ROB 101 - Fall 2021

Matrix Multiplication

September 13, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- How to partition matrices into rows and columns
- How to multiply one matrix times another
- How to swap rows of a matrix

Outcomes

- Multiplying a row vector by a column vector
- Recognizing the rows and columns of a matrix
- Standard definition of matrix multiplication $A \cdot B$ using the rows of A and the columns of B
- Size restrictions when multiplying matrices
- Examples that work and those that don't because the sizes are wrong
- Permutation matrices

Multiplying a Row Vector by a Column Vector

Let $a^{\text{row}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix}$ be a row vector with k elements and let $b^{\text{col}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$ be a column vector with the same number of elements as a^{row}.

Multiplying a Row Vector by a Column Vector

Definition

The *product* of a^{row} and b^{col} is defined as

$$a^{\mathrm{row}} \cdot b^{\mathrm{col}} := \sum_{i=1}^{k} a_i b_i.$$

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For many, the following visual representation is more understandable,

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} := a_1 b_1 + a_2 b_2 + \cdots + a_k b_k.$$

Let $a^{\text{row}} = \begin{bmatrix} 2 & -3 & -1 & 11 \end{bmatrix}$ be a row vector with $k = 4$ elements and let $b^{\text{col}} = \begin{bmatrix} 3 \\ 5 \\ -1 \\ -2 \end{bmatrix}$ be a column vector with $k = 4$ elements. Perform their multiplication if it makes sense.

Because they have the same number of elements, we can form their product and we compute

$$\begin{aligned} a^{\text{row}} \cdot b^{\text{col}} :=& \sum_{i=1}^{4} a_i b_i \\ =&(2)(3) + (-3)(5) + (-1)(-1) + (11)(-2) = -30, \end{aligned}$$

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or, equivalently, we write it like this

$$\begin{aligned} &\begin{bmatrix} 2 & -3 & -1 & 11 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ -1 \\ -2 \end{bmatrix} \\ &= (2)(3) + (-3)(5) + (-1)(-1) + (11)(-2) = -30. \end{aligned}$$

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$$
\begin{aligned}
a^{\text{row}} \cdot b^{\text{col}} :=& \sum_{i=1}^{4} a_i b_i \\
=&(2)(3) + (-3)(5) + (-1)(-1) + (11)(-2) = -30,
\end{aligned}
$$

In [1]:
```
a = [2. -3 -1 11];
b = [3., 5, -1, -2];
a * b
```

Out[1]:
```
1-element Array{Float64,1}:
 -30.0
```

Examples of Row and Column Partitions

Let $A = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right]$ be a 2×3 matrix.

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Then a *partition* of A into *rows* is

$$\begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \end{bmatrix} = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix}, \quad \text{that is,} \quad \begin{aligned} a_1^{\text{row}} &= [1 \quad 2 \quad 3] \\ a_2^{\text{row}} &= [4 \quad 5 \quad 6]. \end{aligned}$$

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Remark

We note that a_1^{row} and a_2^{row} are row vectors of size 1×3; they have the same number of entries as A has columns.

Examples of Row and Column Partitions

A *partition* of $A = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$ into *columns* is

$$\left[\begin{array}{ccc} a_1^{\text{col}} & a_2^{\text{col}} & a_3^{\text{col}} \end{array}\right] = \left[\boxed{\begin{array}{c} 1 \\ 4 \end{array}} \quad \boxed{\begin{array}{c} 2 \\ 5 \end{array}} \quad \boxed{\begin{array}{c} 3 \\ 6 \end{array}}\right],$$

A *partition* of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ into *columns* is

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that is,

$$a_1^{\rm col} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2^{\rm col} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad a_3^{\rm col} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

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that is,

$$a_1^{\text{col}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \ a_2^{\text{col}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \ a_3^{\text{col}} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Remark

We note that a_1^{col}, a_2^{col}, and a_3^{col} are column vectors of size 2×1; they have the same number of entries as A has rows.

Let A be an $n \times m$ matrix. A partition of A into rows is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1^{\mathrm{row}} \\ a_2^{\mathrm{row}} \\ \vdots \\ a_n^{\mathrm{row}} \end{bmatrix} = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2m}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}} \end{bmatrix}.$$

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That is, the i-th row is the $1 \times m$ row vector

$$a_i^{\text{row}} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{im}],$$

where i varies from 1 to n.

A partition of A into columns is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1^{\mathrm{col}} & a_2^{\mathrm{col}} & \cdots & a_m^{\mathrm{col}} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{matrix}} \end{bmatrix}.$$

A partition of A into columns is

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{array}\right] = \left[\boxed{\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{array}} \quad \boxed{\begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{array}} \quad \cdots \quad \boxed{\begin{array}{c} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{array}}\right].$$

That is, the j-th column is the $n \times 1$ column vector

$a_j^{\mathrm{col}} = \left[\begin{array}{c} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{array}\right]$, where j varies from 1 to m.

- Let A be an $n \times k$ matrix, meaning it has n rows and k columns.

