ROB 101 - Fall 2021

LU (Lower-Upper) Factorization

September 20, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- How to reduce a hard problem to two much easier problems.
- The concept of “factoring” a matrix into a product of two simpler matrices that are in turn useful for solving systems of linear equations.

Outcomes

- Our first encounter with an explicit algorithm.
- LU decomposition, where L is a lower triangular matrix and U is an upper triangular matrix.
- Use the LU decomposition to solve linear equations.
- LU factorization with row permutation.

Matrix Multiplication in the Form of Columns Times Rows

Suppose that A is $n \times k$ and B is $k \times m$ so that the two matrices are compatible for matrix multiplication.

Then

$$A \cdot B = \sum_{i=1}^{k} a_i^{\text{col}} \cdot b_i^{\text{row}},$$

the sum of the columns of A multiplied by the rows of B.

Example

Form the matrix product of $A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix}$.

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$$A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_1^{\text{col}} & a_2^{\text{col}} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 4 \end{matrix}} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} b_1^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 5 & \quad 2 \end{matrix}} \\ \boxed{\begin{matrix} 0 & -1 \end{matrix}} \end{bmatrix}.$$

Example

$$a_1^{\text{col}} \cdot b_1^{\text{row}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & 6 \end{bmatrix}$$

$$a_2^{\text{col}} \cdot b_2^{\text{row}} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\begin{aligned} A \cdot B &= \sum_{i=1}^{2} a_i^{\text{col}} \cdot b_i^{\text{row}} = a_1^{\text{col}} \cdot b_1^{\text{row}} + a_2^{\text{col}} \cdot b_2^{\text{row}} \\ &= \begin{bmatrix} 5 & 2 \\ 15 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & 2 \end{bmatrix}. \end{aligned}$$

Q: Given a matrix, working from the top left corner and working down the diagonal, how to successively eliminates columns and rows from a matrix!?

Consider

$$M = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 11 & 17 \\ 6 & 18 & 58 \end{bmatrix}.$$

Our goal is to find a column vector C_1 and a row vector R_1 such that

$$M - C_1 \cdot R_1 = \begin{bmatrix} \color{red}{0} & \color{red}{0} & \color{red}{0} \\ \color{red}{0} & * & * \\ \color{red}{0} & * & * \end{bmatrix}.$$

$*$ means we don't care about the particular values of those entries.

It turns out the following trick works!

- Pick the first column of M and divide by the first diagonal element;

$$C_1 := \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} / 2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- Pick the first row of M and define

$$R_1 := \begin{bmatrix} 2 & 4 & 5 \end{bmatrix}.$$

- $$M = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 11 & 17 \\ 6 & 18 & 58 \end{bmatrix}.$$

- $$C_1 \cdot R_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ 6 & 12 & 15 \end{bmatrix}.$$

We have successfully found C_1 and R_1 that eliminated the first column and row of M.

- $$M - C_1 \cdot R_1 = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 11 & 17 \\ 6 & 18 & 58 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ 6 & 12 & 15 \end{bmatrix}.$$
- $$M - C_1 \cdot R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 6 & 43 \end{bmatrix}$$

Let's repeat this process for the second column and row.

- Pick the second column of $M - C_1 \cdot R_1$ and divide by the second diagonal element;

$$C_2 := \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} / 3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

- Pick the second row of $M - C_1 \cdot R_1$ and define

$$R_2 := \begin{bmatrix} 0 & 3 & 7 \end{bmatrix}.$$

- $$C_2 \cdot R_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 6 & 14 \end{bmatrix}.$$

- $$M - C_1 \cdot R_1 - C_2 \cdot R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 6 & 43 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 6 & 14 \end{bmatrix}.$$

- $$M - C_1 \cdot R_1 - C_2 \cdot R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 43 \end{bmatrix}.$$

And the third ...

