left to right and do the following:

$\{v_1\}$ lin. indep? $\alpha_1 v_1 = 0 \Leftrightarrow \alpha_1 = 0$

$\Leftrightarrow v_1 \neq 0_{n\times1}$

Suppose $\{v_1\}$ is lin. indep.

(If $v_1 = 0_{n\times1}$, discard it)

Next consider

$\{v_1, v_2\}$ lin. indep? $\alpha_1 v_1 + \alpha_2 v_2 = 0$

$\Leftrightarrow \alpha_1 = 0, \alpha_2 = 0$??

Let suppose v_2 is a linear combo. of v_1, so we discard v_2

Next $\{v_1, v_3\}$ lin. indep.?

Suppose yes.

Next check $\{v_1, v_3, v_4\}$ lin. indep?

ETC.

Totally do-able, SUPER TEDIOUS!!!

Is there a clean way to find this number = # lin. indep vectors in a set???

Semi-Pro Tip TFAE

(the Following are Equivalent)

- $\left\{ v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \ldots, v_m = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} \right\}$

  has k linearly indep vectors

- the matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

  has k linearly indep columns

- Define $P \cdot (A^T A) = L\,U$ then U has k linearly indep columns (better for us, because U is upper triangular)

Question: When can we look at diag(U) and simply count the number of non-zero elements?