ROB 101 - Fall 2021

The Vector Space $\mathbb{R}^n$: Part 1

September 27, 2021

UNIVERSITY OF MICHIGAN

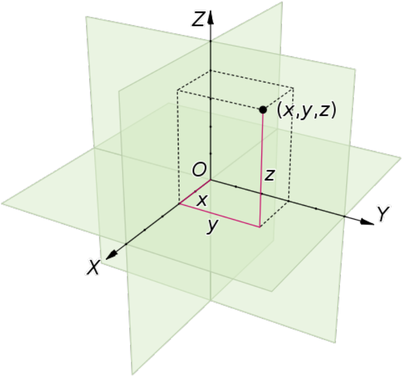
Learning Objectives

- Instead of working with individual vectors, we will work with a collection of vectors.
- Our first encounter with some of the essential concepts in Linear Algebra that go beyond systems of equations.

Outcomes

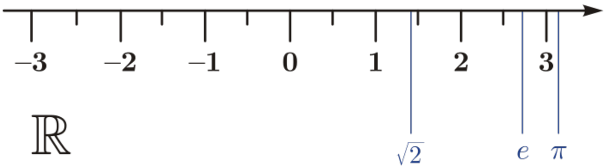
- Vectors as n-tuples of real numbers
- $\mathbb{R}^n$ as the collection of all n-tuples of real numbers
- Linear combinations of vectors
- Linear independence of vectors
- Relation of these concepts to the existence and uniqueness of solutions to $Ax = b$.
- LU Factorization to check the linear independence of a set of vectors, and LDLT to check if one vector is a linear combination of a set of vectors.

Euclidean space is the fundamental space of classical geometry. Ancient Greek geometers introduced Euclidean space for modeling the physical universe.



Real Line $\mathbb{R}$

The real line, or real number line is the line whose points are the real numbers. That is, the real line is the set $\mathbb{R}$ of all real numbers, viewed as the Euclidean space of dimension one.



In [1]:
```
a = -2
```
Out[1]: -2

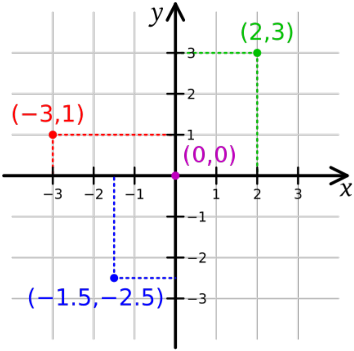
In [2]:
```
b = sqrt(2)
```
Out[2]: 1.4142135623730951

In [3]:
```
c = MathConstants.pi
```
Out[3]: π = 3.1415926535897...

In [4]:
```
d = MathConstants.e
```
Out[4]: ℯ = 2.7182818284590...

The set $\mathbb{R}^2$ of pairs of real numbers is a two-dimensional (2D) Euclidean space. We called this set the 2D plane and mathematically describe it as

$$\mathbb{R}^2 := \{(x,y) | x,y \in \mathbb{R}\}.$$



Two-Dimensional Euclidean Space $\mathbb{R}^2$

In [5]:
```
a = [2; 3] # a is a 2D point with coordinates 2 along the x axis and 3 along the y axis
```

Out[5]:
```
2-element Array{Int64,1}:
 2
 3
```

In [6]:
```
b = [-3, 1] # b is a 2D point with coordinates -3 along the x axis and 1 along the y axi
s
```

Out[6]:
```
2-element Array{Int64,1}:
 -3
  1
```

In [7]:
```
o = [0, 0] # we call this point the origin
```

Out[7]:
```
2-element Array{Int64,1}:
 0
 0
```

In [8]:
```
size(a)
```

Out[8]:
```
(2,)
```

In [9]:
```
size(b)
```

Out[9]:
```
(2,)
```

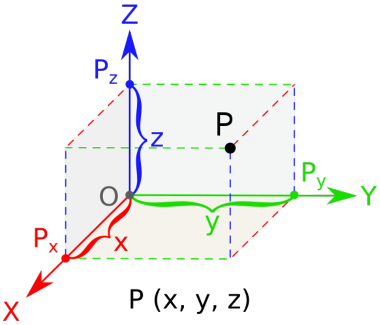
The set $\mathbb{R}^3$ is the three-dimensional (3D) Euclidean space. Mathematically, we describe it as

$$\mathbb{R}^3 := \{(x,y,z) | x,y,z \in \mathbb{R}\}.$$

The set $\mathbb{R}^3$ is the three-dimensional (3D) Euclidean space. Mathematically, we describe it as

$$\mathbb{R}^3 := \{(x,y,z)|x,y,z \in \mathbb{R}\}.$$

The 3D Euclidean space serves as a spatial model of the physical universe in which all known matter exists. We use the following orthogonal coordinate system with O as its origin.



