A Study of Basic Neutral and Hyperbolic Geometries

Introduction

When we think of Non-Euclidean Geometry, our immediate reaction is often one of panic and a sort of discomfort with the apparent vastness of those Euclidean techniques that are no longer at our disposal in our approach to hyperbolic geometric theorems. In fact, Non-Euclidean geometry is simply a theoretical manifestation of the concept of replacing one set of axiomatic principles with another that is, in many ways, incredibly similar to the former. In our approach to proving those conjectures that we will present in this study, for example, we will represent the subjects of our geometric manipulations on a single Euclidean "plane" that can be said to "lie on" the two-dimensional surface of this piece of paper. Additionally, for all intents and purposes, we will continue the practice of taking such terms as "line", "point", "congruent", "passes through", and "lies between" as undefined terms whose properties are assumed to be identical (or at least equivalent) to those we encounter in Euclidean geometries. Finally, we take the liberty of assuming the fundamentality of the first four of Euclid's famous axioms. In other words, the framework for a system henceforth known as "neutral geometry," a set of truths that apply to both Euclidean and non-Euclidean geometries, is based solely upon the following four axioms that ***always*** hold:

1. For every point P and for every point Q that is not equivalent to P, there exists a unique line *l* that passes through both P and Q.

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"Diagrams are NOT on this Copy."

2. For every segment AB and for every segment CD, there exists a point E such that point B lies between A and E and such that segment CD is congruent to segment BE.

3. For every point C and every point O that is not equivalent to C, there exists a circle with center O and radius OC.

4. All right angles are congruent to each other.

Relevant Definitions:

Segment: Given the existence of two distinct points A and B, segment AB is defined as points A and B and the set of all points that lie between points A and B on the unique line defined by points A and B.

Circle: Given the existence of two distinct points C and C', a circle with center

C and radius CC' is defined as the set of all points P such that segment CP is congruent to segment CC'.

Ray: Given the existence of a line defined by two distinct points A and B, the ray that is part of line AB and emanates from A is defined as the set of points lying on line AB that are part of segment AB and all points C such that B is between A and C.

Opposite Rays: Given the existence of ray AB and ray AC, those rays are said to be opposite if they both emanate from the same point A, if they are distinct (non-congruent), and if line AB is equivalent to line AC.

Angle: An angle with vertex V and sides VA and VB is defined as a point V with two non-opposite rays VA and VB that emanate from point V. If the rays *are* opposite, the angle is called a straight angle. If ray VA lies on VB (or vice versa) then the angle is called a zero angle.

Supplementary Angle: Given the existence of two angles WXY and YXZ, those angles are considered supplements of each other if they share a side XY and the other two sides WX and XZ form opposite rays.

Right Angle: An angle is considered a right angle if it has a supplement to which it is congruent.

The necessity for the last five definitions listed stems from our wish to define the term "right angle" such that we avoid the term "degrees", which allows us to state Euclid's fourth postulate such that our only loosely defined term is "congruent".

Neutral Geometry

This section will contain seemingly random theorems obtained by employing the techniques that come from our extensive knowledge of neutral geometry. These neutral geometric theorems have been selected so we may use them in our later attempts to prove hyperbolic geometric theorems. In most cases I will not provide a proof, as neutral geometry is not the focus of this study. I assure you, however, that any theorem I present in this section is a proven result and holds true for both Euclidean and non-Euclidean geometries.

Archimedes' Axiom: For a line *l* and a segment PQ perpendicular to *l* such that point Q lies on *l*, there exists a point R on line *l* such that we may construct segment PR and observe that (angle) QRP° < a° where a is any real number.

Crossbar Theorem: If WZ is between WX and WY, then WZ intersects segment XY.

Theorem: If a triangle exists whose angle sum is 180°, then a rectangle exists. If a rectangle exists, then every triangle has angle sum equal to 180°.

The Parallel Postulate

Here it becomes appropriate to include a discussion of Euclid's infamous fifth postulate, whose unique qualities set it aside from the other four pillars of Euclidean geometry. Often referred to as the "Parallel Postulate", Euclid's fifth axiom is as follows:

> **Parallel Postulate:** For every line l and for every point P that does not lie on l, there exists a unique line m through P that is parallel to l.

It is not immediately clear why such a simple observation would elicit such a highly heated response within the mathematical community and would ultimately give rise to entirely new branches of mathematics. From the time that Euclid published his *Elements*, the five postulates were taken to be true in all cases, and geometry was thought of as the mathematics of the behavior of points, lines, angles, and polygons on a flat and limitless Euclidean plane with the same mathematical properties as, say, an infinite sheet of paper lying on an infinite desk. This is because the Euclidean plane, as I described it, is the most intuitive kind of two-dimensional space, and the most applicable to our lives.

