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Biostatistics 202A (Fall 2017)

# Midterm Exam

Instructions: This is a closed book, closed notes examination. You must SHOW ALL YOUR WORK.

1. (10 points) Suppose we are playing poker, where each player is dealt five cards from a well shuffled deck. What is the probability of getting exactly 3 of a kind (3 cards of the same rank (ace, 2, 3, ..., 10, Jack, Queen, King) and the other two cards of two other ranks)?

$\binom{13}{1}$ ways to choose the rank of the 3 of a kind

$\binom{12}{2}$ ways to choose the ranks of the other 2 cards

$\binom{4}{3}$ ways to choose the 3 cards of the particular rank that will be drawn

$\binom{1}{0}$ ways to choose the card of the particular rank that will not be drawn

$\binom{4}{1}$ ways to choose the one card drawn from the other ranks

$\binom{3}{0}$ ways to choose cards not drawn from other ranks

$\binom{52}{5}$ # of possible hands

$$\boxed{P(3 \text{ of a kind}) = \frac{\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}}$$

$\frac{11}{11}$

$1-F(x) =$

2. (10 points) Let $X$ be a continuous random variable with pdf $f(x)$, which is positive provided $0 < x < b < \infty$, and is equal to zero elsewhere. Derive from basics that

$$E(X) = \int_0^b [1 - F(x)]dx, \quad (1)$$

where $F(x)$ is the cdf of X.

$f(x) = \frac{d}{dx}F(x)$

$\int f(x)dx = F(x)$

We also know that $F(x) = \int_0^x f(x)dx$ so $1 - F(x) = \int_x^b f(x)dx$

~~We define $E(x) = \int_0^b x f(x) dx$~~

We know $\int_0^b f(x)\, dx = 1$, and we define $E(x) = \int_0^b x f(x)\, dx$

~~So $E(x) = \int_0^b x f(x) dx$~~

$\frac{8}{11}$

$-3$

We can observe that $F(x) = \int_0^x f(x)\, dx$, so $1 - F(x) = \int_x^b f(x)dx$

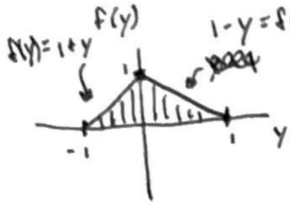
Now if we take the integral over the support of $x$, we get

$$\int_0^b (1 - F(x))dx = \int_0^b \int_x^b f(\underset{t}{x})\, d\underset{t}{x}\, dx$$

need to change order of integration

$$\int_0^b (1 - F(x))\, dx = \int_0^b x f(x)\, dx = E(x) \therefore$$

$\int_0^b f(x)dx - \int_0^x f(x)dx = 1 - F(x)$




$\int_{-1}^{1} f(y)\,dy = 1$, and $f(y) \geq 0$ over the support of $y$,

So $f(y)$ is a legitimate pdf

3. (15 points) If the pdf for $Y$ is $f(y) = 1 - |y|$ for $|y| \leq 1$ and zero elsewhere, find and graph the cdf $F(y)$.

$\int y+1\,dy = \left[\frac{y^2}{2} + y\right]_{-1}^{0} = \frac{1}{2}$

$\int 1-y\,dy = y - \frac{y^2}{2}$

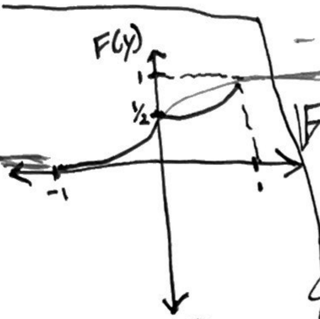
$2y$

$\frac{16}{17}$

$$f(y) = \begin{cases} 1-|y|, & -1 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

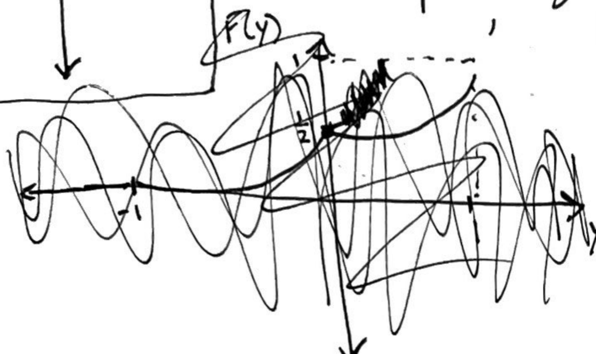
For the cdf, we should split $f(y)$ into two components and observe that

this was correct




$$\cancel{F(y) = \begin{cases} 0, & y \leq -1 \\ \int y+1\,dy, & -1 \leq y \leq 0 \\ \int 1-y\,dy, & 0 < y \leq 1 \\ 1, & y \geq 1 \end{cases}}$$

$$\cancel{\rightarrow F(y) = \begin{cases} 0, & y < -1 \\ \frac{y^2}{2}+y, & -1 \leq y \leq 0 \\ y - \frac{y^2}{2}, & 0 < y \leq 1 \\ 1, & y > 1 \end{cases}}$$




$$\boxed{F(y) = \begin{cases} 0, & y < -1 \\ \int y+1\,dy, & -1 \leq y \leq 0 \\ \int 1-y\,dy, & 0 < y \leq 1 \\ 1, & y > 1 \end{cases} = F(y) = \begin{cases} 0, & y < -1 \\ \frac{y^2}{2}+y+\frac{1}{2}, & -1 \leq y \leq 0 \\ y - \frac{y^2}{2}+\frac{1}{2}, & 0 < y \leq 1 \\ 1, & y > 1 \end{cases}}$$

hard to see, but there were $+\frac{1}{2}$ terms that needed to be added to make it a legitimate cdf

4. (10 points) Assume that the random variable X has the cdf $F_X(\cdot)$. Let a new random variable $Y$ be defined as $Y = F^{-1}(X)$, where $F(\cdot)$ is a an invertible cdf which is not necessarily equal to $F_X(\cdot)$. Derive the cdf of $Y$ in terms of $F_X(\cdot)$ and $F(\cdot)$. Information supplied implies $X = F(Y)$.

