HW # 03 Solutions: FAQ

1. **I was taught the row times column method (also called finger tracing method, sometimes called dot product method) of matrix multiplication. Our alternative way of doing matrix multiplication, summing over columns times rows, is giving me fits. Can you help me?**

Answer: In the following, we'll suppose that you understand well the row times column method and will use it to show you how the summing over columns times rows method works. To do it, we'll assume a matrix A with two columns and a matrix B with two rows

$$A \cdot B = \begin{bmatrix} a_1^{\text{col}} & a_2^{\text{col}} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix}. \tag{1}$$

It is suggested that you replicate the symbolic calculations that follow this box with numbers by taking

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} a_1^{\text{col}} & a_2^{\text{col}} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 3 \\ 5 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 4 \\ 6 \end{matrix}} \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} b_1^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix} = \begin{bmatrix} \boxed{3 \;\; 4} \\ \boxed{2 \;\; 1} \end{bmatrix}. \tag{2}$$

In our textbook, it is shown that

$$A \cdot B = \begin{bmatrix} 3 & 4 \\ 9 & 12 \\ 15 & 20 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 17 & 16 \\ 27 & 26 \end{bmatrix}$$

We first note that

$$\begin{aligned} A &= \begin{bmatrix} a_1^{\text{col}} & 0^{\text{col}} \end{bmatrix} + \begin{bmatrix} 0^{\text{col}} & a_2^{\text{col}} \end{bmatrix} \\ B &= \begin{bmatrix} b_1^{\text{row}} \\ 0^{\text{row}} \end{bmatrix} + \begin{bmatrix} 0^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix}, \end{aligned} \tag{3}$$

where 0^{col} is a column of all zeros and 0^{row} is a row of all zeros. Hence,

$$\begin{aligned} A \cdot B &= \begin{bmatrix} a_1^{\text{col}} & a_2^{\text{col}} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix} \\ &= \left(\begin{bmatrix} a_1^{\text{col}} & 0^{\text{col}} \end{bmatrix} + \begin{bmatrix} 0^{\text{col}} & a_2^{\text{col}} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} b_1^{\text{row}} \\ 0^{\text{row}} \end{bmatrix} + \begin{bmatrix} 0^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix} \right). \end{aligned} \tag{4}$$

After multiplying things out we obtain

$$\begin{aligned} A \cdot B &= \left(\begin{bmatrix} a_1^{\text{col}} & 0^{\text{col}} \end{bmatrix} + \begin{bmatrix} 0^{\text{col}} & a_2^{\text{col}} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} b_1^{\text{row}} \\ 0^{\text{row}} \end{bmatrix} + \begin{bmatrix} 0^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix} \right) \\ &= \begin{bmatrix} a_1^{\text{col}} & 0^{\text{col}} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\text{row}} \\ 0^{\text{row}} \end{bmatrix} + \begin{bmatrix} a_1^{\text{col}} & 0^{\text{col}} \end{bmatrix} \cdot \begin{bmatrix} 0^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix} + \\ &\quad + \begin{bmatrix} 0^{\text{col}} & a_2^{\text{col}} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\text{row}} \\ 0^{\text{row}} \end{bmatrix} + \begin{bmatrix} 0^{\text{col}} & a_2^{\text{col}} \end{bmatrix} \cdot \begin{bmatrix} 0^{\text{row}} \\ b_2^{\text{row}} \end{bmatrix}. \end{aligned} \tag{5}$$

Now, you need to convince yourself that

$$
\begin{aligned}
\begin{bmatrix} a_1^{\rm col} & 0^{\rm col} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\rm row} \\ 0^{\rm row} \end{bmatrix} &= a_1^{\rm col} \cdot b_1^{\rm row} \\
\begin{bmatrix} a_1^{\rm col} & 0^{\rm col} \end{bmatrix} \cdot \begin{bmatrix} 0^{\rm row} \\ b_2^{\rm row} \end{bmatrix} &= 0^{\rm matrix} \\
\begin{bmatrix} 0^{\rm col} & a_2^{\rm col} \end{bmatrix} \cdot \begin{bmatrix} b_1^{\rm row} \\ 0^{\rm row} \end{bmatrix} &= 0^{\rm matrix} \\
\begin{bmatrix} 0^{\rm col} & a_2^{\rm col} \end{bmatrix} \cdot \begin{bmatrix} 0^{\rm row} \\ b_2^{\rm row} \end{bmatrix} &= a_2^{\rm col} \cdot b_2^{\rm row}.
\end{aligned}
\tag{6}
$$

You could try doing the finger tracing method in your head. If that fails, however, you really do need to perform the calculations with the example matrices in (2). Whether you do them by hand or with Julia is up to you.

