University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

Matrix and vector differentiation

Let

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_p \end{pmatrix}$$

be a p-dimensional vector and let $f(\boldsymbol{\theta})$ be a function of $\boldsymbol{\theta}$. When the derivative of $f(\boldsymbol{\theta})$ is taken with respect to the vector $\boldsymbol{\theta}$ we mean that the partial derivative of $f(\boldsymbol{\theta})$ is taken with respect to each element of $\boldsymbol{\theta}$, i.e.

$$\frac{\partial f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{pmatrix} \frac{\partial f(\boldsymbol{\theta})}{\theta_1} \\ \frac{\partial f(\boldsymbol{\theta})}{\theta_2} \\ \vdots \\ \frac{\partial f(\boldsymbol{\theta})}{\theta_p} \end{pmatrix}$$

We will present now two important results of matrix differentiation.

1. Let $\boldsymbol{\theta}$ as defined above and $\mathbf{c}' = (c_1, c_2, \ldots, c_p)$. If $f(\boldsymbol{\theta}) = \mathbf{c}'\boldsymbol{\theta}$ it follows that

$$\frac{\partial f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{c}.$$

Proof

$$\frac{\partial f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{pmatrix} \frac{\partial f(\boldsymbol{\theta})}{\theta_1} \\ \frac{\partial f(\boldsymbol{\theta})}{\theta_2} \\ \vdots \\ \frac{\partial f(\boldsymbol{\theta})}{\theta_p} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{c}'\boldsymbol{\theta}}{\theta_1} \\ \frac{\partial \mathbf{c}'\boldsymbol{\theta}}{\theta_2} \\ \vdots \\ \frac{\partial \mathbf{c}'\boldsymbol{\theta}}{\theta_p} \end{pmatrix} = \begin{pmatrix} \frac{\partial (c_1\theta_1 + \ldots + c_p\theta_p)}{\theta_1} \\ \frac{\partial (c_1\theta_1 + \ldots + c_p\theta_p)}{\theta_2} \\ \vdots \\ \frac{\partial (c_1\theta_1 + \ldots + c_p\theta_p)}{\theta_p} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix} = \mathbf{c}.$$

2. Let $\mathbf{A}$ be a $p \times p$ symmetric matrix and let $\boldsymbol{\theta}$ as define above. Define now the quadratic expression $f(\boldsymbol{\theta}) = \boldsymbol{\theta}'\mathbf{A}\boldsymbol{\theta}$. It follows that

$$\frac{\partial f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 2\mathbf{A}\boldsymbol{\theta}.$$

Proof
Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pp} \end{pmatrix} = \begin{pmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \vdots \\ \mathbf{a}'_p \end{pmatrix}.$$

We can write $f(\boldsymbol{\theta})$ as: $f(\boldsymbol{\theta}) = \boldsymbol{\theta}'\mathbf{A}\boldsymbol{\theta} = \sum_{i=1}^p \theta_i^2 a_{ii} + 2\sum_{i=1}^p \sum_{j \neq i}^p \theta_i\theta_j a_{ij}$.
Take the derivative of $f(\boldsymbol{\theta})$ with respect to θ_1: $\frac{\partial f(\boldsymbol{\theta})}{\theta_1} = 2a_{11}\theta_1 + 2\sum_{j\neq 1}^p a_{1j}\theta_j = 2\sum_{j=1}^p a_{1j}\theta_j = 2\mathbf{a}'_1\boldsymbol{\theta}$.
Take the derivative of $f(\boldsymbol{\theta})$ with respect to θ_2: $\frac{\partial f(\boldsymbol{\theta})}{\theta_2} = 2a_{22}\theta_2 + 2\sum_{j\neq 2}^p a_{2j}\theta_j = 2\sum_{j=1}^p a_{2j}\theta_j = 2\mathbf{a}'_2\boldsymbol{\theta}$.
$\vdots$
Take the derivative of $f(\boldsymbol{\theta})$ with respect to θ_p: $\frac{\partial f(\boldsymbol{\theta})}{\theta_p} = 2a_{pp}\theta_p + 2\sum_{j\neq p}^p a_{pj}\theta_j = 2\sum_{j=1}^p a_{pj}\theta_j = 2\mathbf{a}'_p\boldsymbol{\theta}$.

