ROB 101 - Computational Linear Algebra
Recitation #4

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1 Matrix Determinant

Recall: Determinant Facts

- det(A) is a real number
- $Ax = b$, a system of equations with n equations and n unknowns has a unique solution for any b if an only if $det(A) \neq 0$
- When $det(A) = 0$, the system may have either infinite or no solution
- $det(A) = ad - bc$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

In addition to the previous facts, we also have:

Additional Fact: Determinant of product of matrices

- Let A and B be $n \times n$ matrices $\Rightarrow det(AB) =$

Matrix Determinant via LU Factorization: Given $A = LU$, we now have

$$det(A) =$$

Additional Fact: Determinant of triangular matrices

- Let A be a triangular matrix, then $det(A)$ = product of the elements on the diagonal

Additional Fact: Determinant of triangular matrices

- Let P be a permutation matrix, then $det(P) = \pm 1$

2 Inverse of Matrices

A matrix A is invertible if there exists a matrix A^{-1}such that $AA^{-1} = A^{-1}A = I_n$

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The following statements are equivalent

- A is invertible
- $det(A) \neq 0$
- A is a square matrix

Fact: Inverse of a product of matrices $(AB)^{-1} =$

3 Determinants and Inverses

Additional Fact: Determinant of inverse of a matrix

- Let $det(A^{-1}) = \frac{1}{det(A)}$

Given $PA = LU$ we have $det(A) =$

4 Transpose of a Matrix

Let $A^\intercal$ be the transpose of A. To get $A^\intercal$ we simply take the rows of A and use them as the columns of $A^\intercal$ or we can equivalently take the columns of our A matrix and turn them into the rows of $A^\intercal$. So if A is an nxm matrix, the transpose, $A^\intercal$ is an mxn matrix. Let's take a look at an example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^\intercal = \tag{1}$$

To find the transpose in julia, we can either use $transpose(A)$ or $A^{'}$.
Properties of the Tranpose of a matrix

- $(A^\intercal)^\intercal = A$
- if A is square $det(A^\intercal) = det(A)$
- $(AB)^\intercal = B^\intercal A^\intercal$

5 Linear Combination

A vector, $v \in^n$ is said to be a Linear Combination of vectors $v_1, v_2 \cdots v_m \in^n$ if there exits real numbers $\alpha_1, \alpha_2 \cdots \alpha_m$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

Using this formulation, find if the given vector v, is a linear combination of the vectors $v_1, v_2 \cdots v_m$ in the question, if true, also find the vector of coefficients, α

1. $v = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$
   $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $v = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}$
   $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

6 Linear Independence

The vectors $\{v_1, v_2, ..., v_m\}$ are **linearly independent** if the **only** real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ yielding a linear combination of vectors that adds up to the zero vector,

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0, \tag{2}$$

are $\boldsymbol{\alpha}_1 = 0, \boldsymbol{\alpha}_2 = 0, \ldots, \boldsymbol{\alpha}_m = 0$.

Concise definition of Linear Independence:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0 \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using this Definition, determine if the following vectors are linearly independent.

1. $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

2. $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

7 Solutions of Ax=b and Linear Independence

Existence

The system of equation $Ax = b$ has a solution if:

b is a linear combination of the columns of A and if the columns of A are linearly independent

Using this definition, lets set up the problem below as if we were to find if the following system of equations have a solution:

$$\begin{aligned} -a + 3b + 5c &= 20 \\ -2a - 2c &= -8 \\ -3a + 3b + 4c &= 10 \end{aligned}$$

8 Facts: Linear Independence, Determinant and Inverse

Given $A = [v_1|v_2|v_3|...|v_m]$

- The set of vectors $\{v_1, .., v_m\}$ are linearly independent
- A is invertible
- $det(A^\intercal A) \neq 0$