Project #1 MATLAB Simulator

Dhruvit Manojkumar Mavani | M70570748

Introduction

The objective of the project 1 is to simulate three Dynamic systems namely **DS1**, **DS2** and **DS3** using disparate numerical integration method such as Forward Euler Integration, Adams-Basworth 2 integration, and Range-Kutta 4; and explore the different aspects of numerical integration method such as effect of reducing time step on the characteristic root error and, accuracy of the dynamic system simulation. Furthermore, how to identify or decide which integration method and time step is optimum for any particular simulation is also one of the objective of this project.

Forward Euler Integration

Initially, we will discuss the three different numerical integration method used here, out of which first is; Forward euler integration, which is a first-order numerical integration method, and is used to solve the differential equation of any dynamic system. The difference equation of Euler integration method is

$x_{n+1} = x_n + hf(n)$ .... (4)

The approximate characteristic root error formula for euler integration is

$e_\lambda \approx -\frac{1}{2}(\lambda h), \quad |\lambda h| \ll 1$ .... (5)

Adams Basforth (AB-2) Integration method

Adams Basforth (AB-2) is a second-order numerical integration method used to solve the differential equation of any dynamic system by evaluating two functions, f_{n-1} and f_n. The difference equation of AB-2 integration method is

$x_{n+1} = x_n + \frac{1}{2}h(3f_n - f_{n-1})$ .... (6)

The approximate characteristic root error formula for AB-2 integration method is

$e_\lambda \approx -\frac{5}{12}(\lambda h)^2, \quad |\lambda h| \ll 1$ .... (7)

However, the probelm with AB-2 is that its difference equation includes the term f_{n-1} which is undetermined at time $t = 0$ and $t = h$. so , in this case euler is generally used for the initial time until $t = 2h$.

RK-4 Integration method

RK-4 integration method is also known as Range-Kutta fourth order integration method and it evaluates the four functions f_n, $\hat{f}_{n+\frac{1}{2}}$, $\hat{\hat{f}}_{n+\frac{1}{2}}$, and $\hat{f}_{n+1}$. The differece equation of AB-2 integration method is

$$x_{n+1} = x_n + \frac{h}{6}\left(f_n + 2\hat{f}_{n+\frac{1}{2}} + 2\hat{\hat{f}}_{n+\frac{1}{2}} + \hat{f}_{n+1}\right) \qquad .... (8)$$

The approximate characteristic root error formula for RK-4 integration method is

$$e_\lambda \approx -\frac{6}{720}(\lambda h)^4, \qquad |\lambda h| \ll 1 \qquad .... (9)$$

As we had discussed different numerical integration method, now its time to describe our three dynamic systems DS1, DS2, and DS3 and simulate it.

Dynamic system 1

Dynamic system 1 is a linear time-invariant system having differential equation

$$\dot{\boldsymbol{x}} + 4\boldsymbol{x} = \boldsymbol{u}, \quad x(0) = 2 \qquad .... (10)$$

$\dot{\underline{x}} = \underline{f}\,[-4x + u]$ State Equation

$\underline{y} = \underline{g}\,[x]$ Output Equation

The DS-1 is simulated using **Forward Euler Integration** method at time step, $h = 0.01\,s$.