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- Let B be a $k \times m$ matrix, so that it has k rows and m columns.

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- Let B be a $k \times m$ matrix, so that it has k rows and m columns.
- The values of n, m, and k can be any integers greater than or equal to one.

Definition

When the number of columns of the first matrix A equals the number of rows of the second matrix B, the *matrix product* of A and B is defined and results in an $n \times m$ matrix:

$$[n \times k \ \ \text{matrix}] \cdot [k \times m \ \ \text{matrix}] = [n \times m \ \ \text{matrix}].$$

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- The standard way of doing matrix multiplication $A \cdot B$ involves multiplying the rows of A with the columns of B.
- In general, $A \cdot B \neq B \cdot A$ even when A and B are square matrices of the same size.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \end{bmatrix} = \begin{bmatrix} \boxed{1 \quad 2} \\ \boxed{3 \quad 4} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} b_1^{\text{col}} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 5 \\ 6 \end{matrix}} \end{bmatrix}.$$

The matrix product of A and B is

$$A \cdot B = \begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\text{col}} \end{bmatrix} := \begin{bmatrix} a_1^{\text{row}} \cdot b_1^{\text{col}} \\ a_2^{\text{row}} \cdot b_1^{\text{col}} \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

The matrix product of A and B is

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because

$$a_1^{\text{row}} \cdot b_1^{\text{col}} = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 5 + 12 = 17$$

$$a_2^{\text{row}} \cdot b_1^{\text{col}} = \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 15 + 24 = 39.$$

Example 1

The matrix product of A and B is

$$A \cdot B = \begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\text{col}} \end{bmatrix} := \begin{bmatrix} a_1^{\text{row}} \cdot b_1^{\text{col}} \\ a_2^{\text{row}} \cdot b_1^{\text{col}} \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

In [2]:
```
A = [1 2; 3 4];
B = [5; 6];
A * B
```

Out[2]:
```
2-element Array{Int64,1}:
 17
 39
```

Example 2

We reuse A and B above and ask if we can form the matrix product in the order $B \cdot A$.

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We have that the first matrix B is 2×1 and the second matrix A is 2×2. The number of columns of the first matrix does not match the number of rows of the second matrix, and hence the product cannot be defined in this direction.

Example 2

We reuse A and B above and ask if we can form the matrix product in the order $B \cdot A$.

We have that the first matrix B is 2×1 and the second matrix A is 2×2. The number of columns of the first matrix does not match the number of rows of the second matrix, and hence the product cannot be defined in this direction.

In [3]:
```
# Don't try this at home!
A = [1 2; 3 4];
B = [5; 6];
B * A
```

```
DimensionMismatch("matrix A has dimensions (2,1), matrix B has dimensions (2,2)")
```

Example 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \end{bmatrix} = \begin{bmatrix} \boxed{1 \;\; 2} \\ \boxed{3 \;\; 4} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & -2 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} b_1^{\text{col}} & b_2^{\text{col}} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 5 \\ 6 \end{matrix}} & \boxed{\begin{matrix} -2 \\ 1 \end{matrix}} \end{bmatrix}.$$

The matrix product of A and B is

$$A \cdot B = \begin{bmatrix} \boxed{\begin{matrix} 1 & 2 \end{matrix}} \\ \boxed{\begin{matrix} 3 & 4 \end{matrix}} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} 5 \\ 6 \end{matrix}} & \boxed{\begin{matrix} -2 \\ 1 \end{matrix}} \end{bmatrix}$$

$$= \begin{bmatrix} (1)(5) + (2)(6) & (1)(-2) + (2)(1) \\ (3)(5) + (4)(6) & (3)(-2) + (4)(1) \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 39 & -2 \end{bmatrix}.$$

The matrix product of B and A is

$$B \cdot A = \begin{bmatrix} \boxed{5 \quad -2} \\ \boxed{6 \qquad 1} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 4 \end{matrix}} \end{bmatrix}$$

$$= \begin{bmatrix} (5)(1) + (-2)(3) & (5)(2) + (-2)(4) \\ (6)(1) + (1)(3) & (6)(2) + (1)(4) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 9 & 16 \end{bmatrix}.$$

Example 3

In [4]:
```
A = [1 2; 3 4];
B = [5 -2; 6 1];
A * B
```

Out[4]:
```
2×2 Array{Int64,2}:
 17   0
 39  -2
```

In [5]:
```
B * A
```

Out[5]:
```
2×2 Array{Int64,2}:
 -1   2
  9  16
```

We partition the $n \times k$ matrix A into rows and the $k \times m$ matrix B into columns, as in

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} = \begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \\ \vdots \\ a_n^{\text{row}} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1k}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2k}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nk}} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{km} \end{bmatrix} = \begin{bmatrix} b_1^{\mathrm{col}} & b_2^{\mathrm{col}} & \cdots & b_m^{\mathrm{col}} \end{bmatrix}$$

$$= \left[\boxed{\begin{matrix} b_{11} \\ b_{21} \\ \vdots \\ b_{k1} \end{matrix}} \quad \boxed{\begin{matrix} b_{12} \\ b_{22} \\ \vdots \\ b_{k2} \end{matrix}} \quad \cdots \quad \boxed{\begin{matrix} b_{1m} \\ b_{2m} \\ \vdots \\ b_{km} \end{matrix}} \right],$$

