ROB 101 - Fall 2021

The Vector Space $\mathbb{R}^n$: Part 3 (Basis vectors and Eigenvalues)

October 25, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- Learn how to define coordinates in a subspace of $\mathbb{R}^n$ and understand how many coordinates you need.
- An introduction to eigenvalues and eigenvectors of square matrices.
- Applying to matrices some of the essential concepts in Linear Algebra.

Outcomes

- Basis vectors, dimension, and coordinates.
- Eigenvalues, eigenvectors, and understanding when when eigenvectors provide a basis of $\mathbb{R}^n$.

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- Then $e_i :=$ i-th column of I_n.

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- For example, when $n = 4$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- Looking at $\mathbb{R}^2$, we recall that $\{e_1, e_2\}$ is a linearly independent set, because

$$\left(\alpha_1 e_1 + \alpha_2 e_2 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \iff \left(\alpha_1 = 0, \alpha_2 = 0\right).$$

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- An important property of the set $\{e_1, e_2\} \subset \mathbb{R}^2$ is that any vector $x \in \mathbb{R}^2$ can be written as a linear combination of e_1, e_2.

$$x =: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2.$$

Remark

There is only one linear combination of $\{e_1, e_2\}$ *that yields the point* $x = [x_1, x_2]^\mathsf{T} \in \mathbb{R}^2$

Remark

We can write

$$\begin{aligned}
a &= a_1 e_1 + \cdots + a_n e_n = e_1 a_1 + \cdots + e_n a_n \\
&= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} a_1 + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} a_n \\
&= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = I_n \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}
\end{aligned}$$

Corollary

The columns of the $n \times n$ identity matrix $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$

are vectors and form the standard basis for $\mathbb{R}^n$.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix}.$$

Suppose that V is a subspace of $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots, v_k\}$ is a *basis for* V if

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- $\mathrm{span}\{v_1, v_2, \ldots, v_k\} = V$.

Basis Vectors and Dimension

Suppose that V is a subspace of $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots, v_k\}$ is a *basis for* V if

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- $\mathrm{span}\{v_1, v_2, \ldots, v_k\} = V$.
- The maximum number of vectors in any linearly independent set contained in V is the *dimension* of V (here k).

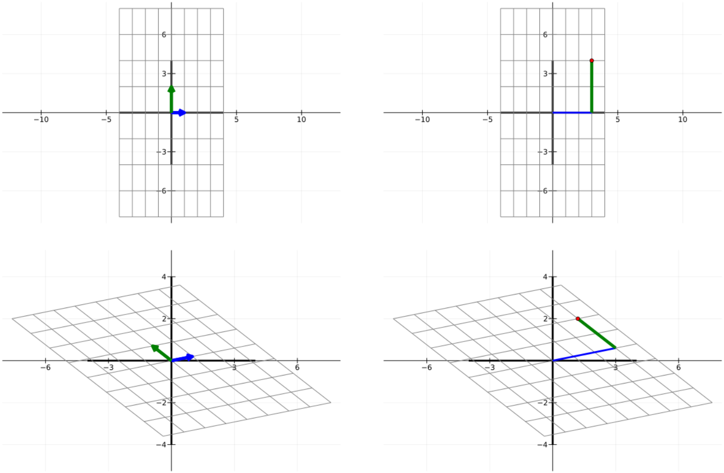
Definition

Let $n \geq 1$ and, as before, define $e_i :=$ i-th column of the $n \times n$ identity matrix, I_n. Then

$$\{e_1, e_2, \ldots, e_n\}$$

is a basis for the vector space $\mathbb{R}^n$.
Its elements e_i are called both natural (standard) basis vectors and canonical basis vectors.

Basis Vectors and Dimension



Vector Space Coordinates and Vector Representation

Suppose that V is a k-dimensional subspace of $\mathbb{R}^n$ with basis $\{v_1, v_2, \ldots, v_k\}$ or all of $\mathbb{R}^n$ itself (in which case, $k = n$).

Vector Space Coordinates and Vector Representation

Suppose that V is a k-dimensional subspace of $\mathbb{R}^n$ with basis $\{v_1, v_2, \ldots, v_k\}$ or all of $\mathbb{R}^n$ itself (in which case, $k = n$).

