ROB 101 - Computational Linear Algebra
Recitation #6

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1 Null Space and Range of a Matrix

For any Matrix $A \in_{m\times n}$, then the following sets (are actually Subspaces!) can be defined:

Def. $\text{null}(A) := \{x \in^m \ \mid Ax = 0_{n\times 1}\}$ is the **null space** of A.

Def. $\text{range}(A) := \{y \in^n \ \mid y = Ax \ \text{ for some } \ x \in^m\}$ is the **range** of A.

Question? What is a null space of a matrix and why is it important?
Source: FAQ Questions `https://umich.instructure.com/courses/475066/files/folder/HW/HW%2006`

2 Basis Vectors

Definition: Suppose that V is a subspace of $\mathbb{R}^n$. Then $\{v_1, v_2, ..., v_k\}$ is a basis for V if

- the set $\{v_1, v_2, ..., v_k\}$ is linearly independent
- span$\{v_1, v_2, ..., v_k\} = \text{V}$

The dimension of V is k, the number of basis vectors.

Remark: Basis vectors provide a simple means to generate all vectors in a vector space or a subspace by forming linear combinations from a finite list of vectors. The three vectors commonly seen in vector calculus and physics $\hat{\imath}, \hat{\jmath}, \hat{k}$ are orthonormal basis vectors!

3 Span of a Set of Vectors

Definition: The span of a set of vectors $S \in \mathbb{R}^n$ is:

$$\text{span}(S) := \{\text{all possible linear combinations of elements in S}\}$$

The span operation is useful for generating a subspace from an arbitrary set vectors in $\mathbb{R}^n$ by the definition of the span (contains zero vector and is closed under linear combination). That is, the result of $\text{span}(S)$ is a subspace of $\mathbb{R}^n$.

4 Column Span of a Matrix

Definition: The column span of an n x m matrix A is:

$$\text{col}(A) := \text{span}(\{a_1^{col}, a_2^{col}, ..., a_m^{col}\})$$

i.e., take the columns of matrix A and form a set S containing m vectors in $\mathbb{R}^n$ ($\{a_1^{col}, a_2^{col}, ..., a_m^{col}\}$) to perform the span(S) operation.
Remark: Ax = b has a solution if, and only if, b is a linear combination of the columns of A. A more elegant way to write this is Ax = b has a solution if, and only if,

$$b \in col(A)$$

We can also discuss, rank and nullity of A here as:

Def. $\text{rank}(A) := \dim \text{col span}\{A\}$.

Def. $\text{nullity}(A) := \dim \text{null}(A)$.

5 Eigen values and Eigen Vectors

Let A be an $n \times n$ matrix. A scalar $\lambda \in \mathbb{R}$ is an **eigenvalue** (e-value) of A, if there exists a non-zero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. Any such vector v is called an **eigenvector** (e-vector) associated with λ.

Eigenvectors are not unique.

- To find e-values, we solve $\det(\lambda I - A) = 0$ because

$$A \cdot v = \lambda v \iff (\lambda I - A) \cdot v = 0 \overset{v \neq 0}{\iff} \det(\lambda I - A) = 0. \tag{1}$$

- To find e-vectors, we find any non-zero $v \in \mathbb{R}^n$ such that

$$(\lambda I - A) \cdot v = 0. \tag{2}$$

Of course, if you prefer, you can solve $(A - \lambda I)v = 0$ when seeking e-vectors.