ROB 101 - Fall 2021

Determinant of Product, Matrix Inverses, and Matrix Transposes

September 22, 2021

UNIVERSITY OF MICHIGAN

Learning Objectives

- Fill in some gaps that we left during our sprint to an effective means for solving large systems of linear equations.

- Whenever two square matrices A and B can be multiplied, it is true that $\det(A \cdot B) = \det(A) \cdot \det(B)$.
- What it means to "invert a matrix," and knowing that you rarely want to actually compute a matrix inverse!
- If $ad - bc \neq 0$, then $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{array}{rr} d & -b \\ -c & a \end{array}\right]$.
- Matrix transpose takes columns of one matrix into the rows of another.

To find the determinant of a product of matrices, we can simply take the product of the determinants.

Fact

Let A and B be $n \times n$ matrices. Then

$$\boxed{\det(AB) = \det(A) \cdot \det(B)}$$

- Now suppose that we have done the LU factorization of a square matrix A.

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- Then, using the previous fact, we have

$$\det(A) = \det(L \cdot U) = \det(L) \cdot \det(U).$$

Recall: Determinant of a Lower Triangular Matrix

Fact

The matrix determinant of a square lower triangular matrix is equal to the product of the elements on the diagonal.

$$A = \begin{bmatrix} 3 & \mathbf{0} & \mathbf{0} \\ 2 & -1 & \mathbf{0} \\ 1 & -2 & 3 \end{bmatrix} \implies \det(A) = 3 \cdot (-1) \cdot 3 = -9 \neq 0.$$

In [1]:
```
using LinearAlgebra

A = [3 0 0; 2 -1 0; 1 -2 3];
det(A)
```

Out[1]:
```
-9.0
```

Recall:Determinant of an Upper Triangular Matrix

Fact

The matrix determinant of a square upper triangular matrix is equal to the product of the elements on the diagonal.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ \mathbf{0} & 2 & 1 \\ \mathbf{0} & \mathbf{0} & 3 \end{bmatrix} \implies \det(A) = 1 \cdot 2 \cdot 3 = 6 \neq 0.$$

In [2]:
```
A = [1 3 2; 0 2 1; 0 0 3];
det(A)
```

Out[2]:
```
6.0
```

Matrix Determinant via LU Factorization

Corollary

Because L and U are triangular matrices, each of their determinants is given by the product of the diagonal elements. Hence, we have a way of computing the determinant for square matrices of arbitrary size.

Q. How about LU with row permutation?

$$PA = LU.$$

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Mathematically (just carrying out calculations based on facts we know), we have

$$\begin{aligned} \det(PA) &= \det(LU) \\ \det(P)\det(A) &= \det(L)\det(U) \\ \det(A) &= \frac{1}{\det(P)} \cdot \det(L)\det(U), \quad \det(P) \neq 0. \end{aligned}$$

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\end{aligned}$$

Fact

For the determinant of a permutation matrix P, we have

$$\det(P) = \pm 1.$$

Example

Compute the matrix determinant of

$$\begin{bmatrix} -2 & -4 & -6 \\ -2 & 1 & -4 \\ -2 & 11 & -4 \end{bmatrix}.$$

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Hence,

$$\det(A) = \underbrace{(1) \cdot (1) \cdot (1)}_{\det(L)} \cdot \underbrace{(-2) \cdot (5) \cdot (-4)}_{\det(U)} = 40.$$

Example

In [1]:

```julia
using LinearAlgebra

# define A
A = [-2. -4 -6; -2 1 -4; -2 11 -4];
# compute LU decomposition
F = lu(A);

@show det(F.L);
@show det(F.U);
@show det(F.P);
@show det(A);
```

```
det(F.L) = 1.0
det(F.U) = -39.99999999999999
det(F.P) = -1.0
det(A) = 39.99999999999999
```

- The identity matrix is a square matrix denoted I that has ones down the main diagonal and zeroes elsewhere.

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- Here are some examples of 1×1, 2×2, 3×3, and 4×4 identity matrices.

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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- The notation I_n means an $n \times n$ identity matrix.

- Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix. Then:

$$A \cdot I_n = A.$$

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$$A \cdot I_n = A.$$
- If I_m is the $m \times m$ identity matrix, $I_m \cdot A = A$.
- See Example 6.2 in ROB 101 book.

- A square $n \times n$ matrix A is said to have an inverse A^{-1} if and only if
$$AA^{-1} = A^{-1}A = I_n.$$
- In this case, the matrix A is called invertible.

Claim

If a matrix has an inverse, it is unique (that is, there is only one of them). If A and B are both $n \times n$, then

$$(A \cdot B = B \cdot A = I_n) \iff B = A^{-1}.$$

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If a matrix has an inverse, it is unique (that is, there is only one of them). If A and B are both $n \times n$, then

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Proof.

Suppose A^{-1} and B both inverses of A. Then we have

$$\begin{aligned} AA^{-1} &= I \\ BAA^{-1} &= BI \\ A^{-1} &= B! \end{aligned}$$

$\square$

- Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose that $\det(A) = a \cdot d - b \cdot c \neq 0$.

Example

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- Then,

$$A^{-1} = \frac{1}{a \cdot d - b \cdot c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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- Applying the above formula for the inverse of a 2×2 matrix immediately gives that

$$\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}.$$

Example

In [1]:
```julia
using LinearAlgebra

A = [4 2; 5 3];
B = [3/2 -1; -5/2 2]; # B is the inverse of A!

@show A * B
@show B * A
```

```
A * B = [1.0 0.0; 0.0 1.0]
B * A = [1.0 0.0; 0.0 1.0]
```

Out[1]:
```
2×2 Array{Float64,2}:
 1.0  0.0
 0.0  1.0
```

Important Facts for the Matrix Determinant (ROB 101 Trivia?)

- Suppose that A is $n \times n$. Because the determinant of a product is the product of the determinants, we have that

$$1 = \det(I_n) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}).$$

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- It follows that if A has an inverse, then $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$

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- It follows that if A has an inverse, then $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$

- If $\det(A) \neq 0$, then it has an inverse (one also says that A^{-1} exists). Putting these facts together gives the next result.

Important Facts for the Matrix Determinant (ROB 101 Trivia?)

Fact

An $n \times n$ *matrix* A *is invertible if, and only if,* $\det(A) \neq 0$.

Important Facts for the Matrix Determinant (ROB 101 Trivia?)

Fact

Another useful fact about matrix inverses is that if A *and* B *are both* $n \times n$ *and invertible, then their product is also invertible and*

$$\boxed{(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}.}$$

Important Facts for the Matrix Determinant (ROB 101 Trivia?)

Fact

Another useful fact about matrix inverses is that if A and B are both $n \times n$ and invertible, then their product is also invertible and

$$\boxed{(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}.}$$

Note that the order is swapped when you compute the inverse. To see why this is true, we note that

$$(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = A \cdot (B \cdot B^{-1}) \cdot A^{-1} = A \cdot (I) \cdot A^{-1} = A \cdot A^{-1} = I.$$

If A is invertible and $A = L \cdot U$ is the LU factorization of A, then

$$A^{-1} = U^{-1} \cdot L^{-1}.$$

Utility of the Matrix Inverse and its Computation

- The primary use of the matrix inverse is that it provides a closed-form solution to linear systems of equations.
- Suppose that A is square and invertible, then

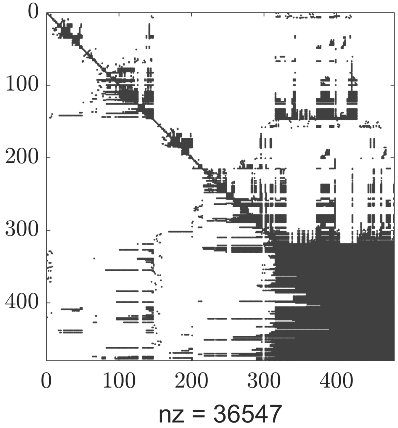
$$\boxed{Ax = b \iff x = A^{-1} \cdot b.}$$

