ROB 101 - Fall 2021

The Vector Space $\mathbb{R}^n$: Part 2 (Subspaces)

October 11, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- A second encounter with some of the essential concepts in Linear Algebra.
- A more abstract view of $\mathbb{R}^n$ as a vector space.

Outcomes

- What is a vector space, a subspace, and the span of a set of vectors.
- Range, column span, and null space of a matrix.

$\mathbb{R}^n$ as a Vector Space

Recall that an n-tuple is an ordered list of n numbers, $(x_1, x_2, \ldots, x_n)$ and identified with column vectors, as in

$$(x_1, x_2, \ldots, x_n) \longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Moreover, we identify $\mathbb{R}^n$ with the set of all n-column vectors with real entries

$$\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}$$

$$\iff \mathbb{R}^n \iff \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_i \in \mathbb{R}, 1 \leq i \leq n \right\}$$

Recall: Properties of Vectors in $\mathbb{R}^n$

For all real numbers α and β, and all vectors x and y in $\mathbb{R}^n$

$$
\begin{aligned}
\alpha x + \beta y &= \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\
&= \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} + \begin{bmatrix} \beta y_1 \\ \beta y_2 \\ \vdots \\ \beta y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix}.
\end{aligned}
$$

- Recall that a set V is a subset of some other set, say W, if $x \in V \implies x \in W$.

- Recall that a set V is a subset of some other set, say W, if $x \in V \implies x \in W$.

- One writes $V \subset W$ to denote V is a subset of W.

- Recall that a set V is a subset of some other set, say W, if $x \in V \implies x \in W$.
- One writes $V \subset W$ to denote V is a subset of W.
- We say that $V = W$ if $V \subset W$ and $W \subset V$ both hold.

Suppose that $V \subset \mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$.

Definition

V is a subspace of $\mathbb{R}^n$ if any linear combination constructed from elements of V and scalars in $\mathbb{R}$ is once again an element of V. One says that V is *closed under linear combinations.*

Suppose that $V \subset \mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$.

Definition

V is a subspace of $\mathbb{R}^n$ if any linear combination constructed from elements of V and scalars in $\mathbb{R}$ is once again an element of V. One says that V is *closed under linear combinations.*

In symbols, $V \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if for all real numbers α and β, and all vectors v_1 and v_2 in V

$$\boxed{\alpha v_1 + \beta v_2 \in V.}$$

Corollary

To check that a subset is a subspace, we can individually check that it is

1. *closed under vector addition,*
2. *and closed under scalar times vector multiplication.*

Claim

Every subspace must contain the zero vector.

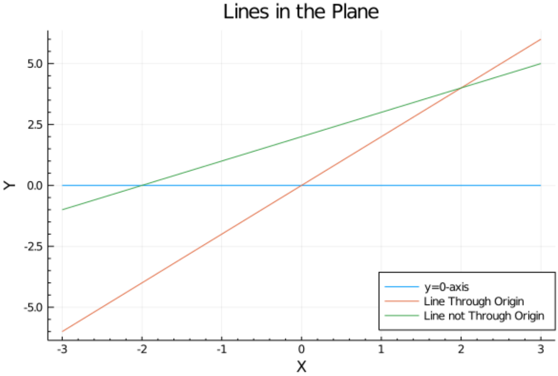
Claim

Every subspace must contain the zero vector.

Proof.

Suppose that $V \subset \mathbb{R}^n$ is a subspace and that $v \in V$. Then $0 \cdot v \in V$ because V is closed under scalar times vector multiplication. But $0 \cdot v = 0$, the zero vector. $\square$

If a line does not pass through the origin, then it does not contain the origin, and hence it cannot be a subspace.



Let $V \subset \mathbb{R}^2$ be the set of all points that lie on a line $y = mx + b$, that is

$$V := \left\{ \begin{bmatrix} x \\ mx + b \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

Then V is a subspace of $\mathbb{R}^2$ if, and only if, $b = 0$, that is, the line must pass through the origin.

V contains the zero vector if, and only, if the y-intercept is zero. But this means that $b = 0$.

