• T-shirts will be ordered today

Summary: Our objective is to have necessary and sufficient conditions for n-linear equations in m-unknowns to have a solution and for it to be unique:

$$\boxed{Ax=b \text{ has unique solution} \iff ???} \quad A = n \times m$$

When $n=m$ (square A): Answer is $\det(A) \neq 0$.

Our tool: $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \,\middle|\, x_i \in \mathbb{R} \right\}$

$\mathbb{R}^n$ is called a vector space

• Elements of $\mathbb{R}^n$ are called vectors

• $\alpha \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, then

$$x+y = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}, \quad \alpha \cdot x = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

• $x+y = y+x$

• $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

• If $z \in \mathbb{R}^n$, then

$$(x+y)+z = x+(y+z)$$

Columns of $A_{n \times m}$ are elements of $\mathbb{R}^n$

$$a_j^{col} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

Today: Linear Combinations
Linear Independence

Def. (Linear Combinations) A vector $v \in \mathbb{R}^n$ is a linear combination of vectors $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$ if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ such that

$$\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = v}$$

Finite sums of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

are called <u>linear combinations</u>.

Examples

$$2\begin{bmatrix}6\\7\\8\end{bmatrix} - 3\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}9\\8\\7\end{bmatrix}$$

Hence, $\begin{bmatrix}9\\8\\7\end{bmatrix}$ is a linear combination of the vectors $\left\{\begin{bmatrix}6\\7\\8\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$.

Also the vector $\begin{bmatrix}3\\3.5\\4\end{bmatrix}$ is a linear combination of $\begin{bmatrix}6\\7\\8\end{bmatrix}$ and $\begin{bmatrix}1\\2\\3\end{bmatrix}$ because

$$\begin{bmatrix} 3 \\ 3.5 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Moreover,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Problem Is $\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$?

Sol. By the definition of linear combination, we seek $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}. \quad (1)$$

Note that (1) is equivalent to

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad (2)$$

and

$$\alpha_1 \cdot 3 + \alpha_2 \cdot 2 = 3 \qquad (3)$$

If α_1 and α_2 satisfy (2), then

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{\det = -3 \neq 0} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$\therefore$ (2) has a unique solution. Some (boring) hand computations yield

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Now do these values of α_1, α_2 also satisfy

$$3\alpha_1 + 2\alpha_2 = 3 .$$

$$3(3) + 2(-1) = 7 \neq 3$$

Hence $\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$ is NOT a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. $\square$

Remark At the present time, checking if $v \in \mathbb{R}^n$ can be written as a linear combination of $\{v_1, \dots, v_m\} \in \mathbb{R}^n$ is tedious (and error prone).

Existence of Solutions to $Ax = b$

Consider a system of n-equations in m-unknowns,

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}}_{A_{n \times m}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{x_{m \times 1}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b_{n \times 1}} \qquad (1)$$

Using our column · times row form of matrix multiplication

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \cdot x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} \cdot x_2 + \cdots + \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix} \cdot x_m = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{\cancel{m}\,n} \end{bmatrix}$$

$$\boxed{x_1 a_1^{col} + x_2 a_2^{col} + \cdots + x_m a_m^{col} = b}$$

Fact: $Ax = b$ has a solution if, and only if, b is a linear combination of the columns of A.

Why: Suppose $A\bar{x} = b$, so $\bar{x}$ is a solution.

Then $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_m \end{bmatrix}$ and we have

$$\bar{x}_1 a_1^{col} + \bar{x}_2 a_2^{col} + \cdots + \bar{x}_m a_m^{col} = b$$

On the other hand, if

$$\alpha_1 a_1^{col} + \alpha_2 a_2^{col} + \cdots + \alpha_m a_m^{col} = b,$$

then $\bar{\bar{x}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \Rightarrow A\bar{\bar{x}} = b$.

Linear Independence

<u>Def.</u> A set of vectors $\{v_1, v_2, .., v_m\} \subset \mathbb{R}^n$ is linearly independent if the only set of real numbers $\alpha_1, \alpha_2, .., \alpha_m \in \mathbb{R}$ satisfying

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

is $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_m = 0$. **Otherwise,** the vectors are linearly dependent.

For Emphasis

Linear Independence

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0 \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Why call the other case linear dependence? If $\{v_1, .., v_m\}$ <u>not</u>

linearly independent, THEN there exist $\alpha_1, \alpha_2, \ldots, \alpha_m$ NOT ALL ZERO such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

One of the α_i is not zero. Let's suppose $\alpha_1 \neq 0$.

$$\alpha_1 v_1 = -\alpha_2 v_2 - \alpha_3 v_3 - \cdots - \alpha_m v_m$$

$$\boxed{v_1 = -\frac{\alpha_2}{\alpha_1} v_2 - \frac{\alpha_3}{\alpha_1} v_3 - \cdots - \frac{\alpha_m}{\alpha_1} v_m}$$

and thus at least one of the vectors can be expressed as a non-trivial linear combination of the other vectors ∇