General Case: What is happening inside Julia

then

$$A \cdot B := \begin{bmatrix} a_1^{\text{row}} \cdot b_1^{\text{col}} & a_1^{\text{row}} \cdot b_2^{\text{col}} & \cdots & a_1^{\text{row}} \cdot b_m^{\text{col}} \\ a_2^{\text{row}} \cdot b_1^{\text{col}} & a_2^{\text{row}} \cdot b_2^{\text{col}} & \cdots & a_2^{\text{row}} \cdot b_m^{\text{col}} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{\text{row}} \cdot b_1^{\text{col}} & a_n^{\text{row}} \cdot b_2^{\text{col}} & \cdots & a_n^{\text{row}} \cdot b_m^{\text{col}} \end{bmatrix}.$$

Another way to see the pattern is like this

$$A \cdot B := \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1k}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2k}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nk}} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} b_{11} \\ b_{21} \\ \vdots \\ b_{k1} \end{matrix}} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ \vdots \\ b_{k2} \end{matrix}} & \cdots & \boxed{\begin{matrix} b_{1m} \\ b_{2m} \\ \vdots \\ b_{km} \end{matrix}} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{k} a_{1i}b_{i1} & \sum_{i=1}^{k} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{k} a_{1i}b_{im} \\ \sum_{i=1}^{k} a_{2i}b_{i1} & \sum_{i=1}^{k} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{k} a_{2i}b_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{k} a_{ni}b_{i1} & \sum_{i=1}^{k} a_{ni}b_{i2} & \cdots & \sum_{i=1}^{k} a_{ni}b_{im} \end{bmatrix}.$$

The Matrix View of Swapping the Order of Equations

This system of equations is neither upper triangular nor lower triangular

$$\begin{aligned} 3x_1 &= 6 \\ x_1 - 2x_2 + 3x_3 &= 2 \\ 2x_1 - x_2 &= -2, \end{aligned}$$

The Matrix View of Swapping the Order of Equations

but if we simply re-arrange the order of the equations, we arrive at the lower triangular equations

$$
\begin{aligned}
3x_1 &= 6 \\
2x_1 - x_2 &= -2 \\
x_1 - 2x_2 + 3x_3 &= 2.
\end{aligned}
$$

The Matrix View of Swapping the Order of Equations

Let's write out the matrix equations for the "unfortunately ordered" equations and then the "nicely" re-arranged system of equations.

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 1 & -2 & 3 \\ 2 & -1 & 0 \end{bmatrix}}_{A_O} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}}_{b_O},$$

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}}_{A_L} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}}_{b_L}.$$

- We see the second and third rows are swapped when we compare A_O to A_L and b_O to b_L.

- We see the second and third rows are swapped when we compare A_O to A_L and b_O to b_L.

Claim

The swapping of rows can be accomplished by multiplying A_O and b_O on the left by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Indeed, we check that

$$P \cdot A_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & -2 & 3 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix} = A_L$$

Indeed, we check that

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and

$$P \cdot b_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} = b_L.$$

Indeed, we check that

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and

$$P \cdot b_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} = b_L.$$

P is called a permutation matrix.

Permutation Matrix

In [1]:
```
AO = [3. 0 0; 1 -2 3; 2 -1 0];
bO = [6.; 2; -2];
P = [1 0 0; 0 0 1; 0 1 0];

AL = P * AO
```

Out[1]:
```
3×3 Array{Float64,2}:
 3.0   0.0  0.0
 2.0  -1.0  0.0
 1.0  -2.0  3.0
```

In [2]:
```
bL = P * bO
```

Out[2]:
```
3-element Array{Float64,1}:
  6.0
 -2.0
  2.0
```

Remark

We note that the permutation matrix P is constructed from the 3×3 identity matrix I by swapping its second and third rows, exactly the rows we wanted to swap in A_O and b_O. This observation works in general.

Suppose we want to do the following rearrangement

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \\ 3 \end{bmatrix},$$

To see the resulting structure at a matrix level, we put the 5×5 identity matrix on the left and the permutation matrix P on the right

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

so that it is apparent that P is just a re-ordering of the rows of I.

Example

In [3]:
```
v = [1; 2; 3; 4; 5];
P = [1 0 0 0 0; 0 1 0 0 0; 0 0 0 0 1; 0 0 0 1 0; 0 0 1 0 0]
```

Out[3]:
```
5×5 Array{Int64,2}:
 1  0  0  0  0
 0  1  0  0  0
 0  0  0  0  1
 0  0  0  1  0
 0  0  1  0  0
```

In [4]:
```
# permute v
P * v
```

Out[4]:
```
5-element Array{Int64,1}:
 1
 2
 5
 4
 3
```

Next Time

- LU (Lower-Upper) Factorization
- Read Chapter 5 of ROB 101 Book