ROB 101 - Fall 2021

Triangular Systems of Equations: Forward and Back Substitution

September 8, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- Equations that have special structure are often much easier to solve

- Some examples to show this.

Outcomes

- Recognize triangular systems of linear equations and distinguish those with a unique answer.
- Learn that the determinant of a square triangular matrix is the product of the terms on its diagonal.
- How to use forward and back substitution.

Example of Lower Triangular (Linear) Systems

This is an example of a square system of linear equations that is *lower triangular*.

$$\begin{aligned} 3x_1 &= 6 \\ 2x_1 - x_2 &= -2 \\ x_1 - 2x_2 + 3x_3 &= 2. \end{aligned}$$

Example of Lower Triangular (Linear) Systems

When we write the system as $Ax = b$, in the lower triangular case we have

$$\begin{aligned} 3x_1 &= 6 \\ 2x_1 - x_2 &= -2 \\ x_1 - 2x_2 + 3x_3 &= 2 \end{aligned} \iff \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}}_{b}.$$

Lower Triangular Matrices

$$A = \begin{bmatrix} 3 & \mathbf{0} & \mathbf{0} \\ 2 & -1 & \mathbf{0} \\ 1 & -2 & 3 \end{bmatrix}$$

- All terms above the diagonal of the matrix A are zero.
- More precisely, the condition is $a_{ij} = 0$ for all $j > i$.
- Such matrices are called *lower-triangular*.

Determinant of a Lower Triangular Matrix

Fact

The matrix determinant of a square lower triangular matrix is equal to the product of the elements on the diagonal.

$$A = \begin{bmatrix} 3 & \mathbf{0} & \mathbf{0} \\ 2 & -1 & \mathbf{0} \\ 1 & -2 & 3 \end{bmatrix} \implies \det(A) = 3 \cdot (-1) \cdot 3 = -9 \neq 0.$$

Determinant of a Lower Triangular Matrix

Fact

The matrix determinant of a square lower triangular matrix is equal to the product of the elements on the diagonal.

$$A = \begin{bmatrix} 3 & \mathbf{0} & \mathbf{0} \\ 2 & -1 & \mathbf{0} \\ 1 & -2 & 3 \end{bmatrix} \implies \det(A) = 3 \cdot (-1) \cdot 3 = -9 \neq 0.$$

In [1]:
```
using LinearAlgebra

A = [3 0 0; 2 -1 0; 1 -2 3];
det(A)
```

Out[1]:
```
-9.0
```

Lower Triangular Systems and Forward Substitution

We will solve the previous example using a method called *forward substitution*.

$$\begin{aligned} 3x_1 &= 6 \\ 2x_1 - x_2 &= -2 \\ x_1 - 2x_2 + 3x_3 &= 2 \end{aligned} \iff \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}}_{b}.$$

Because we have ordered the variables as x_1, x_2, x_3, we isolate x_1 in the first equation, x_2 in the second equation, and x_3 in the third equation by moving the other variables to the right-hand side,

$$\begin{aligned} 3x_1 &= 6 \\ -x_2 &= -2 - 2x_1 \\ 3x_3 &= 2 - x_1 + 2x_2. \end{aligned}$$

Next, we substitute in, working from top to bottom:

$$
\begin{aligned}
x_1 &= \frac{1}{3}6 = 2 \\
x_2 &= 2 + 2x_1 = 6 \\
x_3 &= \frac{1}{3}\left(2 - x_1 + 2x_2\right) = \frac{1}{3}\left(12\right) = 4.
\end{aligned}
$$

This is an example of a square *upper triangular* system of equations.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 6 \\ 2x_2 + x_3 &= -2 \\ 3x_3 &= 4, \end{aligned} \quad \Longleftrightarrow \quad \underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}}_{b}.$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ \mathbf{0} & 2 & 1 \\ \mathbf{0} & \mathbf{0} & 3 \end{bmatrix}$$

- All terms below the diagonal of the matrix A are zero.
- More precisely, the condition is $a_{ij} = 0$ for $i > j$.
- Such matrices are called *upper-triangular*.