- Pick the third column of $M - C_1 \cdot R_1 - C_2 \cdot R_2$ and divide by the third diagonal element;

$$C_3 := \begin{bmatrix} 0 \\ 0 \\ 43 \end{bmatrix} / 43 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- Pick the third row of $M - C_1 \cdot R_1 - C_2 \cdot R_2$ and define

$$R_3 := \begin{bmatrix} 0 & 0 & 43 \end{bmatrix}.$$

- $$C_3 \cdot R_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 43 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 43 \end{bmatrix}.$$
- $$M - C_1 \cdot R_1 - C_2 \cdot R_2 - C_3 \cdot R_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
- or
  $$\boxed{M = C_1 \cdot R_1 + C_2 \cdot R_2 + C_3 \cdot R_3}$$

$$M = C_1 \cdot R_1 + C_2 \cdot R_2 + C_3 \cdot R_3 = \left[\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}\right] \cdot \left[\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\right]$$

- L is *lower triangular*.

$$L := \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

- U is *upper triangular*.

$$U := \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 3 & 7 \\ 0 & 0 & 43 \end{bmatrix}.$$

- $M = L \cdot U$, the product of a lower triangular matrix and an upper triangular matrix.

LU Factorization for Solving Linear Equations

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- Define $U \cdot x =: y$, then

$$\begin{aligned} Ly &= b \\ Ux &= y. \end{aligned}$$

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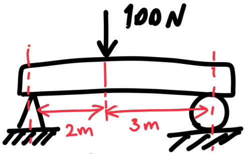
- Define $U \cdot x =: y$, then

$$\begin{aligned} Ly &= b \\ Ux &= y. \end{aligned}$$

- We first solve for y via forward substitution. Given y, we solve for x via back substitution.

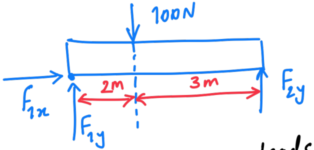
Example



In Statics course, we learn that the conditions for static balance are:

1) $\sum F_x = 0$
2) $\sum F_y = 0$
3) $\sum M_x = 0$
4) $\sum M_y = 0$

Free body diagram:



Applying the constraints leads to the following system.

$$\begin{cases} F_{1x} = 0 \\ F_{1y} + F_{2y} = 100 \\ 5F_{2y} = 200 \end{cases} \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2y} \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 200 \end{bmatrix}$$

$$A \cdot x = b$$

We apply our LU factorization to $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$.

The result is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Example

We have $A \cdot x = b$ and $A = L \cdot U$. We wish to find x.

1. $A \cdot x = L \cdot U \cdot x = b$.
2. Define $U \cdot x =: y$, then we have $L \cdot y = b$.
3. Solve $L \cdot y = b$ by forward substitution.
4. Then given y, solve $U \cdot x = y$ by back substitution.

Solve $L \cdot y = b$ by forward substitution.

- $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 200 \end{bmatrix},$$
- $$\boxed{y = \begin{bmatrix} 0 \\ 100 \\ 200 \end{bmatrix}}$$

Solve $U \cdot x = y$ by back substitution.

- $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 200 \end{bmatrix},$$
- $$\boxed{x = \begin{bmatrix} 0 \\ 60 \\ 40 \end{bmatrix}}$$

LU Factorization with Row Permutations

Q. Are there cases where our simplified LU factorization process fails?

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- **Case 1:** (easiest): C becomes an entire column of zeros.

  Solution: In general, at the k-th step, if C becomes a column of all zeros, set its k-th entry to one and define R as usual.

Find an LU factorization of

$$M = \begin{bmatrix} 2 & -1 & 2 \\ 6 & -3 & 9 \\ 2 & -1 & 6 \end{bmatrix}.$$

- $C_1 := \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} /2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, and $R_1 := \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$.
- $M - C_1 \cdot R_1 = \begin{bmatrix} 2 & -1 & 2 \\ 6 & -3 & 9 \\ 2 & -1 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 2 \\ 6 & -3 & 6 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

Example

- $C_2 := \begin{bmatrix} 0 \\ {\color{red}0} \\ 0 \end{bmatrix} /{\color{red}0}!$
- Solution: $C_{\color{red}2} := \begin{bmatrix} 0 \\ {\color{red}1} \\ 0 \end{bmatrix}$, and $R_2 := \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$.
- $M - C_1 \cdot R_1 - C_2 \cdot R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Example

- $C_3 := \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} /4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $R_2 := \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}$.
- $M - C_1 \cdot R_1 - C_2 \cdot R_2 - C_3 \cdot R_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Assembling all the pieces, we have

- $$M = \begin{bmatrix} 2 & -1 & 2 \\ 6 & -3 & 9 \\ 2 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$
- $$M = L \cdot U.$$
- **Warning:** $\det(M) = \det(U) = 0$ (next lecture); hence, we are in the problematic case of having either no solution or an infinite number of solutions.