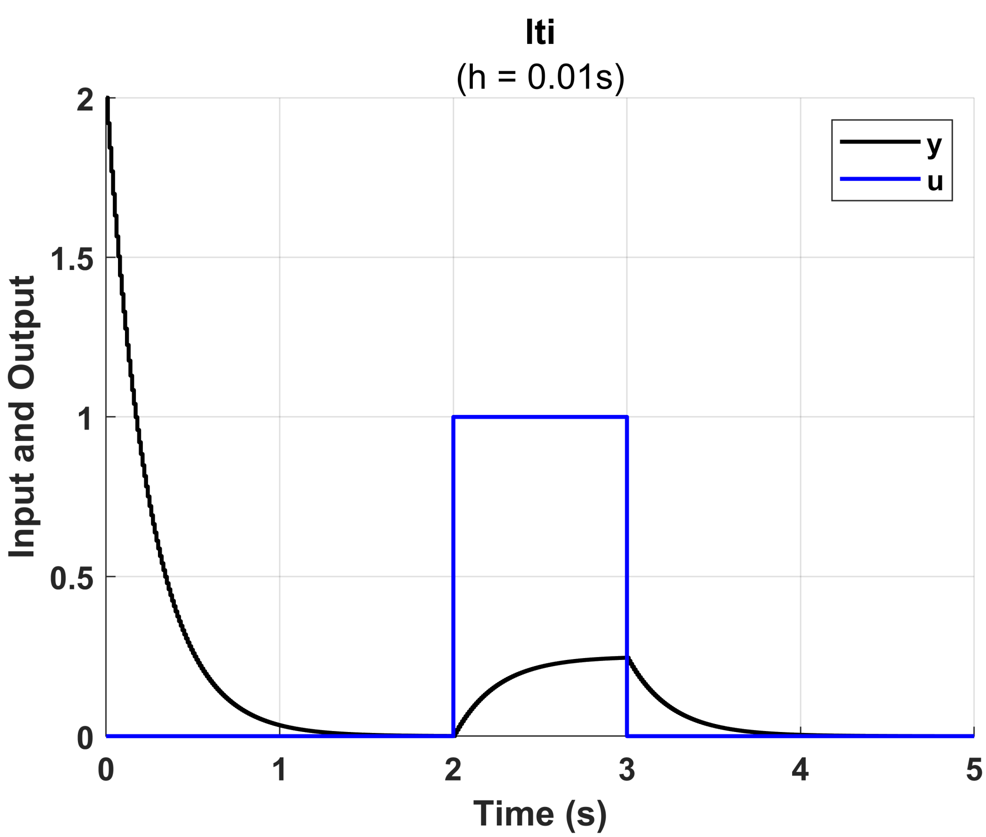

Fig.1 LTI system simulation result (Euler)

Dynamic system 2

Dynamic system 2 is a linear time-varying system with differential equation

$$m\ddot{w} + c\dot{w} + k(t)w = u, \quad w(0) = \dot{w}(0) = 0 \qquad .... (11)$$

where, $m = 1\,\mathrm{kg}$, $c = 0.5$ N/(m/s), and $k(t) = 30 + 3sin(2\pi t)$ N/m, $u(t) = 100\sin(8\pi t)$

$$\dot{\underline{x}} = \underline{f}\begin{bmatrix} x_2 \\ -\frac{k(t)}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u \end{bmatrix} \quad \text{State equation}$$

$$\underline{y} = \underline{g}\,[x_1] \quad \text{Output equation}$$

Linear time-varying system is simulated using **AB-2 Integration** method at time step, $h = 0.01\,s$