One you have convinced yourself that (6) **holds, from** (5)**, we obtain that**

$$\boxed{\boxed{\boxed{A \cdot B = a_1^{\rm col} \cdot b_1^{\rm row} + a_2^{\rm col} \cdot b_2^{\rm row}}}}.$$

2. **I have a hard time keeping track that forward substitution goes with a lower triangular matrix and backward substitution with an upper triangular matrix! How can I do better?**

   Answer 1: How do you keep track that **red** is a color while **read** is the past tense of the infinitive **to read**? Practice! For native speakers, it is easy, but when one is learning a language after the age of 15 of so, things like that drive one crazy, yet people do it! Often, they put (tricky) things they have a hard time remembering on a note card. You can too!

   Answer 2: It's clear that with a triangular system of equations, you start with the isolated variable, which is either at the very top or the very bottom. If you are starting at the top, it's called forward substitution, which you will now mentally picture as TOP-DOWN SUBSTITUTION and if you are starting at the bottom, it is called back substitution, which you will now mentally picture as BOTTOM-UP SUBSTITUTION!

   With a lower triangular matrix, the isolated element is at the top and the base of the triangle is at the bottom.

   With an upper triangular matrix, the isolated element is at the bottom and the base of the triangle is at the top.

   Does any of this help?

3. **With LU Factorization, I was "surprised" by first solving via forward substitution with L then backward substitution with U! Did I miss something?**

   Answer: Maybe! We want to solve $Ax = b$. We are given an LU Factorization of $A = L \cdot U$.

   (a) Suppose that y satisfies $Ly = b$ and x satisfies $Ux = y$. For the moment, let's not care how we obtain x and y, let's just suppose that they magically satisfy
   $$Ly = b \text{ and } Ux = y.$$

   (b) Because $Ux = y$ we can multiply both sides by L and we obtain
   $$L \cdot Ux = Ly.$$

   (c) But from the LU Factorization, we know that $L \cdot U = A$. When we then also substitute in $Ly = b$, we arrive at
   $$L \cdot Ux = Ly \implies Ax = b,$$
   showing that x is indeed a solution to our original equation. **Ta da!**

(d) It is hopefully clear that we must know y before we can solve $Ux = y$. Hence, we must first solve $Ly = b$ for y and then we can solve $Ux = y$. Now, the L in LU stands for lower triangular and the U stands for upper triangular. With a lower triangular matrix, the base is at the bottom, meaning the isolated element is at the top, and therefore we work top-down, which is called forward substitution. Whereas with an upper triangular matrix, the base is at the top, meaning the isolated element is at the bottom, and therefore we work bottom-up, which is called back substitution.

4. **I'm confused with how to deal with zeros on the diagonal of a triangular matrix. Why does that seem so hard to me?**

   Answer: The problem is, you are used to equations having solutions! But that is not always true. Let's note that there is no scalar x satisfying $0x = 4$. In case this seems too trivial, let's consider

$$\begin{aligned} 2x &= 2 \\ 3x + 0y &= 4 \end{aligned} \longleftrightarrow \begin{bmatrix} 2 & 0 \\ 3 & \boxed{0} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

   where we put a box around the zero on the diagonal, $\boxed{0}$. Now, given the first equation says $x = 1$, the second equation is then equivalent to

$$0y = 1,$$

   and once again, there is no solution to the SET of equations.

   On the other hand, let's suppose that the set of equations is

$$\begin{aligned} 2x &= 2 \\ 3x + 0y &= 3 \end{aligned} \longleftrightarrow \begin{bmatrix} 2 & 0 \\ 3 & \boxed{0} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

   This time, while we still have the first equation implying $x = 1$, the second equation is instead equivalent to

$$0y = 0,$$

   for which any value of y is a solution, and hence the set of equations has an infinite number of solutions!

   The bottom line is, when you have a triangular set of equations with a zero on the diagonal, two things are possible:

   (a) the set of equations has no solution, or

   (b) the set of equations has an infinite number of solutions.

   (c) It cannot ever ever ever have a unique solution.

   For tiny sets of equations, one can determine which case applies by careful (though tedious) hand computations. For moderately large or larger sets of equations, we need more theory to determine which case applies and we'll develop that later in the textbook.

5. Normalization of L and pivots in LU Factorization are not clear to me. How can I see what is going on here?

   Answer: When doing LU Factorization, each step begins with a matrix of the form

$$\begin{bmatrix} \boxed{2.0} & 1.0 & 5.0 \\ 6.0 & 2.0 & 20.0 \\ 2.0 & -2.0 & 22.0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & \boxed{-1.0} & 5.0 \\ 0.0 & -3.0 & 17.0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & \boxed{2.0} \end{bmatrix}, \tag{7}$$

   where the **pivots have been boxed**.

