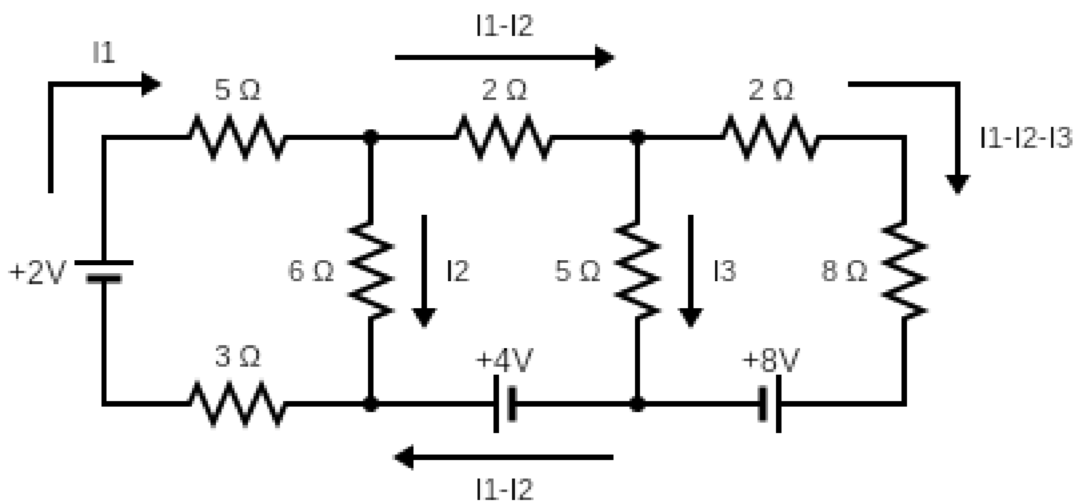
ROB 101 - Computational Linear Algebra
Recitation #2

Justin Yu, Riley Bridges, Tribhi Kathuria

1 Solving $Ax = b$ using LU Decomposition

1. Convert the system of equations into $Ax = b$
2. Decompose the A matrix as $A = LU$.
3. Finally, solve using, Forward Substitution for $Ly = b$ and Backward substitution for $Ux = y$



In order to find the values of currents I_1 through I_3 in the above circuit diagram, we can use Kirchhoff's rules to represent this circuit as the following system of equations:

$$
\begin{aligned}
-5I_1 - 6I_2 - 3I_1 + 2 &= 0 \\
-2(I_1 - I_2) - 5I_3 + 4 + 6I_2 &= 0 \\
-10(I_1 - I_2 - I_3) - 8 + 5I_3 &= 0
\end{aligned}
$$

$$
\begin{aligned}
-5I_1 - 6I_2 - 3I_1 + 2 &= 0 \\
-2(I_1 - I_2) - 5I_3 + 4 + 6I_2 &= 0 \\
-10(I_1 - I_2 - I_3) - 8 + 5I_3 &= 0
\end{aligned}
$$

Solution: First, rearrange the above system of equations to make it easier to represent in matrix form.

$$
\begin{aligned}
8I_1 + 6I_2 + 0I_3 &= 2 \\
2I_1 - 8I_2 + 5I_3 &= 4 \\
-10I_1 + 10I_2 + 15I_3 &= 8
\end{aligned}
$$

$$
A = \begin{bmatrix} 8 & 6 & 0 \\ 2 & -8 & 5 \\ -10 & 10 & 15 \end{bmatrix} \quad x = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}
$$

Now, we perform LU decomposition on matrix A

$$
A = \begin{bmatrix} 8 & 6 & 0 \\ 2 & -8 & 5 \\ -10 & 10 & 15 \end{bmatrix}
$$

Find a column vector C_1 and a row vector R_1 such that $C_1 \cdot R_1 = \begin{bmatrix} 8 & 6 & 0 \\ 2 & * & * \\ -10 & * & * \end{bmatrix}$

$R_1 = \begin{bmatrix} 8 & 6 & 0 \end{bmatrix}$

$C_1 = \begin{bmatrix} 1 \\ 1/4 \\ -5/4 \end{bmatrix}$

$C_1 \cdot R_1 = \begin{bmatrix} 8 & 6 & 0 \\ 2 & 3/2 & 0 \\ -10 & -15/2 & 0 \end{bmatrix}$