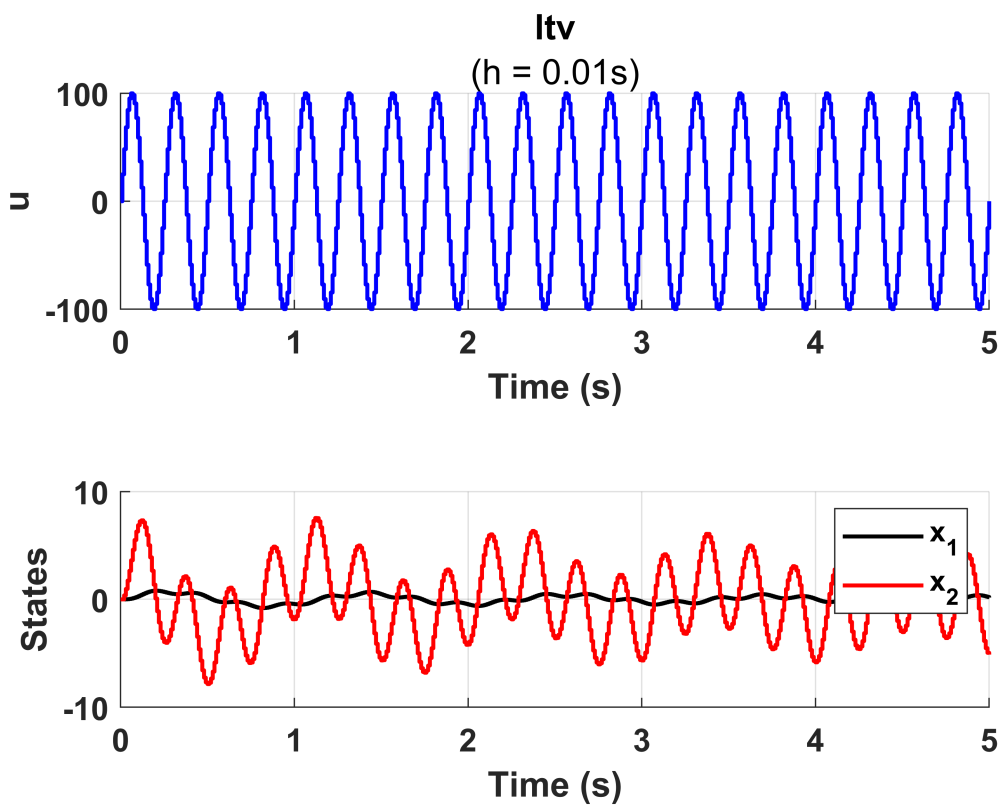

Fig.2 LTV system simulation result (AB-2)

Dynamic system 3

Dynamic system 3 is a nonlinear time-invariant Vanderpol differential equation

$$\ddot{w} - \mu(1 - w^2)\dot{w} + w = u, \quad w(0) = \dot{w}(0) = 0 \qquad .... (12)$$

where, $\mu = 8.53$ and $u(t) = 1.2sin(0.2\pi t)$

$$\dot{\underline{x}} = \underline{f}\begin{bmatrix} x_2 \\ -x_1 + \mu(1 - x_1^2)x_2 + u \end{bmatrix} \quad \text{State Equation}$$

$$\underline{y} = \underline{g}\,[x_1] \quad \text{Output Equation}$$

The dynamic system 3 (VDP) is simulated using **RK-4 (Range-kutta) Integration** method at time step, $h = 0.01s$.

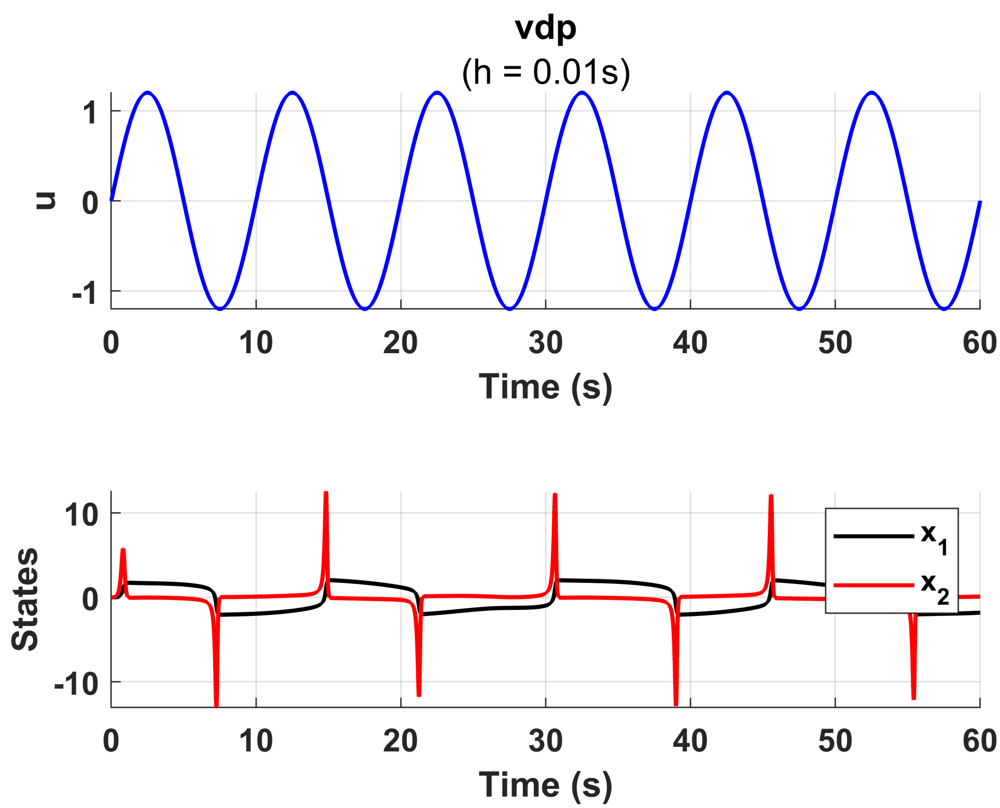

Fig.3 VDP simulation results (RK-4)