   We start with the first matrix A=

$$\begin{bmatrix} \boxed{2.0} & 1.0 & 5.0 \\ 6.0 & 2.0 & 20.0 \\ 2.0 & -2.0 & 22.0 \end{bmatrix} \tag{8}$$

and seek to reduce it to a matrix having its first row and column consisting of all zeros. Following the method in the book, we define the row vector

$$R_1 = \begin{bmatrix} \boxed{2.0} & 1.0 & 5.0 \end{bmatrix} \tag{9}$$

and we seek a column vector C_1 such that

$$A - C_1 \cdot R_1 = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & *.* & *.* \\ 0.0 & *.* & *.* \end{bmatrix},$$

where the asterisks $*.*$ indicate that we do not care about the particular values that go there.

Here, you should get on Julia and try a bunch of different column vectors C_1 until you are convinced that the only one that will work is

$$C_1 = \frac{1}{\boxed{2.0}} \begin{bmatrix} \boxed{2.0} \\ 6.0 \\ 2.0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 3.0 \\ 1.0 \end{bmatrix}. \tag{10}$$

You have to put in the time to try different things to finally convince yourself that our formula is correct and that dividing by the pivot value, $\boxed{2.0}$, is crucially important. In particular, if you do not normalize by the pivot value, it does not work!

With these values for C_1 and R_1, we obtain

$$\begin{aligned} A - C_1 \cdot R_1 &= \begin{bmatrix} \boxed{2.0} & 1.0 & 5.0 \\ 6.0 & 2.0 & 20.0 \\ 2.0 & -2.0 & 22.0 \end{bmatrix} - \begin{bmatrix} 1.0 \\ 3.0 \\ 1.0 \end{bmatrix} \cdot \begin{bmatrix} \boxed{2.0} & 1.0 & 5.0 \end{bmatrix} \\ &= \begin{bmatrix} \boxed{2.0} & 1.0 & 5.0 \\ 6.0 & 2.0 & 20.0 \\ 2.0 & -2.0 & 22.0 \end{bmatrix} - \begin{bmatrix} \boxed{2.0} & 1.0 & 5.0 \\ 6.0 & 3.0 & 15.0 \\ 2.0 & 1.0 & 5.0 \end{bmatrix} \\ &= \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & \boxed{-1.0} & 5.0 \\ 0.0 & -3.0 & 17.0 \end{bmatrix}. \end{aligned}$$

Our new pivot is $\boxed{-1.0}$.

We now seek to continue and create another column and row of zeros. Following the method in the book, we next define the row vector

$$R_2 = \begin{bmatrix} 0 & \boxed{-1.0} & 5.0 \end{bmatrix} \tag{11}$$

and we seek a column vector C_2 such that

$$A - C_1 \cdot R_1 - C_2 \cdot R_2 = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & *.* \end{bmatrix},$$

where once again the asterisk $*.*$ indicates that we do not care about the particular value that goes there.

Because you really want to understand, you get on Julia and try a bunch of different column vectors C_2 until you are convinced that the one that will work is

$$C_2 = \frac{1}{\boxed{-1.0}} \begin{bmatrix} 0.0 \\ \boxed{-1.0} \\ -3.0 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 1.0 \\ 3.0 \end{bmatrix}. \tag{12}$$

In particular, if you do not normalize by the pivot value, it does not work!

With these values for C_2 and R_2, we obtain

$$\begin{aligned}
A - C_1 \cdot R_1 - C_2 \cdot R_2 &= \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & \boxed{-1.0} & 5.0 \\ 0.0 & -3.0 & 17.0 \end{bmatrix} - \begin{bmatrix} 0.0 \\ 1.0 \\ 3.0 \end{bmatrix} \cdot \begin{bmatrix} 0.0 & \boxed{-1.0} & 5.0 \end{bmatrix} \\
&= \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & \boxed{-1.0} & 5.0 \\ 0.0 & -3.0 & 17.0 \end{bmatrix} - \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & \boxed{-1.0} & 5.0 \\ 0.0 & -3.0 & 15.0 \end{bmatrix} \\
&= \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & \boxed{2.0} \end{bmatrix}
\end{aligned}$$

If you have gotten this far, it should be easy for you to do the last step to obtain

$$C_3 = \begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} \text{ and } R_3 = \begin{bmatrix} 0.0 & 0.0 & 2.0 \end{bmatrix},$$

so that

$$A = L \cdot U = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \cdot \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 3.0 & 1.0 & 0.0 \\ 1.0 & 3.0 & 1.0 \end{bmatrix} \cdot \begin{bmatrix} 2.0 & 1.0 & 5.0 \\ 0.0 & -1.0 & 5.0 \\ 0.0 & 0.0 & 2.0 \end{bmatrix}.$$