ROB 101 - Fall 2021

Vectors, Matrices, and Determinants

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UNIVERSITY OF MICHIGAN

Learning Objectives

- Begin to understand the vocabulary of mathematics and programming.
- An introduction to the most important tools of linear algebra: vectors and matrices.
- Find out an easy way to determine when a set of linear equations has a unique answer.

Outcomes

- Scalars vs. array
- Row vectors and column vectors
- Rectangular matrices and square matrices
- Learn new mathematical notation
- Using matrices and vectors to express systems of linear equations
- Determinant of a square matrix and its relation to uniqueness of solutions of systems of linear equations

Scalars

Scalars are simply numbers such as the ones you have been using for a long time: $25.77763, \sqrt{17}, 10, -4, \pi$.

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Let's define some scalars in Julia.

In [1]:
```
# = assigns right hand side to the variable on the left.
a = 25.77763;
b = sqrt(17);
c = 10;
d = -4;
e = π;
println("Hello Scalars!")
print("a= ",a,", b=",b,", c=",c,", d=",d,", e=",e)
```

```
Hello Scalars!
a= 25.77763, b=4.123105625617661, c=10, d=-4, e=π
```

Arrays

- Arrays are scalars that have been organized somehow into lists.
- The lists could be rectangular or not, they could be made of columns of numbers, or rows of numbers.
- If you've ever used a spreadsheet, then you have seen an array of numbers.

Let's define some arrays in Julia.

In [2]:
```
# we only define arrays of numbers.
# a is a 1x5 array
a = [1 -2 4 8.1 π]
```

Out[2]:
```
1×5 Array{Float64,2}:
 1.0  -2.0  4.0  8.1  3.14159
```

In [3]:
```
# b is a 5x1 array
b = [1, -2, 4, 8.1, π]
```

Out[3]:
```
5-element Array{Float64,1}:
  1.0
 -2.0
  4.0
  8.1
  3.141592653589793
```

In [4]:
```
# or
c = [1; -2; 4; 8.1; π]
```

Out[4]:
```
5-element Array{Float64,1}:
  1.0
 -2.0
  4.0
  8.1
  3.141592653589793
```

b and c are equal. We used two different ways of defining a *column array* using comma and semicolon.

Let's check if Julia recognizes b and c are equal.

In [5]:
```
# two arrays are equal if and only if their corresponding
# entries (elements) are equal.
# we can use == to check if b is equal c
b == c
```

Out[5]:
```
true
```

Row Vectors and Column Vectors

For us, a *vector* is a finite ordered list of numbers or variables.

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- A *row vector* has one row and multiple columns. For example, $v = \begin{bmatrix} 1.1 & -3 & 44.7 \end{bmatrix}$, is a row 3-vector (it has three elements in a row!).

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- A *column vector* has multiple rows and one column. For example, $v = \begin{bmatrix} 1.1 \\ -3 \\ 44.7 \end{bmatrix}$, is a column 3-vector.

Row Vectors and Column Vectors

- We denote a general column n-vector

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

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$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

- and a general row n-vector

$$v = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}.$$

Rectangular and Square Matrices

Matrices are generalizations of vectors that allow multiple columns and rows.

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- Here is a 3×2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

In [6]:
```
# Let's define a 3x2 matrix
A = [1 2; 3 4; 5 6]
```

Out[6]:
```
3×2 Array{Int64,2}:
 1  2
 3  4
 5  6
```

- Here is a 2×3 matrix

$$A = \begin{bmatrix} 1.2 & -2.6 & 11.7 \\ 3.1 & \frac{11}{7} & 0 \end{bmatrix}.$$

In [7]:
```
# Let's define a 2x3 matrix
A = [1.2 -2.6 11.7; 3.1 11/7 0]
```

Out[7]:
```
2×3 Array{Float64,2}:
 1.2  -2.6      11.7
 3.1   1.57143   0.0
```

Rectangular and Square Matrices

- A general *rectangular* matrix of size $n \times m$ takes this form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

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- A general *square* matrix of size $n \times n$ takes this form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

- In matrix A, we denote the ij-element a_{ij} that lies on the intersection of the i-th row and j-th column.

The *diagonal* of the square matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \iff \operatorname{diag}(A) = \begin{bmatrix} a_{11} & a_{22} & \cdots & a_{nn} \end{bmatrix}.$$

The Matrix Diagonal

- The diagonal is sometimes called the *main diagonal* of a matrix. What about other elements?

- The diagonal is sometimes called the *main diagonal* of a matrix. What about other elements?
- Let's call them off-diagonal!

Linear Systems of Equations in Matrix Form, $Ax = b$

- We will use the notation $Ax = b$.
- A is an $n \times n$ matrix of the coefficients.
- x is a column n-vector of the variables.
- b is a column n-vector of numbers on the right side of the equation.

