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Problem 1

$$f(x) = \begin{cases} p^x(1-p)^{1-x} & , x=0,1 \\ 0 & , \text{elsewhere} \end{cases}$$

a) $L(p;x) = \prod_i f(x_i) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$

1	6/10
2	5/10
3	3/10
4	2/10
	16/40

$$\log L(p;x) = \sum_{i=1}^{n} \log\left[p^{x_i}(1-p)^{1-x_i}\right]$$

$$= \sum_{i=1}^{n}\left[x_i \log p + (1-x_i)\log(1-p)\right]$$

$$\frac{\partial \log L}{\partial p} = \sum_{i=1}^{n}\left[\frac{x_i}{p} - \frac{(1-x_i)}{1-p}\right]$$

$$= \sum_{i=1}^{n}\left[\frac{x_i(1-p) - p(1-x_i)}{p(1-p)}\right] = \sum_{i=1}^{n}\left[\frac{x_i - p}{p(1-p)}\right]$$

$$= \left(\frac{1}{p(1-p)}\right)\sum_i x_i - \frac{n}{1-p}$$

set equal to 0

$$\hat{p}_{MLE} = \frac{1}{n}\sum_i x_i = \overline{X}_n \quad ✓$$

The MLE is the sample mean, $\overline{X}_n$.

A sum of Bernoullis has a bin(n,p) distribution.

mean np, variance $np(1-p)$

$$\overline{X}_n \xrightarrow{d} N(np, p(1-p))$$

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convergence statements including (n)

b) We want to show $Y_n \xrightarrow{P} p$ for consistency, so we need $\lim_{n\to\infty} P(|Y_n - p| \geq \epsilon) = 0$ $\forall \epsilon > 0$ ✓

$= 1 - P(Y_n - p$

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c) For asymptotic efficiency we would need to show that as $n \to \infty$, the estimator attains the Rao - Cramér lower bound for variance. Given that Y_n is a statistic, we can find the lower bound by

$$Var(Y_n) = \frac{1}{n I(p)}$$

Where we find $I(p)$ by $-E\left[\frac{\partial^2 \log f(p;x)}{\partial p^2}\right]$ or

but it's the harder way to go

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Problem 2

$X_i \sim N(\mu, \sigma^2)$ $\qquad S_n^2 = \frac{1}{n}\sum_i (X_i - \bar{X}_n)^2$ $\qquad \bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

We know by WLLN that $\bar{X}_n \xrightarrow{P} \mu \neq 0$ ✓

So we should be able to state that $\frac{1}{\bar{X}_n} \xrightarrow{P} \frac{1}{\mu}$ ✓

We also know that S_n^2 is a biased estimator of population variance, but that asymptotically the bias disappears. So

$S_n^2 \xrightarrow{P} \sigma^2$, and therefore $S_n^2 \xrightarrow{d} \sigma^2$ ✓

What if we use delta method for

$var((x_i - \mu)^2) = d$

$\sqrt{n}(s^2 - \sigma^2) \xrightarrow{d} N(0, 1)$ ✗

so

$\sqrt{n}(s - \sigma) \xrightarrow{d} N(0, \frac{1}{4})$ ✓

ok $\rightarrow$ simply combine

$$\begin{cases} \bar{X}_n \xrightarrow{P} \mu \\ \sqrt{n}(s - \sigma) \xrightarrow{d} N(0; d/4) \end{cases}$$

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$$S_n^2 = \frac{1}{n}\sum_i (X_i - \overline{X}_n)^2 = \frac{1}{n}\left[\sum_i X_i^2 - \sum_i 2X_i\overline{X}_n + n\overline{X}_n^2\right]$$

$$= \frac{1}{n}\sum_i X_i^2 - 2\overline{X}^2 + \overline{X}_n^2$$

$$= \frac{1}{n}\sum_i X_i^2 - \overline{X}_n^2$$

S_n^2

Problem 3

$Y_i \sim \text{Bernoulli}(p_i)$

a) $E(Y_i)$

So joint distribution is

$f_{Y_n}(y) = \prod_{i=1}^{n} p_i^y (1-p_i)^{1-y}$

b) The score function is what we get when we take the derivative of the log of the density function. I suppose we're meant to use what we derived in problem 1?

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Problem 4

We want to show that changing $Y_n \xrightarrow{P} c$ to $Y_n \xrightarrow{P} Y$ does not strengthen Slutsky's Theorem

If instead of letting Y_n converge to a constant, c, we let it converge to a normally distributed r.v., then we have

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