HW # 02 Solutions: FAQ

1. How to go from equations as used in High School to the matrix-vector form $Ax = b$ used in College courses?

   Answer Part 1: Key things to keep in mind

   (a) Each equation corresponds to a row in the matrix A and the column vector b.

   (b) The number of variables (unknowns) in the equations determines the number of rows in the column vector x.

   (c) If necessary, re-arrange the equations so that all of the unknowns (that is, variables) are on the left-hand side of the equals sign and any constants are summed up and move to the right-hand side. Zero is a valid constant.

   (d) The ij-entry of the matrix A (row i and column j) contains the coefficient of the i-th equation multiplying the j-th entry of x. If the j-th entry of x is not present in that equation, then you put a zero in the ij-th spot of A.

   Answer Part 2: After reading the above, re-read Section 2.5 of the book. Also, some of the examples in the next question may help.

2. Re-arranging equations to have a special structure was confusing to me. How do you go about doing that?

   Answer Part 1: Why triangular is nice: When solving equations using the method of Chapter 1, we need to do substitutions to isolate[1] a single variable, say x_2, and then solve for it and substitute into the remaining equations. If we are lucky, another variable will then be isolated. If not, we do some more substitutions to isolate another variable. This is a hard and tedious approach to solving equations. It does not scale to large sets of equations

   With TRIANGULAR systems, the equations are already in a nice order. This set is UPPER TRIANGULAR

$$\begin{aligned} 3x_1 &= 6 \\ 2x_1 - x_2 &= -2 \\ x_1 - 2x_2 + 3x_3 &= 2 \end{aligned} \iff \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}}_{b}. \tag{1}$$

   We can easily solve the equations from the top down: the first equations has x_1 isolated. The second equation has both x_1 and x_2, but once a value for x_1 is known, then x_2 is isolated and we can solve for it very easily. Similarly, the third equation has x_1, x_2, and x_3, but once values for x_1 and x_2 are are known, then x_3 is isolated and we can solve for it very easily.

   Answer Part 2: Re-arranging This set of equations is NOT TRIANGULAR

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 2 \\ 3x_1 &= 6 \\ 2x_1 - x_2 &= -2 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 3 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}}_{b}. \tag{2}$$

   However, if you move what is now the first equation back to being the third equation, as it was in (1), then the system of equations is once again triangular. This is called **re-arranging the equations**.

   Can all sets of equations be re-arranged so that they become triangular? The answer is a resounding NO! We will learn in Chapter 5, however, how to decompose a given system of equations into two sets of triangular equations. That has a fancy name, LU Factorization. It's super cool and once you have it programmed up in Julia, solving equations becomes a snap.

3. Triangular matrices were hard to understand. Can you say more?

[1] An isolated variable means there is an equation with only one unknown it.

Answer: It helps to understand the origins of different kinds of matrices. A **diagonal** matrix comes from equations that look like this

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{22}x_2 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned} \iff \underbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}. \tag{3}$$

In other words, the only non-zero entries form a diagonal pattern, starting from the upper left corner and going to the lower right corner. It's Ok if some (or even all) of the entries on the diagonal are zero as well. The point is, diagonal matrices correspond to a very special form of linear equations. If the entries on the diaognal are all non-zero, then it is trivial to solve the system of linear equations.

In a similar manner, **triangular matrices** correspond to systems of linear equations in a special form. They can have two "shapes", either **lower triangular** like this one

$$\begin{aligned} 3x_1 &= 6 \\ 2x_1 - x_2 &= -2 \\ x_1 - 2x_2 + 3x_3 &= 2 \end{aligned} \iff \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}}_{b}.$$

or **upper triangular** like this one

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 6 \\ 2x_2 + x_3 &= -2 \\ 3x_3 &= 4, \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}}_{b}.$$

4. Why or how are existence and uniqueness related to determinants? And where does the determinant formula come from?

   Answer: These are two very hard and also deep questions. For a mathematical derivation of the determinant formula, see https://sites.millersville.edu/bikenaga/linear-algebra/det-unique/det-unique.html. To understand it, you probably need to complete the course Math 312 - Applied Modern Algebra.

   In Chapter 7, we will relate the uniqueness of solutions of $Ax = b$ to the "linear independence of the columns of the matrix A". Right now, these words may be so much gibberish to you! Which is totally fine. It was gibberish to your instructors in the beginning too! Once that material is covered, the final step is to relate the determinant to the "volume of a paralellepiped" constructed from the columns of the matrix. What?!? The latter point is explained here, https://textbooks.math.gatech.edu/ila/determinants-volumes.html.

   Here is an example of a matrix where the columns are not "linearly independent". The third column is vector sum of the first two columns.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

   Yeah, that's a lot to follow and it explains why we do not teach a bunch about the determinant in ROB 101. I would venture to say that less than ten faculty in the College of Engineering know the whole story of how the columns of a matrix generate the determinant formula! But, every single one of the faculty "knows" that a square system of linear equations $Ax = b$ has a unique solution if, and only if, $\det(A) \neq 0$. And so do you!

   Now, once we learn more linear algebra, we'll be able to relate uniqueness of solutions to "linear independence" of the columns of A. That part is fairly straightforward. Unwrapping the secrets of the determinant, however, is much more challenging.

5. Determinant of 3×3 and larger matrices. What's up with that?

   Answer: Yeah, beyond 2×2 matrices, the determinant formula is a mess. We do not advocate memorizing it for anything more than a 2×2 matrix, but hey, that's your choice. You can look up many examples on the web.