Dynamic Accuracy of State transition method

State Transition step 1 and step 2

Step 1

First we will start with Eq. 4 which is the first order linear system.

$$\dot{X} = \lambda X + f(t) \quad \text{.... (1)}$$

Now, for initial condition let $f(t) = 0$ at $t = 0$, then the Eq.4 can be written as,

$$\dot{X} = \lambda X \quad \text{.... (2)}$$

Further, from lecture notes we know that the closed form solution of (2) is,

$$X_t = X_0 e^{\lambda t} \quad \text{.... (3)}$$

let $t_0 = nh$ and $t = (n+1)h$ and, replace X_t with X_{n+1} and X_0 with X_n. Also, replace $t = h$.

$$\boldsymbol{X_{n+1} = X_n e^{\lambda h}} \quad \text{.... (4)}$$

Now, we will dive into the Eq.2 of the paper first and then will start deriving Eq.6. So, Eq.2 according to the paper is

$$X(t) = W\left(t - t_0\right)X(t_0) + \int_{t_0}^{t} W(t-\tau)\ BF(\tau)\, d\tau \quad \text{.... (5)}$$

We know that (5) can be written in the form of,

$$X_{n+1} = W(h)X_n + \int_0^h W(h-\tau)\ BF(\tau + nh)\, d\tau \quad \text{.... (6)}$$

let, $W(t) = e^{\lambda t}$, and replacing $t = h$, we get the value of $W(h)$ as,

$$W(h) = e^{\lambda h} \quad \text{.... (7)}$$

Now, replacing the value of $W(h)$ from (6) to the value we got in (7), we get

$$X_{n+1} = X_n e^{\lambda h} + \int_0^h W(h-\tau)\, BF(\tau + nh)\, d\tau \quad \text{.... (8)}$$

Let,

$B = 1$, $F = f$

$\alpha = \tau + nh$

$d\alpha = d\tau$

Replace the values of $(\tau + nh)$ and B in (8)

$$X_{n+1} = X_n e^{\lambda h} + \int_0^h W(h - (\alpha - nh))\, f(\alpha)\, d\alpha$$

$$X_{n+1} = X_n e^{\lambda h} + \int_0^h W((n+1)h - \alpha)\, f(\alpha)\, d\alpha \qquad \text{.... (9)}$$

Now solving (9) for $f = 0$ we get,

$$\boldsymbol{X_{n+1} = X_n e^{\lambda h}} \qquad \text{.... (10)}$$

So, now we can say that the **Eq.6 of the paper is the exact solution of Eq.4** at initial condition $f(t) = 0$.

Step 2

So, as we devised Eq.6 to be the exact solution of Eq.4, now its time to see if the error formulas stands correct for Euler Integration of Eq.4. We know Eq.4

$$\dot{X} = \lambda X + f(t) \qquad \text{.... (11)}$$

let $f(t) = 0$ be the initial condition, then the Eq.4 can be written as,

$$\dot{X} = \lambda X \qquad \text{.... (12)}$$

Now lets write the euler integration difference equation for (12)

$$X_{n+1} = X_n + hf_n$$

$$X_{n+1} = X_n + \lambda h X_n$$

$$X_{n+1} = (1 + \lambda h)X_n \qquad \text{.... (13)}$$

Now, we will use **Approximate error formula for Forward Euler Integration** to calculate approx characteristic root error of (13)

$$e_\lambda \approx -\frac{1}{2}\lambda h \;, \quad |\lambda h| < 0.1$$

let, $\lambda = -3$ and $h = 0.01$,

$$e_\lambda \approx -\frac{1}{2}(-3)(0.01)$$

Therefore, $e_\lambda \approx 0.015$ or 1.5%

Now, we will compare the approximate error to the **exact error formula** and see if the error % comes out to be the same or not.

So, exact error formula for euler integration is

$$e_\lambda = \frac{\ln(1+\lambda h) - \lambda h}{\lambda h} \qquad |\lambda h| \ll 1$$

$$e_\lambda = \frac{\ln(1+(-3)(0.01)) - (-3)(0.01)}{(-3)(0.01)}$$

$$e_\lambda = 0.0153$$

Therefore, using exact error formula the $e_\lambda = 1.53\%$. So, we can say that both the error formulas are correct as errors associated with them are nearly identical.