ROB 101 - Computational Linear Algebra
HW #8

Profs Grizzle and Ghaffari

Due 9 PM = 21:00 ET on Thurs, Nov. 18, 2021

There are six (6) HW problems plus a *jupyter notebook* to complete and turn in. The drill problems this week go against the spirit of ROB 101 in that you are doing by hand things that belong on a computer. However, humans learn better by working out simple examples by hand!

1. Read Chapter 11 of our ROB 101 Booklet, `Notes for Computational Linear Algebra`. Based on your reading of the Chapter, summarize in your own words:

   (a) The purpose of the Chapter;

   (b) Two things you found the most challenging or the most interesting.

2. To make the calculations a snap, we'll do the Bisection Algorithm by hand on $f(x) = 3x + 2$.

   (a) Starting with the bracketing points $a = -2$ and $b = 4$, perform the Bisection Algorithm and fill in the following table

   Table 1: My Bisection Algorithm Results.

k	**a**	**c**	**b**	**f(a)**	**f(c)**	$\mathbf{sign\,(f(a) \cdot f(c))}$
1	-2		4			
2						
3						
4						

   (b) Make a hand sketch of the plot of $f(x)$ for $-3 \leq x \leq 5$. On this plot, indicate your values for a_k, b_k and c_k for $k = 1, 2, 3, 4$ with a bold mark of any kind, say a solid circle or a square. Please label them as a_1, b_1, c_1, etc. If you want to use colors, that is fine. You can use Julia as a calculator if you wish, but do not generate your plot in Julia!

   (c) If you had to make a guess, is there a finite value of k for which $f(c_k) = 0$? Explain your guess. Two short sentences are enough.

3. We consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 3x$. From Calculus, it is known that $\frac{df(x)}{dx} = 2x - 3$. We'll compute forward difference approximations and symmetric difference approximations at $x_0 = 1$, for three values of $h > 0$. While it would be good to try another value of x_0, we don't have the heart to make you do that many hand computations!

   (a) Using the formulas in the notes, **show all of the calculations to determine the first row of Table 2**. By all calculations, it is meant, show the values for $f(x_0 + h)$, $(f(x_0 + h) - f(x_0))$, etc. as you arrive at your numerical approximations of the derivative.

   (b) Complete the table without showing any further calculations

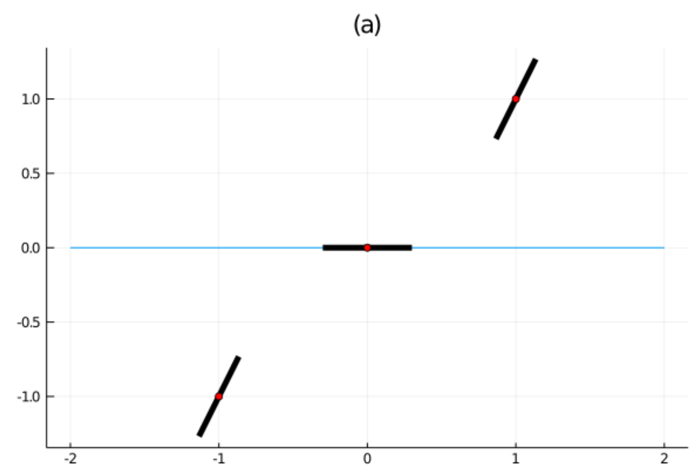
Table 2: My Table Comparing Numerical Derivatives to the Real Thing.