Therefore,

$$\frac{\partial f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{pmatrix} 2\mathbf{a}'_1\boldsymbol{\theta} \\ 2\mathbf{a}'_2\boldsymbol{\theta} \\ \vdots \\ 2\mathbf{a}'_p\boldsymbol{\theta} \end{pmatrix} = 2\begin{pmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_p \end{pmatrix}\boldsymbol{\theta} = 2\mathbf{A}\boldsymbol{\theta}.$$

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Portfolio expected return and risk

Suppose a portfolio consists of n stocks. Let $\bar{R}_i$ and σ_i^2 the expected return and variance of stock i, $i = 1, 2, \ldots, n$. Also, let σ_{ij} the covariance between stocks i and j. Let $x_1, x_2, \ldots, x_n$ the fractions of the investors wealth invested in each one of the n stocks ($\sum_{i=1}^n x_i = 1$). The resulting portfolio is $x_1R_1 + x_2R_2 + \cdots + x_nR_n$ and at time t it has return:

$$R_{pt} = x_1R_{1t} + x_2R_{2t} + \cdots + x_nR_{nt}$$

The expected return of this portfolio is given by:

$$\bar{R}_p = x_1\bar{R}_1 + x_2\bar{R}_2 + \cdots + x_n\bar{R}_n = \sum_{i=1}^n x_i\bar{R}_i = \mathbf{x}'\bar{\mathbf{R}}$$

where,

$$\mathbf{x}' = (x_1, x_2, \cdots, x_n), \text{ and } \bar{\mathbf{R}}' = (\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_n)$$

And its risk (variance) by:

$$\sigma_p^2 = var(x_1R_1 + x_2R_2 + \cdots + x_nR_n) = \sum_{i=1}^n x_i^2\sigma_i^2 + \sum_{i=1}^n\sum_{j\neq i}^n x_ix_j\sigma_{ij} = \sum_{i=1}^n\sum_{j=1}^n x_ix_j\sigma_{ij}$$

Or in matrix form:

$$\sigma_p^2 = \mathbf{x}'\Sigma\mathbf{x}$$

where, Σ is the symmetric, positive definite $n \times n$ variance covariance matrix of the returns of the n stocks as shown below:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24} & \cdots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34} & \cdots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \sigma_{n4} & \cdots & \sigma_n^2 \end{pmatrix}$$

There are n variances and $\frac{n(n-1)}{2}$ covariances.

Why does diversification work?

We will explain here very briefly (Elton et al, 2003) why investing in more than one securities reduces the risk. Suppose in a portfolio there are n securities. Then, the variance of the return on the portfolio (risk) is:

$$\sigma_p^2 = \sum_{i=1}^n x_i^2\sigma_i^2 + \sum_{i=1}^n\sum_{j\neq i}^n x_ix_j\sigma_{ij}$$

Let us consider equal allocation into the n securities. This means that $\frac{1}{n}$ of our wealth will be invested in each security. So, $x_i = \frac{1}{n}$ and the above expression becomes:

$$\sigma_p^2 = \sum_{i=1}^n (\frac{1}{n})^2\sigma_i^2 + \sum_{i=1}^n\sum_{j\neq i}^n (\frac{1}{n})(\frac{1}{n})\sigma_{ij}$$

We can factor out from the first summation $\frac{1}{n}$ and from the second summation $\frac{n-1}{n}$ to get

$$\sigma_p^2 = \frac{1}{n}\sum_{i=1}^n \frac{\sigma_i^2}{n} + \frac{n-1}{n}\sum_{i=1}^n\sum_{j\neq i}^n \frac{\sigma_{ij}}{n(n-1)}$$

and since there are all together $n(n-1)$ covariances we have:

$$\sigma_p^2 = \frac{1}{n}\bar{\sigma}_i^2 + \frac{n-1}{n}\bar{\sigma}_{ij}$$

where $\bar{\sigma}_i^2 = \sum_{i=1}^n \frac{\sigma_i^2}{n}$, and $\bar{\sigma}_{ij} = \sum_{i=1}^n\sum_{j\neq i}^n \frac{\sigma_{ij}}{n(n-1)}$. We see that when n is large the risk of the portfolio is approximately equal the average covariance. The individual risk of securities can be diversified away. Even though equal allocation is not the optimum solution the above can explain the reduction of risk by holding many securities.

