Summary: $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_i \in \mathbb{R} \right\}$ $n \times 1$ Col. Vectors.

- v is linear combination of $\{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$

- $Ax = b$ has a sol. $\iff$ b is a linear combination of the columns of $A_{n \times m}$

$$\begin{bmatrix} a_1^{col} & a_2^{col} & \cdots & a_m^{col} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = b \iff$$

$$x_1 a_1^{col} + x_2 a_2^{col} + \cdots + x_m a_m^{col} = b$$ [col-row def. of multiplication]

- $\{v_1, v_2, \ldots, v_m\} \subset \mathbb{R}^n$ is linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0 \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Today
- Checking linear independence
- Pro Tip!

Example A: Is the set $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independent in $\mathbb{R}^3$?

Solution: Take $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and we seek a solution to

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow$ linear independent

$\square$

Example B Is the set of vectors

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ linearly indep.?

Observe: $\boxed{v_3 = v_2 - v_1}$

$\therefore \quad -v_1 + v_2 - v_3 = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Linearly dependent.

Just to be super super overkill,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

Is there a non-trivial solution to this equation? Yes $\alpha_1 = -1$ $\alpha_2 = 1$, $\alpha_3 = -1$

Example C Is the set of vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

linearly independent?

Apply the definition

Seek $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ 2\alpha_1 - 2\alpha_2 \\ 3\alpha_1 + 4\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If α_1 and α_2 solve the above 3 equations, then they must also solve the first two equations

$$\alpha_1 + \alpha_2 = 0$$
$$2\alpha_1 - 2\alpha_2 = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A) = (1)(-2) - (1)(2) = -4 \neq 0$$

$\therefore$ $\alpha_1 = \alpha_2 = 0$ is the unique solution.

$\therefore$ Linearly Indep. Yeah!

Example D Is the set of vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \\ 5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 6 \\ 2 \\ -3 \end{bmatrix}$$

linearly indep.?

Solution: We have to check if

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution or not!

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 6 \\ 3 & 4 & 2 \\ 1 & 5 & 3 \end{bmatrix}_{4\times3} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times1} \checkmark$$

We could solve this, but we would not enjoy it!

There has to be a better way!

Pro Tip!

How to check linear independance like a pro.

Given $\left\{ v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \dots, v_m = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} \right\}$

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n\times m}$$

Then $\{v_1, v_2, \ldots, v_m\}$ is linearly indep if, and only if, $\det(A^T \cdot A) \neq 0$.

- Note $A^T \cdot A = [m \times n] \cdot [n \times m] = m \times m$

- Proof of the proof of Pro Tip to come end of today or Friday.

Re-do Example (C).

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 4 \end{bmatrix}_{3\times 2}$$

$$A^T \cdot A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 4 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 4 \end{bmatrix}} = \begin{bmatrix} 14 & \overset{+9}{\cancel{-15}} \\ \underset{+9}{\cancel{-15}} & 21 \end{bmatrix}_{2\times 2}$$

$$\det(A^T A) = (14)(21) - \underset{(9)^2}{(\cancel{15})^2} = \overset{283}{\cancel{69}} \neq 0$$

$\therefore$ Linearly Independent!

[Sorry for the calculation mistake ☺]

JWG

Example 4.5 *We apply the Pro Tip to Example 4.2.*

Solution: We use the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 6 \\ 2 \\ -3 \end{bmatrix}$$

and form the matrix

$$A := \begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 6 \\ 3 & 4 & 2 \\ 1 & 5 & -3 \end{bmatrix}.$$

$$A\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We go to Julia and compute that

$$A^\top \cdot A = \begin{bmatrix} 15.0 & 13.0 & 17.0 \\ 13.0 & 45.0 & -19.0 \\ 17.0 & -19.0 & 53.0 \end{bmatrix},$$

and that its LU Factorization is $P \cdot (A^\top \cdot A) = L \cdot U$, where

$$P = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.0 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}, \ L = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.8 & 1.0 & 0.0 \\ 0.9 & 0.5 & 1.0 \end{bmatrix}, \text{ and } U = \begin{bmatrix} 17.0 & -19.0 & 53.0 \\ 0.0 & 59.5 & -59.5 \\ 0.0 & 0.0 & \boxed{-0.0} \end{bmatrix}.$$

We observe that U has a zero on its diagonal and hence the set $\{v_1, v_2, v_3\}$ is linearly dependent. ■

det $= \pm 1$

$\det(A^\top A) = 0 \iff U$ has a zero on its diagonal.

$\therefore \{v_1, v_2, v_3\}$ are linearly dependent!

Why is the Pro Tip True?

What's the secret sauce?

Consider $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$, then

$$y^T \cdot y = [y_1 \ \ y_2 \ \cdots \ y_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= (y_1)^2 + (y_2)^2 + \cdots + (y_n)^2$$

$$\therefore \ y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \iff y^T \cdot y = 0$$

We are looking for non-trivial solutions to $\underset{n \times m}{A} \cdot \alpha_{m \times 1} = 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$

We let $y = A \cdot y$ and . . .