2020 Oct 30 ROB 101

Review: • Intermediate Value Thm:

$a < b$, $f(a) \cdot f(b) < 0$, f cont. $\Rightarrow$ exists $c \in [a, b]$ such that $f(c) = 0$ (root)

• **Bisection:** $c = \dfrac{a+b}{2}$

IF $|f(c)| < tol$
    $x^* = c$; break
ELSEIF $f(a) \cdot f(c) < 0$
    $b = c$
ELSE
    $a = c$
END

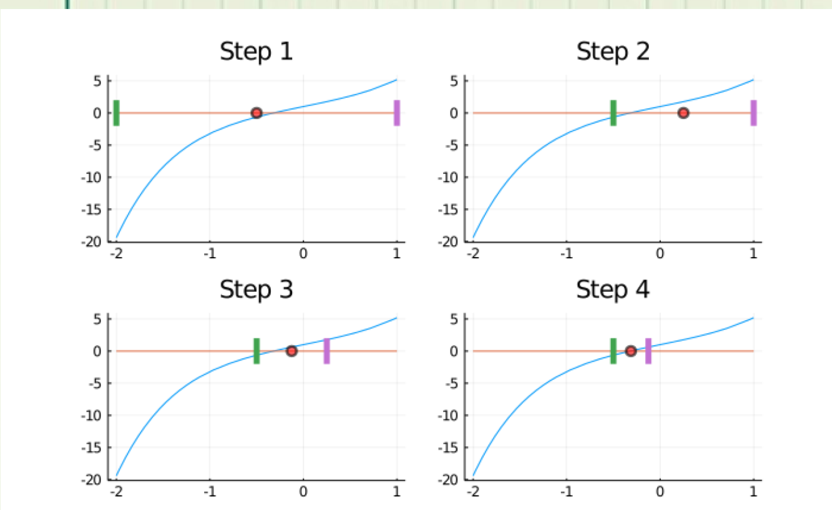


Figure 3: Evolution of the bracketing points a and b as well as the midpoint c in the first four steps of the Bisection Algorithm for finding a root of $0.2x^5 + x^3 + 3x + 1 = 0$. It is very clear that the algorithm hones in on a root!

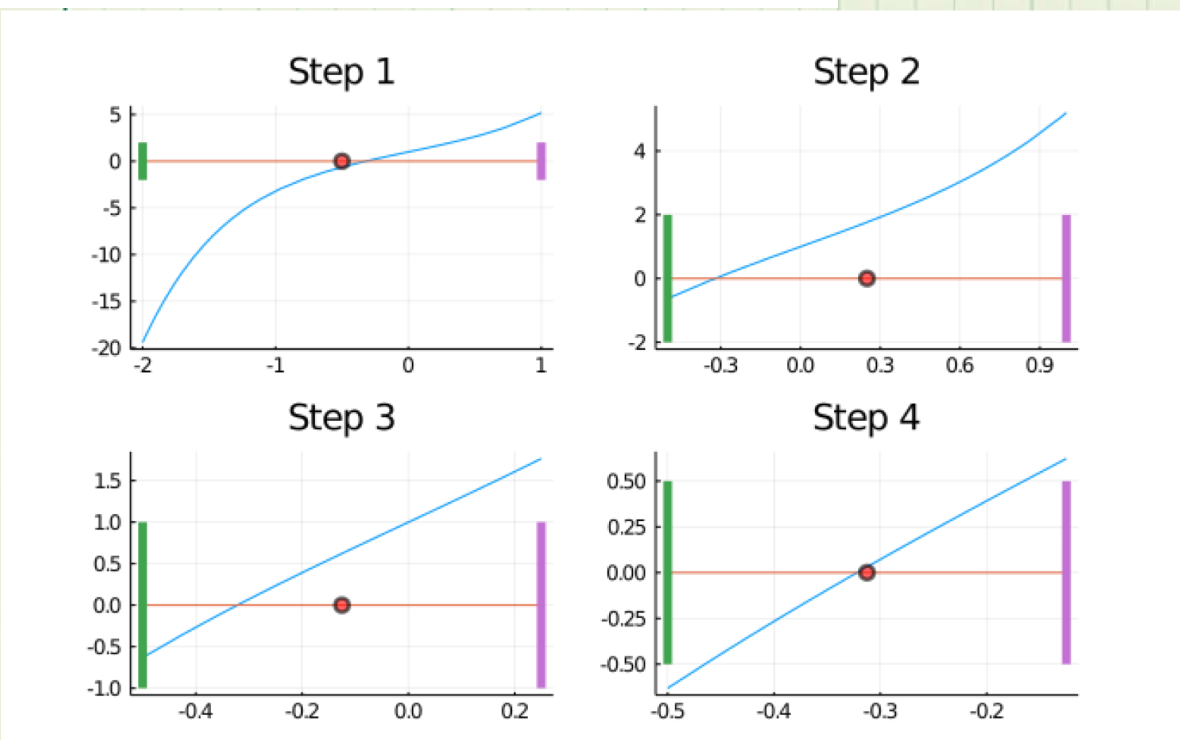
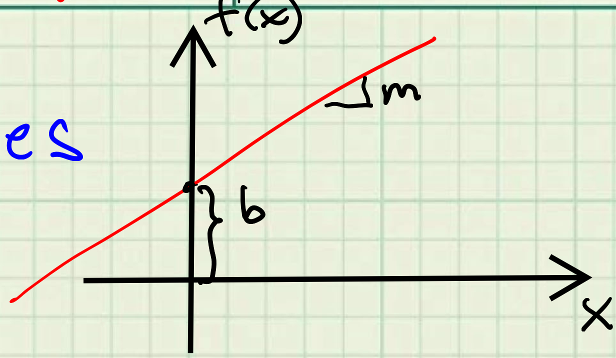