$A - C_1 \cdot R_1 = \begin{bmatrix} 8 & 6 & 0 \\ 2 & -8 & 5 \\ -10 & 10 & 15 \end{bmatrix} - \begin{bmatrix} 8 & 6 & 0 \\ 2 & 3/2 & 0 \\ -10 & -15/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -19/2 & 5 \\ 0 & -35/2 & 15 \end{bmatrix}$

Now find a column vector C_2 and a row vector R_2 such that $C_2 \cdot R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -19/2 & 5 \\ 0 & -35/2 & * \end{bmatrix}$

$R_2 = \begin{bmatrix} 0 & -19/2 & 5 \end{bmatrix}$

$C_2 = \begin{bmatrix} 0 \\ 1 \\ -35/19 \end{bmatrix}$

$C_2 \cdot R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -19/2 & 5 \\ 0 & -35/2 & -175/19 \end{bmatrix}$

$(A - C_1 \cdot R_1) - (C_2 \cdot R_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -19/2 & 5 \\ 0 & -35/2 & 15 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & -19/2 & 5 \\ 0 & -35/2 & -175/19 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 460/19 \end{bmatrix}$

Then the final column vector C_3 and row vector R_3 is just

$R_3 = \begin{bmatrix} 0 & 0 & 460/19 \end{bmatrix}$

$C_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

We combine the row vectors to form our U matrix and the column vectors to form our L matrix:

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -5/4 & -35/19 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 8 & 6 & 0 \\ 0 & -19/2 & 5 \\ 0 & 0 & 460/19 \end{bmatrix}$

Now we use the equations

$Ly = b$

$Ux = y$

Forward Substitution:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -5/4 & -35/19 & 1 \end{bmatrix} y = \begin{bmatrix} x \\ y \\ z \end{bmatrix} b = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -5/4 & -35/19 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$$x = 2$$
$$\frac{x}{4} + y = 4$$
$$\frac{-5x}{4} - \frac{35y}{19} + z = 8$$

$$x = 2$$
$$\frac{x}{4} + y = 4 \to y = 4 - \frac{2}{4} = \frac{7}{2}$$
$$\frac{-5x}{4} - \frac{35y}{19} + z = 8 \to z = 8 + \frac{5(2)}{4} + \frac{35(7/2)}{19} = \frac{322}{19}$$

$$x = 2$$
$$y = \frac{7}{2}$$
$$z = \frac{322}{19}$$

Backward Substitution:

$$U = \begin{bmatrix} 8 & 6 & 0 \\ 0 & -19/2 & 5 \\ 0 & 0 & 460/19 \end{bmatrix} \; x = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \; y = \begin{bmatrix} 2 \\ 7/2 \\ 322/19 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 6 & 0 \\ 0 & -19/2 & 5 \\ 0 & 0 & 460/19 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7/2 \\ 322/19 \end{bmatrix}$$

$$\begin{aligned} 8I_1 + 6I_2 &= 2 \\ \frac{-19I_2}{2} + 5I_3 &= \frac{7}{2} \\ \frac{460I_3}{19} &= \frac{322}{19} \end{aligned}$$

$$\begin{aligned} \frac{460I_3}{19} = \frac{322}{19} &\rightarrow I_3 = \frac{322}{460} = 7/10 \textbf{ A} \\ \frac{-19I_2}{2} + 5I_3 = \frac{7}{2} &\rightarrow I_2 = (7/2 - 7/2) \cdot -2/19 = 0 \textbf{ A} \\ 8I_1 + 6I_2 = 2 &\rightarrow I_1 = (2 - 6(0))/8 = 1/4 \textbf{ A} \end{aligned}$$

