ROB 101 - Computational Linear Algebra
Recitation #4

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1 Matrix Determinant

Recall: Determinant Facts

- det(A) is a real number
- $Ax = b$, a system of equations with n equations and n unknowns has a unique solution for any b if an only if $det(A) \neq 0$
- When $det(A) = 0$, the system may have either infinite or no solution
- $det(A) = ad - bc$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

In addition to the previous facts, we also have:
Additional Fact: Determinant of product of matrices

- Let A and B be $n \times n$ matrices $\Rightarrow det(AB) = det(A)det(B)$
  (Note that the order of the multiplication doesn't matter since $det(A)$ and $det(B)$ are both scalars)

Matrix Determinant via LU Factorization: Given $A = LU$, we now have

$$det(A) = det(L)det(U)$$

Additional Fact: Determinant of triangular matrices

- Let A be a triangular matrix, then $det(A) =$ product of the elements on the diagonal (Note by triangular we mean either upper or lower triangular)

Example: Find the determinant of $A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$

$$det(A) = 3 * 5 * 9 = 135 \tag{1}$$

Additional Fact: Determinant of triangular matrices

- Let P be a permutation matrix, then $det(P) = \pm 1$

2 Inverse of Matrices

A matrix A is invertible if there exists a matrix A^{-1}such that $AA^{-1} = A^{-1}A = I_n$
Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
The following statements are equivalent

- A is invertible
- $det(A) \neq 0$
- A is a square matrix

Fact: Inverse of a product of matrices $(AB)^{-1} = B^{-1}A^{-1}$
(Note: the order of the multiplication matters here)

3 Determinants and Inverses

Additional Fact: Determinant of inverse of a matrix

- Let $det(A^{-1}) = \frac{1}{det(A)}$

Given $PA = LU$ we have

$$\begin{aligned}
PA &= LU \\
P^{-1}PA &= P^{-1}LU \\
A &= P^{-1}LU \\
det(A) &= det(P^{-1}LU) \\
&= det(P^{-1})det(L)det(U) \\
&= \frac{1}{det(P)}det(L)det(U)
\end{aligned}$$

4 Transpose of a Matrix

Let $A^\intercal$ be the transpose of A. To get $A^\intercal$ we simply take the rows of A and use them as the columns of $A^\intercal$ or we can equivalently take the columns of our A matrix and turn them into the rows of $A^\intercal$. So if A is an nxm matrix, the transpose, $A^\intercal$ is an mxn matrix. Let's take a look at an example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^\intercal = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \tag{2}$$

To find the transpose in julia, we can either use $transpose(A)$ or $A^{'}$.
Properties of the Tranpose of a matrix

- $(A^\intercal)^\intercal = A$
- if A is square $det(A^\intercal) = det(A)$
- $(AB)^\intercal = B^\intercal A^\intercal$
  (Note: the order of the multiplication matters here)

5 Linear Combination

A vector, $v \in \mathbb{R}^n$ is said to be a Linear Combination of vectors $v_1, v_2 \cdots v_m \in \mathbb{R}^n$ if there exits real numbers $\alpha_1, \alpha_2 \cdots \alpha_m$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

Using this formulation, find if the given vector v, is a linear combination of the vectors $v_1, v_2 \cdots v_m$ in the question, if true, also find the vector of coefficients, α

1. $v = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$
   $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   Solution: Goal find $\alpha_1, \alpha_2, \alpha_3$ **such that** $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = v$
   $\alpha_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$
   From the system of equations above we get 2 equations:

   (a) $3\alpha_1 + 2\alpha_2 + \alpha_3 = 11$

   (b) $4\alpha_1 + 2\alpha_2 - \alpha_3 = 7$

   From (a) we get: $\alpha_3 = 11 - 3\alpha_1 - 2\alpha_2$
   Substituting $\alpha_3 = 11 - 3\alpha_1 - 2\alpha_2$ **into (b) we get** $7\alpha_1 + 4_2 = 18$
   We now need to pick α_1 **and** α_2 **such that** $7\alpha_1 + 4_2 = 18$
   Let's go with $\alpha_1 = 2$ **and** $\alpha_2 = 1$ **with this choice we get,** $\alpha_3 = 11 - 3 * 2 - 2 * 1 = 3$
   So we now have that v **is a linear combination of** v_1, v_2 **and**v_3 **given** $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 3$

2. $v = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}$
   $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
   Solution: Goal find α_1, α_2 **such that** $\alpha_1 v_1 + \alpha_2 v_2 = v$
   $\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}$
   From the system of equations above we get the following 3 equations