V contains the zero vector if, and only, if the y-intercept is zero. But this means that $b = 0$.

Now, $0 \in V$ is a necessary condition, but not a sufficient condition for V to be a subspace. So, let's check if V, with $b = 0$ is closed under vector addition and scalar times vector multiplication. V is then

$$V := \left\{ \begin{bmatrix} x \\ mx \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

We take

$$v_1 = \begin{bmatrix} x_1 \\ mx_1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} x_2 \\ mx_2 \end{bmatrix}$$

for x_1 and x_2 arbitrary real numbers.

We take

$$v_1 = \begin{bmatrix} x_1 \\ mx_1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} x_2 \\ mx_2 \end{bmatrix}$$

for x_1 and x_2 arbitrary real numbers.

Then

$$v_1 + v_2 = \begin{bmatrix} x_1 + x_2 \\ mx_1 + mx_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ m(x_1 + x_2) \end{bmatrix} \in V,$$

and hence V is closed under vector addition.

Then

$$v_1 + v_2 = \begin{bmatrix} x_1 + x_2 \\ mx_1 + mx_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ m(x_1 + x_2) \end{bmatrix} \in V,$$

and hence V is closed under vector addition.

We note that

$$v_1 + v_2 = \begin{bmatrix} x \\ mx \end{bmatrix}, \text{ for } x = x_1 + x_2,$$

and that is why $v_1 + v_2 \in V$.

We now let $\alpha \in \mathbb{R}$ be arbitrary and check scalar times vector multiplication.

$$\alpha v_1 = \alpha \begin{bmatrix} x_1 \\ m x_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha m x_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ m(\alpha x_1) \end{bmatrix} \in V,$$

and hence V is closed under scalar times vector multiplication.

We now let $\alpha \in \mathbb{R}$ be arbitrary and check scalar times vector multiplication.

$$\alpha v_1 = \alpha \begin{bmatrix} x_1 \\ mx_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha m x_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ m(\alpha x_1) \end{bmatrix} \in V,$$

and hence V is closed under scalar times vector multiplication.

We note that

$$\alpha v_1 = \begin{bmatrix} x \\ mx \end{bmatrix}, \text{ for } x = \alpha x_1,$$

and that is why $\alpha v_1 \in V$.

Three Sources of Subspaces

1. Matrix Null Space
2. Span of a set of Vectors
3. Column Span of a Matrix

Definition

For an $n \times m$ matrix A, its null space is

$$\mathrm{null}(A) := \{x \in \mathbb{R}^m \mid Ax = 0_{n\times 1}\},$$

the set of all solutions (that is, vectors) that result in Ax being the zero vector or the "null vector".

For an $n \times m$ matrix A, show that $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^m$.

Suppose $v_1, v_2 \in \mathrm{null}(A)$, so that $Av_1 = Av_2 = 0_{n\times 1}$.

Suppose $v_1, v_2 \in \mathrm{null}(A)$, so that $Av_1 = Av_2 = 0_{n\times 1}$.

We need to show that for all real numbers α and β,

$$\alpha v_1 + \beta v_2 \in \mathrm{null}(A) \quad \text{(closed under linear combinations).}$$

Example

Suppose $v_1, v_2 \in \mathrm{null}(A)$, so that $Av_1 = Av_2 = 0_{n\times 1}$.

We need to show that for all real numbers α and β,

$$\alpha v_1 + \beta v_2 \in \mathrm{null}(A) \quad \text{(closed under linear combinations).}$$

$A\left(\alpha v_1 + \beta v_2\right) = \alpha A v_1 + \beta A v_2 = \alpha\ 0_{n\times 1} + \beta\ 0_{n\times 1} = 0_{n\times 1}$,
and thus $\alpha v_1 + \beta v_2 \in \mathrm{null}(A)$.

Definition

Suppose that $S \subset \mathbb{R}^n$, then S is a set of vectors. The set of all possible linear combinations of elements of S is called the span of S,

$$\mathrm{span}\{S\} := \{\text{all possible linear combinations of elements of } S\}.$$

Remark

It follows that $\mathrm{span}\{S\}$ *is a subspace of* $\mathbb{R}^n$ *because, by definition, it is closed under linear combinations. This is true for any subset* $S \subset \mathbb{R}^n$.