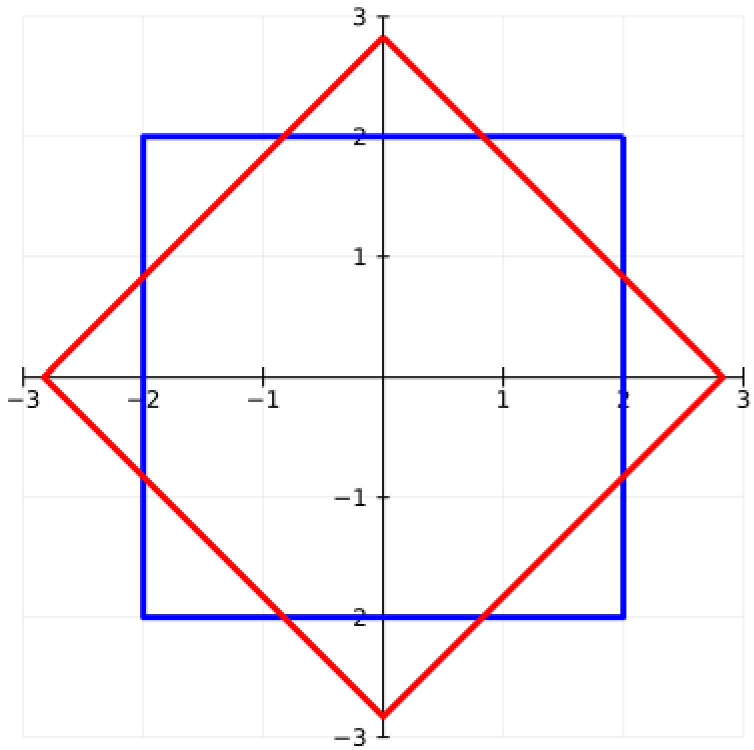
$$\begin{aligned} I_1 &= 1/4 \textbf{ A} \\ I_2 &= 0 \textbf{ A} \\ I_3 &= 7/10 \textbf{ A} \end{aligned}$$

$$A = \begin{bmatrix} 8 & 6 & 0 \\ 2 & -8 & 5 \\ -10 & 10 & 15 \end{bmatrix} \quad x = \begin{bmatrix} 1/4 \\ 0 \\ 7/10 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 6 & 0 \\ 2 & -8 & 5 \\ -10 & 10 & 15 \end{bmatrix} \quad \begin{bmatrix} 1/4 \\ 0 \\ 7/10 \end{bmatrix} \quad = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

2 LU Decomposition with Permutations

1. Convert the system of equations into $PAx = Pb$
2. Decompose the A matrix as $PA = LU$.
3. Finally, solve using Forward Substitution for $Ly = Pb$ and Backward substitution for $Ux = y$



In linear algebra, a rotation matrix is a transformation matrix that is used to perform a rotation on a set of points in Euclidean space about the origin. We will construct the transformation matrix required to rotate the 2-dimensional blue square 45 degrees counter-clockwise so that the result is the red square.

Transforms in 2 dimensions using 2x2 transformation matrices come in the form of

$$Ax = b$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix}$$

Pick two coordinates (x_1, y_1) and (x_2, y_2) on the blue square, and then pick (x_{1new}, y_{1new}) and (x_{2new}, y_{2new}) on the red square such that they correspond with a $45°$ rotation CCW. We can then write out the system of linear equations:

$$\begin{aligned} ax_1 + by_1 &= x_{1new} \\ cx_1 + dy_1 &= y_{1new} \\ ax_2 + by_2 &= x_{2new} \\ cx_2 + dy_2 &= y_{2new} \end{aligned}$$

By examining the system of linear equations, we can form a more familiar x as a column vector with terms a, b, c, d. In order to motivate LU factorization, we will continue with this form for the rest of the solution.

$$A = \begin{bmatrix} x_1 & y_1 & 0 & 0 \\ 0 & 0 & x_1 & y_1 \\ x_2 & y_2 & 0 & 0 \\ 0 & 0 & x_2 & y_2 \end{bmatrix} \quad x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad b = \begin{bmatrix} x_{1new} \\ y_{1new} \\ x_{2new} \\ y_{2new} \end{bmatrix}$$

Verify for yourself that this $Ax = b$ form results in the above system of equations and continue with the problem

Solution: We will take points

$$(x_1, y_1) = (2, 2)$$
$$(x_2, y_2) = (2, -2)$$

from the blue square, giving us an A matrix

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

Additionally, we will take points

$$(x_{1new}, y_{1new}) = (0, 2\sqrt{2})$$
$$(x_{2new}, y_{2new}) = (2\sqrt{2}, 0)$$

from the red square, giving us the b matrix

$$b = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 2\sqrt{2} \\ 0 \end{bmatrix}$$

By permuting rows to ensure there are no zeros along the diagonal, we get

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} b = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 2\sqrt{2} \\ 0 \end{bmatrix} P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we continue with the LU decomposition process.