Simulation Verification

As we have simulated our systems DS-1, DS-2 & DS-3 using different integration methods, we still don't know if our simulation is wrong or right. So,now we will verify our simulation by looking at the errors associated with Euler,AB-2 and RK-4 and see how skewed our simulation is to the exact solution. Here, for Simulation verification the time step is set to $h = 0.01s$. Now, we know that for $|\lambda h| \ll 1$, **euler error** $e_\lambda = -\frac{1}{2}\lambda h$; error for **AB-2 integration** $e_\lambda = -\frac{5}{12}\lambda h^2$; and error induced due to **RK-4 integration** method $e_\lambda = -\frac{6}{720}\lambda h^4$. From Fig.4 we can see that between $t = 0.02$ & $t = 0.03$ the magnitude of Euler error is 0.00303, magnitude of AB-2 error is 0.00143 and magnitude of RK-4 error is $3.25776e^{-9}$.

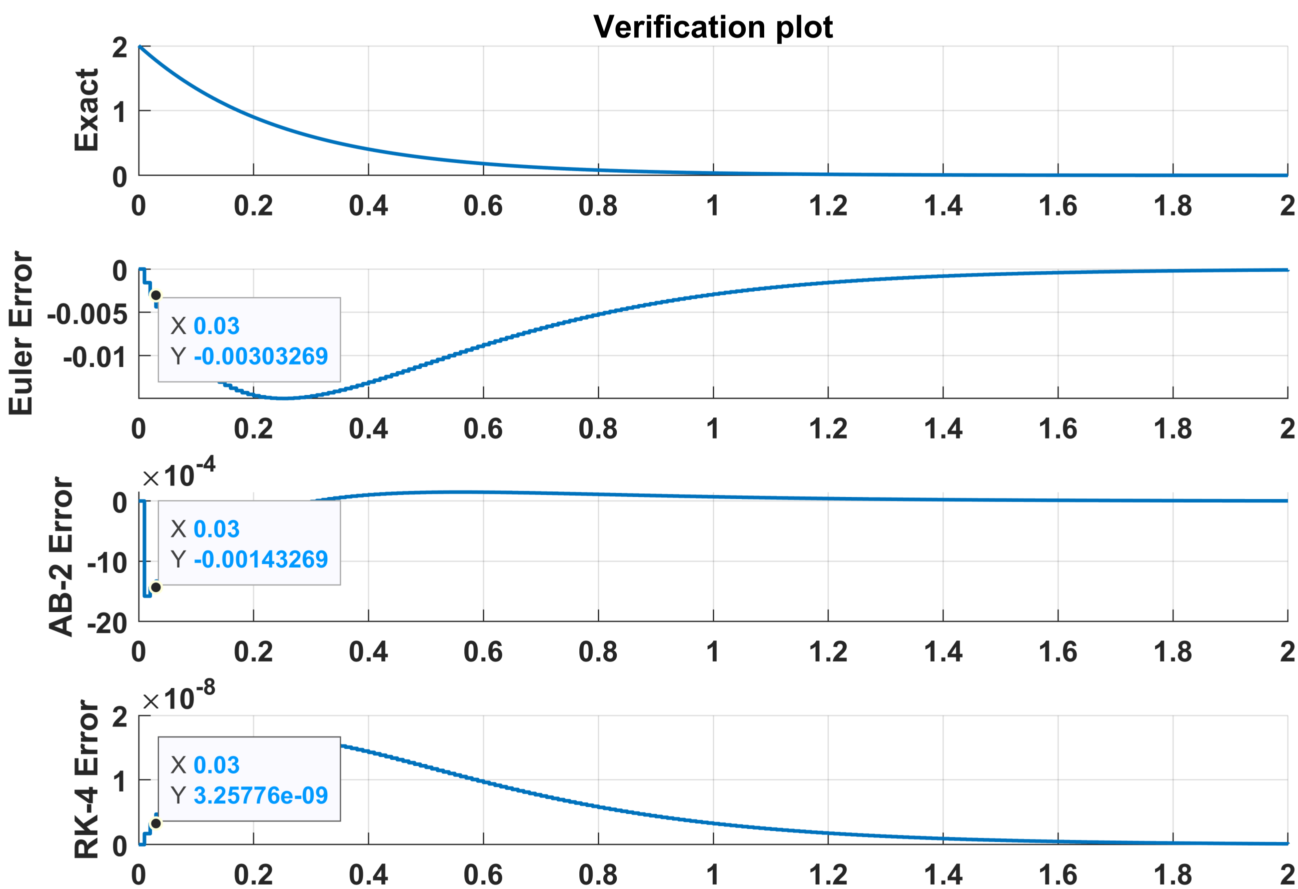

Fig.4 Verification plot

Now we will verify the euler error induced in simulation. Firstly we will derive the characteristic root λ of Dynamic System 1 from the transfer function $X_s = \frac{2}{s+4}$.

So, from transfer function we get characteristic root as **lam**,

```
Characteritic root = -4.000000
```

Now, we will calculate e_λ for euler using the formula mentioned above.

```
e_lambda (Euler) = 0.020000
```

Therefore, we have our $e_\lambda = 0.02$ and using that we will calculate $\hat{\lambda}$.

```
lambda_hat (Euler) = -4.080000
```

```
eulerEr = 0.0030
```

we can see that calculated error between $t = 0.02s$ and $t = 0.03s$ and the error we obtained in our graph are identical (eulerEr = 0.0030 and graph error = 0.00303). Hence, we can say that our simulation is good. However, we are still left with verfication of AB-2 and RK-4.

For AB-2, first we will calculate e_λ using the formula mentioned above,

```
e_lambda (AB-2) = -0.000667

Lambda_hat (AB-2) = -3.997333

AB-2 error = -0.001419
```

Therefore, for AB-2 we can see that error we calculated between $t = 0.02s$ and $t = 0.03s$ is $-1.419e^{-3}$ which is approximately equal to the error mentioned on the graph $(-1.432e^{-3})$. Hence, we can say that the simulation is good to go. Similarly for RK-4 integration method we get calculated error as

```
Lambda_hat (RK-4) = -4.000000

rk4_Error = 4.5410e-09
```

$4.5410e^{-9}$ which is not same as shown in verification plot above that is $3.25776e^{-9}$ but it tends to zero so we can say that this simulation is considerable.

