ROB 101 - Computational Linear Algebra
Recitation #1

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1 Linear Systems

A system of equations with no Non-linearity $(\cos, \sin, \log, x^2$ etc.)

2 Matrix Representation of System of equations & Determinants

Review of determinant facts:

- det(A) is a real number
- $Ax = b$, a system of equations with n equations and n unknowns has a unique solution for any b if an only if $det(A) \neq 0$
- When $det(A) = 0$, the system may have either infinite or no solution
- $det(A) = ad - bc$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let's express these in the matrix form $(Ax = b)$ and determine if the solution is unique

1. $x + 2y = 6$
   $2x - y = 4$

   Solution: $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

   $det(A) = (1)(-1) - (2)(2) = -5 \neq 0$
   Unique solution exists!

2. $x + 2y = 6$
   $3x + 6y = 10$

   Solution: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$

   $det(A) = (1)(6) - (3)(2) = 0$
   No unique solution exists!

3. $x + 2y = 6$
   $3x + 6y = 18$

   Solution: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$

   $det(A) = (1)(6) - (3)(2) = 0$
   No unique solution exists!

4. $4x = 10$
   $x + 6y = 2$

   Solution: $A = \begin{bmatrix} 4 & 0 \\ 1 & 6 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

   $det(A) = (4)(6) - (1)(0) = 24 \neq 0$
   Unique solution exists!

5. $y - 2x = 4$
   $y = 2$

   Solution: $A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

   $det(A) = (-2)(1) - (0)(1) = -2 \neq 0$
   Unique solution exists!

6. $2x - y = 3$
   $6x - 3y = 1$

   Solution: $A = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

   $det(A) = (2)(-3) - (6)(-1) = 0$
   No unique solution exists!

3 Solution of Linear System of Equations

In solving $Ax = b$, we exploit the form of A for finding solutions using Forward and Backward Substitution. However, for us to do that A must be a Triangular Matrix.

3.1 Triangular matrices

Fact: For any Triangular Matrix A, det(A) is the product of all elements on its diagonal.

3.1.1 Upper Triangular Matrices

A matrix, A is upper Triangular, if and only if, for every element a_{ij} in the matrix, i.e. the element at i^{th} row and j^{th} column, the following condition holds true:
$a_{ij} = 0$ if $i > j$
Example:

$$\begin{bmatrix} 2 & 3 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

Method of Solution: **Solution: Backward Substitution**

3.1.2 Lower Triangular Matrices

A matrix, A is Lower Triangular, if and only if, for every element a_{ij} in the matrix, i.e. the element at i^{th} row and j^{th} column, the following condition holds true:
$a_{ij} = 0$ if $j > i$
Example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

Method of Solution: **Solution: Forward Substitution**

Build a matrix of coefficients, A, and check if the Matrix is Triangular or not from the following system of equations:

1.

$$\begin{aligned} 2x_1 + 2x_2 &= 24 \\ 2x_2 &= 14 \end{aligned}$$

Solution: A = $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ $\quad det(A) = (2)(2) = 4$

Remark: **Solution: Unique solution exists**

2.

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 23 \\ 6x_1 + 7x_2 &= 22 \\ 9x_1 &= 18 \end{aligned}$$

Solution: A = $\begin{bmatrix} 4 & 3 & 2 \\ 0 & 7 & 6 \\ 0 & 0 & 9 \end{bmatrix}$ $\quad det(A) = (4)(7)(9) = 252$

Remark: **Solution: Unique solution exists**

3.

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 20 \\ 5x_2 + 6x_3 + 7x_4 &= 34 \\ 8x_3 + 9x_4 &= 25 \\ 10x_4 &= 10 \end{aligned}$$

Solution: A = $\begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ $\quad det(A) = (2)(5)(8)(10) = 800$

Remark: **Solution: Unique solution exists**

4.

$$\begin{aligned} 2x_1 &= 4 \\ 8x_1 + 3x_2 &= 28 \\ 2x_1 + 3x_2 + x_3 &= 22 \\ 2x_1 + 2x_2 + 3x_3 + 2x_4 &= 46 \end{aligned}$$

Solution: A = $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 2 & 3 & 2 \end{bmatrix}$ $\quad det(A) = (2)(3)(1)(2) = 12$