So we know that $Y =$

$$F_Y(Y) = P(Y \leq y) = P(F^{-1}(x) \leq y) = P(X \leq F(y))$$

$$= F_X(F(y)), \text{ so } \boxed{F_Y(Y) = F_X(F(\cdot))}$$

$\frac{11}{11}$

$\frac{87}{100}$

5. Suppose $X_1$ and $X_2$ have the joint pdf $f_{X_1,X_2}(x_1,x_2) = 2e^{-(x_1+x_2)}$, $0 < x_1 < x_2 < \infty$, zero elsewhere.

a. (10 points) Find the marginal pdf of $X_2$.

$$f_{X_2}(x_2) = \int_0^{x_2} f_{X_1,X_2}(x_1, x_2)\, dx_1$$

$$= \int_0^{x_2} 2e^{-(x_1+x_2)}\, dx_1 = \left[-2e^{(-(x_1+x_2))}\right]_0^{x_2}$$

$$= \boxed{-2e^{-2x_2} + e^{-x_2}}$$ X support ?

-3

$\frac{8}{11}$

b. (10 points) Find the conditional expectation $E(X_1|X_2 = 2)$.

$$E(X_1|X_2=2) = \int_0^\infty x_1 f_{1|2}(x_1|x_2=2)\, dx_1$$

$; \quad f_{1|2}(x_1|x_2) = \frac{f_{x_1x_2}(x_1,x_2)}{f_{x_2}(x_2)}$

$f_{1|2}(x_1|x_2=2) = \frac{2e^{-(x_1+x_2)}}{-2e^{-2x_2}+e^{-x_2}}$

support ?

X

$$= \int_0^\infty (x_1)\left(\frac{2e^{-(x_1+2)}}{e^{-2}-2e^{-4}}\right)$$

$$= \frac{1}{e^{-2}-2e^{-4}}\left[2e^{-(x_1+2)} + x_1\left(-2e^{-(x_1+2)}\right)\right]_0^\infty$$

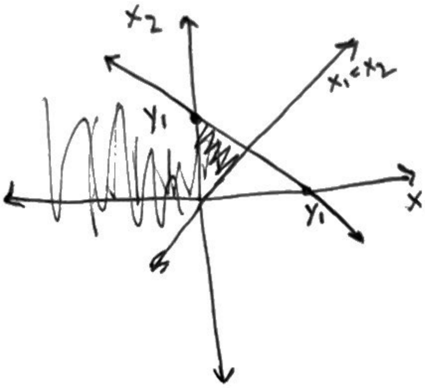
$$= 0 - (2e^{-2} + 0) = \boxed{2e^{-2}}$$

-3

$\frac{8}{11}$

c. (15 points) Find the distribution of $Y_1 = X_1 + X_2$.

$$F(Y_1) = P(Y_1 \le y_1) = P(X_1 + X_2 \le y_1) = P(X_1 \le y_1 - X_2)$$




$$\cancel{= \int_{-\infty}^{y_1/2}\int_{x_1}^{y_1-x_1} f_{X_1X_2}(x_1,x_2)\,dx_2\,dx_1}$$

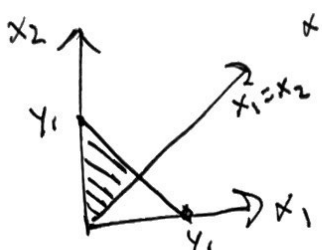
$$\cancel{= \int_{-\infty}^{y_1/2}\left[-2e^{-(x_1+x_2)}\right]_{x_1}^{y_1-x_1} dx_1}$$

$$= \int_0^{\frac{y_1}{2}}\int_{x_1}^{y_1-x_1} f_{X_1X_2}(x_1,x_2)\,dx_2\,dx_1$$

$x_1 = x_2$

$x_1 = y - x_2$

$y = 2x_2 = 2x_1$




$$= \int_0^{\frac{y_1}{2}}\left[-2e^{-(x_1+x_2)}\right]_{x_1}^{y_1-x_1} dx_1 = \int_0^{\frac{y_1}{2}}(-2e^{-y_1}) + 2e^{-2x_1}\,dx_1$$

computational error $-2$ ✗

$$= \left[-2e^{-2x_1}\right]_0^{\frac{y_1}{2}} = (-2e^{-y_1}) - (-2) = \boxed{2 - 2e^{-y_1}}$$

Support? $-1$

$\frac{14}{17}$

6. (10 points) Suppose that one-third of new businesses fail in one year. Of those remaining after one year, two-thirds fail in their second year of operation. Out of a group of 4 new businesses, what is the probability that at least one will survive two years?

P(business survives after 2 years) $= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}$

Let $X$ be a r.v. representing the # of businesses out of a group of 4 that survives after 2 years.

Thus, we seek $P(X \ge 1) = 1 - P(X = 0)$

For Prob. of all 4 businesses surviving, we can assume each event is independent, so that P(business 1 survives) = P(business 2 survives) = etc. and the probability of none surviving is

$\left(\frac{7}{9}\right)^4$ so $\boxed{P(X \ge 1) = 1 - \left(\frac{7}{9}\right)^4 \approx 0.63}$

$\frac{11}{11}$