$\left.\frac{df(x)}{dx}\right\vert_{x=1}$	**h**	**forward difference**	**symmetric difference**
-1.00	0.001		
-1.00	0.01		
-1.00	0.1		

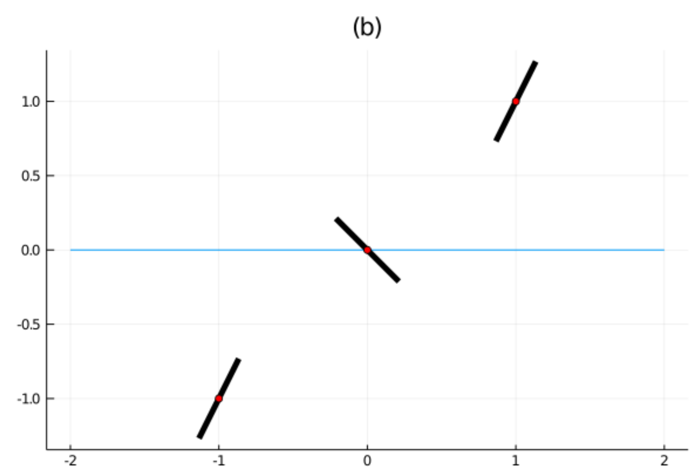
4. **See the HINTS for a worked problem.** The objective here is build your understanding that $\frac{df(x_0)}{dx}$ represents the slope of the function $f(x)$ at the point x_0. You are given three plots that contain function and derivative values. You are to *sketch by hand as* ***simple*** *of a function as you can* that satisfies:

   - the function passes through the given function values indicated by the red dots; and
   - its slope at those points is given by the slope of the corresponding black line segment.

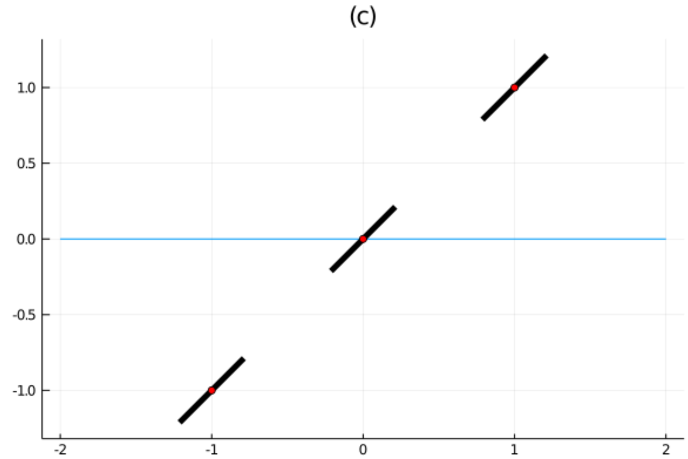
x_0	$f(x_0)$	$\frac{df(x_0)}{dx}$
-1.00	-1.00	2.00
0.00	0.00	0.00
1.00	1.00	2.00



x_0	$f(x_0)$	$\frac{df(x_0)}{dx}$
-1.00	-1.00	2.00
0.00	0.00	-1.00
1.00	1.00	2.00



$\mathbf{x_0}$	$\mathbf{f(x_0)}$	$\mathbf{\frac{df(x_0)}{dx}}$
-1.00	-1.00	1.00
0.00	0.00	1.00
1.00	1.00	1.00



5. For the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 3x = x(x-3)$, you can tell by inspection that its roots are $x^* = 0$ and $x^* = 3$. Nevertheless, we'll use Newton's Algorithm to approximately find at least one of them!

   So that you can focus on the algorithm and NOT on how to numerically compute derivatives, you are given that

$$\frac{df(x)}{dx} = 2x - 3.$$

   (a) For $x_0 = 1$, complete the following Table using Newton's Algorithm. You do not need to show your work.

$\mathbf{k}$	$\mathbf{x_k}$	$\mathbf{f(x_k)}$	$\mathbf{\frac{df(x_k)}{dx}}$
0	1.00	-2.00	-1.00
1			
2			
3			

   (b) For $x_0 = 2$, complete the following Table using Newton's Algorithm

$\mathbf{k}$	$\mathbf{x_k}$	$\mathbf{f(x_k)}$	$\mathbf{\frac{df(x_k)}{dx}}$
0	2.00	-2.00	1.00
1			
2			
3			

6. Using your favorite method for numerically approximating derivatives, compute the Jacobian of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ at the point $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, when

$$f(x_1, x_2) = \begin{bmatrix} \sin(\ (x_1)^7\ ) + (x_2)^3 \\ 4x_1\sqrt{x_2} \end{bmatrix}$$

This is the end of the drill problems. The second part of the HW set is once again a jupyter notebook. **Please go to** the course Canvas Site and complete the assignment titled "juliahw8".