Three-Dimensional (3D) Euclidean Space $\mathbb{R}^3$

In [10]:
```
a = [2; 3; -1] # a is a 3D point with coordinates 2 along the x axis, 3 along the y axi
s, and -1 along the z axis respectively.
```

Out[10]:
```
3-element Array{Int64,1}:
  2
  3
 -1
```

In [11]:
```
b = [-3, 1, 6] # b is a 2D point with coordinates -3 along the x axis, 1 along the y axi
s, and 6 along the z axis respectively.
```

Out[11]:
```
3-element Array{Int64,1}:
 -3
  1
  6
```

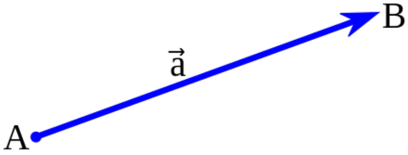
In [12]:
```
o = [0, 0, 0] # we call this point the origin
```

Out[12]:
```
3-element Array{Int64,1}:
 0
 0
 0
```

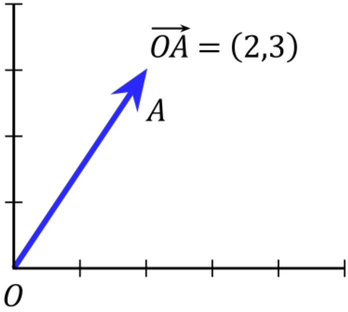
In [13]:
```
println("size (dimension) of point a:  ", size(a))
```

```
size (dimension) of point a:  (3,)
```

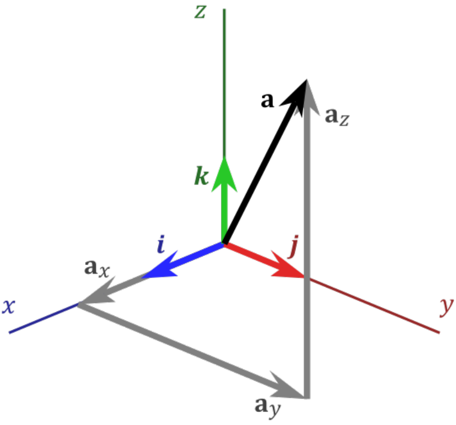
- A Euclidean vector or simply a vector is a geometric object that has magnitude (or length) and direction.
- A vector pointing from A to B is shown in the following figure. A is called the initial point and B is called a terminal point.
- A vector is what is needed to “carry” the point A to the point B; the Latin word vector means “carrier”.



Next figure shows a vector in the Cartesian plane, showing the position of a point A with coordinates $(2, 3)$.



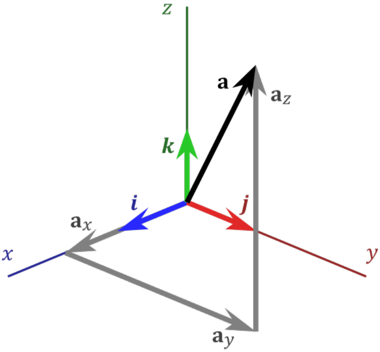
- A common way of representing a vector in $\mathbb{R}^3$ follows the convention shown in the following figure.



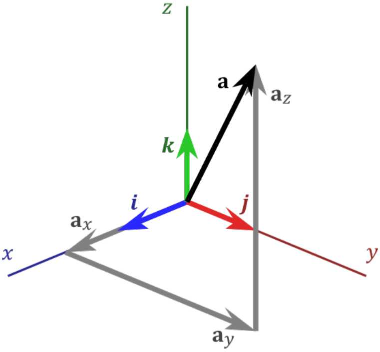
3D Euclidean Vector

- A common way of representing a vector in $\mathbb{R}^3$ follows the convention shown in the following figure.

- We can define unit vectors (vectors with length 1) as $i = (1{,}0{,}0)$, $j = (1{,}0{,}0)$, and $k = (1{,}0{,}0)$; the directions along x, y, and z axes, respectively.



- If a vector such as a has components a_x, a_y, and a_z, then we write $a = a_x i + a_y j + a_z k = (a_x, a_y, a_z)$. It is also common to use bracket instead of parentheses for an array such as $a = [a_x, a_y, a_z]$. Both notations are valid.



Using the mathematical notation we can define $a = (a_1, a_2, a_3)$ using *standard unit vectors (also called basis)* for $\mathbb{R}^3$ as

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Using the mathematical notation we can define $a = (a_1,a_2,a_3)$ using *standard unit vectors (also called basis)* for $\mathbb{R}^3$ as

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

Then

$$\begin{aligned} a &= a_1e_1 + a_2e_2 + a_3e_3 = e_1a_1 + e_2a_2 + e_3a_3 \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} a_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} a_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} a_3 \end{aligned}$$

Remark

Recall that we can also write

$$\begin{aligned} a &= a_1 e_1 + a_2 e_2 + a_3 e_3 = e_1 a_1 + e_2 a_2 + e_3 a_3 \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} a_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} a_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} a_3 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = I_3 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{aligned}$$

Corollary

The columns of the 3×3 identity matrix $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are vectors and form the standard basis for $\mathbb{R}^3$.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}.$$