- **Case 2:** (requires row permutation): C is not all zeros, but the pivot value is zero.

  Solution: It turns out that all we have to do is permute the two rows of our matrix $(M - CR)$ in the current step and the corresponding rows in L.

LU Factorization with Row Permutations

Consider

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

- $C_1 := \left[\begin{array}{c} 0 \\ 1 \end{array} \right] / 0$, and $R_1 := \left[\begin{array}{cc} 0 & 1 \end{array} \right]$.

We're stuck already!

Key idea: We use a permutation matrix, here $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, to swap rows. Recall we construct P from $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by swapping the first and second rows.

LU Factorization with Row Permutations

- $$P \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$
- Then we factorize a new matrix $P \cdot A$ instead of A.
- The result will be LU factorization with row permutations: $P \cdot A = L \cdot U$.
- $$P \cdot A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = L \cdot U.$$

Solving $Ax = b$ via LU with Row Permutation

Solving $Ax = b$ when A is square and $P \cdot A = L \cdot U$.

- $Ax = b \iff P \cdot Ax = P \cdot b \iff L \cdot Ux = P \cdot b$.
- Define $Ux =: y$, then

$$
\begin{aligned}
Ly &= P \cdot b \\
Ux &= y.
\end{aligned}
$$

- We first solve for y via forward substitution. Given y, we solve for x via back substitution.

Remark

You can look into the book for a few examples and algorithmic implementations. From here, we will let Julia compute the factorization for us! We'll cover the algorithms for LU and LU with row permutation in the Lab sessions.

Example

Compute LU factorization of the following matrix.

$$M = \begin{bmatrix} 6 & -2 & 4 & 4 \\ 3 & -1 & 5 & 7 \\ 1 & -1 & 2 & 8 \\ 3 & -1 & -1 & 3 \end{bmatrix}.$$

Example

In [1]:
```
using LinearAlgebra

# define our matrix
M = [6. -2 4 4;
     3 -1 5 7;
     1 -1 2 8;
     3 -1 -1 3];

# compute L, U, and P (Permutation matrix); PM = LU
F = lu(M);

F.L * F.U == F.P * M
```

Out[1]:
```
true
```

Example

In [2]:
```
F.L
```

Out[2]:
```
4×4 Array{Float64,2}:
 1.0        0.0   0.0  0.0
 0.166667   1.0   0.0  0.0
 0.5       -0.0   1.0  0.0
 0.5       -0.0  -1.0  1.0
```

In [3]:
```
F.U
```

Out[3]:
```
4×4 Array{Float64,2}:
 6.0  -2.0       4.0      4.0
 0.0  -0.666667  1.33333  7.33333
 0.0   0.0       3.0      5.0
 0.0   0.0       0.0      6.0
```

In [4]:
```
F.P
```

Out[4]:
```
4×4 Array{Float64,2}:
 1.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0
 0.0  1.0  0.0  0.0
 0.0  0.0  0.0  1.0
```

Solve the following system of linear equations.

$$\begin{bmatrix} 6 & -2 & 4 & 4 \\ 3 & -1 & 5 & 7 \\ 1 & -1 & 2 & 8 \\ 3 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

Example

In [5]:
```
# define A and b
A = [6. -2 4 4;
     3 -1 5 7;
     1 -1 2 8;
     3 -1 -1 3];
b = [1; -1; 3; 7];

# compute L, U, and P (Permutation matrix); PM = LU
F = lu(A);

# solve for y
y = F.L \ (F.P * b);
# given y, solve for x
x = F.U \ y;

# test the solution
A * x - b
```

Out[5]:
```
4-element Array{Float64,1}:
 -4.440892098500626e-16
  0.0
  8.881784197001252e-16
 -1.7763568394002505e-15
```

Next Time

- Determinant, Matrix Inverses, and the Matrix Transpose
- Read Chapter 6 of ROB 101 Book