   (a) $\alpha_1 + 2\alpha_2 = 7$
   (b) $2\alpha_1 + \alpha_2 = 5$
   (c) $3\alpha_1 = 4$

   Starting with (c), we get that $\alpha_1 = \frac{4}{3}$
   Plugging $\alpha_1 = \frac{4}{3}$ **into (b), we get:** $2\frac{4}{3} + \alpha_2 = 5 \Rightarrow \alpha_2 = 5 - \frac{8}{3} = \frac{7}{3}$
   Even though we have α_1 **and** α_2 **we are not done yet. We need to check whether** α_1 **and** α_2 **satisfy (a):**

$$\frac{4}{3} + 2\frac{7}{3} = \frac{18}{3} = 6 \neq 7$$

which they do not $\Rightarrow v$ is not a linear combination of v_1 and v_2

6 Linear Independence

The vectors $\{v_1, v_2, ..., v_m\}$ are **linearly independent** if the **only** real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ yielding a linear combination of vectors that adds up to the zero vector,

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0, \tag{3}$$

are $\boldsymbol{\alpha}_1 = 0, \boldsymbol{\alpha}_2 = 0, \ldots, \boldsymbol{\alpha}_m = 0$.

Concise definition of Linear Independence:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0 \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using this Definition, determine if the following vectors are linearly independent.

1. $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

   Solution: Goal find α_1, α_2 such that $\alpha_1 v_1 + \alpha_2 v_2 = 0$

   $\alpha_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   From the system of equations above, we have 2 equations

   (a) $4\alpha_1 + 2\alpha_2 = 0$

   (b) $\alpha_1 + 3\alpha_2 = 0$

   (a) gives us $\alpha_1 = -3\alpha_2$

   Plugging this into (b) we have $4(-3\alpha_2) + 2\alpha_2 = -12\alpha_2 + 2\alpha_2 = -10\alpha_2 = 0 \Rightarrow \alpha_2 = 0$

   and since $\alpha_1 = -3\alpha_2$ we have $\alpha_1 = 0$

   Therefore since $\alpha_1 = 0$ and $\alpha_2 = 0$, v_1 and v_2 are linearly independent from each other

2. $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

   (Note that this problem is the same as problem 1 in Section 5)

   Solution: Goal find $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

   $\alpha_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 11 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   From the system of equations above, we have 2 equations:

   (a) $3\alpha_1 + 2\alpha_2 + \alpha_3 + 11\alpha_4 = 0$

   (b) $4\alpha_1 + 2\alpha_2 - \alpha_3 + 7\alpha_4 = 0$

   Recall that in problem 1 of Section 5 we found that v was the linear combination of v_1, v_2 and v_3 given $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 3$. Using this as our basis, let's try using $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 3, \alpha_4 = -1$ to solve our system of equations. We get:

   (a) 3*2+2*1-3-11=0

   (b) 4*2+2-3-7=0

   Therefore, with $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 3, \alpha_4 = -1$ we have $\alpha_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 11 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   Since all the $alpha$'s are not 0, v_1, v_2, v_3, and v_4 are not linear independent. In fact, we can conclude in general that if a vector v is a linear combination of a set of vectors $\{v_1, ...v_m\}$, then the set of vectors $\{v_1, ..., v_m, v\}$ are not linearly independent.

7 Solutions of Ax=b and Linear Independence

Existence

The system of equation $Ax = b$ has a solution if:

b is a linear combination of the columns of A and if the columns of A are linearly independent

Using this definition, let's set up the problem below as if we were to find if the following system of equations have a solution:

$$\begin{aligned} -a + 3b + 5c &= 20 \\ -2a - 2c &= -8 \\ -3a + 3b + 4c &= 10 \end{aligned}$$

Solution: From the system of equations above, we have $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 0 & -2 \\ -3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 20 \\ -8 \\ 10 \end{bmatrix}$

Let $v_1 = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$**,** $v_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$**,** $v_3 = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$**,** $v_4 = \begin{bmatrix} 20 \\ -8 \\ -10 \end{bmatrix}$

1. **Are the columns of A linearly independent?**
   Goal: find $\alpha_1, \alpha_2, \alpha_3$ **such that** $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

2. **Is b a linear combination of the columns of A?**
   Goal: find $\alpha_1, \alpha_2, \alpha_3$ **such that** $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = v_4$

8 Facts: Linear Independence, Determinant and Inverse

Given $A = [v_1|v_2|v_3|...|v_m]$

- The set of vectors $\{v_1, .., v_m\}$ are linearly independent
- A is invertible
- $det(A^\intercal A) \neq 0$