# ROB 101 - Computational Linear Algebra
# HW #6

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**Due 9 PM = 21:00 ET on Thurs, Oct 28, 2021**

There are six (6) HW problems plus a *jupyter notebook* to complete and turn in.

1. Read Chapters 9 of our ROB 101 Booklet, `Notes for Computational Linear Algebra`. Based on your reading of the material, summarize in your own words:

   (a) Content and purpose;

   (b) Two things you found the most challenging or interesting in Chapter 9.

2. Check if the following subsets of $\mathbb{R}^3$ are also subspaces of $\mathbb{R}^3$

   (a) $S_1 = \{x \in \mathbb{R}^3 \mid \begin{bmatrix} 2.0 & -1.0 & 11.0 \end{bmatrix} x = 1.0\}$

   (b) $S_2 = \{x \in \mathbb{R}^3 \mid \begin{bmatrix} 2.0 & -1.0 & 11.0 \end{bmatrix} x = 0.0\}$

   (c) $S_3 = \text{span}\{S_1\}$

3. For each of the following matrices, compute their null space.

   (a) $A_1 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}$.

   (b) $A_2 = A_1^\top$, where $A_1$ is given in part (a).

   (c) $A_3$ has LU Factorization $P \cdot A_3 = L \cdot U$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; L = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.5 & 1.0 & 0.0 \\ -0.2 & 0.9 & 1.0 \end{bmatrix}, \; U = \begin{bmatrix} 2.0 & -1.0 & 11.0 \\ 0.0 & 0.0 & 2.0 \\ 0.0 & 0.0 & -12.0 \end{bmatrix}$$

4. The sets $V_1$, $V_2$, $v_3$ below are each subspaces of $\mathbb{R}^3$ because we know that the span of any set is a subspace; you do not have to check that separately in this problem. In each case, find linearly independent vectors that span the given subspaces. For the record, a set of vectors that is linearly independent AND spans a subspace is called a *basis*.

   (a) $V_1 := \text{span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} \}$

   (b) $V_2$ is defined to be the span of the columns of $B_2$, where

$$B_2 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}.$$

   (c) $V_3 := \text{span}\{S_3\}$, where $S_3 = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 = 1.0\}$ (do not miss that $V_3 \subset \mathbb{R}^3$).

5. You are given that the vectors $\{u_1 = \begin{bmatrix} 4 \\ 0 \\ -4 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}\}$ are linearly independent. Find an orthogonal set of vectors $\{v_1, v_2, v_3\}$ satisfying

$$\text{span}\{u_1, u_2, u_3\} = \text{span}\{v_1, v_2, v_3\}.$$

Note that you are NOT asked to normalize the vectors. You are asked, however, to work the problem by hand instead of in Julia, just so you have thought the calculations through once in your life.

6. The goal of the problem is to solve a least squared error problem using the "pipeline" suggested in Chapter 9. The problem is posed as $Ax = b$, with

$$A = \begin{bmatrix} 1.0 & 4.0 \\ 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix}, \quad b = \begin{bmatrix} 2.0 \\ 1.4 \\ -1.0 \end{bmatrix}.$$

   (a) Compute, by hand, a QR Factorization of $A$.

   (b) Solve the problem $Rx = Q^\top \cdot b$.

   (c) Compute the error vector $e := Ax - b$ and its **norm**.

   (d) Verify that you obtain the same solution when you solve $A^\top \cdot Ax = A^\top b$.

   It will be much more fun in Julia!

**This is the end of the drill problems.** The second part of the HW set is once again a jupyter notebook. **Please go to** the course Canvas Site and complete the assignment titled "juliahw6".

## Hints

**Hints: Prob. 2** This is really about applying the definition of being a subspace. Your instructors find it easier to apply the conditions one by one: Is the set closed under vector addition? Is it closed under scalar times vector multiplication?

**Hints: Prob. 3** for (a) and (b), you can apply the definition and find vectors such that $Ax = 0$. You can also use the fact from Chapter 7 that once you write $P \cdot A = L \cdot U$, then $Ax = 0 \iff Ux = 0$. You can also use our Pro Tip, that $Ax = 0 \iff A^\top \cdot Ax = 0$, and thus you can also do the LU Factorization $P \cdot (A^\top \cdot A) = L \cdot U$, and then apply

$$Ax = 0 \iff A^\top \cdot Ax = 0 \iff P \cdot (A^\top \cdot A)x = 0 \iff L \cdot Ux = 0 \iff Ux = 0.$$

You would then be checking $Ux = 0$ where this time, $U$ is from the LU Factorization of $A^\top \cdot A$ instead of $A$.

For part (c), you can directly use $U$ in determining linear independence and the null space, when

$$P \cdot A = L \cdot U.$$

**Hints: Prob. 4** You are seeking vectors that satisfy two conditions: (i) they are linearly independent and (ii) they span the set. In (a) you are already know vectors that span $V_1$, but if the set of vectors is linearly dependent, then you need to eliminate some of the vectors while keeping enough that they still span the set. (b) Is the same as part (a), but easier! For 4-(c), the hint is to find solutions $x \in \mathbb{R}^3$ to the linear equation $x_1 + x_2 = 1.0$ For example, $x = e_1$ is a solution. Each and every one of the solutions is in the set $S_3$, and hence each of the solutions contributes to $V_3$ when you form linear combinations.

**Hints: Prob. 6** It could be that we have not covered this material yet in lecture when you are working on the problem. That is fine. It is treated in the book and we think you can read it on your own and understand it.