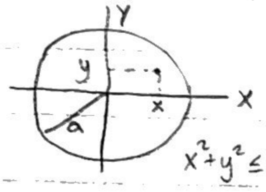
15 — 4.1, 4.2, 4.3 joint distr., indep., bivariate transformations

Example of change of variables: Hitting a target

Fire at a certain target of radius a, let (X,Y) be the coordinates of the hit. We want prob. of hitting within r of center.

Model: $f(x,y)$



→ hitting within r from center

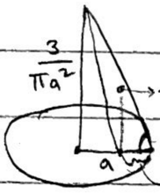
Modeling assumptions:

- Suppose more likely to hit near center than the edge.
- Suppose always hit target (i.e. never hit outside the circle)
- non uniform surface: cone ⇐ (for the joint density function)

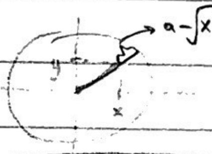
$\iint f(x,y)\,dx\,dy = 1$ ⇒ so select h ∋ volume $= 1$

$x^2+y^2 \le a^2$ Vol. $= \frac{\pi}{3} a^2 h = 1 \Rightarrow h = \frac{3}{\pi a^2}$

next we need $f(x,y)$:



→ consider the tangent of this angle: $\frac{3/\pi a^2}{a} = \frac{f(x,y)}{a-\sqrt{x^2+y^2}}$



$$\Rightarrow f(x,y) = \frac{3}{\pi a^3}\left(a-\sqrt{x^2+y^2}\right)$$

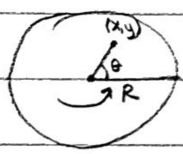
for $x^2+y^2 \le a^2$

⇒ X, Y dependent, (joint distr. doesn't factor)

common switch between $x,y) \Leftrightarrow (r,\theta)$

Change of variables to polar coordinates

Since the prob. we sought canbe characterized easier there, ...



let $R = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{Y}{X}\right)$

$$(x,y) \longrightarrow (r,\theta) = \left(\sqrt{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)$$

need inverse transformations: $x = r\cos\theta$, $y = r\sin\theta$

$$(r,\theta) \longrightarrow (x,y) = (r\cos\theta, r\sin\theta)$$

$$f_{R,\theta}(r,\theta) = f_{X,Y}(r\cos\theta, r\sin\theta)\,|J|$$

$$J = \det \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$$

$$f_{R,\theta}(r,\theta) = \frac{3}{\pi a^3}(a-r)r \qquad \begin{matrix} 0 \le r \le a \\ 0 \le \theta \le 2\pi \end{matrix}$$

R, θ indep. since the joint distr. and support factors completely between r and θ.

$$f_R(r) = \int_0^{2\pi} f_{R,\theta}(r,\theta)\,d\theta = \frac{6r}{a^3}(a-r) \qquad 0 \le r \le a.$$

$$F_R(r) = \frac{6}{a^3}\int_0^r (at - t^2)\,dt = \frac{6}{a^3}\left(a\frac{r^2}{2} - \frac{r^3}{3}\right) = \frac{6r^2}{a^3}\left(\frac{a}{2} - \frac{r}{3}\right)$$

$$= \frac{r^2}{a^2}\left(3 - \frac{2r}{a}\right) = \frac{3r^2}{a^2}\left(1 - \frac{2r}{3a}\right) = \frac{r^2(3a-2r)}{a^3} \qquad \text{for } 0 \le r \le a$$

$$F_R\left(\tfrac{1}{2}a\right) = \frac{a^2(3a-a)}{4a^3} = \boxed{\frac{1}{2}}$$

If (X,Y) were assumed to be uniformly distributed on $x^2+y^2 \le a^2$

$$f(x,y) = \frac{1}{\pi a^2} \qquad x^2+y^2 \le a^2$$

$h = \frac{1}{\pi a^2}$

not a cone but a cylinder.

volume: $\pi a^2 h = 1$

then $P\left(R \le \tfrac{1}{2}a\right) = \frac{\pi a^2/4}{\pi a^2} = \boxed{\frac{1}{4}}$