An n-tuple is a fancy name for an ordered list of n numbers, $(x_1, x_2, \ldots, x_n)$. Mathematically, we write

$$\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}.$$

Or using the column vector convention

$$\mathbb{R}^n := \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_i \in \mathbb{R}, 1 \leq i \leq n \right\}.$$

The standard basis for $\mathbb{R}^n$ are

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, e_2 := \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \ldots, e_n := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1},$$

The standard basis for $\mathbb{R}^n$ are

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, e_2 := \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \ldots, e_n := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1},$$

Then for $a = (a_1, \ldots, a_n)$, we have

$$\begin{aligned} a &= a_1e_1 + \cdots + a_ne_n = e_1a_1 + \cdots + e_na_n \\ &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} a_1 + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} a_n \end{aligned}$$

Remark

We can also write

$$
\begin{aligned}
a &= a_1 e_1 + \dots + a_n e_n = e_1 a_1 + \dots + e_n a_n \\
&= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} a_1 + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} a_n \\
&= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = I_n \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}
\end{aligned}
$$

Corollary

The columns of the $n \times n$ identity matrix $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ *are vectors and form the standard basis for $\mathbb{R}^n$.*

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix}.$$

Columns of Matrices are Vectors and Vice Versa

Suppose that A is an $n \times m$ matrix, then its columns are vectors in $\mathbb{R}^n$ and conversely, given vectors in $\mathbb{R}^n$, we can stack them together and form a matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1^{\text{col}} & a_2^{\text{col}} & \cdots & a_m^{\text{col}} \end{bmatrix}$$

$$\iff a_j^{\text{col}} := \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \in \mathbb{R}^n, 1 \leq j \leq m$$

The following are all vectors in $\mathbb{R}^4$

$$u = \begin{bmatrix} 1 \\ -2 \\ \pi \\ \sqrt{17} \end{bmatrix}, \quad v = \begin{bmatrix} 4.1 \\ -1.1 \\ 0.0 \\ 0.0 \end{bmatrix}. \quad w = \begin{bmatrix} 10^3 \\ 0 \\ 0.7 \\ 1.0 \end{bmatrix}.$$

Use them to make a 4×3 matrix.

The following are all vectors in $\mathbb{R}^4$

$$u = \begin{bmatrix} 1 \\ -2 \\ \pi \\ \sqrt{17} \end{bmatrix}, \quad v = \begin{bmatrix} 4.1 \\ -1.1 \\ 0.0 \\ 0.0 \end{bmatrix}. \quad w = \begin{bmatrix} 10^3 \\ 0 \\ 0.7 \\ 1.0 \end{bmatrix}.$$

Use them to make a 4×3 matrix.

$$A = \begin{bmatrix} 1 & 4.1 & 10^3 \\ -2 & -1.1 & 0.0 \\ \pi & 0.0 & 0.7 \\ \sqrt{17} & 0.0 & 1.0 \end{bmatrix}.$$

The matrix A is a 2×4. Extract its columns to form vectors in $\mathbb{R}^2$.

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right].$$

The matrix A is a 2×4. Extract its columns to form vectors in $\mathbb{R}^2$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}.$$

$$a_1^{\mathrm{col}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \in \mathbb{R}^2, \;\; a_2^{\mathrm{col}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \in \mathbb{R}^2,$$

$$a_3^{\mathrm{col}} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \in \mathbb{R}^2, \;\; a_4^{\mathrm{col}} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \in \mathbb{R}^2.$$

Definition

Consider two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. We define their vector sum by

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix},$$

that is, we sum their respective components or entries.

Scalar Multiplication of a Vector in $\mathbb{R}^n$

Definition

Let α be a real number, i.e., $\alpha \in \mathbb{R}$. We define

$$\alpha x := \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix},$$

that is, to multiply a vector by a real number, we multiply each of the components of the vector by the same real number.

Two vectors x and y are equal if, and only if, they have the same components,

$$x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} =: y \iff x_j = y_j \text{ for all } 1 \leq j \leq n.$$

Said another way,

$$x = y \iff x - y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0_{n \times 1},$$

the zero vector in $\mathbb{R}^n$.

Properties of Vector Addition and Scalar Times Vector Multiplication

1. Addition is commutative: For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

$$x + y = y + x.$$

Properties of Vector Addition and Scalar Times Vector Multiplication

2. Addition is associative: For any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$,

$$(x + y) + z = x + (y + z) \,.$$

Properties of Vector Addition and Scalar Times Vector Multiplication

3. Scalar multiplication is associative: For any $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and any $x \in \mathbb{R}^n$ in

$$\alpha\,(\beta x) = (\alpha\beta)\,x.$$

Properties of Vector Addition and Scalar Times Vector Multiplication

4. Scalar multiplication is distributive: For any $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$,

$$(\alpha + \beta)x = \alpha x + \beta x,$$

and

$$\alpha\left(x + y\right) = \alpha x + \alpha y.$$

Next Time

- The Vector Space $\mathbb{R}^n$: Part 1 (To be continued ...)
- Read Chapter 7 of ROB 101 Book