Hints

Hints: Prob. 2 (a) Here is an example of showing your work for the forward difference approximation of $f(x)$ at $x_0 = 1$, for $h = 0.001$:

$$\frac{df(x_0)}{dx} \approx \frac{f(x_0+h) - f(x_0)}{h} = \frac{f(1+0.001) - f(1)}{0.001} \approx \frac{-2.000999 - (-2)}{0.001} = \frac{-0.000999}{0.001} = -0.999$$

Hints: Prob. 2 (c) Suppose you were trying to find a root of $f(x) = x^2 - 2 = 0$ using bisection and furthermore, suppose you bracket the root $\sqrt{2}$ with two rational numbers a and b, with $f(a) \cdot f(b) < 0$. Then $c = \frac{a+b}{2}$ would be rational and therefore $f(c) \neq 0$. Why, because $\sqrt{2}$ is irrational. Hence, the algorithm must continue. However, at the next step, c replaces one of a or b. But we've already seen that if a and b are rational, then the algorithm does not stop.

The above reasoning does not work for the problem at hand, but something close to it does. That's all the hint we can give!

Hints: Prob. 4 Here is the solution to an example problem.

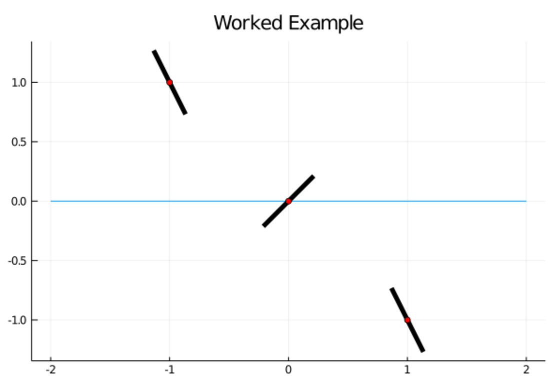


Figure 1: The function values $f(x_0)$ are the red dots and the black line segments represent the "slopes", that is, the slope of each segment is equal to $\frac{df(x_0)}{dx}$. The objective is to sketch a "simple" function that passes through the function values AND has the correct slope (that is, derivative) at those values.

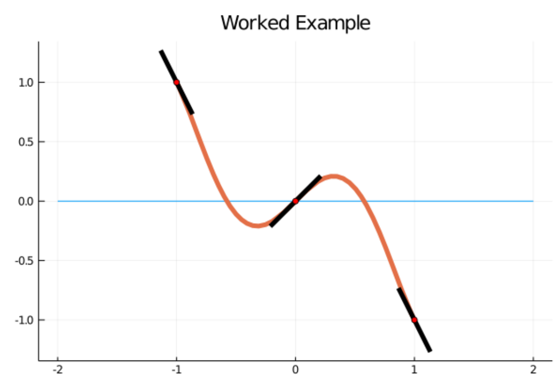


Figure 2: Solution. Yes, we cheated and plotted the real thing! You only need a **hand sketch**. But, at least we worked the hardest one! :-) The method is to sketch a potential graph of the function near the three points where you have the derivative by just extending the lines a bit. Then connect them with as smooth of a curve as you can.

Hints: Prob. 6 We have $f(x_1, x_2) = \begin{bmatrix} \sin(\ (x_1)^7\ ) + (x_2)^3 \\ 4x_1\sqrt{x_2} \end{bmatrix} =: \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$. The Jacobian of $f : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix}.$$

Use forward, backward, or symmetric differences to approximate each term. For example,

$$\frac{\partial f_2(x_1, x_2)}{\partial x_2} \approx \frac{f_2(x_0[1], x_0[2] + h) - f_2(x_0[1], x_0[2])}{h} \approx 1.4140$$