What is the span good for?

What is the span good for?

- The span operation is how one takes an arbitrary set of vectors and generates a subspace from it.

- If S is a set, $\mathrm{span}\{S\}$ is the smallest subspace that contains all of the elements of the set S.

Span of a Set of Vectors

- If a set S is already known to be a subspace of $\mathbb{R}^n$, then taking its span does not add any new vectors because a subspace is closed under linear combinations.

- If a set S is already known to be a subspace of $\mathbb{R}^n$, then taking its span does not add any new vectors because a subspace is closed under linear combinations.

- Hence, $S \subset \mathbb{R}^n$ and S a subspace $\implies \operatorname{span}\{S\} = S$.

- If a set S is already known to be a subspace of $\mathbb{R}^n$, then taking its span does not add any new vectors because a subspace is closed under linear combinations.

- Hence, $S \subset \mathbb{R}^n$ and S a subspace $\implies \operatorname{span}\{S\} = S$.

- When S is not a subspace, then there is at least one linear combination of elements of S that is not in S itself. Then, $\operatorname{span}\{S\}$ and $S \neq \operatorname{span}\{S\}$.

Consider the vector space $\mathbb{R}^3$. Compute $\operatorname{span}\{e_1 + e_3, e_2 + e_3\}$.

We have $S = \{e_1 + e_3, e_2 + e_3\}$. The set $\{e_1 + e_3, e_2 + e_3\}$ is not a subspace because the vector $e_1 - e_2 = (e_1 + e_3) - (e_1 + e_3) \notin S$.

$$
\begin{aligned}
\operatorname{span}\{S\} &:= \{\text{all possible linear combinations of elements of } S\} \\
&= \left\{ \alpha_1(e_1+e_3) + \alpha_2(e_2+e_3) \;\middle|\; \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \right\} \\
&= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 + \alpha_2 \end{bmatrix} \;\middle|\; \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \right\}.
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{span}\{S\} &:= \{\text{all possible linear combinations of elements of } S\} \\
&= \left\{ \alpha_1(e_1 + e_3) + \alpha_2(e_2 + e_3) \;\middle|\; \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \right\} \\
&= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 + \alpha_2 \end{bmatrix} \;\middle|\; \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} x \\ y \\ x + y \end{bmatrix} \;\middle|\; x \in \mathbb{R}, y \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; z = x + y, x \in \mathbb{R}, y \in \mathbb{R} \right\}.
\end{aligned}
$$

$$\operatorname{span}\{S\} := \left\{ \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \;\middle|\; x \in \mathbb{R}, y \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; z = x+y, x \in \mathbb{R}, y \in \mathbb{R} \right\}.$$

Hence, $\operatorname{span}\{e_1 + e_3, e_2 + e_3\}$ is the plane given by $z = x + y$ in $\mathbb{R}^3$.

Definition

Let A be an $n \times m$ matrix. Then its columns are vectors in $\mathbb{R}^n$. Their span is called the column span of A.

$$\text{col span}\{A\} := \text{span}\{a_1^{\text{col}}, \ldots, a_m^{\text{col}}\}.$$

Remark

$Ax = b$ has a solution if, and only if, b is a linear combination of the columns of A. A more elegant way to write this is $Ax = b$ has a solution if, and only if, $b \in \mathrm{col\ span}\{A\}$.

Suppose $A = \begin{bmatrix} 3 & 2 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ -8 \\ 5 \end{bmatrix}$.
Does $Ax = b$ have a solution?

We check that

$$b = -2a_1^{\rm col} + 3a_2^{\rm col} \in \mathrm{span}\{a_1^{\rm col}, a_2^{\rm col}\},$$

and hence b is in the column span of A, and the system of linear equations has a solution, namely,

$$x = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is a solution!

Next Time

- The Vector Space $\mathbb{R}^n$: Part 2 (Dot Product and Orthonormal Vectors)
- Read Chapter 9 of ROB 101 Book