Remark: **Solution: Unique solution exists**

3.2 Forward Substitution

$$\begin{aligned}
2x_1 &= 4 \\
8x_1 + 3x_2 &= 28 \\
2x_1 + 3x_2 + x_3 &= 22 \\
2x_1 + 2x_2 + 3x_3 + 2x_4 &= 46
\end{aligned}$$

Solution: ***First check: no zeros on diagonal***

$$\begin{aligned}
2x_1 &= 4 \\
3x_2 &= 28 - 8x_1 \\
x_3 &= 22 - 2x_1 - 3x_2 \\
2x_4 &= 46 - 2x_1 - 2x_2 - 3x_3
\end{aligned}$$

$$\begin{aligned}
x_1 &= \frac{4}{2} = 2 \\
x_1 &= 2
\end{aligned}$$

$$\begin{aligned}
3x_2 &= 28 - 8(2) \\
x_2 &= \frac{28 - 8(2)}{3} = \frac{12}{3} = 4 \\
x_2 &= 4
\end{aligned}$$

$$\begin{aligned}
x_3 &= 22 - 2(2) - 3(4) = 6 \\
x_3 &= 6
\end{aligned}$$

$$\begin{aligned}
2x_4 &= 46 - 2(2) - 2(4) - 3(6) \\
x_4 &= 46 - 2(2) - 2(4) - 3(6) = \frac{46 - 2(2) - 2(4) - 3(6)}{2} = 8 \\
x_4 &= 8
\end{aligned}$$

3.3 Backward Substitution

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 20 \\ 5x_2 + 6x_3 + 7x_4 &= 34 \\ 8x_3 + 9x_4 &= 25 \\ 10x_4 &= 10 \end{aligned}$$

Solution: ***First check: no zeros on diagonal***

$$\begin{aligned} 10x_4 &= 10 \\ 8x_3 &= 25 - 9x_4 \\ 5x_2 &= 34 - 6x_3 - 7x_4 \\ 2x_1 &= 20 - 2x_2 - 3x_3 - 4x_4 \end{aligned}$$

$$\begin{aligned} x_4 &= \frac{10}{10} = 1 \\ x_4 &= 1 \end{aligned}$$

$$\begin{aligned} 8x_3 &= 25 - 9(1) \\ x_3 &= \frac{25 - 9(1)}{8} = \frac{16}{8} = 2 \\ x_3 &= 2 \end{aligned}$$

$$\begin{aligned} 5x_2 &= 34 - 6(2) - 7(1) \\ x_2 &= \frac{34 - 6(2) - 7(1)}{5} = \frac{15}{5}3 \\ x_2 &= 3 \end{aligned}$$

$$\begin{aligned} 2x_1 &= 20 - 2(3) - 3(2) - 4(1) \\ x_1 &= \frac{20 - 2(3) - 3(2) - 4(1)}{2} = 2 \\ x_1 &= 2 \end{aligned}$$

4 Appendix

Review: The general form of a lower triangular system with a non-zero determinant is

$$\begin{aligned} a_{11}x_1 &= b_1 \quad (a_{11} \neq 0) \\ a_{21}x_1 + a_{22}x_2 &= b_2 \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \quad (a_{nn} \neq 0) \end{aligned} \tag{1}$$

and the solution proceeds from top to bottom, like this

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \quad (a_{11} \neq 0) \\ x_2 &= \frac{b_2 - a_{21}x_1}{a_{22}} \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ x_n &= \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}}{a_{nn}} \quad (a_{nn} \neq 0). \end{aligned} \tag{2}$$

Similarly, The general form of an upper triangular system with a non-zero determinant is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \quad (a_{11} \neq 0) \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \quad (a_{22} \neq 0) \\ a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \quad (a_{33} \neq 0) \\ \vdots &= \vdots \\ a_{nn}x_n &= b_n \quad (a_{nn} \neq 0), \end{aligned} \tag{3}$$

and the solution proceeds from bottom to top, like this,

$$\begin{aligned} x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} && (a_{11} \neq 0) \\ x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} && (a_{22} \neq 0) \\ \vdots &= \vdots && \vdots \\ x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} && (a_{n-1,n-1} \neq 0) \\ x_n &= \frac{b_n}{a_{nn}} && (a_{nn} \neq 0), \end{aligned} \tag{4}$$