Find a column vector C_1 and a row vector R_1 such that $C_1 \cdot R_1 = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$

$R_1 = \begin{bmatrix} 2 & 2 & 0 & 0 \end{bmatrix}$

$C_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$C_1 \cdot R_1 = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A - C_1 \cdot R_1 = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$

Now, find a column vector C_2 and a row vector R_2 such that $C_2 \cdot R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$

$R_2 = \begin{bmatrix} 0 & -4 & 0 & 0 \end{bmatrix}$

$C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$C_2 \cdot R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$(A - C_1 \cdot R_1) - (C_2 \cdot R_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$

Now, find a column vector C_3 and a row vector R_3 such that $C_3 \cdot R_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & * \end{bmatrix}$

$R_3 = \begin{bmatrix} 0 & 0 & 2 & 2 \end{bmatrix}$

$C_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$C_3 \cdot R_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

$((A - C_1 \cdot R_1) - C_2 \cdot R_2) - (C_3 \cdot R_3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

giving us
$R_4 = \begin{bmatrix} 0 & 0 & 0 & -4 \end{bmatrix}$

$C_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

and finally our LU matrices can be constructed,

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

Now we use the equations

$Ly = P \cdot b$

$Ux = y$

Forward Substitution:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} y = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} b = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 2\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 2\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{aligned} w &= 0 \\ w + x &= 2\sqrt{2} \\ y &= 2\sqrt{2} \\ y + z &= 0 \end{aligned}$$

$$\begin{aligned} w &= 0 \\ x &= 2\sqrt{2} \\ y &= 2\sqrt{2} \\ z &= -2\sqrt{2} \end{aligned}$$

Backward Substitution:

$$U = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix} x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} y = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

$$\begin{aligned} 2a + 2b &= 0 \\ -4b &= 2\sqrt{2} \\ 2c + 2d &= 2\sqrt{2} \\ -4d &= -2\sqrt{2} \end{aligned}$$

$$\begin{aligned} -4b = 2\sqrt{2} \rightarrow b &= \frac{2\sqrt{2}}{-4} = -\frac{1}{\sqrt{2}} \\ 2a + 2b = 0 \rightarrow a &= -b = \frac{1}{\sqrt{2}} \\ -4d = -2\sqrt{2} \rightarrow d &= \frac{-2\sqrt{2}}{-4} = \frac{1}{\sqrt{2}} \\ 2c + 2d = 2\sqrt{2} \rightarrow c &= \sqrt{2} - d = \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} a &= 1/\sqrt{2} \\ b &= -1/\sqrt{2} \\ c &= 1/\sqrt{2} \\ d &= 1/\sqrt{2} \end{aligned}$$

Our final transformation matrix in the 2x2 form is then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Remark 1: The general transform matrix for a clockwise rotation by angle θ about the origin is given by

$$T = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$$

Verify that

$$\begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

for $\theta = -45°$ is true!

Remark 2: We aren't limited to transforming 2 ordered pairs as we did in the problem. Nor do we have to rearrange the system of equations every time to result in a column vector x. Recall that the number of columns in matrix A has to equal the number of rows in matrix x. With more coordinates we would simply continue stacking columns of (x_n, y_n) to the right of matrix x and b.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix} \quad b = \begin{bmatrix} x_{1new} & x_{2new} & \dots & x_{nnew} \\ y_{1new} & x_{2new} & \dots & y_{nnew} \end{bmatrix}$$

You can now verify that

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & -2 & -2 \\ 2 & -2 & -2 & 2 \end{bmatrix} \quad = \begin{bmatrix} 0 & 2\sqrt{2} & 0 & -2\sqrt{2} \\ 2\sqrt{2} & 0 & -2\sqrt{2} & 0 \end{bmatrix}$$

is true!

Remark 3: Try using Julia to rotate your own matrix of ordered pairs, multiplied by the general rotation transform matrix with your own option of θ and plot it!