Remark

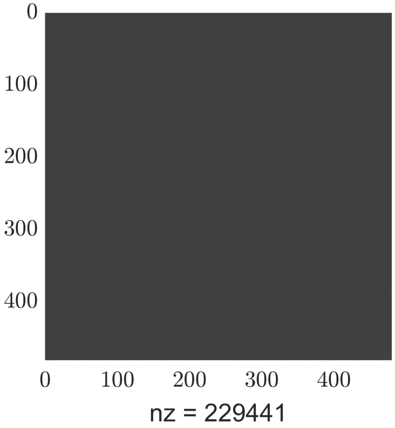
It is much better to solve $Ax = b$ by factoring $A = L \cdot U$ and using back and forward substitution, than to first compute A^{-1} and then multiply A^{-1} and b. Explicitly computing A^{-1} can lead to numerical instability.

Remark

If A has any special structure such as sparsity, A^{-1} in general will not preserve it.



(a) A is sparse.



(b) A^{-1} is dense.

Problem

Tell me how to invert an $n \times n$ (invertible) matrix A, without telling me to invert it explicitly!

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Tell me how to invert an $n \times n$ (invertible) matrix A, without telling me to invert it explicitly! Hint: use LU factorization of A and I_n.

The transpose takes the rows of a matrix and turns them into the columns of a new matrix.

- Equivalently, you can view the matrix transpose as taking each column of one matrix and laying the elements out as rows in a new matrix
- The (i,j)-entry of A becomes the (j,i)-entry of A^{T}.

In [1]:
```
using LinearAlgebra

# define A
A = [-2 -4 -6; -2 1 -4]
```

Out[1]:
```
2×3 Array{Int64,2}:
 -2  -4  -6
 -2   1  -4
```

In [2]:
```
# A transpose
A'
```

Out[2]:
```
3×2 Adjoint{Int64,Array{Int64,2}}:
 -2  -2
 -4   1
 -6  -4
```

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and r and s scalars.

- Applying the transpose twice we get $(A^{\mathsf{T}})^{\mathsf{T}} = A$.
- If A is square, $\det(A^{\mathsf{T}}) = \det(A)$.
- Transpose changes the order of matrix multiplication $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$.
- Reach the chapter for discussions about these properties.

- Matrices that consist of all ones and zeros, with each row and each column having a single one, are called permutation matrices.

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- We put the 5×5 identity matrix on the left and the corresponding permutation matrix P on the right

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \leftrightarrow P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 3 \to 1 \\ 2 \to 2 \\ 5 \to 3 \\ 1 \to 4 \\ 4 \to 5 \end{bmatrix}.$$

P is still just a re-ordering of the rows of I. You can check that $P^{\mathsf{T}} \cdot P = P \cdot P^{\mathsf{T}} = I$.

Remark

Hence, the inverse of a permutation matrix is its transpose!

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Remark

Hence, the inverse of a permutation matrix is its transpose!

Corollary

For the determinant of a permutation matrix P, we have

$$\begin{aligned} P \cdot P^{\mathsf{T}} &= I \\ \det(P \cdot P^{\mathsf{T}}) &= \det(I) \\ \det(P)\det(P^{\mathsf{T}}) &= 1 \\ \det(P)^2 &= 1 \\ \boxed{\det(P) = \pm 1} \end{aligned}$$

In [3]:
```
# construct our permutation matrix
ids = [3,2,5,1,4];
P = zeros(5,5) + I;
P = P[ids,:]
```

Out[3]:
```
5×5 Array{Float64,2}:
 0.0  0.0  1.0  0.0  0.0
 0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  1.0
 1.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  1.0  0.0
```

In [4]:
```
# verify its inverse is its transpose!
P * P'
```

Out[4]:
```
5×5 Array{Float64,2}:
 1.0  0.0  0.0  0.0  0.0
 0.0  1.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0  0.0
 0.0  0.0  0.0  1.0  0.0
 0.0  0.0  0.0  0.0  1.0
```

Next Time

- The Vector Space $\mathbb{R}^n$: Part 1
- Read Chapter 7 of ROB 101 Book