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Combinations of two risky assets: Short sales not allowed

Define:

x_A is the fraction of available funds invested in asset A.
x_B is the fraction of available funds invested in asset B.
$\bar{R}_A$ is the expected return on the asset A.
$\bar{R}_B$ is the expected return on the asset B.
$\bar{R}_p$ is the expected return on the portfolio.
σ_A^2 is the variance of the return on asset A.
σ_B^2 is the variance of the return on asset B.
σ_{AB} $(cov(R_A, R_B))$ is the covariance between the returns on asset A and asset B.
ρ_{AB} is the correlation coefficient between the returns on asset A and asset B.
σ_p is the standard deviation of the return on the portfolio.

Correlation coefficient and the efficient frontier

The inputs of portfolio are:

- Expected return for each stock.
- Standard deviation of the return of each stock.
- Covariance between two stocks

The correlation coefficient (ρ) between stocks A, B is always between -1, 1 and it is equal to:

$$\rho = \frac{cov(R_A, R_B)}{\sigma_A \sigma_B} \Rightarrow cov(R_A, R_B) = \rho \sigma_A \sigma_B$$

Expected return of the portfolio:

$$E(X_A R_A + X_B R_B) = X_A \bar{R}_A + X_B \bar{R}_B$$

Variance of the portfolio:

$$var(X_A R_A + X_B R_B) = X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2 X_A X_B \rho \sigma_A \sigma_B$$

Or

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2 X_A (1 - X_A) \rho \sigma_A \sigma_B$$

The standard deviation of the portfolio:

$$\sigma_p = \left[X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2 X_A (1 - X_A) \rho \sigma_A \sigma_B\right]^{\frac{1}{2}}$$

Next pages we explore the shape of the efficient frontier for different values of the correlation coefficient.

What do you observe when $\rho = 1$, $\rho = -1$?

Suppose $\rho = +1$

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B \qquad (1)$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2 X_A (1 - X_A) \sigma_A \sigma_B$$

or

$$\sigma_p^2 = [X_A \sigma_A + (1 - X_A) \sigma_B]^2$$

$$\sigma_p = X_A \sigma_A + (1 - X_A) \sigma_B \Rightarrow X_A = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B} \qquad (2)$$

By combining (1) and (2) we get

$$\bar{R}_p = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B} \bar{R}_A + \frac{\sigma_A - \sigma_p}{\sigma_A - \sigma_B} \bar{R}_B$$

Finally:

$$\bar{R}_p = \frac{\sigma_A \bar{R}_B - \sigma_B \bar{R}_A}{\sigma_A - \sigma_B} + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B} \sigma_p$$

Conclusion: All the portfolios (combinations of stocks A and B) will be found on this line.

Suppose $\rho = -1$:

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B \qquad (3)$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 - 2 X_A (1 - X_A) \sigma_A \sigma_B$$

Therefore,

$$\sigma_p^2 = [X_A \sigma_A - (1 - X_A) \sigma_B]^2$$

$$\sigma_p = X_A \sigma_A - (1 - X_A) \sigma_B \Rightarrow X_A = \frac{\sigma_p + \sigma_B}{\sigma_A + \sigma_B} \qquad (4)$$

or

$$\sigma_p^2 = [-X_A \sigma_A + (1 - X_A) \sigma_B]^2$$

$$\sigma_p = -X_A \sigma_A + (1 - X_A) \sigma_B \Rightarrow X_A = \frac{\sigma_p - \sigma_B}{-\sigma_A - \sigma_B} \qquad (5)$$

By combining (3), (4), and (5) we get two equations:

$$\bar{R}_p = \frac{\sigma_B \bar{R}_A + \sigma_A \bar{R}_B}{\sigma_A + \sigma_B} + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A + \sigma_B} \sigma_p$$

and

$$\bar{R}_p = \frac{\sigma_B \bar{R}_A + \sigma_A \bar{R}_B}{\sigma_A + \sigma_B} + \frac{\bar{R}_B - \bar{R}_A}{\sigma_A + \sigma_B} \sigma_p$$

Conclusion: All the portfolios (combinations of stocks A and B) will be found on these two lines.