Linear Systems of Equations in Matrix Form, $Ax = b$

Consider the following system of linear equations with two unknowns.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Linear Systems of Equations in Matrix Form, $Ax = b$

We can also write it as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where:

$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b := \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

We now write a general system of linear equations as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

We can write this system as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where:

$$A := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Example

Express the system of linear equations in matrix form:

$$\begin{aligned} x_1 + 5x_2 &= 4 \\ 3x_1 - x_2 + 7x_3 &= 2 \\ -x_1 + 2x_3 &= -5 \end{aligned}$$

Remark

When one of the x variables is missing from a row, the coefficient is zero, which we have to include in the matrix.

Example

$$
\begin{aligned}
x_1 + 5x_2 + 0x_3 &= 4 \\
3x_1 - x_2 + 7x_3 &= 2 \\
-x_1 + 0x_2 + 2x_3 &= -5
\end{aligned}
$$

Example

$$
\begin{aligned}
x_1 + 5x_2 + 0x_3 &= 4 \\
3x_1 - x_2 + 7x_3 &= 2 \\
-x_1 + 0x_2 + 2x_3 &= -5
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 5 & 0 \\ 3 & -1 & 7 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}
$$

Let's define A and b in Julia.

In [1]:
```
A = [1 5 0; 3 -2 7; -1 0 2]
```

Out[1]:
```
3×3 Array{Int64,2}:
  1   5  0
  3  -2  7
 -1   0  2
```

In [2]:
```
b = [4; 2; -5]
```

Out[2]:
```
3-element Array{Int64,1}:
  4
  2
 -5
```

Look at the systems and declare matrix A and vector b such that $Ax = b$.

Problem (1)

$$\begin{aligned} x_1 + 6x_3 - x_4 &= 2 \\ 2x_1 - x_2 + 3x_3 + 6x_4 &= -9 \\ -x_1 + 2x_3 &= 0 \\ x_2 + 4x_3 - 2x_4 &= 12 \end{aligned}$$

Problem (2)

Note: These variables are not in a "nice" order. Pay attention to the coefficients, and make sure they are in the correct order.

$$\begin{aligned} x_4 - 3x_2 + 6x_5 &= 7 \\ 2x_3 - x_1 &= -5 \\ x_1 + 4x_5 - 3x_2 + x_4 - 3x_3 &= 17 \\ 9x_2 - 3x_1 &= 8 \\ 4x_5 + 2x_1 - 7x_3 &= -12 \end{aligned}$$

We take an operational approach to define the determinant (a tricky topic in linear algebra).

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- The determinant is a function that maps a square matrix to a real number.

We take an operational approach to define the determinant (a tricky topic in linear algebra).

- The determinant is a function that maps a square matrix to a real number.
- The determinant of a 1×1 matrix is the scalar value that defines the matrix.

- The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A) := ad - bc$$

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$$\det(A) := ad - bc$$

- This notation is another way to express the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

A formula for the determinant of a 3×3 matrix is

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) := a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Q. Is there a way to compute determinant of any square matrix painlessly?

Q. Is there a way to compute determinant of any square matrix painlessly?

```
# Run this block to enable the determinant function
using Pkg
Pkg.add("LinearAlgebra")
using LinearAlgebra

A = [1 2 3; 4 5 6; 7 8 9];
det(A)
```

Why does the determinant matter?

- A square system of linear equations $Ax = b$ will have a unique solution x when $\det(A) \neq 0$.
- If $\det(A) = 0$, the system either has no possible solutions or infinite possible solutions.
- The determinant is only defined for square matrices.

For the following problems, convert the system of equations to the matrix form $Ax = b$, then find the determinant to see if x has a unique solution. Do not solve for x! We haven't learned that yet.

```
# Declare your matrix for A
A =
# take the determinant of A
det(A)
```

Problem (1)

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 7 \\
2x_1 - x_2 + x_3 &= 0.5 \\
x_1 + 4x_3 &= 7 \\
x_4 + 2x_3 - 5x_5 &= 11 \\
-4x_2 + 12x_4 &= 0
\end{aligned}
$$

Problem (2)

$$
\begin{aligned}
x_2 + 2x_5 &= 7 \\
-14x_5 + -7x_2 &= 0.5 \\
-5x_1 + 4x_3 &= 7 \\
x_1 + 2x_2 + 3x_3 + 4x_5 + 5x_5 &= 11 \\
-4x_2 + 12x_4 &= 0
\end{aligned}
$$

Next Time

- Triangular Systems of Equations
- Forward and Back Substitution
- Read Chapter 3 of ROB 101 Book