Mathematicians, however, soon became unsatisfied, as they frequently are, with their perception of geometry. They began to question the inherent validity of the parallel postulate and attempted to prove it using Euclid's other postulates. In other words, it became evident that uniqueness of parallels is a property that exists only on that familiar, flat and infinite plane with the properties of a boundless sheet of paper. There is no reason, though, that we shouldn't consider other types of planes in our analysis of geometry. If, for example, we take a *new* plane on which a "straight" line would in fact have properties of curvature that cannot apply to our Euclidean notion of a "straight" line, the parallel postulate may no longer hold.

Developing Hyperbolic Geometry

Hyperbolic geometry will henceforth be the primary focus of our study. Hyperbolic geometry is a result of our rejection of the parallel postulate. We will replace our Euclidean plane, therefore, with the same two-dimensional hyperbolic space we discussed earlier, in which we claim that the following statement is axiomatic:

> **Hyperbolic Axiom:** In hyperbolic geometry there exist a line *l* and a point P not on *l* such that at least two distinct lines parallel to *l* pass through P.

In itself, the Hyperbolic Axiom is quite unsatisfying and tragically pathetic. All it tells of two-dimensional hyperbolic space is that in *one* instance-- that is, for *one* line *l* and *one* point P not on that line-- there are at least two distinct lines parallel to *l* that pass through P. It could be that the remainder of hyperbolic space replicates those properties of plane geometry that we hold near and dear. The more impressive feat, therefore, would be to develop a stronger hypothesis. What if we could make a statement about hyperbolic geometry powerful enough to match the strength of Euclid's fifth postulate? Then, couldn't we develop hyperbolic geometry to be just as broad as Euclidean geometry and equally valid? Yes! So our current goal is to prove what we call the Universal Hyperbolic Theorem (UHT). Remember that our diagrams are simply visual representations of our dealings in hyperbolic space reflected on a Euclidean plane. DO NOT TRUST YOUR EYES!

Universal Hyperbolic Theorem: In hyperbolic geometry, for *every* line *l* and *every* point P that does not lie on *l* there pass through P at least two distinct lines that are parallel to *l*.

Proof of UHT: Let *l* be a line and P a point not on *l*. Drop perpendicular PQ to *l* such that point Q lies on *l*. Construct line *m* through P so that *m* is perpendicular to PQ. Thus, $m \| l$.

Let R be some other point on *l*. Construct line *u* through point R such that *u* (perpendicular) *l*. Drop perpendicular PS to *u* such that point S lies on *u*. We know for a fact, now, that PS is parallel to *l* because both *l* and PS are perpendicular to *u*. Thus, we know that *m* and PS are both parallels to *l* that pass through P.

Now the trick is to prove that *m* and PS are *distinct* lines. In order to do so, let us observe the case where PS and *m* are the same line. In that instance, (Quadrilateral) PQRS must be regarded as a rectangle. We proved earlier that if a rectangle exists, then all triangles must have angle sum 180°. In Euclidean geometry, it may seem preposterous to suggest that there may be a triangle with angle sum less than 180°, but in hyperbolic geometry we have not yet studied the properties of triangles. Therefore, if we want to prove the UHT, then it is necessary to prove the following lemma:

Lemma: There exists, in hyperbolic geometry, a triangle whose angle sum is less than 180°.

Proof of Lemma: Consider the case referenced in the hyperbolic axiom. Let *l* be a line and P a point not on *l* such that there are two distinct parallel lines to *l* that pass through P, called *n* and *m*. Let *m* be the parallel constructed by dropping perpendicular PQ to *l* and and constructing line *m* perpendicular to PQ. Furthermore, let PX be a ray of *n* lying between PQ and a ray PY of *m*.

Using the principle we discussed earlier (an application of Archimedes' axiom), we can find some point R on line *l* such that (angle) QRP is any value we desire. In this case, let us select R such that (angle) QRP < (angle) XPY.

We can rest assured that, no matter where on *l* we choose to place R, PR will always lie in the interior of (angle) QPX, because if this was not true, then PX would have to lie in the interior of (angle) QPR. The Crossbar Theorem, however, states that *n* would then have to intersect *l*, which is impossible because *l*||*n*.

By that logic, (angle) RPQ° < (angle) XPQ°. Adding our inequalities yields (angle) QRP° + (angle) RPQ° < 90°. Therefore, the angle sum of (triangle) QRP *must* be less than 180°. This implies that our lemma is true, and, therefore, that we have proven the Universal Hyperbolic Theorem.

For completeness, we should include a brief discussion of the implications of our lemma. It is now clear that all triangles in hyperbolic space have angle sum less than 180°. However, it is possible that not all triangles have the same angle sum. We, therefore, refer to the positive number obtained by subtracting the angle sum of a triangle from 180° as the defect of that triangle. We should note that our lemma also implies that, in hyperbolic geometry, all convex quadrilaterals have angle sum less than 360°. This is because if we take (quadrilateral) ABCD and draw in segment AC, we can simply add the angle sums of the two resulting triangles and observe that the angle sum of (quadrilateral) ABCD is less than 360° as is shown below

Representing the Hyperbolic Plane

There is no easy way to visualize the properties of a two-dimensional hyperbolic plane without synthesizing a method by which we might understand what it means to have some geometric entity on that plane. As a result, there are two dominant models to aid us in our attempts to fathom such occurrences.