Determinant of an Upper Triangular Matrix

Fact

The matrix determinant of a square upper triangular matrix is equal to the product of the elements on the diagonal.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ \mathbf{0} & 2 & 1 \\ \mathbf{0} & \mathbf{0} & 3 \end{bmatrix} \implies \det(A) = 1 \cdot 2 \cdot 3 = 6 \neq 0.$$

Determinant of an Upper Triangular Matrix

Fact

The matrix determinant of a square upper triangular matrix is equal to the product of the elements on the diagonal.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ \mathbf{0} & 2 & 1 \\ \mathbf{0} & \mathbf{0} & 3 \end{bmatrix} \implies \det(A) = 1 \cdot 2 \cdot 3 = 6 \neq 0.$$

In [2]:
```
A = [1 3 2; 0 2 1; 0 0 3];
det(A)
```

Out[2]:
```
6.0
```

We solve the upper triangular systems using a method called *back substitution*.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 6 \\ 2x_2 + x_3 &= -2 \\ 3x_3 &= 4, \end{aligned} \quad \Longleftrightarrow \quad \underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}}_{b}.$$

We develop a solution by starting at the bottom and work our way to the top via back substitution.

$$
\begin{aligned}
x_1 &= 6 - (3x_2 + 2x_3) \\
x_2 &= \frac{1}{2}\,(-2 - x_3) \\
x_3 &= \frac{4}{3},
\end{aligned}
$$

Next, we do back substitution, from bottom to top, sequentially plugging in numbers from the previous equations.

$$
\begin{aligned}
x_1 &= 6 - (3x_2 + 2x_3) = 6 - (3 \cdot (-\frac{5}{3}) + 2 \cdot \frac{4}{3}) = \frac{11}{3} \\
x_2 &= \frac{1}{2} \cdot (-2 - x_3) = \frac{1}{2} \cdot \left(-2 - \frac{4}{3}\right) = -\frac{5}{3} \\
x_3 &= \frac{4}{3}.
\end{aligned}
$$

The general form of a lower triangular system with a non-zero determinant is

$$
\begin{aligned}
a_{11}x_1 &= b_1 \quad (a_{11} \neq 0) \\
a_{21}x_1 + a_{22}x_2 &= b_2 \quad (a_{22} \neq 0) \\
\vdots &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \quad (a_{nn} \neq 0)
\end{aligned}
$$

and the solution proceeds from top to bottom, like this

$$
\begin{aligned}
x_1 &= \frac{b_1}{a_{11}} \quad (a_{11} \neq 0) \\
x_2 &= \frac{b_2 - a_{21}x_1}{a_{22}} \quad (a_{22} \neq 0) \\
\vdots &= \vdots \\
x_n &= \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}}{a_{nn}} \quad (a_{nn} \neq 0).
\end{aligned}
$$

The general form of an upper triangular system with a non-zero determinant is

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \quad (a_{11} \neq 0)\\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \quad (a_{22} \neq 0)\\
a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \quad (a_{33} \neq 0)\\
\vdots &= \vdots\\
a_{nn}x_n &= b_n \quad (a_{nn} \neq 0),
\end{aligned}
$$

and the solution proceeds from bottom to top, like this,

$$
\begin{aligned}
x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} && (a_{11} \neq 0) \\
x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} && (a_{22} \neq 0) \\
\vdots &= \vdots && \vdots \\
x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} && (a_{n-1,n-1} \neq 0) \\
x_n &= \frac{b_n}{a_{nn}} && (a_{nn} \neq 0),
\end{aligned}
$$

Practice Problems

In HW you will develop Julia code to solve triangular systems of equations, whether upper or lower triangular!

Next Time

- Matrix Multiplication
- Read Chapter 4 of ROB 101 Book