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$$a) L(p; X) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}$$

$$f(x) = \begin{cases} p^x (1-p)^{1-x} & , x=0,1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$\log L(p; X) = \sum_{i=1}^n \log [p^{x_i} (1-p)^{1-x_i}]$$

$$= \sum_{i=1}^n [x_i \log p + (1-x_i) \log (1-p)]$$

$$\frac{\partial \log L}{\partial p} = \sum_{i=1}^n \left[\frac{x_i}{p} - \frac{(1-x_i)}{1-p} \right]$$

$$= \sum_{i=1}^n \left[\frac{x_i(1-p) - p(1-x_i)}{p(1-p)} \right] = \sum_{i=1}^n \left[\frac{x_i - p}{p(1-p)} \right]$$

$$= \left(\frac{1}{p(1-p)} \right) \sum_{i=1}^n x_i - \frac{n}{1-p}$$

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2	5/10
3	3/10
4	2/10
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Set equal to 0

$$\hat{p}_{MLE} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{X}_n \quad \checkmark$$

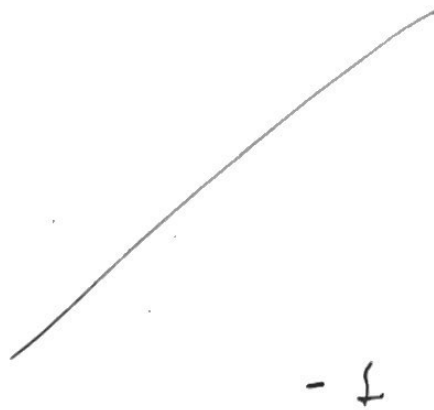
The MLE is the sample mean, \bar{X}_n .A sum of Bernoullis has a $\text{bin}(n, p)$ distribution.mean np , variance $np(1-p)$.

$$\bar{X}_n \xrightarrow{d} N(np, p(1-p)) \quad -1$$

convergence statements including (n)
 $\forall \epsilon > 0$

b) We want to show $\bar{Y}_n \xrightarrow{P} p$ for consistency, so we need $\lim_{n \rightarrow \infty} P(|\bar{Y}_n - p| \geq \epsilon) = 0$

$$= 1 - P(\bar{Y}_n - p)$$



c) For asymptotic efficiency we would need to show that as $n \rightarrow \infty$, the estimator attains the Rao - Cramér lower bound for variance. Given that Y_n is a statistic, we can find the lower bound by

$$\text{Var}(Y_n) = \frac{1}{nI(p)}$$

Where we find $I(p)$ by $-E\left[\frac{\partial^2 \log f(p; X)}{\partial p^2}\right]$ or

but it's the harder way to go

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Problem 2

$$X_i \sim N(\mu, \sigma^2) \quad S_n^2 = \frac{1}{n} \sum_i (X_i - \bar{X}_n)^2 \quad \bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$$

We know by WLLN that $\bar{X}_n \xrightarrow{P} \mu \neq 0$ ✓

So we should be able to state that $\frac{1}{\bar{X}_n} \xrightarrow{P} \frac{1}{\mu}$ ✓

We also know that S_n^2 is a biased estimator of population variance, but that asymptotically the bias disappears. So

$$S_n^2 \xrightarrow{P} \sigma^2, \text{ and therefore } S_n^2 \xrightarrow{d} \sigma^2 \quad \checkmark$$

What if we use delta method for $\text{var}((X_i - \mu)^2) = d$

$$\sqrt{n}(S^2 - \sigma^2) \xrightarrow{d} N(0, 1) \quad \checkmark$$

$$\text{so} \quad \sqrt{n}(S - \sigma) \xrightarrow{d} N(0, \frac{1}{4}) \quad \checkmark$$

OK \rightarrow simply combine

$$\begin{cases} \bar{X}_n \xrightarrow{P} \mu \\ \sqrt{n}(S - \sigma) \xrightarrow{d} N(0; d/4) \end{cases}$$

$$\begin{aligned} S_n^2 &= \frac{1}{n} \sum_i (X_i - \bar{X}_n)^2 = \frac{1}{n} \left[\sum_i X_i^2 - \sum_i 2X_i \bar{X}_n + n \bar{X}_n^2 \right] \\ &= \frac{1}{n} \sum_i X_i^2 - 2\bar{X}^2 + \bar{X}_n^2 \\ &= \frac{1}{n} \sum_i X_i^2 - \bar{X}_n^2 \end{aligned}$$

S_n^2

Problem 3a) $E(Y_L)$

$$Y_L \sim \text{Bernoulli}(p_L)$$

So joint distribution is

$$f_{Y_n}(y) = \prod_{i=1}^n p_i^{y_i} (1-p_i)^{1-y_i} \quad \checkmark$$

b) The score function is what we get when we take the derivative of the log of the density function. I suppose we're meant to use what we derived in problem 1?

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Problem 4

We want to show that changing $Y_n \xrightarrow{P} c$ to $Y_n \xrightarrow{P} Y$
does not strengthen Slutsky's Theorem

If instead of letting Y_n converge to a constant, c , we
let it converge to a normally distributed r.v., then we have

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