Vanderpol Dynamic system comparison

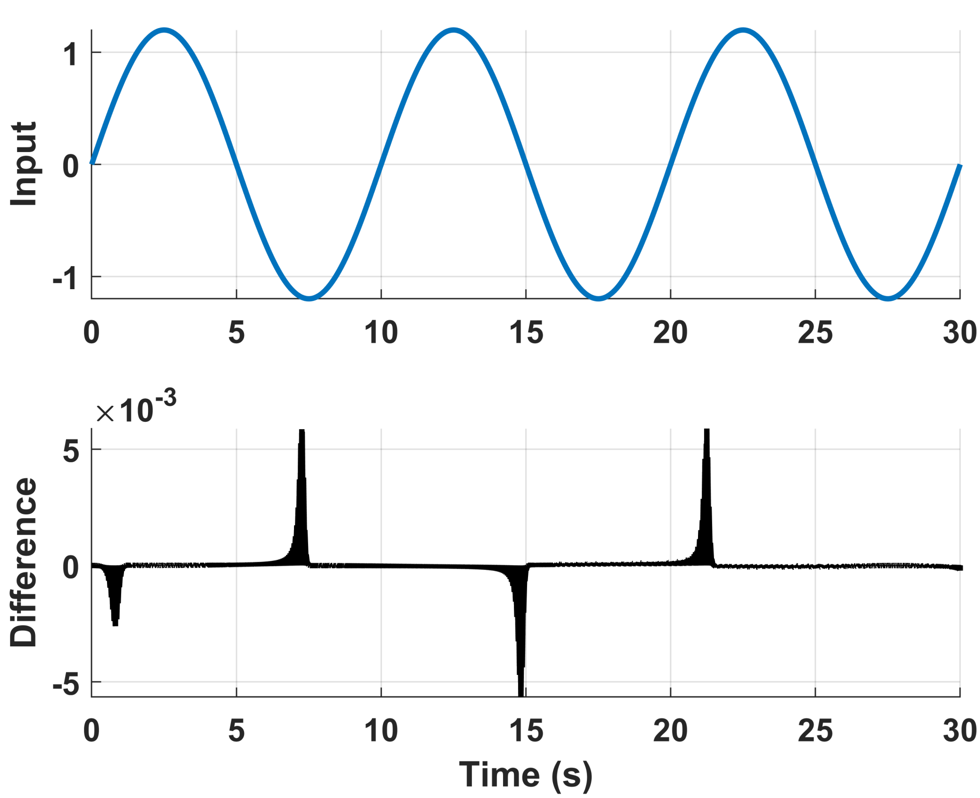

Fig.5 hGoodRK analysis/comparison

In VDP comparison the time step **hGoodRk** is determined such that difference between **hGoodRK** and **hGoodRK/10** is less than 0.01 and the hGoodRK I choose is 0.0005, as it gives me the difference of around 0.0058. Similarly, I found hGoodEuler = 0.000001. Now, we will see the difference betwee hGoodRK and hGoodEuler.

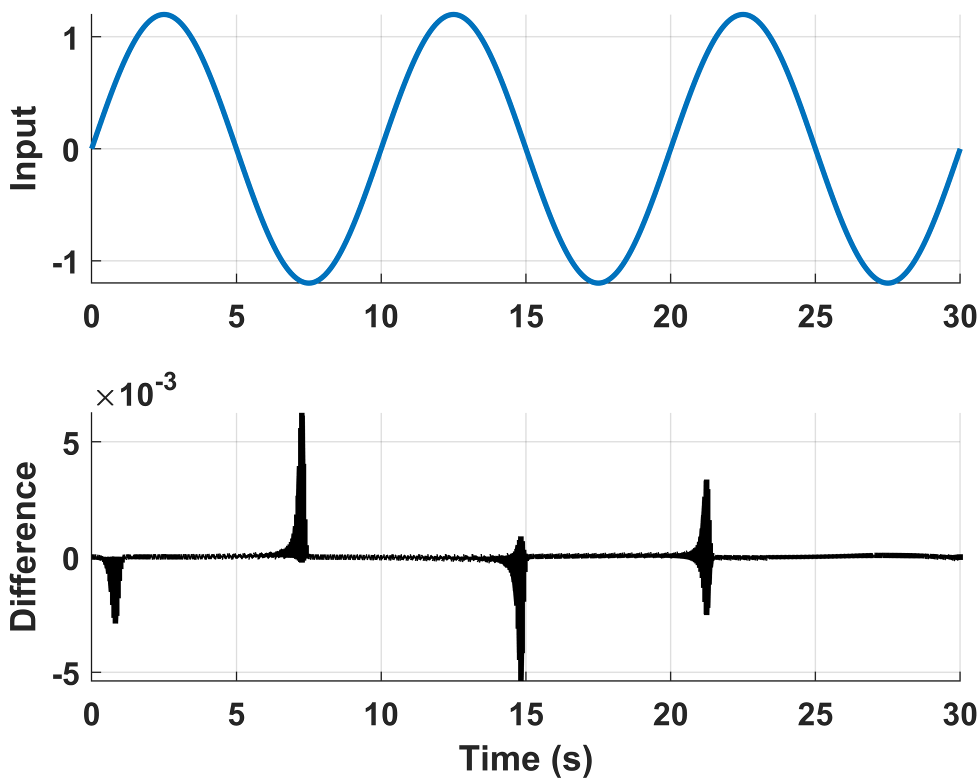

Fig.6 hGoodRK and hGoodEuler comaprison

The difference between hGoodRK and hGoodEuler is somewhat similar to the comparison of hGoodRK and is less than 0.01. Therefore, we can use RK-4 integration method instead of euler for DS-3 (Vanderpol) as it has upper hand when it comes to the error accuracy or small error Vs step time.

Conclusion

Selecting an integraton method and timestep is an important part of conducting the simulation of any dynamic system. The level of accuracy one wants, or the error magnitude one desire for in their simulation, all these factors plays a role while selecting an integration method and time step for it. The analysis executed in this project showed that, selecting higher order integration method such as RK-4 can make us select bit larger time step with minimal error magnitude, whereas, selecting lower order integration method such as Euler, can produce similar error magnitude but with quite smaller time step, wich might be stressful for your computer, or computation time can be get higher. Therefore, choosing optimal Integration method with appropriate time step is a key to the successful simulation.