- Then each $x \in V$ can be expressed (uniquely) as a linear combination of basis vectors

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k.$$

Vector Space Coordinates and Vector Representation

Suppose that V is a k-dimensional subspace of $\mathbb{R}^n$ with basis $\{v_1, v_2, \ldots, v_k\}$ or all of $\mathbb{R}^n$ itself (in which case, $k = n$).

- $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$.
- Stacking the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ into a column vector yields

$$[x]_{\{v_1,\ldots,v_k\}} := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix},$$

which is called the *representation of* x in the basis $\{v_1, v_2, \ldots, v_k\}$.

Vector Space Coordinates and Vector Representation

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- $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$.
- The k-tuple
$$\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_k)$$
forms the *coordinates of* x associated with the basis $\{v_1, v_2, \ldots, v_k\}$.

We consider the subspace of $\mathbb{R}^3$ defined by

$$V := \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \right\}.$$

Show that

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis for V and hence V is a two dimensional subspace of $\mathbb{R}^3$. In addition, show that

$$v := \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} \in V$$

and find its coordinates on V.

To show that $\{v_1, v_2\}$ is a basis for V, we need that to check that

- $\{v_1, v_2\} \subset V$,
- the set $\{v_1, v_2\}$ is linearly independent, and
- $\operatorname{span}\{v_1, v_2\} = V$.

v_1 and v_2 are in V and they are linearly independent!

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To show the span property, we need to show that all vectors in V can be written as a linear combination of v_1 and v_2.

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$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in V \iff x_1 + x_2 + x_3 = 0$$

$$\iff x_3 = -(x_1 + x_2) \iff x = \begin{bmatrix} x_1 \\ x_2 \\ -(x_1 + x_2) \end{bmatrix}.$$

Taking $x_1 = 1$ and $x_2 = 0$ gives v_1, while taking $x_1 = 0$ and $x_2 = 1$ gives v_2.

Example

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in V \iff x = \begin{bmatrix} x_1 \\ x_2 \\ -(x_1 + x_2) \end{bmatrix}$$
$$\iff x = x_1 v_1 + x_2 v_2 \iff x \in \operatorname{span}\{v_1, v_2\}.$$

The dimension follows from the number of elements in the basis.

To complete the problem, we first verify that $v^{\mathsf{T}} = \left[\begin{array}{ccc} 3 & -4 & 1 \end{array}\right]^{\mathsf{T}}$ is in V because the sum of its components equals zero.

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Next, we check that

$$v := \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} = 3v_1 - 4v_2$$

and hence its coordinates are $(3, -4)$ in the basis $\{v_1, v_2\}$.

We let A be an $n \times n$ matrix. The following statements are equivalent.

1. $\det(A) \neq 0$.
2. The columns of A are linearly independent.
3. The columns of A form a basis for $\mathbb{R}^n$.

Remark

As a special case, we can take $A = I_n$, the columns of which give the canonical basis vectors.

Determine if the vectors $\{v_1, \ldots, v_5\}$ form a basis for $\mathbb{R}^5$.

$$v_1 = \begin{bmatrix} 1.0 \\ 2.0 \\ 0.0 \\ 0.0 \\ 2.0 \end{bmatrix}, v_2 = \begin{bmatrix} -1.0 \\ 0.0 \\ 0.0 \\ 1.0 \\ 1.0 \end{bmatrix}, v_3 = \begin{bmatrix} 1.0 \\ 2.0 \\ -2.0 \\ 0.0 \\ 2.0 \end{bmatrix}$$

$$v_4 = \begin{bmatrix} 0.0 \\ 2.0 \\ -2.0 \\ 0.0 \\ 0.0 \end{bmatrix}, v_5 = \begin{bmatrix} -1.0 \\ 2.0 \\ 0.0 \\ 0.0 \\ 2.0 \end{bmatrix}.$$

We define

$$A = \begin{bmatrix} 1.0 & -1.0 & 1.0 & 0.0 & -1.0 \\ 2.0 & 0.0 & 2.0 & 2.0 & 2.0 \\ 0.0 & 0.0 & -2.0 & -2.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 2.0 & 1.0 & 2.0 & 0.0 & 2.0 \end{bmatrix}_{5\times 5} .$$

In Julia, we compute $\det(A) = 16.0$ and hence the set of vectors $\{v_1, \ldots, v_5\}$ does form a basis for $\mathbb{R}^5$.

