Eigenvalues and Eigenvectors of $n \times n$ real matrices

Complex numbers

- $x^2+1=0$ has no real solutions, hence we introduce a new number $i := \sqrt{-1}$. The imaginary number i is really defined by $\boxed{(i)^2 := -1}$

- The set of complex numbers is

$$\mathbb{C} := \{a + ib \mid a \in \mathbb{R},\ b \in \mathbb{R}\}$$

Examples

$$\left.\begin{aligned} z_1 &= 2 + i3 \\ z_2 &= -6 + i\sqrt{2} \\ z_3 &= \pi - i\sqrt{17} \end{aligned}\right\} \text{complex numbers}$$

Def. If $z = a + ib \in \mathbb{C}$, then

$a := \mathrm{real}(z)$ real part of z

$b := \mathrm{imag}(z)$ imaginary part of z

Note: $\text{real}(i) = 0$ $\quad i = 0 + 1 \cdot i$

$\text{imag}(i) = 1$

$$\boxed{\text{Note: } z = a + ib \in \mathbb{C} \Leftrightarrow a \in \mathbb{R},\ b \in \mathbb{R} \ !!}$$

Arithmetic of Complex Numbers

Addition: $(a_1 + ib_1) + (a_2 + ib_2) := (a_1 + a_2) + i(b_1 + b_2)$

We simply add the corresponding real and imaginary parts, similar to how we add vectors $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$

Multiplication:

$$(a_1 + ib_1)(a_2 + ib_2) := (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Why? $(x_1 + y_1)(x_2 + y_2) = x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2$

Let $x_1 = a_1 \quad y_1 = ib_1$

$x_2 = a_2 \quad y_2 = ib_2$

$x_1 x_2 = a_1 a_2$ real

$x_1 y_2 = a_1 (i b_2) = i\, a_1 b_2$ imag

$x_2 y_1 = a_2 (i b_1) = i\, a_2 b_1$ imag

$y_1 y_2 = (i b_1)(i b_2) = (i)^2 b_1 b_2 = -b_1 b_2$ real

Example $(2+i3)(-4-i6) =$

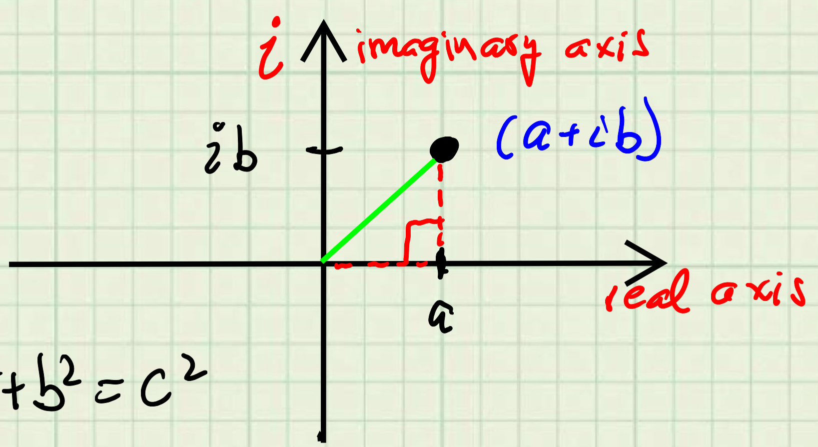
$$= \big((2)(-4) - (3)(-6)\big) + i\big((2)(-6) + (3)(-4)\big)$$

$$= 10 - i\,24$$

$\boxed{\text{Division?}}$ A bit more painful!

Def. Let $z = a+ib \in \mathbb{C}$. Then the magnitude of z is $|z| := \sqrt{a^2+b^2}$.

Why?



Pythagorean Thm: $a^2+b^2=c^2$

For us: $c = |z|$

Example $z = 3 + i(-4)$

$$|z|^2 = (3)^2 + (-4)^2 = 25$$

$$|z| = \sqrt{25} = 5$$

Def. Let $z = a + ib \in \mathbb{C}$. The complex conjugate of z is $z^* := a - ib$.

Example $z = 3 + i4 \Rightarrow z^* = 3 - i4$

$$z = 3 - i4 \Rightarrow z^* = 3 + i4$$

$$\boxed{\therefore (z^*)^* = z}$$

Fact: Let $z = a + ib \in \mathbb{C}$. Then

$$z \cdot z^* = |z|^2$$

Why? $|z|^2 = a^2 + b^2$

$$z^* = a - ib$$

$$z \cdot z^* = (a+ib)(a-ib)$$

$$= \Big(\underbrace{a^2 - (b)(-b)}_{a^2+b^2} \Big) + i \Big(\underbrace{(a)(-b) + (b)(a)}_{0} \Big)$$

Def. $z = a + ib \in \mathbb{C}$ is the zero complex number if $|z| = 0$ $\left(\Leftrightarrow a = 0 \ \& \ b = 0 \right)$

Division: Suppose $z_1 = a_1 + ib_1 \in \mathbb{C}$ and $z_2 = a_2 + ib_2 \in \mathbb{C}$, and $z_2 \neq 0$. Then

$$\frac{z_1}{z_2} := \frac{z_1 \cdot (z_2^*)}{|z_2|^2} = \frac{(a_1a_2 + b_1b_2) + i\,(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

and thus $\frac{z_1}{z_2} \in \mathbb{C}$.

Why this definition?

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} = \frac{(z_1)(z_2^*)}{(z_2)(z_2^*)} = \frac{z_1(z_2^*)}{|z_2|^2}$$

$$\frac{2+i3}{1+i} = \frac{(2+i3)}{(1+i)} \cdot \frac{(1-i)}{(1-i)}$$

$$= \frac{5+i}{2} = \frac{5}{2} + i\frac{1}{2}$$

Complex Numbers in Julia

DEMO

Recall $\mathbb{R}^n := \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_k \in \mathbb{R} \right\}$

Def: $\mathbb{C}^n : \left\{ \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \middle| \; z_k \in \mathbb{C} \right\}$

Column vectors of complex numbers.

- $\mathbb{C}^n$ is a vector space
- all of our notions of linear combinations and linear independence carry over!

Linear Difference Equations

$x_{k+1} = a\, x_k$, $a \in \mathbb{R}$, $x \in \mathbb{R}$, k = time

Going

$x[k+1] = A\, x[k]$ $A = n \times n$ matrix

$x[n] = A^n\, x[0] = A^n x_0$

$x[k+1] = A\, \underline{x[k]} + \underline{B\, u[k]}$