The Beltrami-Klein model is perhaps the most basic representation of hyperbolic space, where all the points in the interior of a set circle *C* represent all points in the hyperbolic plane. In the Beltrami-Klein model, we replace lines with what we call open chords. An open chord is essentially a segment joining two points on *C*, excluding its endpoints. Open chords, in this model, represent the lines in the hyperbolic plane. Because we define two lines to be parallel if they have no points in common, we can easily create a visual representation of a general case of the Universal Hyperbolic Theorem as is shown below.

As it turn out, this model has one immediately noticeable flaw: The way the angles are represented is inaccurate. We know that, in hyperbolic space, lines certainly are not straight.

How do we resolve this issue, then? The Poincaré disk is another model for a hyperbolic plane. Similarly to the Beltrami-Klein model, the Poincaré disk uses a circle *C*, the interior of which represents all points on the hyperbolic plane. This time, however, only lines that pass through the center of the circle O are represented by open chords in *C*. All other lines are represented by open arcs of circles that are orthogonal to *C* as is shown below.

In this model, we use the Euclidean sense of the terms "lie on" and "between". Furthermore, the definition of parallel remains the same when we are working within the Poincaré disk. Two Poincaré lines, then, are parallel if they have no points in common. Additionally, the Poincaré model is conformal, meaning that it accurately represents angles. How, you may ask, do we measure angles between the curved Poincaré lines? The answer is simple: The number of degrees in the angle that two Poincaré lines make is the same as the number of degrees in the angle formed by the tangents to each arc at their common point, as is shown below.

Briefly consider two parallel Poincaré lines *l* and *m* with a common perpendicular PQ such that P lies on *m* and Q lies on *l* where Q is the center of our Poincaré disk. Now construct another perpendicular *r* to *l*. We can now justifiably consider this a graphical representation of a quadrilateral (technically a Lambert quadrilateral) in hyperbolic space. We can actually see that the angle created by lines *r* and *m* is acute, so this is one such example of our corollary that all quadrilaterals in hyperbolic space must have less than 360°.

Obviously, there are multiple legitimate representations of hyperbolic space and, as it turns out, these models are all isomorphic. The problem, however, no longer concerns how we represent hyperbolic space. The more pertinent question concerns the consistency of hyperbolic geometry-- after all, if hyperbolic geometry is inconsistent, why spend any more time on its development? From the numerous models of hyperbolic space and the rigorous metamathematical analysis of the relationship between Euclidean and hyperbolic geometries, a groundbreaking theorem emerged:

Metamathematical Theorem 1: If Euclidean geometry is consistent, then so is hyperbolic geometry.

What follows from such a hypothesis is a series of questions: How can we know that Euclidean geometry is consistent? Does this mean that we should be able to prove the parallel postulate when we work on a Euclidean plane and contradictory, non-Euclidean axioms when we work in other types of space? To answer those questions, let's examine what happens when we assume that a proof of the parallel postulate exists.

Hyperbolic geometry, in this instance, would be inconsistent because it would contradict a proven argument (namely, the parallel postulate). Metamathematical Theorem 1, however, states that hyperbolic geometry must be consistent when Euclidean geometry is consistent. Because of that contradiction, we can safely argue that our original assumption was false and that there exists no proof of the parallel postulate. Furthermore, because we take Euclidean geometry to be consistent, we are also assured that there is no disproof of the parallel postulate! Therefore, we are assured that both Euclidean and hyperbolic geometries are consistent. And so we continue on…

Properties of Triangles

The next natural step for us to take in this study is to examine the properties of triangles in hyperbolic space. For instance, how do we determine whether two triangles are congruent or similar? Observing a triangle in the Poincaré disk model of hyperbolic space, it would appear that increasing the length of the sides of a triangle would actually result in a *decrease* in the angle sum of that triangle. Equivalently, such an action would cause an *increase* in the triangle's defect. It seems, therefore, that it is nonsensical to refer to triangles that are similar, but not congruent in hyperbolic geometry. In order to assure ourselves of this phenomenon, let us propose a theorem:

Theorem: In hyperbolic geometry if two triangles are similar, they are also congruent.

Proof: Let us assume that there exist triangles ABC and A'B'C' which are similar but not congruent. No corresponding side of these triangles could possibly be congruent, because then we would meet ASA, a criterion for congruent triangles.

Conclusion

If we were looking to develop our understanding of hyperbolic geometry even further, we would proceed in much the same way we would for any other field of mathematics-- one theorem at a time. As it turns out, hyperbolic geometry is just as broad as Euclidean geometry and equally valid. I sincerely wish that I had more time to improve upon the knowledge I gained in writing this study, and I aim to do just that in the future. Meanwhile, I hope you enjoyed my presentation of basic neutral and hyperbolic geometry.