ROB 101 - Computational Linear Algebra
Recitation #5

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1 Linear Independence and Dependence (review + pro tips)

Definition: The set of vectors $\{v_1, v_2, ..., v_m\} \in \mathbb{R}^n$ is **linearly dependent** if there exist real numbers $\alpha_1, \alpha_2, ..., \alpha_m \in \mathbb{R}$ NOT ALL ZERO yielding a linear combination of vectors that adds up to the zero vector, that is:

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m = 0_{nx1}$$

Conversely, the set of vectors $\{v_1, v_2, ..., v_m\} \in \mathbb{R}^n$ is **linearly independent** if the **only** real numbers $\alpha_1, \alpha_2, ..., \alpha_m \in \mathbb{R}$ yielding a linear combination of vectors that adds up to the zero vector

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m = 0_{nx1}$$

are zeros: $\alpha_1 = 0, \alpha_2 = 0, ..., \alpha_m = 0$

Pro Tip: The following statements regarding linear independence are equivalent:

- The set of vectors $\{v_1, v_2, ..., v_m\}$ is linearly independent.
- The mxm matrix $A^T \cdot A$ is invertible. That is, it is possible to find $(A^T \cdot A)^{-1}$
- $\det(A^T \cdot A) \neq 0$.
- For any LU Factorization P $\cdot$ $(A^T \cdot A) =$ L $\cdot$ U of $A^T \cdot A$, the mxm upper triangular matrix U has no zeros on its diagonal.

2 Subspaces

Definition: A subspace of $\mathbb{R}^n$ is a subset $V \subset \mathbb{R}^n$ satisfying the following conditions:

- **Non-emptiness:** The zero vector is in V
- **Closure under addition:** If vectors u and v are in V, then $u + v$ is also in V
- **Closure under scalar multiplication:** If vector v is in V and $c \in \mathbb{R}$, then cv is also in V

Remark 1: The last two requirements can be combined to give rise to a property known as **closure under linear combination**, that if vectors u and v are in V, then $\alpha u + \beta v$ is also in V where $\alpha, \beta \in \mathbb{R}$

Remark 2: The set $\mathbb{R}^n$ is a subspace of itself: it contains the zero vector, and is closed under linear combination (you can take the linear combination of any vectors in the set and form a new vector still in the set).

Subsets versus Subspaces: A subset of $\mathbb{R}^n$ is any collection of vectors in $\mathbb{R}^n$ whatsoever. For instance, the set of vectors defined by the unit circle $c = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a subset of $\mathbb{R}^2$ but it is not a subspace. A subspace is a subset that happens to satisfy the above three additional defining properties. Verify for yourself that the unit circle is in fact not a subspace using the properties. A subspace of $\mathbb{R}^n$ is necessarily a subset of $\mathbb{R}^n$ by definition, but a subset of $\mathbb{R}^n$ is not necessarily a subspace of $\mathbb{R}^n$.

2.1 Null Space of a Matrix

Definition: The null space of an n x m matrix A is:

$$\text{null}(A) := \{x \in \mathbb{R}^m \mid Ax = 0_{n\text{x}1}\}$$

or the set of all solutions $x \in \mathbb{R}^m$ that result in Ax equalling the zero vector or "null vector".
Remark: null(A) is a supspace of $\mathbb{R}^m$

Question? What is a null space of a matrix and why is it important?
Source: FAQ Questions `https://umich.instructure.com/courses/475066/files/folder/HW/HW%2006`

2.2 Span of a Set of Vectors

Definition: The span of a set of vectors $S \in \mathbb{R}^n$ is:

$$\text{span}(S) := \{\text{all possible linear combinations of elements in S}\}$$

The span operation is useful for generating a subspace from an arbitrary set vectors in $\mathbb{R}^n$ by the definition of the span (contains zero vector and is closed under linear combination). That is, the result of $\text{span}(S)$ is a subspace of $\mathbb{R}^n$.

2.3 Column Span of a Matrix

Definition: The column span of an n x m matrix A is:

$$\text{col}(A) := \text{span}(\{a_1^{col}, a_2^{col}, ..., a_m^{col}\})$$

i.e., take the columns of matrix A and form a set S containing m vectors in $\mathbb{R}^n$ $(\{a_1^{col}, a_2^{col}, ..., a_m^{col}\})$ to perform the span(S) operation.
Remark: Ax = b has a solution if, and only if, b is a linear combination of the columns of A. A more elegant way to write this is Ax = b has a solution if, and only if,

$$b \in col(A)$$

2.4 Basis Vectors

Definition: Suppose that V is a subspace of $\mathbb{R}^n$. Then $\{v_1, v_2, ..., v_k\}$ is a basis for V if

- the set $\{v_1, v_2, ..., v_k\}$ is linearly independent
- span$\{v_1, v_2, ..., v_k\}$ = V

The dimension of V is k, the number of basis vectors.

Remark: Basis vectors provide a simple means to generate all vectors in a vector space or a subspace by forming linear combinations from a finite list of vectors. The three vectors commonly seen in vector calculus physics $\hat{i}, \hat{j}, \hat{k}$ are orthonormal basis vectors!

3 QR factorization & Gram Schmidt Process

3.1 The Gram Schmidt Process

Motivation: Applying Gram-Schmidt to the columns of a matrix yields the QR Factorization, which is one of the most advanced numerical methods for solving systems of linear equations.

Suppose that that the set of vectors $\{u_1, u_2, ..., u_m\}$ is linearly independent. You can generate a new set of orthonormal vectors by

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 \\
v_3 &= u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{u_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2 \\
&\vdots \\
v_k &= u_k - \sum_{i=1}^{k-1} \left(\frac{u_k \cdot v_i}{v_i \cdot v_i}\right) v_i
\end{aligned}$$

Then the set of vectors $\{v_1, v_2, ..., v_m\}$ is

- orthogonal, meaning for $i \neq j \rightarrow v_i \cdot v_j = 0$
- span preserving, meaning that, for all $1 \leq k \leq m, span\{v_1, v_2, ..., v_k\} = span\{u_1, u_2, ..., u_k\}$
- and linearly independent

We have now made an orthonormal basis!

3.2 QR Factorization

Suppose that A is an nxm matrix with linearly independent columns. Then there exists an nxm matrix Q with orthonormal columns and an upper triangular, mxm, invertible matrix R such that $A = Q \cdot R$.

Q and R are constructed as follows:

- Let $\{u_1, ..., u_m\}$ be the columns of A with their order preserved so that

$$A = [u_1 \; u_2 \; \dots \; u_m]$$

- Q is constructed by applying the Gram-Schmidt Process to the columns of A and normalizing their lengths to one,

$$\{u_1, u_2, ..., u_m\} \xrightarrow[\text{Process}]{\text{Gram-Schmidt}} \{v_1, v_2, ..., v_m\}$$
$$Q := \left[\frac{v_1}{||v_1||} \frac{v_2}{||v_2||} \cdots \frac{v_m}{||v_m||}\right]$$

- Because $Q^T Q = I_m$, it follows that $A = Q \cdot R \leftrightarrow R = Q^T \cdot A$.

Questions from HW 06 FAQ

https://umich.instructure.com/courses/475066/files/folder/HW/HW%2006

1. Orthogonality: Why is it important?

2. What's the difference between orthogonal and orthonormal Matrices?