Figure 4: Zooms of the first four steps of the Bisection Algorithm for finding a root of $0.2x^5 + x^3 + 3x + 1 = 0$ that lies between -1 and 2. Observe that as we zoom into the function at a point, it looks more and more like a straight line!

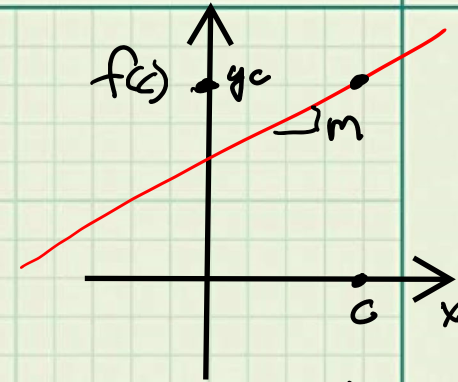
ZOOM

TODAY

Lines



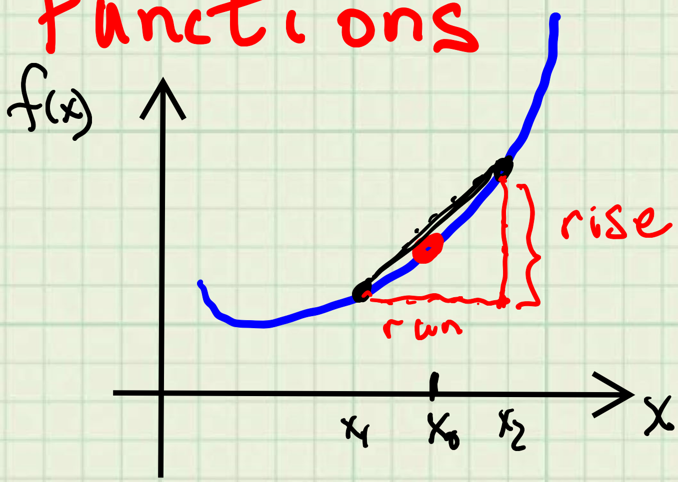
$y = mx + b$



$y = y_c + m(x - c)$

$c = 0 \Rightarrow y_c = b =$ the y-intercept

Numerical Derivatives & Linear Approximations to Nonlinear Functions



"slope" of a function at a point

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Question: How to select x_2 & x_1?

Calculus Answer: $x_1 = x_0$ and $x_2 = x_0 + h$ where $h > 0$ is very very small, but not zero.

Calculus Notation:

$$\frac{df(x_0)}{dx} = \text{"} \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \text{"}$$

Called the derivative of f at x_0.