Let A be an $n \times n$ real matrix. An *eigenvector* v satisfies $Av = \lambda v$, with λ being a scalar called the *eigenvalue*.

Whenever $\lambda \neq 0$

$$\operatorname{span}\{Av\} = \operatorname{span}\{v\}.$$

Eigenvalues and Eigenvectors

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Whenever $\lambda \neq 0$

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Is this even possible!?

Multiply the matrix

$$A := \begin{bmatrix} -8.0 & 10.0 & 10.0 \\ -2.0 & 5.0 & 2.0 \\ -10.0 & 9.0 & 12.0 \end{bmatrix}$$

times each of the vectors $\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{bmatrix} 1.0 \\ 0.0 \\ 1.0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0.0 \\ -1.0 \\ 1.0 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 5.0 \\ 2.0 \\ 4.0 \end{bmatrix}.$$

Turning to Julia we obtain

$$Av_1 = \begin{bmatrix} 2.0 \\ 0.0 \\ 2.0 \end{bmatrix} = \mathbf{2}v_1, \;\; Av_2 = \begin{bmatrix} 0.0 \\ -3.0 \\ 3.0 \end{bmatrix} = \mathbf{3}v_2,$$

$$\text{and } Av_3 = \begin{bmatrix} 20.0 \\ 8.0 \\ 16.0 \end{bmatrix} = \mathbf{4}v_3.$$

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Remark

Hence, when A acts on this set of vectors, all it does is scale the vector by a factor of 2, 3 or 4, respectively. There is no "rotation" of the vector. That seems kind of magical.

Eigenvalues and Eigenvectors: Temporary Definitions

Definition

Let A be an $n \times n$ real matrix. A scalar $\lambda \in \mathbb{R}$ is an *eigenvalue* of A, if there exists a non-zero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$.

Eigenvalues and Eigenvectors: Temporary Definitions

Definition

Let A be an $n \times n$ real matrix. A scalar $\lambda \in \mathbb{R}$ is an *eigenvalue* of A, if there exists a non-zero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$.

Any such vector v is called an *eigenvector* associated with λ. We note that if v is an eigenvector, then so is αv for any $\alpha \neq 0$, and therefore, eigenvectors are not unique.

- To find eigenvalues, we need to have conditions under which there exists $v \in \mathbb{R}^n$, $v \neq 0$, such that $Av = \lambda v$. We first note that

$$
\begin{aligned}
Av = \lambda v &\iff \lambda v - Av = 0_{n\times 1} \\
&\iff \lambda I v - Av = 0_{n\times 1} \iff (\lambda I - A)v = 0_{n\times 1}.
\end{aligned}
$$

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- We then note that there exists $v \neq 0_{n\times 1}$ such that $(\lambda I - A)v = 0_{n\times 1}$ if, and only if

$$\det(\lambda I - A) = 0.$$

Let A be the 2×2 real matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Determine, if any, its eigenvalues and eigenvectors.

Example

To find eigenvalues, we need to solve

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) - 6 = \lambda^2 - 3\lambda - 4 = 0.$$

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We compute the roots with the quadratic formula to be $\lambda_1 = -1$ and $\lambda_2 = 4$.

To determine an eigenvector associated with $\lambda_1 = -1$, we need to find $v_1 \in \mathbb{R}^2$ such that

$$(A - \lambda_1 I_2)v_1 = 0_{2\times 1}$$

$$\Updownarrow$$

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_{1a} \\ v_{1b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_{1a} \\ v_{1b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} v_{1a} \\ v_{1b} \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \alpha_1 \neq 0.$$

Similarly, to determine an eigenvector associated with $\lambda_2 = 4$, we need to find $v_2 \in^2$ such that

$$(A - \lambda_2 I_2)v_2 = 0_{2\times 1}$$
$$\Updownarrow$$
$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_{2a} \\ v_{2b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} v_{2a} \\ v_{2b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} v_{2a} \\ v_{2b} \end{bmatrix} = \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \; \alpha_2 \neq 0.$$

Next Time

- The Vector Space $\mathbb{R}^n$: Part 3
- Read Chapter 10 of ROB 101 Book