Three Numerical Approximations to a derivative of a function

$$\frac{df}{dx}(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

forward difference approx

$$\frac{df}{dx}(x_0) \approx \frac{f(x_0) - f(x_0-h)}{h}$$

backward diff. approximation

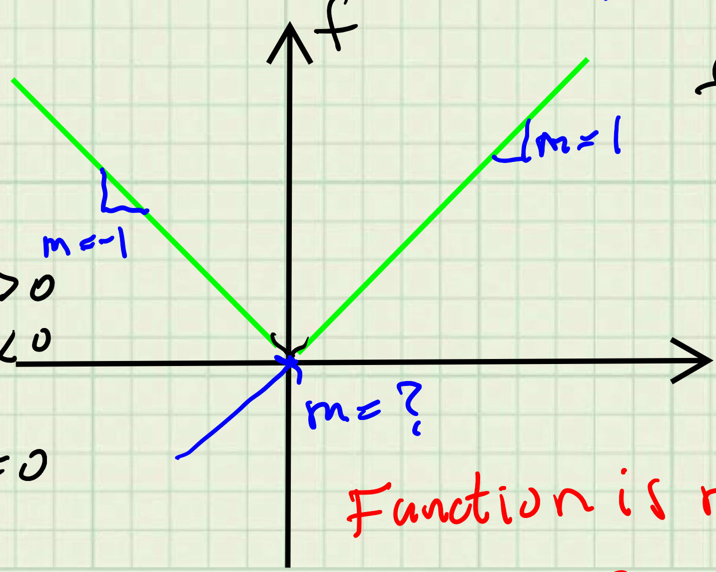
$$\frac{df}{dx}(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$

symmetric diff. approximation

Remark: The forward and backward difference approximations to the analytical derivative are exact for linear functions, while the symmetric difference approximation is exact for quadratic functions ($f(x) = a_0 + a_1 x + a_2 x^2$).

Question: Can you always associate a "slope" (derivative) to a function at a point?

NO.

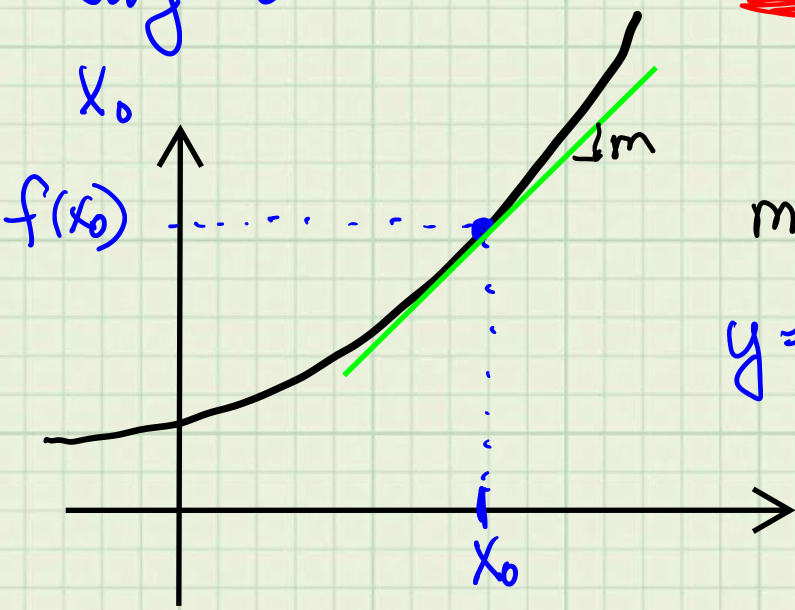


$$\frac{df}{dx}(x) = \begin{cases} +1 & x>0 \\ -1 & x<0 \\ \text{undef} & x=0 \end{cases}$$

Function is not differentiable at $x_0 = 0$.

Linear Approximation of a Function at a Point

Assumption: We can associate a $\boxed{\text{slope to a point } x_0}$. We say that $f(x)$ is <u>differentiable</u> at x_0



$m = \frac{df}{dx}(x_0)$

$y = y_c + m(x-c)$

$c = x_0$

$m = \frac{df}{dx}(x_0)$

$y_c = f(x_0)$

$$\boxed{f(x) \approx f(x_0) + \frac{df}{dx}(x_0)\left(x - x_0\right)}$$

Most important formula coming from Calc I.

Newton's Method (1669)

Very clever way to find roots of nonlinear functions

Suppose x_k is our current estimate of a root of $f(x)$, $f: \mathbb{R} \to \mathbb{R}$

Seek next x_k, call it x_{k+1}, that is (we hope) a better approx of the root:

Write $f(x) \approx \underbrace{f(x_k) + \frac{df}{dx}(x_k)(x - x_k)}_{\text{linear}}$

We want to choose x_{k+1} such that $f(x_{k+1}) \approx 0$

If we TRUST our linear approx then this what we do.

$f(x_{k+1}) \approx 0 \Leftrightarrow 0 = f(x_k) + \frac{df}{dx}(x_k)(x_{k+1} - x_k)$

Solve for x_{k+1} !

If $\frac{df}{dx}(x_k) \neq 0$, then

$$x_{k+1} = x_k - \frac{f(x_k)}{\frac{df}{dx}(x_k)}$$

$$\boxed{x_{k+1} = x_k - \left[\frac{df(x_k)}{dx}\right]^{-1} f(x_k)}$$